Projection and Minimum Residual Methods for MDPs

Reference: K. Judd (1998)

Numerical Methods in Economics

1. Residual Operators For an MDP problem, define a *residual operator* $R : B \to B$ by:

$$R(V)(s) = V(s) - \Gamma(V)(s) \tag{1}$$

Note that the value function V is the unique zero of $R(V)$, i.e.

$$R(V)(s) = 0 \quad \text{if} V = \Gamma(V) \tag{2}$$

2. Parametric Approximations Let V_θ denote a *parametric approximation* to $V \in B$, i.e. a mapping from $\Theta \subset R^k$ into B. A classic example is a *polynomial series approximation*

$$V_\theta(s) = \sum_{i=1}^{k} \theta_i p_i(s) \tag{3}$$

We consider a sequence of parametric approximations that is *ultimately dense* in B, i.e. given any $V \in B$ we have:

$$\underset{k\to\infty}{lim} \; \underset{\theta \in R^k}{inf} \; \|V_\theta - V\| = 0 \tag{4}$$

Comment: Parametric approximations need not be "linear in parameters" such as the series approximation in (3). Neural network approximations are an example of parametric approximations which are nonlinear functions of θ.

3. How to Determine θ There are two main strategies: *projection methods* and *minimum weighted residual methods*. Both attempt to find a zero of the parametrized residual operator, $R(V_\theta)$, i.e. to find a $\hat{\theta}$ such that $R(V_{\hat{\theta}})$ is as close to the 0 element of B as possible.

A. **Projection Methods** Suppose we have an inner product defined on B, i.e. $< f, g >_\mu = \int f(s)g(s)\mu(ds)$. If θ is a $k \times 1$ vector, and $\{\phi_i\}$ is a basis for B, then choose $\hat{\theta}$ to solve the following system of k equations in k unknowns:

$$< R(V_{\hat{\theta}}), \phi_i >_\mu \equiv \int \left[V_{\hat{\theta}}(s) - \hat{\Gamma}_\sigma(V_{\hat{\theta}})(s) \right] \phi_i(s)\mu(ds) = 0. \quad (4)$$

Comment: Equation (4) states that the residual function $R(V_{\hat{\theta}})$ should be *orthogonal* to the first k basis functions, $\{\phi_1, \ldots, \phi_k\}$. If $R(V)$ is orthogonal to *all* basis functions, $\{\phi_i\}$ then $R(V) = 0$.

Comment: The *Galerkin Method* is a special case of the projection method when $\phi_i = p_i$, i.e. the basis functions

Comment: The *Collocation Method* is a special case when $\phi_i = \delta_{s_i}$ for some $s_i \in S$, the *Dirac Delta Function* located at s_i. Then the projection condition reduces to:

$$R(V_{\hat{\theta}})(s_i) = 0, \quad i = 1, \ldots, k \quad (5)$$

B. **Minimum Residual Methods** Given some metric on B, e.g. a norm, then choose $\hat{\theta}$ to minimize $R(V_{\hat{\theta}})$ in this norm.

$$\hat{\theta} = \underset{\theta \in R^k}{argmin} \, \|R(V_\theta)\|_p \equiv \left[\int |V_\theta(s) - \Gamma(V_\theta)(s)|^p \, \mu(ds)\right]^{\frac{1}{p}} \tag{6}$$

Comment: The function μ entering equation (6) represents a probability measure reflecting the analyst's view of the relative importance of deviations in the residual function at various values of s. For this reason, the MR method is also known as the method of minimum *weighted* residuals (MWR).

Comment: In practical applications L_2 norm is typically used. Must numerically approximate integral defining L_2 norm and the integrals, maximizations, etc. defining the Bellman operator $\Gamma(V)$.

$$\hat{\theta} = \underset{\theta \in R^k}{argmin} \sum_{i=1}^{N} \left|V_\theta(s_i) - \hat{\Gamma}(V_\theta)(s_i)\right|^2 w_i. \tag{7}$$

Comment: In DDPs to insure differentiability in θ in (7), may want to use *smoothed Bellman operator* Γ_σ to smooth out the "kinks" induced by the max operator. Recall that Γ_σ is given by:

$$\Gamma_\sigma(V)(s) = \sigma \, log \left[\sum_{a \in A(s)} exp\left\{\frac{1}{\sigma}\left[u(s,a) + \beta \int V(s')p(s'|s,a)ds'\right]\right\}\right] \tag{8}$$

Note that $lim_{\sigma \to 0} \, \Gamma_\sigma(V) = \Gamma(V)$.

Comment: Making $R(V)$ small in L_2 norm, $\|R(V)\|_2$, does not generally gaurantee that $R(V)$ is small in L_∞ or sup norm. However if B is a *Sobolev Space* of bounded, continuously differentiable functions of order r, then the *Rellich-Kondrachev Theorem* gaurantees that for each $p \geq 2$ there exists $c_p > 0$ such that:

$$\|\hat{V} - V\|_p \leq c_p \|\hat{V} - V\|_2, \quad p \geq 2, \quad V, \hat{V} \in B. \tag{9}$$