

George Hall · John Rust

# The $(S,s)$ policy is an optimal trading strategy in a class of commodity price speculation problems

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**Abstract** This paper introduces a model of commodity price speculation and proves that the optimal trading strategy is of the  $(S, s)$  form when a *no expected loss condition* holds. A strong form of this condition is that the retail price charged to consumers at time  $t$  exceeds the expected wholesale price of the commodity at time  $t + 1$ , i.e.  $p_t^r \geq \beta E\{p_{t+1}|p_t, x_t\}$ , where  $\beta \in (0, 1)$  is the speculator's discount factor.

**Keywords** Commodity price speculation · Inventory investment ·  $K$ -concavity ·  $(S, s)$  policy

**JEL Classification Numbers** C13–15

## 1 Introduction

This paper introduces a model of commodity price speculation and proves that the optimal trading strategy is of the  $(S, s)$  form. We consider a speculator who can purchase inventories of a durable commodity in a wholesale market at price  $p_t$  for

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G. Hall  
Department of Economics, Yale University,  
New Haven, CT 06520, USA  
E-mail: george.hall@yale.edu

J. Rust (✉)  
Department of Economics, University of Maryland,  
College Park, MD 20910, USA  
E-mail: jrust@gemini.econ.umd.edu

subsequent resale to retail customers at price  $p_{t+j}^r$  on business day  $t + j$ ,  $j \geq 0$ . A trading strategy is a rule for purchasing the commodity  $q_t^o$  that depends on the information available to the speculator at the start of business day  $t$  when purchase decisions are assumed to be updated. This information includes the current wholesale market “spot price”  $p_t$ , the level of inventories carried over from yesterday  $q_t$ , and a vector of other information  $x_t$  affecting retail demand, prices, or storage costs. We prove that the optimal trading strategy takes the form of an  $(S, s)$  policy in which the optimal order quantity  $q^o(p, q, x)$  is given by  $q^o(p, q, x) = S(p, x) - q$  if  $q < s(p, x)$  and  $q^o(p, q, x) = 0$  otherwise. Following Scarf (1960), the key to proving that the optimal trading strategy is of the  $(S, s)$  form is to show that the value function  $V(p, q, x)$  (representing the conditional expected present discounted value of following an optimal trading strategy) is  $K$ -concave in  $q$ . We show that a sufficient condition for the  $K$ -concavity of  $V$  is that a *no expected loss condition* holds:

$$p_t^r - \beta E\{p_{t+1}|p_t, x_t\} \geq 0, \tag{1}$$

where  $\beta \in (0, 1)$  is the speculator’s discount factor. This condition states that the retail price of the commodity is always at least as high as the expected discounted wholesale price on business day  $t + 1$ . This latter quantity represents the expected discounted cost of replacing the unit sold on business day  $t$ . This seems to be a mild restriction that would ordinarily be satisfied in practice. If the speculator has the power to set prices, we would normally expect this condition to be satisfied. If the retail market is competitive, the speculator may have little control over  $p_t^r$ , however if the no expected loss condition didn’t hold speculators would leave the market which would tend to drive up  $p_t^r$  and drive down  $p_{t+1}$  until the condition is satisfied.

Indeed, a fundamental “no arbitrage condition” from the commodity pricing literature (see, e.g. Working 1949; Williams and Wright 1991), is that wholesale prices are set so that

$$p_t = c^h + \beta E\{p_{t+1}|p_t, x_t\}, \tag{2}$$

where  $c^h$  is the per unit holding cost. If  $c^h \geq 0$  (i.e. there are positive holding costs, as opposed to the case of “convenience yields” where  $c^h < 0$ ), then the no expected loss condition will hold if  $p_t^r \geq p_t$ , i.e. retail prices charged to individual consumers are not less than the wholesale price of the commodity.

We prove that the no expected loss condition implies that  $V$  is  $K$ -concave in  $q$  for any  $(p, x)$ , and this in turn implies that an  $(S, s)$  policy is optimal. We show that  $V = \max[V^o, V^n]$  where  $V^n$  is a  $K$ -concave function representing the value of not ordering any new inventory, and  $V^o$  is a linear function representing the value of restoring inventories to the optimal level  $S(p, x)$ . The optimal inventory level  $S(p, x)$  is the value of  $q$  that equates the shadow price of an extra unit of  $q$  to its marginal cost,  $\nabla_q V^n(p, S(p, x), x) = p$ . The purchase threshold  $s(p, x)$  is the point at which  $V^n$  and  $V^o$  first intersect. We show that a sufficient condition for  $S(p, x)$  to be decreasing in  $p$  is that the shadow value of inventory increases at a slower rate than its marginal cost at  $q = S(p, x)$ :  $\nabla_p p^s(p, q, x) \equiv \nabla_{pq}^2 V^n(p, q, x) < 1$ .

Our work builds upon and helps to unify two previously separate literatures on optimal inventory investment (Arrow, Harris, and Marschak 1951; Holt et al. 1960; Scarf 1960), and on the rational expectations commodity storage model (see,

e.g. Working 1949; Williams and Wright 1991; Miranda and Rui 1999). The latter literature studies the role of commodity storage at an aggregate level, analyzing how the collective behavior of speculators affects prices in the commodity market. However, this literature has not explicitly studied the decision problem faced by individual speculators in the commodity market. The literature on optimal inventory investment does focus on the decision problem faced by individual agents, but apparently the connection between this literature and the literature on commodity price speculation has gone unnoticed. All of the inventory-theoretic models that we are aware of focus on the role of inventory decisions in production problems, rather than on inventory management in commodity price speculation problems. Although many previous authors have conjectured that generalized forms of the  $(S, s)$  policy (such as where the  $S$  and  $s$  bands are functions of other state variables) might be an optimal, we are not aware of a proof of this result – at least in a context that is sufficiently general to be applied to the class of problems we are studying here.

The only previous work that we are aware of that anticipates some of the results in this paper is some recent work in operations research on generalizations of optimal inventory policy with Markovian demands (Sethi and Cheng 1997; Cheng and Sethi 1999). The articles by Sethi and Cheng use the traditional cost-minimization formulation of the inventory problem but introduce a finite state Markov chain, whose current realized state affects the demand for and cost of acquisition of the commodity. They present a sufficient condition for the  $K$ -convexity of the value function and the optimality of the  $(S, s)$  policy that is remarkably similar to our no expected loss condition. Their sufficient condition requires that the marginal shortage cost exceed the expected unit ordering cost less an expected marginal inventory holding cost. We became aware of the Cheng and Sethi result after we wrote this paper. Our formulation of the problem is significantly different, since it was inspired by and was directly tailored to the problem of optimal commodity price speculation introduced in Hall and Rust (2000). We believe that the problem of optimal commodity price speculation is more naturally specified as profit maximization rather than as a cost minimization problem. Our formulation contains a more general specification of the Markov process affecting the demand and acquisition cost of the commodity than the discrete Markov chain formulation considered by Cheng and Sethi. We allow general transition probabilities for the underlying “forcing process”  $\{p_t, x_t\}$  that can accommodate continuous, discrete, or mixed discrete/continuous laws of motion for these variables.

## 2 Motivation and notation

We work with a generalization of the  $(S, s)$  model of commodity price speculation introduced by Hall and Rust (2000, 2005). This model, developed from an empirical study of the observed trading behavior of a particular speculator in the steel market, characterizes the optimal trading policy of a commodity price speculator who is able to purchase bulk quantities of a durable commodity in a wholesale market at price  $p_t$ . Time is discrete and indexes successive business days. We assume that it is prohibitively costly for the speculator to resell his inventory in the wholesale market, but he can sell it in a retail market at a price  $p_t^r$ . Purchases in the wholesale market are also costly, requiring the speculator to incur a fixed transactions

cost  $K$  to purchase any positive amount of the commodity, which we denote by  $q^o > 0$ . The transactions cost discourages the speculator from making frequent small purchases in the wholesale market. In all other respects the wholesale market is perfectly competitive, and the speculator has no ability to affect  $p_t$ . However we do allow for the possibility that the speculator might have a limited amount of market power in the retail market. Due to substantial informational frictions, the retail market can be conceptualized as a “telephone market” where sales result from private bilateral negotiations. The search frictions provide an opportunity for the speculator to charge his retail customers a potentially randomly varying markup over the wholesale market price  $p_t$ . While it seems reasonable to expect that the retail price  $p_t^r$  should satisfy  $p_t^r \geq p_t$  with probability 1, if it is impossible or prohibitively costly for the speculator to re-sell the commodity in the wholesale market, then under certain circumstances it could be optimal for the speculator to set  $p_t^r < p_t$ . Thus, we do not rule out the possibility that the speculator, even if behaving fully optimally, might incur ex post losses in his trading. For this reason the trading problem we model is best described as speculation rather than arbitrage.

We characterize the optimal trading strategy of a speculator who behaves strategically in the wholesale market by optimally choosing the level of new inventory purchases in the wholesale market, but behaves passively with respect to his sales decisions in the retail market. Let  $p_t^r$  denote the retail price at time  $t$ . This retail price could either represent the “going price” under the assumption that the retail market is perfectly competitive, or it could represent a price chosen by the speculator under the assumption that the retail market is imperfectly competitive, affording the speculator some control over retail prices. In either case we assume that  $p_t^r$  is a draw from a conditional distribution  $\gamma(\cdot | p_t, x_t)$ . The essence of a passive retail sales policy is that the speculator should be willing to sell his entire inventory  $q_t + q_t^o$  to his retail customers at price  $p_t^r$ . To see why it might be reasonable to treat  $p_t^r$  as a random variable with respect to the ex ante information available to the speculator at the beginning of business day  $t$ , note that the retail price  $p_t^r$  the speculator will ultimately charge his retail customers will generally depend on additional signals that the speculator receives about his customers and the state of the retail market during the course of business day  $t$ , which are random with respect to the information  $(p_t, q_t, x_t)$  available to the speculator at the start of the day. We assume that  $(p_t, x_t)$  is a sufficient statistic for these additional signals, so the speculator’s beliefs about the retail price  $p_t^r$  he will subsequently charge during day  $t$  are given by a conditional probability distribution  $\gamma(\cdot | p_t, x_t)$  that depends on  $(p_t, x_t)$  but not  $q_t$ . The reason why we exclude  $q_t$  as an element of this conditioning set will be clear shortly.

After the retail price  $p_t^r$  is set, there is a conditional probability  $\eta(p_t^r, p_t, x_t)$  that no customer will arrive, and with the complementary probability, one or more customers will place orders for the commodity at the quoted retail price. A sale of a unit yields a revenue of  $p_t^r$ , but there is also an opportunity cost of having to replace the unit on some subsequent day  $t + j$ . The *no expected loss condition* states that the return from selling a unit of the commodity, net of the discounted expected cost of replacing that unit on day  $t + 1$ , is non-negative:

$$\int [p_t^r - \beta E\{p_{t+1} | p_t, x_t\}] [1 - \eta(p_t^r, p_t, x_t)] \gamma(dp_t^r | p_t, x_t) \geq 0. \quad (3)$$

The expectation in the expression above is taken with respect to  $\gamma(dp_t^r | p_t, x_t)$  which represents the conditional distribution of ex post retail prices  $p_t^r$  given the ex ante information at the start of the day  $(p_t, x_t)$ . This version of the no expected loss condition is slightly weaker than the version presented in equation (1), which specified that  $[p_t^r - \beta E\{p_{t+1} | p_t, x_t\}] \geq 0$  with probability 1, and not just in expectation as in the weaker version given above.

As noted above, the no expected loss condition is not automatically satisfied in all circumstances, e.g. as a necessary condition for profit maximization. The reason is that under imperfect market conditions, such as when the speculator has power to set prices and/or when there are “irreversible investment constraints” that prevent the speculator from being able to liquidate inventories at the prevailing spot market price, it is not necessarily the case that it is always optimal to replace units of the commodity that are sold on day  $t$  via wholesale market purchases on day  $t + 1$ . Thus, the expected opportunity cost of selling a unit of the commodity today may not always equal  $\beta E\{p_{t+1} | p_t, x_t\}$ . For example, if the speculator is overstocked, then the opportunity cost of selling a unit of the commodity today could well be less than  $\beta E\{p_{t+1} | p_t, x_t\}$ . However, as noted above, in well functioning “competitive” commodity markets, retail prices will not be under the speculator’s control and the intertemporal no arbitrage condition from the rational expectations commodity pricing literature does imply that the no expected loss condition will hold. Thus, the no expected loss condition is an economically meaningful restriction that needs to be verified on a case by case basis.

However, we caution readers that while the  $(S, s)$  policy might appear to be a very natural and robust trading strategy, it is not hard to change assumptions in ways that destroy its optimality. For example,  $(S, s)$  is unlikely to remain optimal if the speculator faces significant quantity discounts, or other types of non-linear pricing schedules in the wholesale market. Characterizing the form of optimal speculative trading strategies under these conditions remains a topic for future research.

**Assumption 0** (Timing of the speculator’s information and actions)

1. At the start of day  $t$  the speculator knows his inventory level  $q_t$ , the current spot price  $p_t$ , and the values of the other state variables  $x_t$ .
2. Given  $(q_t, p_t, x_t)$  the speculator orders additional inventory  $q_t^o$  for immediate delivery.
3. Given  $(p_t, x_t)$  the speculator sets a retail price  $p_t^r$  that may depend on information that the speculator observes but which is unobserved from the standpoint of other observers. Thus,  $p_t^r$  is modeled as a random draw from a conditional distribution  $\gamma(\cdot | p_t, x_t)$ .
4. Given  $(p_t^r, p_t, x_t)$  the speculator observes a realized retail demand for the commodity,  $q_t^r$ , modeled as a draw from a distribution  $H(q_t^r | p_t, p_t^r, x_t)$  with a point mass at  $q_t^r = 0$ , representing the probability that there is no retail demand for the commodity on day  $t$ .
5. The speculator cannot sell more of the commodity than he has on hand, so the actual quantity sold satisfies

$$q_t^s = \min [q_t + q_t^o, q_t^r]. \quad (4)$$

6. The sales in period  $t$  determine the level of inventories on hand at the start of the next business day,  $t + 1$ , by the standard inventory identity:

$$q_{t+1} = q_t + q_t^o - q_t^s. \quad (5)$$

7. New values of  $(p_{t+1}, x_{t+1})$  are drawn from a Markov transition probability  $g(p_{t+1}, x_{t+1} | p_t, x_t)$ .

Note that Assumption 0 implies that the speculator does not face any delivery lags and cannot backlog unfilled orders. Thus, whenever demand exceeds quantity on hand, the residual unfilled demand is lost. This implies that the amount of the commodity sold each period is the *minimum* of retail demand  $q_t^r$  and quantity on hand  $q_t + q_t^o$  as given in equation (4).

For technical reasons, it is convenient to assume that the state space for the DP problem is compact.

**Assumption 1** The speculator has a maximum storage capacity equal to  $\bar{q} \leq \infty$ . Negative orders and inventories (representing backlogs) are not allowed, so  $q_t^o$  is restricted to the interval  $[0, \bar{q} - q_t]$  and  $q_t$  must lie in the interval  $[0, \bar{q}]$  with probability 1. The joint Markov process  $\{p_t, x_t\}$  has support  $P \times X$  where  $X$  is a compact subset of  $R^k$  and  $P = [\underline{p}, \bar{p}]$  where  $\underline{p} > 0$  and  $\bar{p} < \infty$ .

To understand the implications of Assumptions 0 and 1, we need to describe the speculator’s retail sales and revenue in a bit more detail. We assume that the speculator’s retail sales on business day  $t$  is a random draw  $q_t^r$  from a conditional distribution  $H(q_t^r | p_t^r, p_t, x_t)$  that depends on the retail price  $p_t^r$ , the current spot price  $p_t$ , and the values of the other observed state variables  $x_t$ . Let  $\eta(p^r, p, x) = H(0 | p^r, p, x)$  be the probability that the speculator will not make any retail sales on a particular day. We assume that there are no other mass points in the distribution function for quantity demanded,  $H$ , so it can be represented as follows.

**Assumption 2** The conditional probability distribution for the speculator’s retail sales on day  $t$  is given by:

$$H(q^r | p^r, p, x) = \eta(p^r, p, x) + [1 - \eta(p^r, p, x)] \int_0^{q^r} h(dq | p^r, p, x), \quad (6)$$

where  $\eta(p^r, p, x) \in [0, 1)$ , and  $h$  is a continuous probability density function over the interval  $[0, \infty)$  satisfying  $h(q | p^r, p, x) \geq \epsilon > 0$  for all  $q \in [0, \bar{q}]$ , and all  $(p^r, p, x) \in R^+ \times P \times X$ .

Since the quantity demanded has support on the  $[0, \infty)$  interval, equation (4) implies that there is always a positive probability of a *stockout* given by:

$$\delta(q + q^o, p, p^r, x) = 1 - H(q + q^o | p^r, p, x). \quad (7)$$

When a stockout occurs, the speculator may incur per unit “goodwill cost”  $c^g(p^r, p, x) \geq 0$  on the amount of unsatisfied demand. We let  $EG$  denote the

speculator’s ex ante expectation of these goodwill costs at the beginning of a business day, conditional on the information  $(p, x)$  and inventory on hand of  $q$

$$\begin{aligned}
EG(p, q, x) &= E\{c^g(p^r, p, x) \max[(q^r - q), 0] | p, q, x\} \\
&= \int_0^\infty \left[ c^g(p^r, p, x) [1 - \eta(p^r, p, x)] \int_q^\infty (q^r - q) h(q^r | p^r, p, x) dq^r \right] \gamma(dp^r | p, x).
\end{aligned}
\tag{8}$$

The key to the solution of the speculator’s optimal trading is the expected per period retail sales revenue  $ES(p, q, x)$ . This is just the conditional expectation of realized sales revenue  $p^r q^s$ , the product of the retail price  $p^r$  times the quantity actually sold  $q^s = \min[q + q^o, q^r]$ , given the current spot price  $p$ , quantity on hand  $q$ , and other information  $x$ :

$$\begin{aligned}
ES(p, q, x) &= E\{p^r q^s | p, q, x\} \\
&= E\{p^r E\{\min[q, q^r] | p^r, p, q, x\} | p, q, x\} \\
&= \int_0^\infty p^r [1 - \eta(p^r, p, x)] \left[ \int_0^q q^r h(q^r | p^r, p, x) dq^r \right. \\
&\quad \left. + q \int_q^\infty h(q^r | p^r, p, x) dq^r \right] \gamma(dp^r | p, x).
\end{aligned}
\tag{9}$$

**Lemma 1** *If Assumptions 0–2 hold,  $ES(p, q, x)$  is a strictly increasing and concave function of  $q$  for each  $(p, x)$  and  $EG(p, q, x)$  is a strictly decreasing and convex function of  $q$  for each  $(p, x)$ .*

*Proof* It is straightforward to verify via direct differentiation that

$$\nabla_q ES(p, q, x) = \int_0^\infty p^r [1 - \eta(p^r, p, x)] \int_q^\infty h(q^r | p^r, p, x) dq^r \gamma(dp^r | p, x) > 0,
\tag{10}$$

and

$$\nabla_{qq}^2 ES(p, q, x) = - \int_0^\infty p^r [1 - \eta(p^r, p, x)] h(q | p^r, p, x) \gamma(dp^r | p, x) < 0,
\tag{11}$$

since  $h(q | p^r, p, x) \geq \epsilon > 0$  and  $\eta(p^r, p, x) < 1$  for all  $(p^r, p, x) \in R^+ \times P \times X$  by Assumption 3. The properties of  $EG(p, q, x)$  can be verified via a similar calculation.  $\square$

**Assumption 3** The speculator incurs a period physical storage cost  $c^h(p, q, x)$  of holding inventory which is a non-decreasing and convex function of  $q$  for all  $(p, x) \in P \times X$ .

We assume the speculator incurs a cost of ordering  $q^o$  units of the commodity for inventory given by a function  $c^o(p, q^o, x)$  that is linear in  $p$ , but discontinuous at  $q^o = 0$  due to a fixed transactions cost of placing orders in the wholesale market.

**Assumption 4** The cost of purchasing  $q^o$  units of the commodity in the wholesale market is given by:

$$c^o(p, q^o, x) = \begin{cases} K + pq^o & \text{if } q^o > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

where  $K \geq 0$  is a fixed transaction cost associated with placing any order, regardless of the quantity ordered.

This specification can be easily modified to account for constant per unit shipping costs  $\rho$ , and to allow both  $\rho$  and  $K$  to depend on  $x$ . All of the results below hold under this more general specification, but since the notation becomes more complex, we will initially ignore shipping costs and assume  $K$  is independent of  $x$ . For notational simplicity we assume that any per unit shipping costs are already embodied in the spot price  $p$ , so that at least in this respect the simplified specification of order costs given in Assumption 4 involves no loss of generality.

Under these assumptions, the speculator’s single-period profits  $\pi$  equals its sales revenues, less any goodwill costs due to unsatisfied demand, less the cost of new orders for inventory  $c^o(p, q^o, x)$  and inventory holding costs  $c^h(p, q, x)$ :

$$\pi(p, p^r, q^r, q + q^o, x) = p^r \min[q^r, q + q^o] - c^g(p^r, p, x) \max[q^r - q - q^o, 0] - c^o(p, q^o, x) - c^h(p, q + q^o, x). \tag{13}$$

The speculator’s inventory investment behavior is governed by the decision rule:

$$q_t^o = q^o(p_t, q_t, x_t), \tag{14}$$

where the function  $q^o$  is the solution to:

$$V(p_t, q_t, x_t) = \max_{q^o} E \left\{ \sum_{j=t}^{\infty} \beta^{(j-t)} \pi(p_j, p_j^r, q_j^r, q_j^o + q_j, x_j) \mid p_t, q_t, x_t \right\}. \tag{15}$$

The value function  $V(p, q, x)$  is given by the unique solution to Bellman’s equation:

$$V(p, q, x) = \max_{0 \leq q^o \leq \bar{q} - q} \left[ W(p, q + q^o, x) - c^o(p, q^o, x) \right], \tag{16}$$

where:

$$W(p, q, x) \equiv \left[ ES(p, q, x) - EG(p, q, x) - c^h(p, q, x) + \beta EV(p, q, x) \right], \tag{17}$$



and  $EV$  denotes the conditional expectation of  $V$  given by:

$$\begin{aligned}
 EV(p, q, x) &= E\{V(p, \max[0, q - q^r], x)|p, q, x\} \\
 &= \int_{p'} \int_{x'} \int_{p^r} \eta(p^r, p, x)V(p', q, x')\gamma(dp^r|p, x)g(dp', dx'|p, x) \\
 &\quad + \int_{p'} \int_{x'} \int_{p^r} [1 - \eta(p^r, p, x)]V(p', 0, x') \\
 &\quad \times \int_q^\infty h(q^r|p^r, p, x)dq^r\gamma(dp^r|p, x)g(dp', dx'|p, x) \\
 &\quad + \int_{p'} \int_{x'} \int_{p^r} [1 - \eta(p^r, p, x)] \int_0^q V(p', q - q^r, x') \\
 &\quad \times h(q^r|p^r, p, x)dq^r\gamma(dp^r|p, x)g(dp', dx'|p, x). \tag{18}
 \end{aligned}$$

The optimal decision rule  $q^o(p, q, x)$  is given by:

$$q^o(p, q, x) = \inf \operatorname{argmax}_{0 \leq q^o \leq \bar{q} - q} [W(p, q + q^o, x) - c^o(p, q^o, x)]. \tag{19}$$

Note that we invoke the  $\inf$  operator in the definition of the optimal decision rule in equation (19) to handle the case where there are multiple maximizing values of  $q^o$ . This could arise if  $W$  is not strictly concave in  $q$ .

**Definition 0** An  $(S, s)$  rule is a trading strategy of the form:

$$q^o(p, q, x) = \begin{cases} 0 & \text{if } q \geq s(p, x) \\ S(p, x) - q & \text{otherwise,} \end{cases} \tag{20}$$

where  $S$  and  $s$  are functions satisfying  $S(p, x) \geq s(p, x)$  for all  $p$  and  $x$ .

Candidate functions for the upper and lower bands of the generalized  $(S, s)$  policy can be defined in terms of the optimal decision rule  $q^o(p, q, x)$ . The upper band  $S(p, x)$  is defined as the optimal order quantity when the speculator has no inventory on hand:

$$S(p, x) = q^o(p, 0, x). \tag{21}$$

The lower band  $s(p, x)$  is the smallest value of  $q$  such that desired inventory investment is zero:

$$s(p, x) = \inf \{ q \in [0, \bar{q}] \mid q^o(p, q, x) = 0 \}. \tag{22}$$

Clearly, desired inventory investment at  $S(p, x)$  is  $0 : q^o(p, S(p, x), x) = 0$ . Since  $s(p, x)$  is the *smallest* value of  $q$  satisfying  $q^o(p, q, x) = 0$  it follows that  $s(p, x) \leq S(p, x)$ . We show that the speculator is indifferent between ordering and not ordering at  $q = s(p, x)$  provided  $s(p, x) > 0$ . Defining  $q^o$  in terms of the functions  $S$  and  $s$  (which are defined in turn from  $q^o$ ) may appear circular, but the

$(S, s)$  rule does amount to a real restriction on the trading strategy  $q^o$ . Substituting for  $S(p, x)$  in equation (20) we have

$$q^o(p, q, x) = q^o(p, 0, x) - q, \tag{23}$$

when  $q^o(p, q, x) > 0$  and  $q^o(p, q, x) = 0$  otherwise. The  $(S, s)$  rule further restricts the set of  $(p, q, x)$  for which  $q^o(p, q, x) = 0$  to be a set of the form  $\{(p, q, x) | q \geq s(p, x)\}$  for some function  $s(p, x)$ . Thus, it should be clear that  $(S, s)$  rules are indeed a restricted subset of admissible trading strategies and our definition is not tautological.

The weakest known sufficient condition for the optimality of the  $(S, s)$  is the  $K$ -concavity condition introduced by Scarf (1960). For convenience, we re-state its definition below.

**Definition 1** A function  $f : [0, \bar{q}] \rightarrow R$  is  $K$ -concave if and only if for all  $q, z, b \in R$  satisfying  $0 \leq q - b \leq q \leq q + z \leq \bar{q}$  we have:

$$f(q + z) - K \leq f(q) + z \left[ \frac{f(q) - f(q - b)}{b} \right]. \tag{24}$$

A function is  $K$ -concave if the secant approximation to  $f(q + z)$  given on the right hand side of equation (24) exceeds  $f(q + z)$  less the constant  $K$ . Clearly a concave function is 0-concave, and thus  $K$ -concave for all  $K \geq 0$ . A function  $W(p, q, x)$  is  $K$ -concave in  $q$  if the inequality (24) also holds for  $W$  as a function of  $q$  for all  $(p, x)$ . Scarf (1960) actually defined the property of  $K$ -convexity, but just as for ordinary convex functions, it is easy to show that  $f$  is  $K$ -concave if and only if  $-f$  is  $K$ -convex.

The following Lemma summarizes the key properties of  $K$ -concave functions. It can be proved via trivial modifications to the proof of an analogous result characterizing properties of  $K$ -convex functions (see, e.g. Lemma 2.1 in Bertsekas 1995) and is therefore omitted.

- Lemma 2**
1. A concave function is 0-concave and hence  $K$ -concave for all  $K \geq 0$ .
  2. If  $f_1(q)$  and  $f_2(q)$  are  $K_1$ -concave and  $K_2$ -concave, respectively, for constants  $K_1 \geq 0$  and  $K_2 \geq 0$ , then  $\alpha f_1(q) + \beta f_2(q)$  is  $(\alpha K_1 + \beta K_2)$ -concave for any  $\alpha > 0$  and  $\beta > 0$ .
  3. If  $\{f_n(q)\}$  is a sequence of  $K$ -concave functions and  $f = \lim_{n \rightarrow \infty} f_n$  is the pointwise limit of these functions, and if  $|f(q)| < \infty$  for all  $q \in R$ , then  $f$  is  $K$ -concave.
  4. If  $f$  is  $K$ -concave and  $w$  is a random variable for which  $E\{|f(q - w)|\} < \infty$  for all  $q$ , then  $g(q) = E\{f(q - w)\}$  is  $K$ -concave.
  5. If  $f$  is a continuous,  $K$ -concave function on the interval  $[0, \bar{q}]$ , then there exists scalars  $0 \leq s \leq S \leq \bar{q}$  such that
    - (a)  $f(S) \geq f(q)$  for all  $q \in [0, \bar{q}]$ .
    - (b) Either  $s = 0$  and  $f(S) - K \leq f(0)$  or  $s > 0$  and  $f(S) - K = f(s) > f(q)$  for all  $q \in [0, s)$ .
    - (c)  $f$  is strictly increasing for  $q \in [0, s)$ .
    - (d)  $f(z) - K \leq f(q)$  for all  $z$  and  $q$  satisfying  $s \leq q \leq z \leq \bar{q}$ .

Scarf established the optimality of the  $(S, s)$  policy via an inductive proof that the value function  $W$  is  $K$ -convex in  $q$ . We will prove an analogous result for  $K$ -concave functions, i.e., that the Bellman operator maps the class of continuous  $K$ -concave functions,  $\mathcal{F}_{KC}$ , into itself. Then part (3) of Lemma 2 implies that  $V$  and  $W$  are uniform limits of sequences of continuous,  $K$ -concave functions, and must also be continuous and  $K$ -concave. Lemma 3 verifies the analog of Scarf’s basic result for  $K$ -convex functions in our setting, namely that  $K$ -concavity of  $W$  implies the optimality of the  $(S, s)$  rule.

**Lemma 3** *Suppose the function  $W(p, q, x)$  is continuous and  $K$ -concave in  $q$  for all  $(p, x)$ . Let  $V$  be given by:*

$$V(p, q, x) = \max_{0 \leq q^o \leq \bar{q} - q} [W(p, q + q^o, x) - c^o(p, q^o, x)], \tag{25}$$

and let the  $(S, s)$  bands be given by:

$$\begin{aligned} S(p, x) &= \inf \operatorname{argmax}_{0 \leq q^o \leq \bar{q}} [W(p, q^o, x) - pq^o]. \\ s(p, x) &= \inf \{q \in [0, S(p, x)] \mid W(p, q, x) \\ &\geq W(p, S(p, x), x) - p[S(p, x) - q] - K\}. \end{aligned} \tag{26}$$

Then there is a solution  $q^o(p, q, x)$  to problem (25) that is of the  $(S, s)$  form with the functions  $S(p, x)$  and  $s(p, x)$  given above. The value function  $V$  can be expressed in terms of  $W$  and  $(S, s)$  as:

$$V(p, q, x) = \begin{cases} W(p, S(p, x), x) - p[S(p, x) - q] - K & \text{if } q \in [0, s(p, x)] \\ W(p, q, x) & \text{if } q \in [s(p, x), \bar{q}]. \end{cases} \tag{27}$$

*Proof* To help the reader follow the proof, we illustrate the determination of the functions  $S(p, x)$  and  $s(p, x)$  in Fig. 1 below, where for illustrative purposes, we have plotted the  $W$  function as a strictly concave function of  $q$  holding its other two arguments  $(p, x)$  fixed. However, as we will see, the proof below does not require  $W$  to be strictly concave in  $q$ . It is sufficient for  $W$  to be  $K$ -concave. Define the functions  $V^n$  and  $V^o$  as follows:

$$V^n(p, q, x) = W(p, q, x) \tag{28}$$

$$V^o(p, q, x) = \begin{cases} W(p, S(p, x), x) - p[S(p, x) - q] - K & \text{if } q \in [0, s(p, x)] \\ W(p, q, x) & \text{if } q \in (s(p, x), \bar{q}]. \end{cases} \tag{29}$$

$V^n(p, q, x)$  represents the value of not ordering, whereas  $V^o(p, q, x)$  represents the value of ordering the target inventory level  $S(p, x)$  if  $q \leq s(p, x)$  and not ordering otherwise. It is not hard to see that  $V = \max[V^n, V^o]$ .

The left hand panel of Fig. 1 illustrates how  $S(p, x)$  is determined. Since  $S(p, x) = \operatorname{argmax}_q W(p, q, x) - pq$ , it is located at the point of tangency between the straight line with slope  $p$ , illustrated by the dashed blue line in the left hand panel of Fig. 1, and the function  $W(p, q, x)$  (the concave green curve in Fig. 1).

The right hand panel of Fig. 1 illustrates how  $s(p, x)$  is determined. The function  $V^o$  [the value of ordering to the optimal order quantity  $S(p, x)$  given in equation

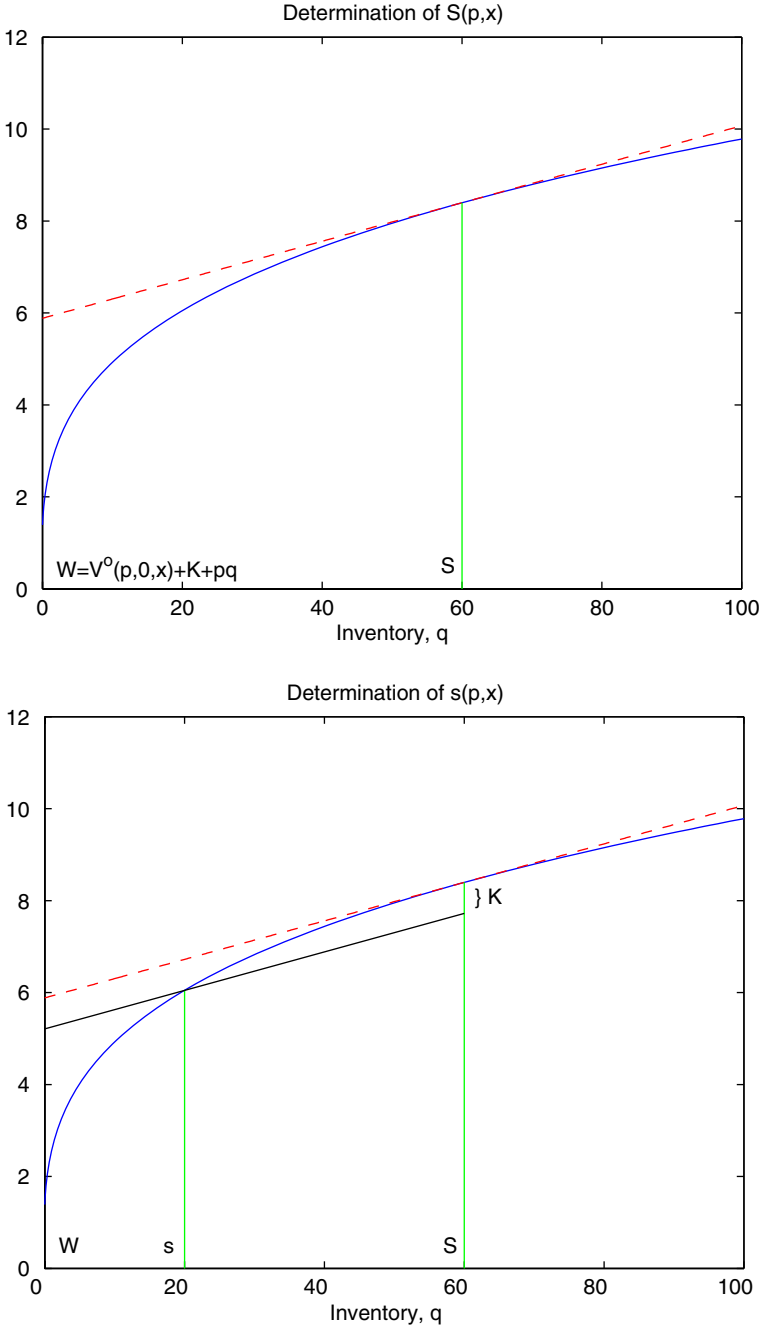


Fig. 1 Determination of the functions  $S(p, x)$  and  $s(p, x)$

(29) above] is the maximum of the red line representing the actual value of ordering the quantity  $S(p, x) - q$  to reach the target  $S(p, x)$  when the initial inventory is  $q$ , less the fixed cost  $K$  of placing the order. The latter is represented by a parallel downward shift in the dashed blue line by  $K$  units. The “value of ordering up to  $S(p, x)$ ” (i.e. the  $V^o$  function) is actually only defined on the interval  $[0, S(p, x)]$ , since if  $q > S(p, x)$  the speculator’s inventory already exceeds the optimal level. In this region, we define the “value of ordering” to coincide with the value of not ordering, i.e.  $V^o(p, q, x) = W(p, q, x)$  for  $q > S(p, x)$ . Thus in the right hand panel of Fig. 1, the graph of  $V^o$  is equal to the red linear segment on the interval  $[0, s(p, x)]$  and is equal to the green  $W(p, q, x)$  function on the interval  $[s(p, x), \bar{q}]$ .

The optimal order threshold is the value of  $q$  where the speculator is indifferent between ordering and not ordering. Thus,  $q = s(p, x)$  is the solution to the equation  $V^o(s(p, x), p, x) = V^n(s(p, x), p, x)$ , or more explicitly in terms of the function  $W$ ,  $W(p, s(p, x), x) = W(p, S(p, x), x) - p[S(p, x) - s(p, x)] - K$ . This point is represented by the first intersection of the functions  $V^o$  and  $V^n$  in the right hand panel of Fig. 1. If there is no positive value of  $q$  for which  $V^o = V^n$ , then  $s(p, x) = 0$  and it would never be optimal for the speculator to order new inventory. This would correspond to a “shut down” scenario where the speculator gradually sells off existing inventory and then goes out of business.

The function  $S(p, x)$  exists and is well defined since the maximum of a continuous function over a compact set exists by the Theorem of the Maximum. The function  $s(p, x)$  exists because the set of  $q$  satisfying  $W(p, q, x) \geq W(p, S(p, x), x) - p[S(p, x) - q] - K$  is non-empty [for example  $q = S(p, x)$  trivially satisfies this inequality]. Now we wish to show that the  $(S, s)$  rule is indeed optimal. Suppose that  $s(p, x) > 0$ . Then for any  $q < s(p, x)$  we must have  $W(p, q, x) < W(p, S(p, x), x) - p[S(p, x) - q] - K$  otherwise we would have a contradiction of the definition of  $s(p, x)$  as the smallest  $q$  satisfying  $W(p, q, x) \geq W(p, S(p, x), x) - p[S(p, x) - q] - K$ . But this implies that the speculator would prefer to order  $q^o = S(p, x) - q$  units than to order none. By definition of  $S(p, x)$  there is no other order quantity that would yield strictly higher expected discounted profits, so it follows that  $q^o(p, q, x) = S(p, x) - q$  is indeed optimal when  $q < s(p, x)$ . By continuity this also holds at  $q = s(p, x)$ . Now consider the case when  $s(p, x) < q < S(p, x)$ . Lemma 2 implies that if  $W$  is  $K$ -concave, then so is the function  $W(p, q, x) - pq$ . By the definition of  $K$ -concavity we have:

$$\begin{aligned} & W(p, S(p, x), x) - pS(p, x) - K \\ & \leq W(p, q, x) - pq + \frac{S(p, x) - q}{q - s(p, x)} \\ & \quad \times [W(p, q, x) - pq - W(p, s(p, x), x) + ps(p, x)]. \end{aligned} \quad (30)$$

It is easy to see that the above inequality can be rewritten as

$$\begin{aligned} & W(p, S(p, x), x) - pS(p, x) - K + \frac{S(p, x) - q}{q - s(p, x)} \times [W(p, s(p, x), x) - ps(p, x)] \\ & \leq \frac{S(p, x) - s(p, x)}{q - s(p, x)} [W(p, q, x) - pq]. \end{aligned} \quad (31)$$

However, the definition of  $s(p, x)$  implies that

$$W(p, s(p, x), x) - ps(p, x) \geq W(p, S(p, x), x) - pS(p, x) - K. \quad (32)$$

Combining the above inequality with inequality (31) we conclude that

$$\begin{aligned} & \frac{S(p, x) - s(p, x)}{q - s(p, x)} [W(p, S(p, x), x) - pS(p, x) - K] \\ & \leq \frac{S(p, x) - s(p, x)}{q - s(p, x)} [W(p, q, x) - pq]. \end{aligned} \quad (33)$$

The above inequality is algebraically equivalent to the inequality

$$W(p, S(p, x), x) - p[S(p, x) - q] - K \leq W(p, q, x), \quad (34)$$

which says that it is not optimal for the speculator to order when  $s(p, x) < q < S(p, x)$ . Thus, the  $(S, s)$  rule yields the optimal decision  $q^o(p, q, x) = 0$  in this case. It is easy to see that the optimal order quantity is also zero when  $q = S(p, x)$ . The final case to consider is when  $q \in (S(p, x), \bar{q}]$ . By  $K$ -concavity, for any  $z \in [0, \bar{q} - q]$  we have

$$\begin{aligned} & W(p, q + z, x) - p(q + z) - K \\ & \leq W(p, q, x) - pq + \frac{z}{q - S(p, x)} \\ & \quad \times [W(p, q, x) - pq - W(p, S(p, x), x) + pS(p, x)]. \end{aligned} \quad (35)$$

By the definition of  $S(p, x)$ , the term on the right hand side of the above inequality is non-positive, so rearranging we have

$$W(p, q + z, x) - pz - K \leq W(p, q, x). \quad (36)$$

However this implies that  $q^o(p, q, x) = 0$ , so the  $(S, s)$  rule also yields the correct decision in this case.  $\square$

**Lemma 4** *Under the assumptions of Lemma 3,  $V$  can be represented as:*

$$V(p, q, x) = \max [V^n(p, q, x), V^o(p, q, x)]. \quad (37)$$

In order to help the reader understand the proofs of the following lemmas, Fig. 2 further illustrates the functions  $V^n$ ,  $V^o$ , and  $V$ . In this illustration, the value of not ordering,  $V^n$ , is drawn as a strictly concave function of  $q$ . As noted above, strict concavity in  $q$  is not required for the proofs below, only the weaker condition of  $K$ -concavity. The right hand panel of Fig. 2 shows that the value function  $V$  is the maximum of the piece-wise linear concave function  $V^o$  [the value of ordering up to  $S(p, x)$ ], and the strictly concave function  $V^n$ . Obviously the maximum of two concave functions is not necessarily concave, and we see this in the right hand panel of Fig. 2, where there is a non-concave kink point in  $V$  at  $q = S(p, x)$ . Thus, while  $V$  is not concave in  $q$ , we will now show that if the no expected loss condition holds, it is  $K$ -concave in  $q$ .

**Definition 2** Let  $\mathcal{F}_{KC}$  denote the class of functions  $V(p, q, x)$  which are continuous and  $K$ -concave as a function of  $q$  for all  $(p, x) \in P \times X$ .

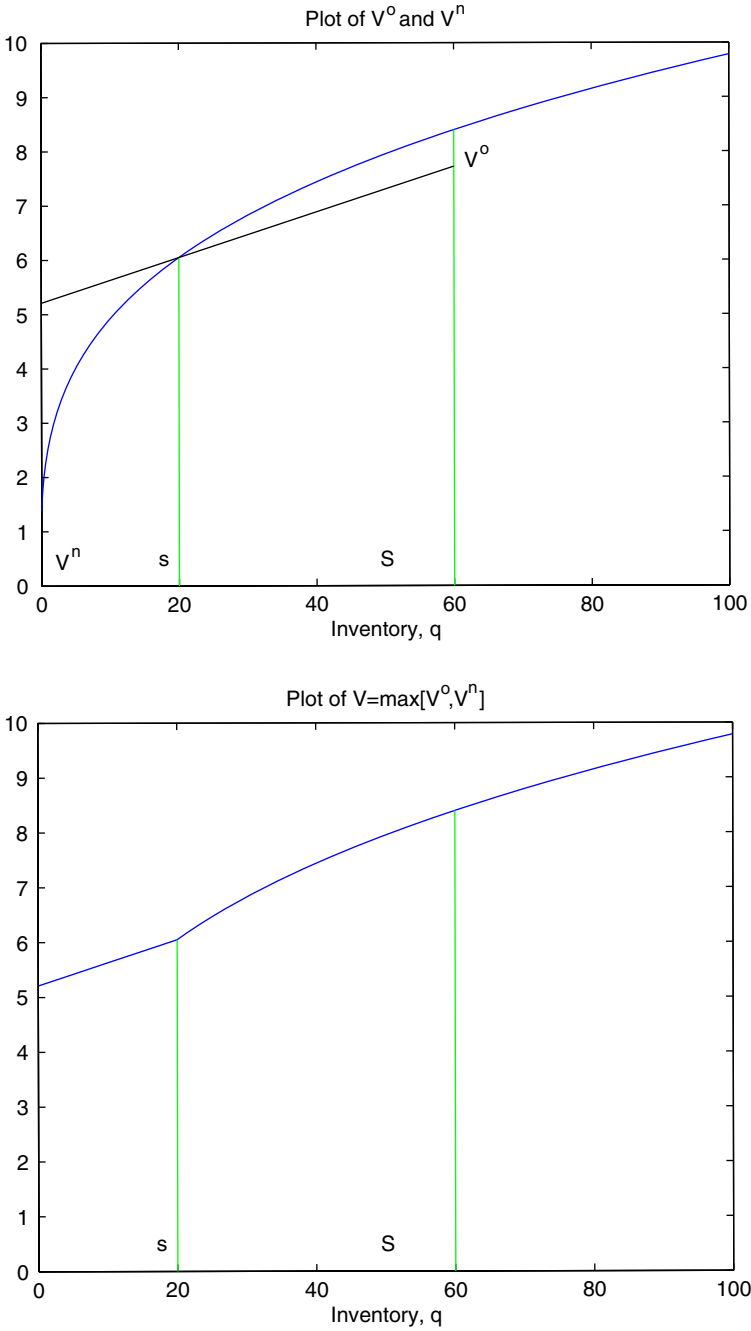


Fig. 2 Illustrations of the functions  $V^o$ ,  $V^n$  and  $V = \max[V^n, V^o]$

Let  $B$  denote the Banach space of all continuous functions  $W(p, q, x)$  mapping  $P \times [0, \bar{q}] \times X$  into  $R$  under the usual sup-norm  $\| \cdot \|$ . It is not difficult to show that the Bellman equation for the value function  $V(p, q, x)$  can be represented as a fixed point of a contraction mapping on  $B$ , and hence  $V$  exists and is unique. We now characterize its properties. In particular, we show that both  $V$  and  $W$  are  $K$ -concave in  $q$ . We do this by showing that the Bellman operator can be represented as the composition of two operators  $\Gamma : B \rightarrow B$  and  $\Lambda : B \rightarrow B$  given by:

$$\Gamma(W)(p, q, x) = \max_{0 \leq q^o \leq \bar{q} - q} [W(p, q + q^o, x) - c^o(p, q^o, x)], \tag{38}$$

$$\Lambda(V)(p, q, x) = ES(p, q, x) - EG(p, q, x) - c^h(p, q, x) + \beta EV(p, q, x). \tag{39}$$

**Lemma 5** *The value function  $V$  is the unique fixed point of the composition operator,  $\Gamma \circ \Lambda : B \rightarrow B$  given by:*

$$V = \Gamma \circ \Lambda(V) \equiv \Gamma(\Lambda(V)). \tag{40}$$

*The function  $W$  is the unique fixed point to the composition operator  $\Lambda \circ \Gamma : B \rightarrow B$  given by:*

$$W = \Lambda \circ \Gamma(W) \equiv \Lambda(\Gamma(W)). \tag{41}$$

The representation of the Bellman operator in Lemma 5 suggests that we can prove that  $V$  and  $W$  are  $K$ -concave in  $q$  in two steps: (1) first we demonstrate that  $\Gamma : \mathcal{F}_{KC} \rightarrow \mathcal{F}_{KC}$  and (2) we demonstrate that  $\Lambda \circ \Gamma : \mathcal{F}_{KC} \rightarrow \mathcal{F}_{KC}$ . This will enable us to establish the key induction step of our argument, which will imply that the fixed point  $V$  is a uniform limit of functions in  $\mathcal{F}_{KC}$ , and hence will also be a member of this class.

**Lemma 6** *Assumptions 1–5 imply that  $\Gamma : \mathcal{F}_{KC} \rightarrow \mathcal{F}_{KC}$ . That is, if  $W$  is continuous and  $K$ -concave in  $q$ , then  $V = \Gamma(W)$  is continuous and  $K$ -concave in  $q$ .*

*Proof* By the theorem of the maximum if  $W$  is continuous in  $q$  then  $\Gamma(W)$  given in equation (38) is also continuous in  $q$ . The proof is completed by showing that for any  $(p, x) \in P \times X$  and any points  $q - b, q$  and  $q + z$  satisfying  $0 \leq q - b \leq q \leq q + z \leq \bar{q}$ , the function  $V = \Gamma(W)$  satisfies the definition of  $K$ -concavity

$$V(p, q + z, x) - K \leq V(p, q, x) + \frac{z}{b} [V(p, q, x) - V(p, q - b, x)]. \tag{42}$$

Let  $S(p, x)$  and  $s(p, x)$  be the  $(S, s)$  bands defined in Lemma 3. There are three cases to consider, depending on which side of the kink point at  $s(p, x)$  the points  $q - b, q$  and  $q + z$  lay on. Since  $V$  is linear in  $q$  for  $q \leq s(p, x)$ , if  $q + z \leq s(p, x)$  then all these points lie in the interval  $[0, s(p, x)]$  where  $V$  is linear, and thus  $K$ -concave. Similarly, if  $s(p, x) \leq q - b$  all of the points lie on the interval  $[s(p, x), \bar{q}]$  where  $V = W$ , and since  $W$  is  $K$ -concave, then so is  $V$ . So the only remaining case to consider is where the points  $q - b, q$  and  $q + z$  straddle the kink in  $V$  at  $s(p, x)$ . In this case we have  $0 \leq q - b < s(p, x)$  and  $q + z > s(p, x)$ . Equation (27) implies that  $V(p, q + z, x) = W(p, q + z, x)$  and

$$\begin{aligned} V(p, q - b, x) &= W(p, S(p, x), x) - p[S(p, x) - (q - b)] - K \\ &= W(p, s(p, x), x) - p[s(p, x) - (q - b)], \end{aligned} \tag{43}$$



where we have used the fact since  $s(p, x) > 0$  we have

$$W(p, s(p, x), x) = W(p, S(p, x), x) - p[S(p, x) - s(p, x)] - K, \quad (44)$$

by result 5-b of Lemma 2. If  $q < s(p, x)$ , then it is easy to see that  $V(p, q, x) - V(p, q - b, x) = pb$ , so that inequality (42) characterizing the  $K$ -concavity of  $V$  reduces to:

$$V(p, q + z, x) - K \leq W(p, S(p, x), x) - p(S(p, x) - q) - K + pz, \quad (45)$$

which can be rearranged into an equivalent inequality

$$V(p, q + z, x) - p(q + z) \leq V(p, S(p, x), x) - pS(p, x), \quad (46)$$

which necessarily holds via the definition of  $S(p, x)$  as the argmax of  $W(p, q, x) - pq$  in  $q$ . Now if  $q > s(p, x)$ , then Lemma 3 implies that  $V(p, q, x) = W(p, q, x)$ . Suppose that  $q$  is such that

$$W(p, q, x) - pq \leq W(p, s(p, x), x) - ps(p, x). \quad (47)$$

Via some simple algebra, we see that this inequality is equivalent to the inequality

$$\begin{aligned} & \frac{z}{q - s(p, x)} [W(p, q, x) - W(p, s(p, x), x)] \\ & \leq \frac{z}{b} [W(p, q, x) - W(p, s(p, x), x) + p(s(p, x) - q + b)], \end{aligned} \quad (48)$$

which holds for any  $z \geq 0$ . Since  $W$  is  $K$ -concave, we have

$$W(p, q + z, x) - K \leq W(p, q, x) + \frac{z}{s(p, x)} [W(p, q, x) - W(p, s(p, x), x)]. \quad (49)$$

Using this inequality and inequality (48) we have

$$\begin{aligned} & W(p, q + z, x) - K \leq W(p, q, x) \\ & + \frac{z}{b} [W(p, q, x) - W(p, s(p, x), x) + p(s(p, x) - q + b)]. \end{aligned} \quad (50)$$

Using the identity (44) and the definition of  $V$  in (27) we see that the above inequality is equivalent to the inequality defining  $K$ -concavity of  $V$  in (42). The final case is where  $q > s(p, x)$  and  $q$  satisfies

$$W(p, q, x) - pq > W(p, s(p, x), x) - ps(p, x). \quad (51)$$

Using this inequality and the definition of  $S(p, x)$  as the argmax of  $W(p, q, x) - pq$  in  $q$  we have

$$\begin{aligned} & W(p, q + z, x) - p(q + z) - K \\ & \leq W(p, S(p, x), x) - pS(p, x) - K \\ & = W(p, s(p, x), x) - ps(p, x) \\ & < W(p, q, x) - pq \\ & < W(p, q, x) - pq \\ & + \frac{z}{b} [W(p, q, x) - pq - W(p, s(p, x), x) + ps(p, x)]. \end{aligned} \quad (52)$$

Rearranging terms in the last inequality we obtain

$$W(p, q + z, x) - K < W(p, q, x) + \frac{z}{b} [W(p, q, x) - W(p, s(p, x), x) + p(s(p, x) - q + b)], \quad (53)$$

which is equivalent to the inequality defining the  $K$ -concavity of  $V$  in (42). The next key result, that  $\Lambda \circ \Gamma : \mathcal{F}_{KC} \rightarrow \mathcal{F}_{KC}$ , is a harder to establish, and requires an additional condition. Although it is possible prove this using a weaker sufficient condition (which we will discuss following our proof of Lemma 7), we prefer to use the no expected loss condition below since it is easy to verify and has a simple economic interpretation.  $\square$

**Assumption 5** (No expected loss condition) With probability 1 the following inequality holds:

$$\int_{p^r} \left[ p^r - \beta \int_{p'} \int_{x'} p' g(dp', dx'|p, x) \right] [1 - \eta(p^r, p, x)] \gamma(dp^r|p, x) \geq 0, \quad (54)$$

i.e. the set of  $(p, x)$  for which the conditional expectation above is non-negative has probability 1 for the Markov process  $\{p_t, x_t\}$  in each time period  $t$ .

**Lemma 7** *Assumptions 1–5 imply that  $\Lambda \circ \Gamma : \mathcal{F}_{KC} \rightarrow \mathcal{F}_{KC}$ . That is, if  $U$  is  $K$ -concave in  $q$ , then  $W = \Lambda(\Gamma(U))$  is  $K$ -concave in  $q$ .*

*Proof* By Lemma 6 if  $U \in \mathcal{F}_{KC}$ , then  $\Gamma(U) \in \mathcal{F}_{KC}$ . By Lemma 3, there exist functions  $S : P \times X \rightarrow R$  and  $s : P \times X \rightarrow R$  satisfying  $0 \leq s(p, x) \leq S(p, x) \leq \bar{q}$  for which  $\Gamma(U)$  can be represented as

$$\Gamma(U)(p, q, x) = \begin{cases} U(p, S(p, x), x) - p[S(p, x) - q] - K & \text{if } q \in [0, s(p, x)) \\ U(p, q, x) & \text{if } q \in [s(p, x), \bar{q}]. \end{cases} \quad (55)$$

Although  $\Gamma(U)$  is defined for  $q \in [0, \bar{q}]$ , it can be extended to a function  $V$  defined on  $(-\infty, \bar{q}]$  by

$$V(p, q, x) = \begin{cases} \Gamma(U)(p, 0, x) + pq & \text{if } q \in (-\infty, 0] \\ \Gamma(U)(p, q, x) & \text{otherwise.} \end{cases} \quad (56)$$

It is not difficult to see that the proof of Lemma 6 implies that  $V$  is  $K$ -concave over the entire interval  $(-\infty, \bar{q}]$ . Now consider the function  $\int_0^\infty V(p, q - q^r, x)h(q^r|p, x)dq^r$ . Since each translate  $V(p, q - q^r, x)$  is  $K$ -concave in  $q$  over the interval  $(-\infty, \bar{q}]$ , and since positive linear combinations and pointwise limits of  $K$ -concave functions are  $K$ -concave by Lemma 2, it follows that  $\int_0^\infty V(p, q - q^r, x)h(q^r|p, x)dq^r$  is  $K$ -concave in  $q$  on the interval  $(-\infty, \bar{q}]$ . We have

$$\begin{aligned}
& \int_0^{\infty} V(p, q - q^r, x)h(q^r | p, x)dq^r \\
&= \int_0^q V(p, q - q^r, x)h(q^r | p, x)dq^r + \int_q^{\infty} V(p, q - q^r, x)h(q^r | p, x)dq^r \\
&= \int_0^q V(p, q - q^r, x)h(q^r | p, x)dq^r \\
&\quad + V(p, 0, x) \int_q^{\infty} h(q^r | p, x)dq^r + p \int_q^{\infty} (q - q^r)h(q^r | p, x)dq^r.
\end{aligned}$$

Using equations (18) and (56), we have

$$\begin{aligned}
\Lambda \circ \Gamma(U)(p, q, x) &= \Lambda(V)(p, q, x) \\
&= ES(p, q, x) - EG(p, q, x) - c^h(p, q, x) + \beta EV(p, q, x) \\
&= ES(p, q, x) - EG(p, q, x) - c^h(p, q, x) \\
&\quad + \beta \int_{p'} \int_{x'} \int_{p'} \eta(p^r, p, x)V(p', q, x')\gamma(dp^r | p, x)g(dp', dx' | p, x) \\
&\quad + \beta \int_{p'} \int_{x'} \int_{p'} [1 - \eta(p^r, p, x)]V(p', 0, x') \\
&\quad \times \int_q^{\infty} h(q^r | p^r, p, x)dq^r \gamma(dp^r | p, x)g(dp', dx' | p, x) \\
&\quad + \beta \int_{p'} \int_{x'} \int_{p'} [1 - \eta(p^r, p, x)] \\
&\quad \times \int_0^q V(p', q - q^r, x')h(q^r | p^r, p, x)dq^r \gamma(dp^r | p, x)g(dp', dx' | p, x) \\
&\quad - \beta \int_{p'} \int_{x'} \int_{p'} [1 - \eta(p^r, p, x)] \\
&\quad \times \int_q^{\infty} p'(q - q^r)h(q^r | p, x)dq^r g(dp', dx' | p, x). \tag{57}
\end{aligned}$$

The sum of the fourth, fifth and sixth terms in the last equation in (57) (i.e. the three triple integrals except for the last one) is  $K$ -concave since they are a limits of convex combinations of  $K$ -concave functions (see Lemma 2). Since  $EG(p, q, x)$  is a convex function of  $q$  by Lemma 1 and  $c^h(p, q, x)$  is a convex function of  $q$  by

Assumption 3, a sufficient condition for the  $K$ -concavity of  $\Lambda \circ \Gamma(U)(p, q, x)$  is that the function

$$\begin{aligned}
 & ES(p, q, x) - \beta \int_{p'} \int_{x'} \int_{p'} [1 - \eta(p^r, p, x)] \\
 & \quad \times \int_q^\infty p'(q - q^r)h(q^r | p, x)dq^r g(dp', dx' | p, x), \tag{58}
 \end{aligned}$$

is concave in  $q$ . It is easy to see this function is continuously differentiable in  $q$  with second derivative given by:

$$\begin{aligned}
 & \nabla_{qq}^2 \left[ ES(p, q, x) - \beta \int_{p'} \int_{x'} \int_{p'} [1 - \eta(p^r, p, x)] \right. \\
 & \quad \left. \times \int_q^\infty p'(q - q^r)h(q^r | p, x)dq^r g(dp', dx' | p, x) \right] \\
 & = - \int_{p'} \left[ p^r - \beta \int_{p'} \int_{x'} p' g(dp', dx' | p, x) \right] \\
 & \quad \times [1 - \eta(p^r, p, x)] h(q | p^r, p, x) \gamma(dp^r | p, x). \tag{59}
 \end{aligned}$$

However,  $h(q | p^r, p, x) \geq \epsilon > 0$  by Assumption 2, so the no expected loss condition, Assumption 5, guarantees that expression on the right hand side of equation (59) is non-positive, and this enables us to conclude that  $W = \Lambda(V) = \Lambda\Gamma U$  is  $K$ -concave in  $q$  for any  $(p, x) \in P \times X$ .

Note that we could have proven Lemma 7 under the weaker condition that the function

$$\begin{aligned}
 & ES(p, q, x) - EG(p, q, x) - c^h(p, q, x) \\
 & \quad - \beta \int_{p'} \int_{x'} \int_{p'} [1 - \eta(p^r, p, x)] \int_q^\infty p'(q - q^r)h(q^r | p, x)dq^r g(dp', dx' | p, x), \tag{60}
 \end{aligned}$$

is concave in  $q$  for all  $(p, x) \in P \times X$ . Assuming that  $c^h$  is twice continuously differentiable, a sufficient condition for this to hold is that the hessian of this function is negative. This leads to the following more general version of the no expected loss condition:

$$\begin{aligned}
 & \int_{p'} \left[ p^r + c^g(p^r, p, x) - \beta \int_{p'} \int_{x'} p' g(dp', dx' | p, x) \right] \\
 & \quad \times [1 - \eta(p^r, p, x)] \gamma(dp^r | p, x) - \nabla_{qq} c^h(p, q, x) \geq 0. \tag{61}
 \end{aligned}$$

By Assumption 3,  $c^g(p^r, p, x) \geq 0$ , and  $\nabla_{qq}c^h(p, q, x) \leq 0$ , so we see that our formulation of the no expected loss condition in Assumption 5 is stronger than necessary to prove our result. Thus, we do not claim that we have found the weakest possible conditions under which our results can be proved, however, since it is easier to verify and provide a simple economic interpretation for the more restrictive form of the no expected loss condition in Assumption 5, we have opted to stress this version since as we have noted above it is likely to be satisfied in many situations.  $\square$

**Lemma 8** *Under Assumptions 0–5, the functions  $V$  and  $W$ , the unique fixed points of the contraction mappings given in Lemma 5, are  $K$ -concave functions of  $q$  for all  $(p, x) \in P \times X$ .*

*Proof* We prove this by induction using Lemmas 6 and 7. Since  $\Gamma \circ \Lambda$  is a contraction mapping, the fixed point  $V = \Gamma \circ \Lambda(V)$  can be uniformly approximated by the method of successive approximations starting from an initial guess,  $V_0 = 0$ . We have  $\Lambda V_0(p, q, x) = ES(p, q, x) - c^h(p, q, x)$  is concave in  $q$  by Assumption 2 and Lemma 1. Since concave functions are automatically  $K$ -concave, we have that  $\Lambda V_0 \in \mathcal{F}_{KC}$ . Lemma 6 implies that  $V_1 = \Gamma \Lambda V_0 \in \mathcal{F}_{KC}$ . Lemma 7 implies that  $\Lambda V_1 = (\Lambda \circ \Gamma)\Lambda V_0 \in \mathcal{F}_{KC}$ . Continuing inductively, we see that for each  $t \geq 0$  in the sequence of successive approximations,  $V_t \in \mathcal{F}_{KC}$ . Since the fixed point  $V$  is a uniform limit of functions in  $\mathcal{F}_{KC}$  it follows that  $V \in \mathcal{F}_{KC}$ . Since  $W = \Lambda \Gamma V$ , Lemma 7 also implies that  $W \in \mathcal{F}_{KC}$ .

Lemmas 1–8 constitute the proof of our main result.  $\square$

**Theorem 1** *Consider the function  $W(p, q + q^o, x)$  defined in equation (17), where  $W$  is defined in terms of the unique solution  $V$  to Bellman’s equation (16). Under Assumptions 0–5, for any  $(p, x) \in P \times X$  the functions  $V$  and  $W$  are  $K$ -concave in  $q$ , and the speculator’s optimal inventory investment policy  $q^o(p, q, x)$  takes the form of an  $(S, s)$  policy. That is, there exist a pair of functions  $(S, s)$  satisfying  $S(p, x) \geq s(p, x)$  where  $S(p, x)$  is the target inventory level and  $s(p, x)$  is the inventory order threshold, i.e.*

$$q^o(p, q, x) = \begin{cases} 0 & \text{if } q \geq s(p, x) \\ S(p, x) - q & \text{otherwise,} \end{cases} \tag{62}$$

where  $S(p, x)$  is given by:

$$S(p, x) = \operatorname{argmax}_{0 \leq q^o \leq \bar{q} - q} [W(p, q^o, x) - c^o(p, q^o, x)], \tag{63}$$

and the lower inventory order limit  $s(p, x)$  is the value of  $q$  that makes the speculator indifferent between ordering and not ordering more inventory:

$$s(p, x) = \inf \{q \in [0, \bar{q}] | W(p, q, x) \geq W(p, S(p, x), x) - p[S(p, x) - q] - K\}. \tag{64}$$

**Corollary** *If fixed costs of placing orders are zero,  $K = 0$ , then the minimum order size is 0, i.e.*

$$S(p, x) = s(p, x). \tag{65}$$

**Theorem 2** *Suppose that  $W(p, q, x)$  is twice continuously differentiable in  $(p, q)$ . Then  $S(p, x)$  is a decreasing function of  $p$  iff the shadow price of inventory increases at a slower rate than the wholesale price  $p$ , i.e. iff*

$$1 > \nabla_{pq}^2 W(p, q, x) \quad \text{at } q = S(p, x). \quad (66)$$

*Proof* Totally differentiating the first order (Euler) equation for  $S(p, x)$  and solving for  $\nabla_p S(p, x)$  we get

$$\nabla_p S(p, x) = \frac{1 - \nabla_{pq}^2 W(p, q, x)}{\nabla_q^2 W(p, q, x)}, \quad q = S(p, x). \quad (67)$$

The denominator is strictly negative since  $q = S(p, x)$  is a global optimum of the  $W$  function. Thus the sign of  $\nabla_p S(p, x)$  depends on the sign of  $1 - \nabla_{pq}^2 W(p, q, x)$ .  $\square$

One can interpret  $S(p, x)$  as the “target demand function” for inventory. As with ordinary demand functions, we would expect that the target demand function should be downward sloping in price. Theorem 2 provides a sufficient condition for this to be the case. Note that  $\nabla_q W$  is the shadow value of an additional unit of inventory. Thus,  $\nabla_{pq}^2 W$  represents how this shadow value changes when the underlying wholesale price  $p$  increases. Theorem 2 tells us that if the shadow value of inventory holdings increases at a slower rate than the wholesale price at which new inventories can be purchased,  $\nabla_{pq}^2 W < 1$ , then  $S(p, x)$  is a decreasing function of  $p$ . The intuition for this result is that if purchases of the commodity are not subject to “free disposal” (i.e. speculator cannot sell back any excess inventory at the current wholesale price  $p$ ), then it is not necessarily the case that  $W(p, q, x) \geq pq$ . If there are also costly delays to waiting for retail customers to purchase any excess inventory the speculator may have previously acquired (due, for example, to costs of holding inventory), then we would also expect that  $\nabla_{pq}^2 W < 1$ , i.e. an increase in the unit wholesale price of the commodity will increase the incremental value of an additional unit of inventory by less than the increase in the wholesale price. If this is the case, then as wholesale prices increase, the marginal cost of acquiring an additional unit of inventory is not matched by a corresponding increase in the marginal value of being able to sell that unit in the retail market. This implies that the speculator will want to hold less inventory as the wholesale price increases, i.e. the target inventory demand function  $S(p, x)$  is downward sloping in  $p$ .

### 3 Example: Scarf’s inventory model

We conclude the paper by illustrating how our results apply to the model of optimal inventory investment studied by Scarf (1960). Scarf formulated the inventory problem as a cost minimization problem. However, if it is recast as a profit maximization problem, then it is easily seen to be a special case of our framework where the vector of state variables  $x$  does not enter the model,  $\eta = 0$ , the wholesale price  $p$  is a non-random constant, and the retail price equals a non-random constant  $p^r$ . Thus the only state variable is  $q$ .

Scarf considered the case where unfilled inventory can be backlogged, represented by negative inventory levels  $q < 0$ . We assume orders cannot be backlogged

and impose the non-negativity constraint  $q \geq 0$ . An  $(S, s)$  policy in this case is simply two scalars  $s$  and  $S$  satisfying  $s \leq S$  where  $S$  is given by

$$S = \operatorname{argmax}_{0 \leq q \leq \bar{q}} [W(q) - pq], \quad (68)$$

and  $s$  is given by

$$s = \inf \{q \in [0, S] \mid W(q) \geq W(S) - p[S - q] - K\}. \quad (69)$$

It is easy to see that the expected sales and value functions  $ES$  and  $EV$  are given by

$$ES(q) = p^r \left[ \int_0^q q^r h(q^r) dq^r + q \int_q^\infty h(q^r) dq^r \right],$$

$$EV(q) = V(0) \int_q^\infty h(q^r) dq^r + \int_0^q V(q - q^r) h(q^r) dq^r. \quad (70)$$

The function  $W$ , the value of holding inventory  $q$ , is given by:

$$W(q) = ES(q) - EG(q) - c^h(q) + \beta EV(q). \quad (71)$$

Bellman's equation is given by

$$V(q) = \max_{0 \leq q^o \leq \bar{q} - q} [W(q + q^o) - c^o(q^o)], \quad (72)$$

where  $c^o$  is the order cost function (Assumption 4). The no expected loss condition (Assumption 5) reduces to

$$p^r \geq \beta p. \quad (73)$$

If the other regularity conditions in Assumptions 1–5 hold, Theorem 1 guarantees that  $V$  and  $W$  will be  $K$ -concave in  $q$  and the  $(S, s)$  policy is an optimal trading strategy.

## References

- Arrow, K.J., Harris, T., Marschak, J.: Optimal inventory policy. *Econometrica* **19**(3), 250–272 (1951)
- Bertsekas, D.P.: Dynamic programming and optimal control. Belmont, MA: Athena Scientific 1995
- Cheng, F., Sethi, S.: Optimality of state-dependent  $(s, S)$  policies in inventory models with markov-modulated demand and lost sales. *Prod Oper Manage* **8**(2), 183–192 (1999)
- Hall, G. and J. Rust: An empirical model of inventory investment by durable commodity intermediaries. *Carn Roch Conf Ser Publ Policy* **52**, 171–214 (2000)
- Hall, G., Rust, J.: Simulated minimum distance estimation of a model of commodity price speculation with endogenously sampled prices manuscript. New Haven, Yale University (2005)
- Holt, C. Modigliani, F., Muth, J., Simon, H.: Planning production, inventories and work force. Englewood Cliffs, NJ: Prentice Hall 1960
- Miranda, M., Rui, X.: An empirical reassessment of the nonlinear rational expectations commodity storage model manuscript. Ohio, Ohio State University (1999)

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- Scarf, H.: The optimality of  $(S, s)$  policies in the dynamic inventory problem. In: Arrow, K., Karlin, S., Suppes, P. (eds.) *Mathematical methods in the social sciences*. Stanford, CA: Stanford University Press 1960
- Sethi, S., Cheng, F.: Optimality of  $(s, S)$  policies in inventory models with markovian demand. *Oper. Res.* **45**(6), 931–939 (1997)
- Williams, J.C., Wright, B.: *Storage and commodity markets*. New York: Cambridge University Press 1991
- Working, H.: Theory of price and storage. *Am Econ Rev* **39**, 1254–1262 (1949)