When to Sell the Car? Actual Replacement Decisions of a Rental Car Company versus Predictions of an Optimal Stopping Model

Sungjin Cho
Hanyang University
sungjcho@hanyang.ac.kr

John Rust
University of Maryland
jrust@gemini.econ.umd.edu
1.1 Summary
What we did

1. We analyzed the vehicle replacement decisions by a rental car company.
1. We analyzed the vehicle replacement decisions by a rental car company.

2. Our goal: to “test” whether this firm is profit-maximizing
What we did

1. We analyzed the vehicle replacement decisions by a rental car company.

2. Our goal: to “test” whether this firm is profit-maximizing

3. The firm is highly successful: its pre-tax IRR on car investments $\sim 50\%$
What we did

1. We analyzed the vehicle replacement decisions by a rental car company.
2. Our goal: to “test” whether this firm is profit-maximizing
3. The firm is highly successful: its pre-tax IRR on car investments $\sim 50\%$
4. Nevertheless, we present evidence that the firm is *not* maximizing profits
1. We analyzed the vehicle replacement decisions by a rental car company.
2. Our goal: to “test” whether this firm is profit-maximizing.
3. The firm is highly successful: its pre-tax IRR on car investments $\sim 50\%$
4. Nevertheless, we present evidence that the firm is \textit{not} maximizing profits.
5. We show that an alternative operating strategy can increase profits from 6 to 140\%, depending on vehicle type.
What we did

1. We analyzed the vehicle replacement decisions by a rental car company.
2. Our goal: to “test” whether this firm is profit-maximizing.
3. The firm is highly successful: its pre-tax IRR on car investments $\sim 50\%$
4. Nevertheless, we present evidence that the firm is \textit{not} maximizing profits.
5. We show that an alternative operating strategy can increase profits from 6 to 140%, depending on vehicle type.
6. The alternative strategy: keep cars longer, and allow customers to choose new or old vehicles from a “menu” with discounts for older vehicles.
1. We model the history of a rental car as a realization of a \textit{semi-Markov process}.
1. We model the history of a rental car as a realization of a semi-Markov process.

2. A rental car can be in one of three possible states:
1. We model the history of a rental car as a realization of a *semi-Markov process*.

2. A rental car can be in one of three possible states:
   - In a *lot spell*, waiting to be rented
1. We model the history of a rental car as a realization of a *semi-Markov process*.

2. A rental car can be in one of three possible states:
   - In a **lot spell**, waiting to be rented
   - In a **short term rental spell**
Conceptual Framework

1. We model the history of a rental car as a realization of a semi-Markov process.

2. A rental car can be in one of three possible states:
   - In a *lot spell*, waiting to be rented
   - In a *short term rental spell*
   - In a *long term rental spell*
1. We model the history of a rental car as a realization of a semi-Markov process.

2. A rental car can be in one of three possible states:
   - In a **lot spell**, waiting to be rented
   - In a **short term rental spell**
   - In a **long term rental spell**

3. We analyze three different types of vehicles in the company fleet
1. We model the history of a rental car as a realization of a *semi-Markov process*.

2. A rental car can be in one of three possible states:
   - In a *lot spell*, waiting to be rented
   - In a *short term rental spell*
   - In a *long term rental spell*

3. We analyze three different types of vehicles in the company fleet
   - A compact vehicle
Conceptual Framework

1. We model the history of a rental car as a realization of a semi-Markov process.

2. A rental car can be in one of three possible states:
   - In a **lot spell**, waiting to be rented
   - In a **short term rental spell**
   - In a **long term rental spell**

3. We analyze three different types of vehicles in the company fleet
   - A compact vehicle
   - A luxury sedan
Conceptual Framework

1. We model the history of a rental car as a realization of a *semi-Markov process*.

2. A rental car can be in one of three possible states:
   - In a *lot spell*, waiting to be rented
   - In a *short term rental spell*
   - In a *long term rental spell*

3. We analyze three different types of vehicles in the company fleet
   - A compact vehicle
   - A luxury sedan
   - A recreational vehicle (RV)
Conceptual Framework

1. We model the history of a rental car as a realization of a *semi-Markov process*.

2. A rental car can be in one of three possible states:
   - In a *lot spell*, waiting to be rented
   - In a *short term rental spell*
   - In a *long term rental spell*

3. We analyze three different types of vehicles in the company fleet
   - A compact vehicle
   - A luxury sedan
   - A recreational vehicle (RV)

4. Unfortunately, due to confidentiality restrictions on the data, we are not at liberty to disclose the name of the company and cannot provide much more detail on the exact makes/models of these cars or their locations.
1. We use econometric methods for *duration* and *transition* models
1. We use econometric methods for *duration* and *transition* models

2. We estimate hazard functions for spell durations non-parametrically
Econometric Methodology

1. We use econometric methods for *duration* and *transition* models

2. We estimate hazard functions for spell durations non-parametrically

3. We estimate transitions between spells using a *trinomial logit* model
Econometric Methodology

1. We use econometric methods for *duration* and *transition* models
2. We estimate hazard functions for spell durations non-parametrically
3. We estimate transitions between spells using a *trinomial logit* model
4. We use regression analysis to predict resale prices of vehicles
Econometric Methodology

1. We use econometric methods for duration and transition models

2. We estimate hazard functions for spell durations non-parametrically

3. We estimate transitions between spells using a trinomial logit model

4. We use regression analysis to predict resale prices of vehicles

5. We use a parametric (Erlang) distribution to model vehicle usage (i.e. kilometers driven) during rental contracts.
1. We use econometric methods for duration and transition models
2. We estimate hazard functions for spell durations non-parametrically
3. We estimate transitions between spells using a trinomial logit model
4. We use regression analysis to predict resale prices of vehicles
5. We use a parametric (Erlang) distribution to model vehicle usage (i.e. kilometers driven) during rental contracts.
6. We also model maintenance costs,
Econometric Methodology

1. We use econometric methods for *duration* and *transition* models
2. We estimate hazard functions for spell durations non-parametrically
3. We estimate transitions between spells using a *trinomial logit* model
4. We use regression analysis to predict resale prices of vehicles
5. We use a parametric (Erlang) distribution to model vehicle usage (i.e. kilometers driven) during rental contracts.
6. We also model *maintenance costs*,
7. and estimate a binomial logit model of the firm’s *selling decision*
1. We use econometric methods for *duration* and *transition* models

2. We estimate hazard functions for spell durations non-parametrically

3. We estimate transitions between spells using a *trinomial logit* model

4. We use regression analysis to predict resale prices of vehicles

5. We use a parametric (Erlang) distribution to model vehicle usage (i.e. kilometers driven) during rental contracts.

6. We also model *maintenance costs*

7. and estimate a binomial logit model of the firm’s *selling decision*

8. We then have all objects necessary to *simulate* the rental operations of this company
1.2 Our Findings
Main Findings

1. We find that our simulation model provides a good approximation to the actual outcomes for this firm.
Main Findings

1. We find that our simulation model provides a good approximation to the actual outcomes for this firm.

2. In particular, our simulation model matches the high internal rates of return that this company earns under the *status quo*. 
1. We find that our simulation model provides a good approximation to the actual outcomes for this firm.

2. In particular, our simulation model matches the high internal rates of return that this company earns under the status quo.

3. However the value of having an econometric/simulation model is that we can evaluate the profitability of a wide range of alternative operating strategies.
Main Findings

1. We find that our simulation model provides a good approximation to the actual outcomes for this firm.

2. In particular, our simulation model matches the high internal rates of return that this company earns under the status quo.

3. However, the value of having an econometric/simulation model is that we can evaluate the profitability of a wide range of alternative operating strategies.

4. We formulate the optimal replacement problem and show that it is equivalent to a regenerative optimal stopping problem.
Main Findings

1. We find that our simulation model provides a good approximation to the actual outcomes for this firm.

2. In particular, our simulation model matches the high internal rates of return that this company earns under the status quo.

3. However the value of having an econometric/simulation model is that we can evaluate the profitability of a wide range of alternative operating strategies.

4. We formulate the optimal replacement problem and show that it is equivalent to a regenerative optimal stopping problem.

5. We solve the stopping problem numerically and characterize compare the optimal replacement policy to the firm’s current replacement policy.
1. We find that the predictions of the optimal stopping model are sensitive to the specification of *aging effects*.
1. We find that the predictions of the optimal stopping model are sensitive to the specification of *aging effects*.

2. A key aging effect is the *rapid depreciation in vehicle resale values*.
1. We find that the predictions of the optimal stopping model are sensitive to the specification of aging effects.

2. A key aging effect is the rapid depreciation in vehicle resale values.

3. However over the range of our observations, it is difficult to detect other significant aging effects.
Vehicle Aging Effects are ...

1. We find that the predictions of the optimal stopping model are sensitive to the specification of **aging effects**.

2. A key aging effect is the *rapid depreciation in vehicle resale values*.

3. However over the range of our observations, it is difficult to detect other significant aging effects.

4. In particular, rental rates, maintenance costs, and durations of lot spells and rental spells show no evidence of aging effects.
1. We find that the predictions of the optimal stopping model are sensitive to the specification of \textit{aging effects}.

2. A key aging effect is the \textit{rapid depreciation in vehicle resale values}.

3. However over the range of our observations, it is difficult to detect other significant aging effects.

4. In particular, rental rates, maintenance costs, and durations of lot spells and rental spells show no evidence of aging effects.

5. The only aging effect that we can detect is a \textit{rental contract composition effect}.
1. That is, new cars tend to start off in long term rental contracts, but as they age, there is an increasing chance the contracts will switch from long term to short term.
1. That is, new cars tend to start off in long term rental contracts, but as they age, there is an increasing chance the contracts will switch from long term to short term.

2. Durations on the lot tend to be longer between short term rentals since long term rentals have a high chance of rollover.
1. That is, new cars tend to start off in long term rental contracts, but as they age, there is an increasing chance the contracts will switch from long term to short term.

2. Durations on the lot tend to be longer between short term rentals since long term rentals have a high chance of rollover.

3. Thus, the fraction of time spent on the lot tends to increase as a vehicle ages (i.e. as its odometer increases).
1. That is, new cars tend to start off in long term rental contracts, but as they age, there is an increasing chance the contracts will switch from long term to short term.

2. Durations on the lot tend to be longer between short term rentals since long term rentals have a high chance of *rollover*.

3. Thus, the fraction of time spent on the lot tends to increase as a vehicle ages (i.e. as its odometer increases).

4. It is difficult to detect aging effects for very old cars, because few such cars can be observed under the *status quo*. 
hard to detect in young cars

1. That is, new cars tend to start off in long term rental contracts, but as they age, there is an increasing chance the contracts will switch from long term to short term.

2. Durations on the lot tend to be longer between short term rentals since long term rentals have a high chance of rollover.

3. Thus, the fraction of time spent on the lot tends to increase as a vehicle ages (i.e. as its odometer increases).

4. It is difficult to detect aging effects for very old cars, because few such cars can be observed under the status quo.

5. The average age at sale is about 3 years, and the mean odometer at time of sale is about 70,000 kilometers.
1. That is, new cars tend to start off in long term rental contracts, but as they age, there is an increasing chance the contracts will switch from long term to short term.

2. Durations on the lot tend to be longer between short term rentals since long term rentals have a high chance of rollover.

3. Thus, the fraction of time spent on the lot tends to increase as a vehicle ages (i.e. as its odometer increases).

4. It is difficult to detect aging effects for very old cars, because few such cars can be observed under the status quo.

5. The average age at sale is about 3 years, and the mean odometer at time of sale is about 70,000 kilometers.

6. We observe only a very few cars that are over 5 years old or whose odometers have more than 140,000 kilometers.
1.3 Implications
1. If the only aging effects are resale price depreciation and the gradual switch from long term to short term contracts,
Implications for Replacement

1. If the only aging effects are resale price depreciation and the gradual switch from long term to short term contracts,

2. *the optimal policy is to never sell the car!*
Implications for Replacement

1. If the only aging effects are resale price depreciation and the gradual switch from long term to short term contracts,
2. the optimal policy is to never sell the car!
3. Intuition for result:
Implications for Replacement

1. If the only aging effects are resale price depreciation and the gradual switch from long term to short term contracts,
2. the optimal policy is to never sell the car!
3. Intuition for result:
   ■ replacement of cars is a costly “investment” precisely due to the rapid price depreciation.
1. If the only aging effects are resale price depreciation and the gradual switch from long term to short term contracts,

2. *the optimal policy is to never sell the car!*

3. Intuition for result:
   - replacement of cars is a costly “investment” precisely due to the rapid price depreciation.
   - if rental rates and maintenance costs do not decline with age/odometer, then it is optimal to “amortize” the initial investment in a vehicle by keeping and maintaining it indefinitely.
Implications for Replacement

1. If the only aging effects are resale price depreciation and the gradual switch from long term to short term contracts,

2. *the optimal policy is to never sell the car!*

3. Intuition for result:
   - replacement of cars is a costly “investment” precisely due to the rapid price depreciation.
   - if rental rates and maintenance costs do not decline with age/odometer, then it is optimal to “amortize” the initial investment in a vehicle by keeping and maintaining it indefinitely.

4. Of course, it is unreasonable to suppose that rental rates would not decrease if the company kept its vehicle stock indefinitely
1. If the only aging effects are resale price depreciation and the gradual switch from long term to short term contracts,

2. *the optimal policy is to never sell the car!*

3. Intuition for result:
   - replacement of cars is a costly “investment” precisely due to the rapid price depreciation.
   - if rental rates and maintenance costs do not decline with age/odometer, then it is optimal to “amortize” the initial investment in a vehicle by keeping and maintaining it indefinitely.

4. Of course, it is unreasonable to suppose that rental rates would not decrease if the company kept its vehicle stock indefinitely

5. *Customers prefer new cars, all other things equal!*
1. Thus, we face an *extrapolation problem*: we do not observe the company keeping cars very long in the data.
1. Thus, we face an *extrapolation problem*: we do not observe the company keeping cars very long in the data.

2. Lacking data on maintenance costs and rental histories for very old cars, it is hazardous to make *out of sample policy predictions* about what will have if the company keeps cars far longer than it currently does.
1. Thus, we face an *extrapolation problem*: we do not observe the company keeping cars very long in the data.

2. Lacking data on maintenance costs and rental histories for very old cars, it is hazardous to make *out of sample policy predictions* about what will have if the company keeps cars far longer than it currently does.

3. Our approach: make *pessimistic assumptions* about increases in maintenance costs and required *discounts* on older cars.
The Extrapolation Problem

1. Thus, we face an *extrapolation problem*: we do not observe the company keeping cars very long in the data.

2. Lacking data on maintenance costs and rental histories for very old cars, it is hazardous to make *out of sample policy predictions* about what will have if the company keeps cars far longer than it currently does.

3. Our approach: make *pessimistic assumptions* about increases in maintenance costs and required *discounts* on older cars.

4. Assume that beyond 130,000 kilometers average daily maintenance costs increase rapidly, increasing by a factor of 11 by the time the odometer reaches 400,000 kilometers.
The Extrapolation Problem

1. Thus, we face an *extrapolation problem*: we do not observe the company keeping cars very long in the data.

2. Lacking data on maintenance costs and rental histories for very old cars, it is hazardous to make *out of sample policy predictions* about what will have if the company keeps cars far longer than it currently does.

3. Our approach: make *pessimistic assumptions* about increases in maintenance costs and required *discounts* on older cars.

4. Assume that beyond 130,000 kilometers average daily maintenance costs increase rapidly, increasing by a factor of 11 by the time the odometer reaches 400,000 kilometers.

5. We also assume that with appropriate *odometer-based discounts* on rental vehicles, customers can be induced to rent older vehicles.
The “Pessimistic Scenario”

1. Assume that daily rental rates are flat until 130,000 kilometers, but decrease linearly with odometer thereafter, until rates hit 0 at 400,000 kilometers.
The “Pessimistic Scenario”

1. Assume that daily rental rates are flat until 130,000 kilometers, but decrease linearly with odometer thereafter, until rates hit 0 at 400,000 kilometers.

2. Even under these unrealistically pessimistic assumptions, we find that it is still optimal to keep vehicles roughly twice as long as the company currently keeps them.
The “Pessimistic Scenario”

1. Assume that daily rental rates are flat until 130,000 kilometers, but decrease linearly with odometer thereafter, until rates hit 0 at 400,000 kilometers.

2. Even under these unrealistically pessimistic assumptions, we find that it is still optimal to keep vehicles roughly twice as long as the company currently keeps them.

3. The *optimal replacement threshold* in the pessimistic scenario is 150,000 kilometers, about twice as large as under the *status quo*. 
1. Assume that daily rental rates are flat until 130,000 kilometers, but decrease linearly with odometer thereafter, until rates hit 0 at 400,000 kilometers.

2. Even under these unrealistically pessimistic assumptions, we find that it is still optimal to keep vehicles roughly twice as long as the company currently keeps them.

3. The optimal replacement threshold in the pessimistic scenario is 150,000 kilometers, about twice as large as under the status quo.

4. Expected discounted profits increase significantly. Depending on the type of car, we predict profits will be between 18-240 percent larger than the status quo if it adopts the optimal replacement policy.
1. We recommend that the company undertake an *experiment* with cars assigned to the *treatment group* kept longer and rental rates are discounted after a certain age/odometer threshold.
1. We recommend that the company undertake an experiment with cars assigned to the *treatment group* kept longer and rental rates are discounted after a certain age/odometer threshold.

2. The *treatment effect* is the increase in profit/internal rate of return in the treatment group relative to the *control group* (i.e. the *status quo* operating policy).
1. We recommend that the company undertake an experiment with cars assigned to the treatment group kept longer and rental rates are discounted after a certain age/odometer threshold.

2. The treatment effect is the increase in profit/internal rate of return in the treatment group relative to the control group (i.e. the status quo operating policy).

3. Drawback of experiments: they are costly and time consuming, and may contaminate customers who receive discounts, leading them to expect similar discounts at other locations.
1. We recommend that the company undertake an **experiment** with cars assigned to the **treatment group** kept longer and rental rates are discounted after a certain age/odometer threshold.

2. The **treatment effect** is the increase in profit/internal rate of return in the treatment group relative to the **control group** (i.e. the *status quo* operating policy).

3. Drawback of experiments: they are costly and time consuming, and may **contaminate** customers who receive discounts, leading them to expect similar discounts at other locations.

4. In the absence of experimental data, we believe **model-based predictions** such as ours, can be useful devices to help a company evaluate the profitability of its current operating strategy.
1.4 Outline of Talk
Section 2: Description of the Data, Reduced-form Approaches to Evaluating Profitability
Road Map

- Section 2: Description of the Data, Reduced-form Approaches to Evaluating Profitability

- Section 3: Econometric Model and Empirical Results
1. Introduction

1.4 Outline of Talk

- Road Map

- Section 2: Description of the Data, Reduced-form Approaches to Evaluating Profitability

- Section 3: Econometric Model and Empirical Results

- Section 4: Validation/Goodness of Fit: Comparison of Simulated and Actual Outcomes
Section 2: Description of the Data, Reduced-form Approaches to Evaluating Profitability

Section 3: Econometric Model and Empirical Results

Section 4: Validation/Goodness of Fit: Comparison of Simulated and Actual Outcomes

Section 5: Characterization of the Optimal Replacement Policy
Road Map

- Section 2: Description of the Data, Reduced-form Approaches to Evaluating Profitability
- Section 3: Econometric Model and Empirical Results
- Section 4: Validation/Goodness of Fit: Comparison of Simulated and Actual Outcomes
- Section 5: Characterization of the Optimal Replacement Policy
- Section 6: Comparison of Actual versus the Optimal Strategy in the “Pessimistic Scenario”
Road Map

- Section 2: Description of the Data, Reduced-form Approaches to Evaluating Profitability
- Section 3: Econometric Model and Empirical Results
- Section 4: Validation/Goodness of Fit: Comparison of Simulated and Actual Outcomes
- Section 5: Characterization of the Optimal Replacement Policy
- Section 6: Comparison of Actual versus the Optimal Strategy in the “Pessimistic Scenario”
- Section 7: Extending the Analysis: Rental Fleet “Portfolio Management”
Section 2: Description of the Data, Reduced-form Approaches to Evaluating Profitability

Section 3: Econometric Model and Empirical Results

Section 4: Validation/Goodness of Fit: Comparison of Simulated and Actual Outcomes

Section 5: Characterization of the Optimal Replacement Policy

Section 6: Comparison of Actual versus the Optimal Strategy in the “Pessimistic Scenario”

Section 7: Extending the Analysis: Rental Fleet “Portfolio Management”

Section 8: Conclusions
2.1 Data Description
1. Computerized records on over 3900 vehicles acquired after 1999
What the company gave us

1. Computerized records on over 3900 vehicles acquired after 1999

2. The company generally acquires cars via *individual purchases* rather than *block acquisitions*
What the company gave us

1. Computerized records on over 3900 vehicles acquired after 1999
2. The company generally acquires cars via *individual purchases* rather than *block acquisitions*
3. The data consist of
1. Computerized records on over 3900 vehicles acquired after 1999

2. The company generally acquires cars via *individual purchases* rather than *block acquisitions*

3. The data consist of
   - date and purchase price for each vehicle
What the company gave us

1. Computerized records on over 3900 vehicles acquired after 1999
2. The company generally acquires cars via individual purchases rather than block acquisitions
3. The data consist of
   - date and purchase price for each vehicle
   - date and odometer value when the vehicle was sold
What the company gave us

1. Computerized records on over 3900 vehicles acquired after 1999
2. The company generally acquires cars via *individual purchases* rather than *block acquisitions*
3. The data consist of
   - date and purchase price for each vehicle
   - date and odometer value when the vehicle was sold
   - the complete history of maintenance and rentals between the purchase and sale dates.
1. While the contract data record the correct begin and end dates for each contract,
Problems with the data

1. While the contract data record the correct begin and end dates for each contract,

2. the in and out odometer values were frequently rounded, missing, or “guestimated”
Problems with the data

1. While the contract data record the correct begin and end dates for each contract,

2. the in and out odometer values were frequently rounded, missing, or “guestimated”

3. The problem is especially acute for long term contracts which frequently “roll over”: rental agents appear to supply rough guess of odometer values at the rollover dates
1. While the contract data record the correct begin and end dates for each contract,

2. the in and out odometer values were frequently rounded, missing, or “guestimated”

3. The problem is especially acute for long term contracts which frequently “roll over”: rental agents appear to supply rough guess of odometer values at the rollover dates

4. As a result, we did not trust most of the in/out odometer readings in the company’s rental records.
1. While the contract data record the correct begin and end dates for each contract,

2. the in and out odometer values were frequently rounded, missing, or “guestimated”

3. The problem is especially acute for long term contracts which frequently “roll over”: rental agents appear to supply rough guess of odometer values at the rollover dates.

4. As a result, we did not trust most of the in/out odometer readings in the company’s rental records.

5. In order to infer the driving patterns and number of kilometers typically travelled during each rental contract we relied on some (we believe reasonable) econometric modelling assumptions that we will describe shortly.
When to sell the car?

1. Introduction

2. Data

2.1 Data Description

● What the company gave us
● Problems with the data
● Accident data

3. An Econometric Model

4. Simulation Results

5. Optimal Replacement Policy

6. Numerical Results

7. Extensions

8. Conclusions

Accident data

1. The company also provided us with records on the date of accidents and the cost of repairing accident damage, as well as decisions to scrap totalled vehicles.
1. The company also provided us with records on the date of accidents and the cost of repairing accident damage, as well as decisions to scrap totalled vehicles.

2. Although 2543 of the 3908 vehicles in the data set experienced one or more accidents over the service lives, only 123 vehicles were sufficiently badly damaged that they had to be scrapped.
1. The company also provided us with records on the date of accidents and the cost of repairing accident damage, as well as decisions to scrap totalled vehicles.

2. Although 2543 of the 3908 vehicles in the data set experienced one or more accidents over the service lives, only 123 vehicles were sufficiently badly damaged that they had to be scrapped.

3. In almost all cases where accidents have occurred, the cost of repairing the damage to the vehicle is covered by the insurance of the renter (if the renter was at fault), the insurance of the other party to the accident (if they were at fault), or by the company’s insurance (if the party at fault has no insurance).
Accident data

1. The company also provided us with records on the date of accidents and the cost of repairing accident damage, as well as decisions to scrap totalled vehicles.

2. Although 2543 of the 3908 vehicles in the data set experienced one or more accidents over the service lives, only 123 vehicles were sufficiently badly damaged that they had to be scrapped.

3. In almost all cases where accidents have occurred, the cost of repairing the damage to the vehicle is covered by the insurance of the renter (if the renter was at fault), the insurance of the other party to the accident (if they were at fault), or by the company’s insurance (if the party at fault has no insurance).

4. As a result, our analysis ignores accidents since these losses are nearly fully insured.
2.2 Analyzing Rentals
1. The firm rents its cars on two types of contracts
1. The firm rents its cars on two types of contracts
   - *long term contracts* with typical durations of 30 days,
1. The firm rents its cars on two types of contracts

- **long term contracts** with typical durations of 30 days,
- **short term contract** with typical durations of 3-4 days.
1. The firm rents its cars on two types of contracts
   - *long term contracts* with typical durations of 30 days,
   - *short term contract* with typical durations of 3-4 days.

2. Customers are allowed to *roll over* a 30 day long term contract into a *defacto* equivalent of a long term lease.
Rental Contracts

1. The firm rents its cars on two types of contracts
   - *long term contracts* with typical durations of 30 days,
   - *short term contract* with typical durations of 3-4 days.

2. Customers are allowed to *roll over* a 30 day long term contract into a *defacto* equivalent of a long term lease.

3. There is a penalty for early returns of vehicles in long term contracts, generally equal to 20% of the lost rental revenue for the unfinished remaining days in the contract.
Typical Rental Histories

Rental History for Compact, urban location (service life: 1115 days, IRR=78.6%)

Rental History for Luxury 1, urban location (service life: 995 days, IRR=57.6%)

Rental History for RV, tourist location (service life: 810 days, IRR=96.6%)
1. Recall, the IRR is the $r$ that solves

$$0 = \sum_{t=0}^{T} \exp\left\{-a_t r / 365\right\} c_t,$$

where $c_t$ is the cash inflow (if positive) or outflow (if negative), and $a_t$ is the number of days after the initial purchase of the vehicle that the $t^{th}$ cash flow occurred.
1. Recall, the IRR is the $r$ that solves

$$0 = \sum_{t=0}^{T} \exp\{-a_t r/365\} c_t,$$

where $c_t$ is the cash inflow (if positive) or outflow (if negative), and $a_t$ is the number of days after the initial purchase of the vehicle that the $t^{th}$ cash flow occurred.

2. Thus, $c_0 < 0$ and $a_0 = 0$ represent the initial purchase of the car, $a_T$ is the service life and $c_T$ is the resale price the company receives from selling the car in the used car market, or at an auction.
1. Recall, the IRR is the $r$ that solves

$$0 = \sum_{t=0}^{T} \exp\left\{-\frac{a_t r}{365}\right\} c_t,$$

where $c_t$ is the cash inflow (if positive) or outflow (if negative), and $a_t$ is the number of days after the initial purchase of the vehicle that the $t^{th}$ cash flow occurred.

2. Thus, $c_0 < 0$ and $a_0 = 0$ represent the initial purchase of the car, $a_T$ is the service life and $c_T$ is the resale price the company receives from selling the car in the used car market, or at an auction.

3. We see that for each of the cars illustrated in figure 1, the realized rates of return are extraordinarily high. *These high returns are not atypical.*
2.3 Analyzing Returns
Return distribution: Compact

Distribution of Internal Rates of Return on Compact, all locations

Mean 0.76776
Median 0.65936
Minimum -0.017
Maximum 9.2275
Std dev 0.72254
Observations 167
Distribution of Internal Rates of Return on Luxury 2, all locations

Mean  0.49204
Median 0.48693
Minimum 0.2859
Maximum 0.79523
Std dev 0.10407
Observations 40
Return distribution: RV

Distribution of Internal Rates of Return on RV, all locations

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.53362</td>
</tr>
<tr>
<td>Median</td>
<td>0.48909</td>
</tr>
<tr>
<td>Minimum</td>
<td>-0.20801</td>
</tr>
<tr>
<td>Maximum</td>
<td>1.148</td>
</tr>
<tr>
<td>Std dev</td>
<td>0.2826</td>
</tr>
<tr>
<td>Observations</td>
<td>31</td>
</tr>
</tbody>
</table>
1. Which factors affect/determine the return on a rental car?
Analysis of returns

1. When to sell the car?

1. Introduction

2. Data

2.3 Analyzing Returns

- Analysis of returns
- Regression results: IRR
- Discussion of results
- Why no age/odometer effect?

3. An Econometric Model

4. Simulation Results

5. Optimal Replacement Policy

6. Numerical Results

7. Extensions

8. Conclusions

100

200

300

400

500

600

700

800

0

0.2

0.4

0.6

0.8

1

1.2

1.4

1.6

1.8

2

Sungjin Cho and John Rust When to Sell the Car? - slide #29

1. Which factors affect/determine the return on a rental car?

- Rental rates
1. Which factors affect/determine the return on a rental car?

- Rental rates
- Capacity utilization
1. Which factors affect/determine the return on a rental car?

- Rental rates
- Capacity utilization
- New purchase price and resale value
1. Which factors affect/determine the return on a rental car?

- Rental rates
- Capacity utilization
- New purchase price and resale value
- service life of vehicle
Analysis of returns

1. Which factors affect/determine the return on a rental car?
   - Rental rates
   - Capacity utilization
   - New purchase price and resale value
   - *service life of vehicle*

2. Daily rental rates for short contracts are typically significantly higher than for long term contracts.
1. Which factors affect/determine the return on a rental car?
   - Rental rates
   - Capacity utilization
   - New purchase price and resale value
   - *service life of vehicle*

2. Daily rental rates for short contracts are typically significantly higher than for long term contracts.

3. Counterbalancing this, is that cars are more likely to be on the lot between successive short term rental spells.
1. Which factors affect/determine the return on a rental car?
   - Rental rates
   - Capacity utilization
   - New purchase price and resale value
   - *service life of vehicle*

2. Daily rental rates for short contracts are typically significantly higher than for long term contracts.

3. Counterbalancing this, is that cars are more likely to be on the lot between successive short term rental spells.

4. *Which contract is more profitable: long or short term?*
### Regression results: IRR

<table>
<thead>
<tr>
<th>Variable</th>
<th>Compact</th>
<th>Luxury</th>
<th>RV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.575*</td>
<td>−0.006</td>
<td>0.999</td>
</tr>
<tr>
<td>Utilization Rate</td>
<td>0.003</td>
<td>0.522*</td>
<td>1.366*</td>
</tr>
<tr>
<td>Fraction Rented Long Term</td>
<td>−0.220*</td>
<td>−0.076</td>
<td>−0.876*</td>
</tr>
<tr>
<td>Total Maintenance costs ($000)</td>
<td>−7.46e−5*</td>
<td>−2.00e−5</td>
<td>6.978e−6</td>
</tr>
<tr>
<td>Odometer (000 km)</td>
<td>0.0007</td>
<td>−0.0004</td>
<td>−0.001</td>
</tr>
<tr>
<td>Age at Sale (years)</td>
<td>0.151*</td>
<td>0.072</td>
<td>−0.154</td>
</tr>
<tr>
<td>New Price ($000)</td>
<td>−0.104*</td>
<td>−0.036*</td>
<td>−0.082*</td>
</tr>
<tr>
<td>Sale Price ($000)</td>
<td>0.008</td>
<td>−0.002</td>
<td>0.063</td>
</tr>
<tr>
<td>Short term rental rate</td>
<td>0.003*</td>
<td>0.0006*</td>
<td>0.004*</td>
</tr>
<tr>
<td>Long term rental rate</td>
<td>0.037**</td>
<td>0.020**</td>
<td>0.009</td>
</tr>
<tr>
<td>Observations, $R^2$</td>
<td>167.81%</td>
<td>40.78%</td>
<td>31.86%</td>
</tr>
</tbody>
</table>
1. Regression results generally confirm our expectations:
1. Regression results generally confirm our expectations:
   - higher purchase prices reduce the IRR, resale prices increase it
Discussion of results

1. Regression results generally confirm our expectations:

- higher purchase prices reduce the IRR, resale prices increase it

- the estimate of utilization rate is positive and statistically significant
1. Regression results generally confirm our expectations:
   - higher purchase prices reduce the IRR, resale prices increase it
   - the estimate of utilization rate is positive and statistically significant
   - the daily rental rates are also positive and generally significant.
1. Regression results generally confirm our expectations:

- higher purchase prices reduce the IRR, resale prices increase it
- the estimate of utilization rate is positive and statistically significant
- the daily rental rates are also positive and generally significant.

- the fraction of the time the car was rented long term has a negative coefficient, suggesting long term contracts are less profitable than short term contracts
Discussion of results

1. Regression results generally confirm our expectations:
   - higher purchase prices reduce the IRR, resale prices increase it
   - the estimate of utilization rate is positive and statistically significant
   - the daily rental rates are also positive and generally significant.
   - the fraction of the time the car was rented long term has a negative coefficient, suggesting long term contracts are less profitable than short term contracts

2. However estimates of maintenance costs ambiguous,
Discussion of results

1. Regression results generally confirm our expectations:
   - higher purchase prices reduce the IRR, resale prices increase it
   - the estimate of utilization rate is positive and statistically significant
   - the daily rental rates are also positive and generally significant.
   - the fraction of the time the car was rented long term has a negative coefficient, suggesting long term contracts are less profitable than short term contracts

2. However estimates of maintenance costs ambiguous,

3. and coefficients on age and odometer are insignificantly different from 0.
1. There are several possible reasons why effect of age, odometer, and maintenance costs are insignificantly different from 0.
Why no age/odometer effect?

1. There are several possible reasons why effect of age, odometer, and maintenance costs are insignificantly different from 0.

2. One reason is *multicollinearity* especially between age and odometer.
1. There are several possible reasons why effect of age, odometer, and maintenance costs are insignificantly different from 0.

2. One reason is *multicollinearity* especially between age and odometer.

3. However if the regression and includes only age or odometer individually, the results are still generally insignificant.
Why no age/odometer effect?

1. There are several possible reasons why effect of age, odometer, and maintenance costs are insignificantly different from 0.

2. One reason is *multicollinearity* especially between age and odometer.

3. However if the regression and includes only age or odometer individually, the results are still generally insignificant.

4. Only in one case, for the luxury vehicle, are both age and maintenance significant
1. There are several possible reasons why effect of age, odometer, and maintenance costs are insignificantly different from 0.

2. One reason is *multicollinearity* especially between age and odometer.

3. However if the regression and includes only age or odometer individually, the results are still generally insignificant.

4. Only in one case, for the luxury vehicle, are both age and maintenance significant.

5. In the luxury case, age has a positive coefficient and total maintenance cost has a negative coefficient.
1. There are several possible reasons why effect of age, odometer, and maintenance costs are insignificantly different from 0.

2. One reason is **multicollinearity** especially between age and odometer.

3. However if the regression and includes only age or odometer individually, the results are still generally insignificant.

4. Only in one case, for the luxury vehicle, are both age and maintenance significant.

5. In the luxury case, age has a positive coefficient and total maintenance cost has a negative coefficient.

6. But even in this case, the effect of age on IRR is small: the regression results predict that keeping a luxury car for 100 more days increases the IRR by 0.03.
2.4 Is regression enough?
A sign of optimality?

1. Let $\Pi(o)$ denote the expected discounted profits from selling a car at odometer value $o$
1. Let $\Pi(o)$ denote the expected discounted profits from selling a car at odometer value $o$.

2. If the firm chooses an optimal threshold $o^*$, then

$$\frac{\partial \Pi}{\partial o}(o^*) = 0.$$  

so small variations in $o$ should not affect profits and IRR.
A sign of optimality?

1. Let $\Pi(o)$ denote the expected discounted profits from selling a car at odometer value $o$

2. If the firm chooses an optimal threshold $o^*$, then

   \[
   \frac{\partial \Pi}{\partial o}(o^*) = 0.
   \]

so small variations in $o$ should not affect profits and IRR.

3. **Problem 1: the range of odometer values and ages at which vehicles are replaced is very wide.**
A sign of optimality?

1. Let $\Pi(o)$ denote the expected discounted profits from selling a car at odometer value $o$

2. If the firm chooses an optimal threshold $o^*$, then

$$\frac{\partial \Pi}{\partial o}(o^*) = 0. \quad (2)$$

so small variations in $o$ should not affect profits and IRR.

3. **Problem 1**: the range of odometer values and ages at which vehicles are replaced is very wide.

4. **Problem 2**: vehicle age and odometer values are endogenous Unobserved factors that lead a car to be more profitable, could also lead the firm to keep it longer.
A sign of optimality?

1. Let $\Pi(o)$ denote the expected discounted profits from selling a car at odometer value $o$.

2. If the firm chooses an optimal threshold $o^*$, then

$$\frac{\partial \Pi}{\partial o}(o^*) = 0.$$ (2)

so small variations in $o$ should not affect profits and IRR.

3. **Problem 1:** the range of odometer values and ages at which vehicles are replaced is very wide.

4. **Problem 2:** vehicle age and odometer values are **endogenous** Unobserved factors that lead a car to be more profitable, could also lead the firm to keep it longer.

5. If so, we would expect a **positive correlation** between vehicle age and odometer values and the error term in the regression, so the estimates for these coefficients could be **upward biased**.
1. What we need is an *instrumental variable* that causes *exogenous* shifts in the age at which the company replaced some of its vehicles.
Can we find an Instrument?

1. What we need is an *instrumental variable* that causes *exogenous* shifts in the age at which the company replaced some of its vehicles.

2. An example of such a variable might be a *recall dummy*. 
1. What we need is an *instrumental variable* that causes *exogenous* shifts in the age at which the company replaced some of its vehicles.

2. An example of such a variable might be a *recall dummy*.

3. That is, if there was some major problem in one of the types of cars that the company owned that resulted in mass recalls or prompted the company to sell the cars “prematurely” this might constitute a valid instrument.
Can we find an Instrument?

1. What we need is an *instrumental variable* that causes *exogenous* shifts in the age at which the company replaced some of its vehicles.

2. An example of such a variable might be a *recall dummy*.

3. That is, if there was some major problem in one of the types of cars that the company owned that resulted in mass recalls or prompted the company to sell the cars “prematurely” this might constitute a valid instrument.

4. However a better “instrument” is a *treatment dummy*. 
Can we find an Instrument?

1. What we need is an *instrumental variable* that causes *exogenous* shifts in the age at which the company replaced some of its vehicles.

2. An example of such a variable might be a *recall dummy*.

3. That is, if there was some major problem in one of the types of cars that the company owned that resulted in mass recalls or prompted the company to sell the cars “prematurely” this might constitute a valid instrument.

4. However a better “instrument” is a *treatment dummy*.

5. That is, if the company had undertaken a randomized experiment, keeping some cars longer than it would have otherwise.
Can we find an Instrument?

1. What we need is an *instrumental variable* that causes *exogenous* shifts in the age at which the company replaced some of its vehicles.

2. An example of such a variable might be a *recall dummy*.

3. That is, if there was some major problem in one of the types of cars that the company owned that resulted in mass recalls or prompted the company to sell the cars “prematurely” this might constitute a valid instrument.

4. However a better “instrument” is a *treatment dummy*.

5. That is, if the company had undertaken a randomized experiment, keeping some cars longer than it would have otherwise.

6. Unfortunately, we do not have either of these instrumental variables in our data set.
1. In the absence of any good instruments, and if the company has not undertaken any experiments, how can we proceed to test the hypothesis, i.e. is the company maximizing profits?
How to proceed?

1. In the absence of any good instruments, and if the company has not undertaken any experiments, how can we proceed to test the hypothesis, i.e. is the company maximizing profits?

2. Our approach: to create a *model* of the firm’s rental operations.
1. In the absence of any good instruments, and if the company has not undertaken any experiments, how can we proceed to test the hypothesis, i.e., is the company maximizing profits?

2. Our approach: to create a model of the firm’s rental operations.

3. We estimate the unknown parameters of this model using the company’s data.
How to proceed?

1. In the absence of any good instruments, and if the company has not undertaken any experiments, how can we proceed to test the hypothesis, i.e. is the company maximizing profits?

2. Our approach: to create a *model* of the firm’s rental operations.

3. We estimate the unknown parameters of this model using the company’s data.

4. Once the model is estimated, we can *simulate* it.
How to proceed?

1. In the absence of any good instruments, and if the company has not undertaken any experiments, how can we proceed to test the hypothesis, i.e. is the company maximizing profits?

2. Our approach: to create a *model* of the firm’s rental operations.

3. We estimate the unknown parameters of this model using the company’s data.

4. Once the model is estimated, we can *simulate* it.

5. We will simulate the model under the *status quo* and show it provides a good approximation to the data we observe.
1. In the absence of any good instruments, and if the company has not undertaken any experiments, how can we proceed to test the hypothesis, i.e. is the company maximizing profits?

2. Our approach: to create a *model* of the firm’s rental operations.

3. We estimate the unknown parameters of this model using the company’s data.

4. Once the model is estimated, we can *simulate* it.

5. We will simulate the model under the *status quo* and show it provides a good approximation to the data we observe.

6. Then we use the model to compute and simulate *alternative rental policies*. We show that certain alternative policies result in *significantly higher profits*. 
3.1 Overview
In this model, a car can be in one of four possible states at any given point in time:
In this model, a car can be in one of four possible states at any given point in time:

1. In a long term rental contract (i.e. a "long term rental spell"),
In this model, a car can be in one of four possible states at any given point in time:

1. In a long term rental contract (i.e. a “long term rental spell”),
2. In a short term rental contract (i.e. a “short term rental spell”),
In this model, a car can be in one of four possible states at any given point in time:

1. In a long term rental contract (i.e. a “long term rental spell”),
2. In a short term rental contract (i.e. a “short term rental spell”),
3. In the lot waiting to be rented, where the previous rental state was a long term rental spell,
In this model, a car can be in one of four possible states at any given point in time:

1. In a long term rental contract (i.e. a “long term rental spell”),
2. In a short term rental contract (i.e. a “short term rental spell”),
3. In the lot waiting to be rented, where the previous rental state was a long term rental spell,
4. In the lot waiting to be rented, where the previous rental state was a short term rental spell.
A Semi-Markov Model

- In this model, a car can be in one of four possible states at any given point in time:
  1. In a long term rental contract (i.e. a “long term rental spell”),
  2. In a short term rental contract (i.e. a “short term rental spell”),
  3. In the lot waiting to be rented, where the previous rental state was a long term rental spell,
  4. In the lot waiting to be rented, where the previous rental state was a short term rental spell.

- We refer to the latter two states, 3 and 4, as *lot spells*. 
Other State Variables

- in addition to the *rental state* $r_t \in \{1, 2, 3, 4\}$ other relevant state variables for modeling the decisions of the rental company are:
in addition to the *rental state* $r_t \in \{1, 2, 3, 4\}$ other relevant state variables for modeling the decisions of the rental company are:

1. *odometer value* $o_t$
Other State Variables

- in addition to the *rental state* $r_t \in \{1, 2, 3, 4\}$ other relevant state variables for modeling the decisions of the rental company are:

1. *odometer value* $o_t$
2. *duration in the current rental state* $d_t$
Other State Variables

- in addition to the rental state \( r_t \in \{1, 2, 3, 4\} \) other relevant state variables for modeling the decisions of the rental company are:
  1. **odometer value** \( o_t \)
  2. **duration in the current rental state** \( d_t \)

- Thus, we seek to model the joint stochastic process \( \{r_t, o_t, d_t\} \).
in addition to the rental state $r_t \in \{1, 2, 3, 4\}$ other relevant state variables for modeling the decisions of the rental company are:

1. **odometer value** $o_t$
2. **duration in the current rental state** $d_t$

Thus, we seek to model the joint stochastic process \( \{r_t, o_t, d_t\} \).

The other potential state variable of interest, the vehicle’s age $a_t$ creates complications due to **non-stationarity**.
Other State Variables

- in addition to the rental state \( r_t \in \{1, 2, 3, 4\} \) other relevant state variables for modeling the decisions of the rental company are:

  1. **odometer value** \( o_t \)
  2. **duration in the current rental state** \( d_t \)

- Thus, we seek to model the joint stochastic process \( \{r_t, o_t, d_t\} \).

- The other potential state variable of interest, the vehicle’s age \( a_t \) creates complications due to non-stationarity.

- Since age and odometer are highly correlated, we feel that not much damage is done from excluding \( a_t \) as an explicit state variable, and *deriving* the implied distribution of vehicle ages from our model.
Once we estimate the stochastic process $\{r_t, o_t, d_t\}$, we can derive/simulate other variables of interest, including...
Once we estimate the stochastic process \( \{r_t, o_t, d_t\} \), we can derive/simulate other variables of interest, including:

1. *rental revenues*
Once we estimate the stochastic process \( \{r_t, o_t, d_t\} \), we can derive/simulate other variables of interest, including:

1. *rental revenues*

2. *maintenance costs*
Once we estimate the stochastic process \( \{r_t, o_t, d_t\} \), we can derive/simulate other variables of interest, including:

1. *rental revenues*
2. *maintenance costs*
3. *rental profits and internal rates of return*
Once we estimate the stochastic process $\{r_t, o_t, d_t\}$, we can derive/simulate other variables of interest, including:

1. rental revenues
2. maintenance costs
3. rental profits and internal rates of return

However to do this, we also need econometric models for a vehicle’s *resale price* and a model of the *the timing of the replacement decision*. 
Implied State Variables

Once we estimate the stochastic process \( \{r_t, o_t, d_t\} \), we can derive/simulate other variables of interest, including:

1. rental revenues
2. maintenance costs
3. rental profits and internal rates of return

However to do this, we also need econometric models for a vehicle’s resale price and a model of the timing of the replacement decision.

With this information, we can construct a complete econometric model of the company’s rental operations, and conduct stochastic simulations of the model to see how accurately it can represent the company’s actual operations.
1. A model of *resale prices*
Model Components

1. A model of \textit{resale prices}

2. A \textit{duration model} for the random durations of a car in each of the rental and lot states,
Model Components

1. A model of *resale prices*
2. A *duration model* for the random durations of a car in each of the rental and lot states,
3. A *transition model* for a vehicle’s transitions between rental states at the end of the current rental spell
Model Components

1. A model of *resale prices*
2. A *duration model* for the random durations of a car in each of the rental and lot states,
3. A *transition model* for a vehicle’s transitions between rental states at the end of the current rental spell
4. A *utilization model* for the kilometers driven during a long or short term rental contract,
Model Components

1. A model of *resale prices*
2. A *duration model* for the random durations of a car in each of the rental and lot states,
3. A *transition model* for a vehicle’s transitions between rental states at the end of the current rental spell
4. A *utilization model* for the kilometers driven during a long or short term rental contract,
5. A *model for maintenance costs* incurred by the company over the life of the car,
Model Components

1. A model of *resale prices*

2. A *duration model* for the random durations of a car in each of the rental and lot states,

3. A *transition model* a a vehicle’s transitions between rental states at the end of the current rental spell

4. A *utilization model* for the kilometers driven during a long or short term rental contract,

5. A *model for maintenance costs* incurred by the company over the life of the car,

6. A model of the company’s *replacement decision*, i.e. the factors that motivate it to sell a given car at a particular point in time.
3.2 Resale Price Model
We have data on both the new price $\overline{P}(\tau)$ as well as the realized sales price $P_t(o_t, \tau)$ of each car, where $\tau$ denotes a particular make and model of vehicle, which we will also call a *car type*. 
The resale price model

- We have data on both the new price $\bar{P}(\tau)$ as well as the realized sales price $P_t(o_t, \tau)$ of each car, where $\tau$ denotes a particular make and model of vehicle, which we will also call a *car type*.

- For each of the three car types $\tau \in \{\text{compact, luxury, RV}\}$, we estimated a simple linear regression model with the log depreciation rate, $\bar{P}(\tau)/P_t(o_t, \tau)$, as the dependent variable:

$$\log(\bar{P}(\tau)/P_t(o_t, \tau)) = \alpha_1(\tau) + \alpha_2(\tau) o_t + \epsilon_t. \quad (3)$$
The resale price model

- We have data on both the new price $P(\tau)$ as well as the realized sales price $P_t(o_t, \tau)$ of each car, where $\tau$ denotes a particular make and model of vehicle, which we will also call a car type.

- For each of the three car types $\tau \in \{\text{compact, luxury, RV}\}$, we estimated a simple linear regression model with the log depreciation rate, $P(\tau)/P_t(o_t, \tau)$, as the dependent variable

  \[
  \log(P(\tau)/P_t(o_t, \tau)) = \alpha_1(\tau) + \alpha_2(\tau)o_t + \epsilon_t. 
  \]

- The type-specific “depreciation coefficients” $(\alpha_1(\tau), \alpha_2(\tau))$ are used to predict resale prices.
The resale price model

- We have data on both the new price $\bar{P}(\tau)$ as well as the realized sales price $P_t(o_t, \tau)$ of each car, where $\tau$ denotes a particular make and model of vehicle, which we will also call a *car type*.

- For each of the three car types $\tau \in \{\text{compact, luxury, RV}\}$, we estimated a simple linear regression model with the log depreciation rate, $\bar{P}(\tau)/P_t(o_t, \tau)$, as the dependent variable

\[
\log(\bar{P}(\tau)/P_t(o_t, \tau)) = \alpha_1(\tau) + \alpha_2(\tau) o_t + \epsilon_t.
\]

- The type-specific “depreciation coefficients” $(\alpha_1(\tau), \alpha_2(\tau))$ are used to predict resale prices.

- We also estimated regressions that included vehicle age and other variables such as the number of accidents and the total accident repair cost.
### Resale price regression

<table>
<thead>
<tr>
<th>Variable</th>
<th>Compact</th>
<th>Luxury</th>
<th>RV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>$-0.4789^*$</td>
<td>$-0.6201^{**}$</td>
<td>$-0.8521^*$</td>
</tr>
<tr>
<td>Age (days)</td>
<td>$-0.0001^*$</td>
<td>$-0.0004^*$</td>
<td>$-0.0004^*$</td>
</tr>
<tr>
<td>Odometer (000 km)</td>
<td>$-0.0007^*$</td>
<td>$-0.0011^*$</td>
<td>0.0016</td>
</tr>
<tr>
<td>Number of Accidents</td>
<td>$-0.0112$</td>
<td>0.0006</td>
<td>0.0371</td>
</tr>
<tr>
<td>Accident Repair Costs</td>
<td>$-0.888e^{-6}$</td>
<td>$-4.672e^{-6}$</td>
<td>$-1.654e^{-6}$</td>
</tr>
<tr>
<td>Internal Rate of Return</td>
<td>0.1629^{**}</td>
<td>0.067</td>
<td>0.394^{*}</td>
</tr>
<tr>
<td>Maintenance Cost per Day</td>
<td>0.0092</td>
<td>$-0.0039$</td>
<td>$-0.0053$</td>
</tr>
<tr>
<td>$N, R^2$</td>
<td>288, 38.9%</td>
<td>91, 42.0%</td>
<td>41, 48.1%</td>
</tr>
</tbody>
</table>
Predicted versus Actual Resale Prices for Compact, all locations

- Mean Price New: $9669
- Mean Sale Price: $5478
- Max Sale Price: $7290
- Min Sale Price: $3500
- Mean Odometer at Sale: 77914
- Max Odometer at Sale: 153143
- Min Odometer at Sale: 10500
- Number of Observations: 288
Resale prices: Luxury

Predicted versus Actual Resale Prices for Luxury 2, all locations

- Mean Price New: $23,389
- Mean Sale Price: $12,014
- Max Sale Price: $16,550
- Min Sale Price: $7,500
- Mean Odometer at Sale: 74,684
- Max Odometer at Sale: 187,800
- Min Odometer at Sale: 35,000
- Number of Observations: 91

Odometer (thousands of kilometers) vs. Resale price of car (thousands of dollars)
Resale prices: RV

Predicted versus Actual Resale Prices for RV, all locations

- Mean Price New: 18774
- Mean Sale Price: 7751
- Max Sale Price: 10550
- Min Sale Price: 4000
- Mean Odometer at Sale: 88829
- Max Odometer at Sale: 163860
- Min Odometer at Sale: 38000
- Number of Observations: 41
1. The constant term in the regressions measures the instantaneous depreciation in car prices the minute it leaves the new car lot.
1. The constant term in the regressions measures the \textit{instantaneous depreciation} in car prices the minute it leaves the new car lot.

2. We see that the instantaneous depreciation is huge for all three vehicle types: \(62\% = \exp(-0.48)\) for the compact, \(52\%\) for the luxury vehicle, \(43\%\) for the RV.
1. The constant term in the regressions measures the *instantaneous depreciation* in car prices the minute it leaves the new car lot.

2. We see that the instantaneous depreciation is huge for all three vehicle types: \( 62\% = \exp(-0.48) \) for the compact, \( 52\% \) for the luxury vehicle, \( 43\% \) for the RV.

3. We did not feel we could trust the regression extrapolations for used vehicle prices for age or odometer values very close to zero. Therefore we made a simple, but *ad hoc*, linear extrapolation for odometer values less than 20,000 kilometers, so that the instantaneous depreciation is only 5% rather than the regression estimates.
1. The constant term in the regressions measures the the *instantaneous depreciation* in car prices the minute it leaves the new car lot.

2. We see that the instantaneous depreciation is huge for all three vehicle types: $62\% = \exp(-0.48)$ for the compact, $52\%$ for the luxury vehicle, $43\%$ for the RV.

3. We did not feel we could trust the regression extrapolations for used vehicle prices for age or odometer values very close to zero. Therefore we made a simple, but *ad hoc*, linear extrapolation for odometer values less than 20,000 kilometers, so that the instantaneous depreciation is only 5% rather than the regression estimates.

4. However predictions of the optimal replacement policy are not sensitive to our assumptions about the precise shape of the depreciation curve for cars with odometer values of less than 20,000 kilometers.
1. *Conclusion 1*: beyond age and odometer value (and implicitly the car’s characteristics, as represented by its make and model), there are few other significant explanatory variables for the resale value of a car.
1. *Conclusion 1:* beyond age and odometer value (and implicitly the car’s characteristics, as represented by its make and model), there are few other significant explanatory variables for the resale value of a car.

2. Our regressions can explain only between 40 to 50% of the variation in the resale values of the cars the company sells: there is a lot of “residual variance” that leads one car to sell for much more than another car that from our standpoint is “observationally equivalent” to it.
1. **Conclusion 1**: beyond age and odometer value (and implicitly the car’s characteristics, as represented by its make and model), there are few other significant explanatory variables for the resale value of a car.

2. Our regressions can explain only between 40 to 50% of the variation in the resale values of the cars the company sells: there is a lot of “residual variance” that leads one car to sell for much more than another car that from our standpoint is “observationally equivalent” to it.

3. **Conclusion 2**: the rapid initial depreciation implies that vehicle replacement is a significant investment that can be *amortized* by keeping the vehicle sufficiently long before next replacement.
3.3 Vehicle Usage Model
1. As noted above, the firm frequently does not accurately record in/out odometer values for rental spells.
1. As noted above, the firm frequently does not accurately record in/out odometer values for rental spells.

2. To circumvent this problem we make a *functional form assumption*. Let \( F(o' \mid o, d, r) \) denote the conditional distribution of odometer value of a car returning from a rental contract of type \( r \) that lasted \( d \) days when the out odometer value was \( o \).
Vehicle Usage Model

1. As noted above, the firm frequently does not accurately record in/out odometer values for rental spells.

2. To circumvent this problem we make a functional form assumption. Let $F(o' | o, d, r)$ denote the conditional distribution of odometer value of a car returning from a rental contract of type $r$ that lasted $d$ days when the out odometer value was $o$.

3. Thus, $\nabla o = o' - o$ is the number of kilometers driven by the customer during the rental spell.
Vehicle Usage Model

1. As noted above, the firm frequently does not accurately record in/out odometer values for rental spells.

2. To circumvent this problem we make a functional form assumption. Let $F(o' \mid o, d, r)$ denote the conditional distribution of odometer value of a car returning from a rental contract of type $r$ that lasted $d$ days when the out odometer value was $o$.

3. Thus, $\nabla o = o' - o$ is the number of kilometers driven by the customer during the rental spell.

4. We assume that the number of kilometers travelled each day by a rental customer are IID draws from an exponential distribution with parameter $\lambda_r$. 

Vehicle Usage Model

1. As noted above, the firm frequently does not accurately record in/out odometer values for rental spells.

2. To circumvent this problem we make a functional form assumption. Let \( F(o'|o, d, r) \) denote the conditional distribution of odometer value of a car returning from a rental contract of type \( r \) that lasted \( d \) days when the out odometer value was \( o \).

3. Thus, \( \nabla o = o' - o \) is the number of kilometers driven by the customer during the rental spell.

4. We assume that the number of kilometers travelled each day by a rental customer are \( IID \) draws from an exponential distribution with parameter \( \lambda_r \).

5. Conditional on spell length \( d \), it follows that \( F(o'|d, r) \) is a gamma distribution, since a sum of \( IID \) exponential random variables has a gamma distribution.
1. Suppose that at time of sale, a rental car had been rented for $N^s$ days under short term rental contracts and $N^l$ days under long term rental contracts. Then the odometer value on the car at time of sale, $\tilde{o}$, is given by

$$\tilde{o} = \sum_{i=1}^{N^l} \nabla o^l_i + \sum_{i=1}^{N^s} \nabla o^s_i$$
1. Suppose that at time of sale, a rental car had been rented for $N^s$ days under short term rental contracts and $N^l$ days under long term rental contracts. Then the odometer value on the car at time of sale, $\tilde{o}$, is given by

$$\tilde{o} = \sum_{i=1}^{N^l} \nabla o^l_i + \sum_{i=1}^{N^s} \nabla o^s_i$$

2. Thus, we have

$$E\{\tilde{o}|N^l, N^s\} = \lambda_1 N^l + \lambda_2 N^s.$$
1. Since we do accurately observe $N^l_i$ and $N^s_i$ for each rental car, we can estimate $\lambda_1$ and $\lambda_2$ as coefficients on a simple linear regression

$$o_i = \lambda_1 N^l_i + \lambda_2 N^s_i + \epsilon_i$$

where $o_i$ is the odometer at time of sale on the $i^{th}$ rental car sold by the company, and $N^s_i$ and $N^l_i$ are the number of days the $i^{th}$ car had been in short and long term rentals over its service life.
1. Since we do accurately observe $N^l$ and $N^s$ for each rental car, we can estimate $\lambda_1$ and $\lambda_2$ as coefficients on a simple linear regression

\[ o_i = \lambda_1 N^l_i + \lambda_2 N^s_i + \varepsilon_i \]  

where $o_i$ is the odometer at time of sale on the $i^{\text{th}}$ rental car sold by the company, and $N^s_i$ and $N^l_i$ are the number of days the $i^{\text{th}}$ car had been in short and long term rentals over its service life.

2. Estimation results

<table>
<thead>
<tr>
<th>Variable</th>
<th>Compact</th>
<th>Luxury</th>
<th>RV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>78.7</td>
<td>86.6</td>
<td>95.4</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>157.1</td>
<td>140.8</td>
<td>167.7</td>
</tr>
</tbody>
</table>
3.4 The Replacement Model
1. Let $s_t$ denote a binary variable for the selling decision with $s_t = 1$ if the company sells the car and $s_t = 0$ if the company keeps the car.
1. Let \( s_t \) denote a binary variable for the selling decision with \( s_t = 1 \) if the company sells the car and \( s_t = 0 \) if the company keeps the car.

2. We estimated the company’s decision to sell the car using a binary logit model

\[
Pr \{ s_t = 1 | x_t \} = \frac{\exp\{ x_t \theta \}}{1 + \exp\{ x_t \theta \}}.
\]
1. Let $s_t$ denote a binary variable for the selling decision with $s_t = 1$ if the company sells the car and $s_t = 0$ if the company keeps the car.

2. We estimated the company’s decision to sell the car using a binary logit model

$$\Pr \{s_t = 1 | x_t\} = \frac{\exp\{x_t \theta\}}{1 + \exp\{x_t \theta\}}.$$ \hspace{1cm} (7)

3. Among the variables in the vector $x_t$ are the vehicle’s age and predicted odometer value (based on the regression estimate $\hat{\sigma}_t$ using the observed values of $N^l_t$ and $N^s_t$ from the rental contract data, as discussed above), duration in the lot, average daily maintenance costs, and utilization rate.
Logit Estimation Results

<table>
<thead>
<tr>
<th>Variable</th>
<th>Compact</th>
<th>Luxury</th>
<th>RV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>$-13.06^{**}$</td>
<td>$-12.27^{*}$</td>
<td>$-14.67^{*}$</td>
</tr>
<tr>
<td>Age (days)</td>
<td>$0.0077^{*}$</td>
<td>$0.0011$</td>
<td>$0.0125^{*}$</td>
</tr>
<tr>
<td>Odometer (km)</td>
<td>$0.0050$</td>
<td>$0.0987^{*}$</td>
<td>$-0.038$</td>
</tr>
<tr>
<td>Duration, Age &lt; 500</td>
<td>$0.0206^{**}$</td>
<td>$-11.99^{**}$</td>
<td>$-6.069^{**}$</td>
</tr>
<tr>
<td>Duration, Age $\in [500, 1000)$</td>
<td>$0.0867^{**}$</td>
<td>$0.0471^{*}$</td>
<td>$0.0399^{*}$</td>
</tr>
<tr>
<td>Duration, Age $&gt; 1000$</td>
<td>$0.1362^{**}$</td>
<td>$0.1736^{*}$</td>
<td>$0.1744^{*}$</td>
</tr>
<tr>
<td>Maintenance Cost</td>
<td>$0.00003^{*}$</td>
<td>$0.2030$</td>
<td>$-0.0188$</td>
</tr>
<tr>
<td>Utilization Rate</td>
<td>$0.4049$</td>
<td>$-1.616$</td>
<td>$1.989$</td>
</tr>
<tr>
<td>$N, \log(L)/N$</td>
<td>$36262, -0.017$</td>
<td>$6445, -0.022$</td>
<td>$7192, -0.017$</td>
</tr>
</tbody>
</table>
1. Due to the collinearity between age and odometer value, it is difficult to identify the separate effects of age versus odometer value on the firm’s decision to sell a vehicle.
1. Due to the collinearity between age and odometer value, it is difficult to identify the separate effects of age versus odometer value on the firm’s decision to sell a vehicle.

2. Model fits about as well if age is excluded from the logit.
1. Due to the collinearity between age and odometer value, it is difficult to identify the separate effects of age versus odometer value on the firm’s decision to sell a vehicle.

2. Model fits about as well if age is excluded from the logit.

3. Besides age and odometer, the only variable whose coefficient estimates are statistically significant and has signs that are (generally) consistent with our *a priori* expectations is duration.
Empirical CDF of Kilometers at Sale: Compact, all locations

- Mean: 77914
- Median: 77560
- Min: 10500
- Max: 153143
- Obs: 288
Empirical CDF of Age at Sale: Compact, all locations

<table>
<thead>
<tr>
<th>Statistic</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>2.81</td>
</tr>
<tr>
<td>Median</td>
<td>2.84</td>
</tr>
<tr>
<td>Min</td>
<td>1.53</td>
</tr>
<tr>
<td>Max</td>
<td>4.03</td>
</tr>
<tr>
<td>Obs</td>
<td>288</td>
</tr>
</tbody>
</table>
Empirical CDF of Kilometers at Sale: Luxury 2, all locations

- Mean: 74684
- Median: 71300
- Min: 35000
- Max: 187800
- Obs: 91
Age at Sale: Luxury

Empirical CDF of Age at Sale: Luxury 2, all locations

Mean 2.92
Median 2.80
Min 1.92
Max 4.90
Obs 91
Empirical CDF of Kilometers at Sale: RV, all locations

Mean: 88829
Median: 91200
Min: 38000
Max: 163860
Obs: 41

Odometer (thousands of kilometers) vs. Fraction of Cars Sold
Age at Sale: RV

Empirical CDF of Age at Sale: RV, all locations

Mean 2.72
Median 2.72
Min 1.92
Max 3.73
Obs 41

Fraction of Cars Sold vs. Age of Car (years)
3.5 The Duration Model
1. The remaining objects to be estimated to implement our econometric model are the *spell durations* and the *state transition probabilities*. 
1. The remaining objects to be estimated to implement our econometric model are the *spell durations* and the *state transition probabilities*.

2. Let \( h(d, r) \) denote the *hazard rate* for the rental state \( r \).
1. The remaining objects to be estimated to implement our econometric model are the *spell durations* and the *state transition probabilities*.

2. Let \( h(d, r) \) denote the *hazard rate* for the rental state \( r \).

3. The duration distribution \( f(d|r) \) implied by the hazard function \( h(d, r) \) is

\[
f(d|r) = \begin{cases} 
  f(1|r) = h(0, r) \\
  f(d|r) = \prod_{j=0}^{d-2} [1 - h(j, r)] h(d - 1, r) & d \geq 2 
\end{cases}
\]
1. The remaining objects to be estimated to implement our econometric model are the \textit{spell durations} and the \textit{state transition probabilities}.

2. Let \( h(d, r) \) denote the \textit{hazard rate} for the rental state \( r \).

3. The duration distribution \( f(d | r) \) implied by the hazard function \( h(d, r) \) is

\[
f(d | r) = \begin{cases} 
  f(1 | r) = h(0, r) \\
  f(d | r) = \prod_{j=0}^{d-2} [1 - h(j, r)] h(d - 1, r) & d \geq 2 
\end{cases}
\]

(8)

4. Since we have sufficiently many observations of rental spells, we were able to estimate the hazard functions for these spells \textit{non-parametrically}. 
1. We have far fewer observations on lot spell durations, especially for type 3 lot spells due to the high probability of roll overs in longer term contracts.
1. We have far fewer observations on lot spell durations, especially for type 3 lot spells due to the high probability of roll overs in longer term contracts.

2. With fewer observations, our nonparametrically estimated hazard functions are quite jagged.
Lot Spell Durations

1. We have far fewer observations on lot spell durations, especially for type 3 lot spells due to the high probability of rollovers in longer term contracts.

2. With fewer observations, our nonparametrically estimated hazard functions are quite jagged.

3. Also, unlike rental contracts, there is no *a priori* upper bound on the duration of a lot spell.
Lot Spell Durations

1. We have far fewer observations on lot spell durations, especially for type 3 lot spells due to the high probability of roll overs in longer term contracts.

2. With fewer observations, our nonparametrically estimated hazard functions are quite jagged.

3. Also, unlike rental contracts, there is no \textit{a priori} upper bound on the duration of a lot spell.

4. As a result we needed some method of extrapolation to predict durations given that we have only a small number of cases with extremely long lot durations.
Lot Spell Durations

1. We have far fewer observations on lot spell durations, especially for type 3 lot spells due to the high probability of roll overs in longer term contracts.

2. With fewer observations, our nonparametrically estimated hazard functions are quite jagged.

3. Also, unlike rental contracts, there is no *a priori* upper bound on the duration of a lot spell.

4. As a result we needed some method of extrapolation to predict durations given that we have only a small number of cases with extremely long lot durations.

5. We assume that the hazard function is constant after $d = 31$ days, which implies that the upper tail for the distribution of lot spells is geometric.
Lot Spell Durations

1. We have far fewer observations on lot spell durations, especially for type 3 lot spells due to the high probability of roll overs in longer term contracts.

2. With fewer observations, our nonparametrically estimated hazard functions are quite jagged.

3. Also, unlike rental contracts, there is no \( a_{priori} \) upper bound on the duration of a lot spell.

4. As a result we needed some method of extrapolation to predict durations given that we have only a small number of cases with extremely long lot durations.

5. We assume that the hazard function is constant after \( d = 31 \) days, which implies that the upper tail for the distribution of lot spells is geometric.

6. We also \textit{regression smoothed} the non-parametric hazard estimates.
Lot Durations: Compact

Lot Spell Hazard Functions for Compact, all locations

- Smoothed hazard: previous long term rental
- Nonparametric hazard: previous long term rental
- Smoothed hazard: previous short term rental
- Nonparametric hazard: previous short term rental
3.6 Transition Models
1. When a spell in a given rental state ends, there is a transition to a new rental state.
1. When a spell in a given rental state ends, there is a transition to a new rental state.

2. Let $\pi(r'|r, d, o)$ denote probability the new rental state for a car will be $r'$ given that the current rental state is $r$, the odometer value is $o$, and the duration in state $r$ is $d$. 
Transition Probabilities

1. When a spell in a given rental state ends, there is a transition to a new rental state.

2. Let \( \pi(r'|r, d, o) \) denote probability the new rental state for a car will be \( r' \) given that the current rental state is \( r \), the odometer value is \( o \), and the duration in state \( r \) is \( d \).

3. We call \( \pi \) the *rental state transition probability*.
Transition Probabilities

1. When a spell in a given rental state ends, there is a transition to a new rental state.

2. Let $\pi(r' | r, d, o)$ denote probability the new rental state for a car will be $r'$ given that the current rental state is $r$, the odometer value is $o$, and the duration in state $r$ is $d$.

3. We call $\pi$ the *rental state transition probability*.

4. We rule out "self transitions" to lot spells, i.e. $\pi(r | r, d, o) = 0$ for $r > 2$. 
1. When a spell in a given rental state ends, there is a transition to a new rental state.

2. Let \( \pi(r'|r, d, o) \) denote probability the new rental state for a car will be \( r' \) given that the current rental state is \( r \), the odometer value is \( o \), and the duration in state \( r \) is \( d \).

3. We call \( \pi \) the \textit{rental state transition probability}.

4. We rule out “self transitions” to lot spells, i.e. \( \pi(r|r, d, o) = 0 \) for \( r > 2 \).

5. However for rental spells, there is a conceptual distinction between a rental spell that terminates with an immediate transition to a new rental spell versus the case where an existing rental contract continues for one more day.
Transition Probabilities

1. When a spell in a given rental state ends, there is a transition to a new rental state.

2. Let $\pi(r' | r, d, o)$ denote probability the new rental state for a car will be $r'$ given that the current rental state is $r$, the odometer value is $o$, and the duration in state $r$ is $d$.

3. We call $\pi$ the rental state transition probability.

4. We rule out “self transitions” to lot spells, i.e. $\pi(r | r, d, o) = 0$ for $r > 2$.

5. However for rental spells, there is a conceptual distinction between a rental spell that terminates with an immediate transition to a new rental spell versus the case where an existing rental contract continues for one more day.

6. The former case can be viewed as an immediate “roll over” of one rental contract to another one.
1. Since there are three possible destination states for transitions out of rental spells (i.e. long term contract, short term contract, or lot spell), we used a trinomial logit model to estimate these probabilities.

\[
\pi(r'|r, d, o) = \frac{\exp\{v(r, d, o)\theta_{r'}\}}{\sum_{\rho \in \{1, 2, l(r)\}} \exp\{v(r, d, o)\theta_{\rho}\}},
\]

(9)
1. Since there are three possible destination states for transitions out of rental spells (i.e. long term contract, short term contract, or lot spell), we used a trinomial logit model to estimate these probabilities.

\[
\pi(r' | r, d, o) = \frac{\exp\{v(r, d, o)\theta_{r'}\}}{\sum_{\rho\in\{1, 2, l(r)\}} \exp\{v(r, d, o)\theta_{\rho}\}},
\]

2. \(v(r, d, o)\) is a vector-valued function of the variables \((r, d, o)\) and \(\theta_{\rho}\) is an alternative-specific vector of parameters, for \(\rho = \{1, 2, l(r)\}\) (where \(l(r)\) denotes a lot spell, either of type 3 if \(r = 1\) or type 4 if \(r = 2\)) with the same dimension as \(v\).
1. Since there are three possible destination states for transitions out of rental spells (i.e. long term contract, short term contract, or lot spell), we used a trinomial logit model to estimate these probabilities.

\[
\pi(r'|r, d, o) = \frac{\exp\{v(r, d, o)\theta_{r'}\}}{\sum_{\rho \in \{1, 2, l(r)\}} \exp\{v(r, d, o)\theta_{\rho}\}},
\]

2. \(v(r, d, o)\) is a vector-valued function of the variables \((r, d, o)\) and \(\theta_{\rho}\) is an alternative-specific vector of parameters, for \(\rho = \{1, 2, l(r)\}\) (where \(l(r)\) denotes a lot spell, either of type 3 if \(r = 1\) or type 4 if \(r = 2\)) with the same dimension as \(v\).

3. As is well known, it is not possible to identify all three of the \(\theta_{\rho}\) vectors. Therefore we make an identifying normalization that \(\theta_1 = 0\), i.e. we normalized the parameters for transition to long term contract to zero.
Trinomial Logit Estimates

<table>
<thead>
<tr>
<th>Variable</th>
<th>Compact</th>
<th>Luxury</th>
<th>RV</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimates of $\theta_2$ (from short term rental)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>4.60**</td>
<td>3.01**</td>
<td>4.14**</td>
</tr>
<tr>
<td>Odometer, $o$ (000 km)</td>
<td>0.011*</td>
<td>0.002</td>
<td>0.001</td>
</tr>
<tr>
<td>Duration, $d$</td>
<td>$-0.068^*$</td>
<td>$-0.039^*$</td>
<td>$-0.087^*$</td>
</tr>
<tr>
<td>$I{d \geq 29}$</td>
<td>$-0.421$</td>
<td>0.006</td>
<td>0.079</td>
</tr>
<tr>
<td>$I{r = 1}$</td>
<td>$-6.65^{**}$</td>
<td>$-6.29^{**}$</td>
<td>$-6.33^{**}$</td>
</tr>
<tr>
<td></td>
<td>Estimates of $\theta_{l(r)}$ (from lot spell)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>3.88**</td>
<td>3.70**</td>
<td>4.36**</td>
</tr>
<tr>
<td>Odometer, $o$ (000 km)</td>
<td>0.0204*</td>
<td>0.007*</td>
<td>0.010*</td>
</tr>
<tr>
<td>Duration, $d$</td>
<td>$-0.077^{**}$</td>
<td>$-0.082^{**}$</td>
<td>$-0.120^{**}$</td>
</tr>
<tr>
<td>$I{d \geq 29}$</td>
<td>$-1.50^{**}$</td>
<td>$-1.07^{*}$</td>
<td>$-0.77$</td>
</tr>
<tr>
<td>$I{r = 1}$</td>
<td>$-4.49^{**}$</td>
<td>$-3.44^{**}$</td>
<td>$-3.77^{**}$</td>
</tr>
<tr>
<td>$N$, log($L$)/$N$</td>
<td>16246, $-0.606$</td>
<td>3617, $-0.484$</td>
<td>2142, $-0.583$</td>
</tr>
</tbody>
</table>
1. For transitions out of lot spells, since we have ruled out the possibility of “self-transitions” there are only two possible destinations: long term rental spells and short term rental spells.
Lot Spell Transitions

1. For transitions out of lot spells, since we have ruled out the possibility of “self-transitions” there are only two possible destinations: long term rental spells and short term rental spells.

2. We use a binomial logit model to estimate these probabilities

\[
\pi(r' = 1|r, d, o) = \frac{\exp\{v(o, d)\theta_r\}}{1 + \exp\{v(o, d)\theta_r\}}, \quad r \in \{3, 4\}
\]
Lot Spell Transitions

1. For transitions out of lot spells, since we have ruled out the possibility of “self-transitions” there are only two possible destinations: long term rental spells and short term rental spells.

2. We use a binomial logit model to estimate these probabilities

\[
\pi(r' = 1 | r, d, o) = \frac{\exp\{v(o, d)\theta_r\}}{1 + \exp\{v(o, d)\theta_r\}}, \quad r \in \{3, 4\}
\]

3. Note that there are far fewer observations on transitions out of type 3 lot spells due to the high frequency of roll over of long term rental contracts.
## Binomial Logit Estimates

<table>
<thead>
<tr>
<th>Variable</th>
<th>Compact</th>
<th>Luxury</th>
<th>RV</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Estimates of $\theta_3$ (previous rental long term)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>2.26*</td>
<td>1.59*</td>
<td>2.34*</td>
</tr>
<tr>
<td>Odometer, $o$ (000 km)</td>
<td>0.010</td>
<td>−0.002</td>
<td>−0.009</td>
</tr>
<tr>
<td>Duration, $d$</td>
<td>−0.05*</td>
<td>−0.004</td>
<td>−0.038</td>
</tr>
<tr>
<td>$N$, log($L$)/$N$</td>
<td>173, −0.326</td>
<td>181, −0.490</td>
<td>43, −0.511</td>
</tr>
<tr>
<td><strong>Estimates of $\theta_4$ (previous rental short term)</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Constant</td>
<td>3.63*</td>
<td>1.94*</td>
<td>4.54*</td>
</tr>
<tr>
<td>Odometer, $o$ (000 km)</td>
<td>0.021*</td>
<td>0.013*</td>
<td>−0.009</td>
</tr>
<tr>
<td>Duration, $d$</td>
<td>−0.06*</td>
<td>−0.003</td>
<td>−0.03*</td>
</tr>
<tr>
<td>$N$, log($L$)/$N$</td>
<td>5162, −0.077</td>
<td>961, −0.683</td>
<td>922, −0.090</td>
</tr>
</tbody>
</table>
Conclusions

- There are two key points to take away from the transition probability estimates
Conclusions

- There are two key points to take away from the transition probability estimates

1. for all car types, there is a very high probability that cars will be initially rented in long term contracts,
There are two key points to take away from the transition probability estimates:

1. for all car types, there is a very high probability that cars will be initially rented in long term contracts,
2. the results provide clear evidence of “contract age effects”.
Conclusions

- There are two key points to take away from the transition probability estimates
  1. for all car types, there is a very high probability that cars will be initially rented in long term contracts,
  2. the results provide clear evidence of “contract age effects”.

- That is, as the odometer value increases (i.e. the age of the car increases) the probability of transitions into long term rental contracts decreases and the probability of transitions into short term rental contracts increases.
Conclusions

- There are two key points to take away from the transition probability estimates
  1. for all car types, there is a very high probability that cars will be initially rented in long term contracts,
  2. the results provide clear evidence of “contract age effects”.

- That is, as the odometer value increases (i.e. the age of the car increases) the probability of transitions into long term rental contracts decreases and the probability of transitions into short term rental contracts increases.

- **However we find no other aging effects in spell durations or in maintenance costs.**
3.7 Probability Graphs
Transition from Long Term

Estimated Transition Probabilities from Long Term Rental Spell

- Long Term Rental Spell
- Short Term Rental Spell
- Lot Spell
Transition from Short Term

Estimated Transition Probabilities from Short Term Rental Spell

- Long Term Rental Spell
- Short Term Rental Spell
- Lot Spell

Odometer, (thousands of kilometers)

Probability of New Spell

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9

0 50 100 150

Sungjin Cho and John Rust
Transition from Lot Type 3

Estimated Transition Probabilities from Lot Spell, Type 3

Probability of Entering Long Term Rental Spell

Odometer, (thousands of kilometers)

- \( d=1 \)
- \( d=10 \)
- \( d=30 \)
Transition from Lot Type 4

Estimated Transition Probabilities from Lot Spell, Type 4

Odometer, (thousands of kilometers)

Probability of Entering Long Term Rental Spell

- $d=1$
- $d=10$
- $d=30$
3.8 No Other Age Effects
No Aging in Maintenance
No Aging in Rental Durations

Scatterplot of Short Term Rental Durations for Compact, all locations
Mean: 2.5
Std dev: 5.1
Observations: 10390

Scatterplot of Long Term Rental Durations for Compact, all locations
Mean: 28.8
Std dev: 5.4
Observations: 2689
No Aging in Lot Durations

Durations of Lot Spells for Compact, all locations, Previous Rental Spell was Long Term

Mean duration 9.3
Mean odometer 43.0
Observations 174

Durations of Lot Spells for Compact, all locations, Previous Rental Spell was Short Term

Mean duration 3.3
Mean odometer 41.6
Observations 5165
3.0 Simulation Methodology
1. Draw a *random selling threshold* $\overline{o}$ from the *empirical distribution of odometer at sale* $\hat{G}(o)$,
Simulating a Car’s Life

1. Draw a random selling threshold $\overline{g}$ from the empirical distribution of odometer at sale $\hat{G}(o)$.

2. Record the purchase price $\overline{P}$ and start the car in an initial lot spell.
Simulating a Car’s Life

1. Draw a random selling threshold $\tilde{c}$ from the empirical distribution of odometer at sale $\hat{G}(o)$,

2. Record the purchase price $\bar{P}$ and start the car in an initial lot spell.

3. Draw a random duration in the lot, $\tilde{d}$, from the duration distribution $f(d|r_0)$ implied by the estimated hazard function $\hat{h}(d, r_0)$ (a mixture of the hazards for type 3 and 4 lot spells designed to match the initial lot duration, $\hat{h}(d, r_0) = \alpha \hat{h}(d, 3) + (1 - \alpha) \hat{h}(d, 4)$).
Simulating a Car’s Life

1. Draw a random selling threshold $\bar{\sigma}$ from the empirical distribution of odometer at sale $\hat{G}(\sigma)$,

2. Record the purchase price $\bar{P}$ and start the car in an initial lot spell.

3. Draw a random duration in the lot, $\tilde{d}$, from the duration distribution $f(d|r_0)$ implied by the estimated hazard function $\hat{h}(d, r_0)$ (a mixture of the hazards for type 3 and 4 lot spells designed to match the initial lot duration, $\hat{h}(d, r_0) = \hat{\alpha}\hat{h}(d, 3) + (1 - \hat{\alpha})\hat{h}(d, 4)$).

4. Draw a random initial rental state $\tilde{r}_{\tilde{d}}$ from the transition probability $\hat{\pi}(r|r_0, \tilde{d}, 0)$. 
Simulating a Car’s Life

1. Draw a random selling threshold $\overline{\theta}$ from the empirical distribution of odometer at sale $\hat{G}(o)$,

2. Record the purchase price $\overline{P}$ and start the car in an initial lot spell.

3. Draw a random duration in the lot, $\tilde{d}$, from the duration distribution $f(d|r_0)$ implied by the estimated hazard function $\hat{h}(d, r_0)$ (a mixture of the hazards for type 3 and 4 lot spells designed to match the initial lot duration,

$\hat{h}(d, r_0) = \alpha \hat{h}(d, 3) + (1 - \alpha) \hat{h}(d, 4)$).

4. Draw a random initial rental state $\tilde{r}_\tilde{d}$ from the transition probability $\hat{\pi}(r|r_0, \tilde{d}, 0)$.

5. The car is now in its first rental spell.
1. The duration $\tilde{d}$ is drawn from the $f(d|r)$ implied by $\hat{h}(d, r)$.
Simulating Rental Spells

1. The duration $\tilde{d}$ is drawn from the $f(d|r)$ implied by $\hat{h}(d, r)$.

2. Draw a random in-odometer value $\tilde{o}$ from the Erlang (Gamma) distribution $f(\tilde{o}|\hat{\lambda}_r, \tilde{d}, o)$ that we assumed for vehicle usage.
Simulating Rental Spells

1. The duration $\tilde{d}$ is drawn from the $f(d|r)$ implied by $\hat{h}(d, r)$.

2. Draw a random in-odometer value $\tilde{o}$ from the Erlang (Gamma) distribution $f(\tilde{o}|\hat{\lambda}_r, d, o)$ that we assumed for vehicle usage.

3. If $\tilde{o} > \bar{o}$, the car is sold. A random selling price $\tilde{P}(\tilde{o})$ is drawn from the lognormal distribution implied by our estimated price depreciation model.
Simulating Rental Spells

1. The duration $\tilde{d}$ is drawn from the $\hat{f}(d|r)$ implied by $\hat{h}(d, r)$.
2. Draw a random in-odometer value $\tilde{o}$ from the Erlang (Gamma) distribution $f(\tilde{o}|\hat{\lambda}_r, \tilde{d}, o)$ that we assumed for vehicle usage.
3. If $\tilde{o} > \bar{o}$, the car is sold. A random selling price $\tilde{P}(\bar{o})$ is drawn from the lognormal distribution implied by our estimated price depreciation model.
4. Otherwise, the car enters state $\tilde{r} \tilde{d}$ drawn from our trinomial logit model, $\pi(\tilde{r} \tilde{d}|r, \tilde{d}, \tilde{o})$. 
Simulating Rental Spells

1. The duration \( \tilde{d} \) is drawn from the \( f(d|\tilde{r}) \) implied by \( \hat{h}(d, \tilde{r}) \).

2. Draw a random in-odometer value \( \tilde{o} \) from the Erlang (Gamma) distribution \( f(\tilde{o}|\hat{\lambda}(\tilde{d}, \tilde{r})) \) that we assumed for vehicle usage.

3. If \( \tilde{o} > \bar{o} \), the car is sold. A random selling price \( \tilde{P}(\tilde{o}) \) is drawn from the lognormal distribution implied by our estimated price depreciation model.

4. Otherwise, the car enters state \( \tilde{r} \tilde{d} \) drawn from our trinomial logit model, \( \pi(\tilde{r} \tilde{d}|r, \tilde{d}, \tilde{o}) \).

5. If this new state is a lot state, the duration in the lot is simulated as described on the previous slide.
Simulating Rental Spells

1. The duration $\tilde{d}$ is drawn from the $\hat{f}(d|r)$ implied by $\hat{h}(d, r)$.
2. Draw a random in-odometer value $\tilde{o}$ from the Erlang (Gamma) distribution $f(\tilde{o}|\hat{\lambda}_r, \tilde{d}, o)$ that we assumed for vehicle usage.
3. If $\tilde{o} > \bar{o}$, the car is sold. A random selling price $\tilde{P}(\tilde{o})$ is drawn from the lognormal distribution implied by our estimated price depreciation model.
4. Otherwise, the car enters state $\tilde{r}\tilde{d}$ drawn from our trinomial logit model, $\pi(\tilde{r}\tilde{d}|r, \tilde{d}, \tilde{o})$.
5. If this new state is a lot state, the duration in the lot is simulated as described on the previous slide.
6. If the new state is a rental spell, the simulation repeats as described in steps 1-4 above.
Simulated Histories

Compact, all locations realization 43 (service life: 1171 days, IRR=69.4)

Luxury 2, all locations realization 87 (service life: 979 days, IRR=50.1)

RV, all locations realization 88 (service life: 705 days, IRR=44.6)
3.1 Odometer and Age
Distribution of Kilometers at Sale (in thousands) Luxury 2, all locations

- **Actual**
  - Mean: 74.7
  - Median: 71.3
  - Minimum: 35.0
  - Maximum: 187.8
  - Std dev: 24.8
  - N: 91

- **Simulated**
  - Mean: 71.1
  - Median: 69.4
  - Minimum: 31.5
  - Maximum: 139.4
  - Std dev: 20.8
  - N: 100
Age at Replacement

Distribution of Age at Sale (Years) Luxury 2, all locations

Actual
Mean 2.7
Median 2.6
Minimum 1.9
Maximum 4.0
Std dev 0.4
N 40

Simulated
Mean 2.6
Median 2.5
Minimum 0.8
Maximum 5.1
Std dev 0.8
N 100
4.2 Number of Spells
Number of Long Term Rentals

Distribution of Number of Long Term Rentals Luxury 2, all locations

Actual
Mean 24.7
Median 24.0
Minimum 6.0
Maximum 48.0
Std dev 9.3
N 40

Simulated
Mean 22.8
Median 23.0
Minimum 2.0
Maximum 39.0
Std dev 7.0
N 100
Number of Short Term Rentals

Distribution of Number of Short Term Rentals Luxury 2, all locations

- **Actual**
  - Mean: 15.2
  - Median: 8.0
  - Minimum: 0.0
  - Maximum: 73.0
  - Std dev: 18.7
  - N: 40

- **Simulated**
  - Mean: 18.4
  - Median: 12.5
  - Minimum: 0.0
  - Maximum: 89.0
  - Std dev: 18.3
  - N: 100
Number of Lot Spells

Distribution of Number of Lot Spells Luxury 2, all locations

- Actual
  - Mean: 13.9
  - Median: 9.0
  - Minimum: 2.0
  - Maximum: 59.0
  - Std dev: 14.2
  - N: 40

- Simulated
  - Mean: 14.2
  - Median: 11.0
  - Minimum: 1.0
  - Maximum: 69.0
  - Std dev: 12.8
  - N: 100
4.3 Days in Spells
Days in Long Term Rentals

Distribution of Days in Long Term Rentals Luxury 2, all locations

Actual
Mean 696.8
Median 715.0
Minimum 146.0
Maximum 1252.0
Std dev 270.7
N 40

Simulated
Mean 642.6
Median 628.5
Minimum 174.0
Maximum 1454.0
Std dev 215.2
N 100
Days in Short Term Rentals

Distribution of Days in Short Term Rentals Luxury 2, all locations

- Actual:
  - Mean: 100.3
  - Median: 55.5
  - Minimum: 0.0
  - Maximum: 588.0
  - Std dev: 123.6
  - N: 40

- Simulated:
  - Mean: 101.0
  - Median: 64.5
  - Minimum: 0.0
  - Maximum: 429.0
  - Std dev: 103.6
  - N: 100
Days in Lot Spells

Distribution of Days on the Lot Luxury 2, all locations

- Actual
  - Mean: 192.6
  - Median: 144.0
  - Minimum: 8.0
  - Maximum: 544.0
  - Std dev: 141.2
  - N: 40

- Simulated
  - Mean: 129.2
  - Median: 103.0
  - Minimum: 3.0
  - Maximum: 615.0
  - Std dev: 109.8
  - N: 100
4.4 Financial Outcomes
Vehicle Sales Proceeds

Distribution of Revenue from Sale of Car Luxury 2, all locations

- Actual
  - Mean: 12283.2
  - Median: 12180.0
  - Minimum: 7500.0
  - Maximum: 16550.0
  - Std dev: 1666.6
  - N: 40

- Simulated
  - Mean: 12109.4
  - Median: 12197.7
  - Minimum: 7954.2
  - Maximum: 16881.9
  - Std dev: 1709.8
  - N: 100
Distribution of Total Maintenance Costs Luxury 2, all locations

- **Actual**
  - Mean: 956.4
  - Median: 748.9
  - Minimum: 135.0
  - Maximum: 5866.7
  - Std dev: 932.1
  - N: 40

- **Simulated**
  - Mean: 1085.7
  - Median: 1068.8
  - Minimum: 471.9
  - Maximum: 2348.8
  - Std dev: 359.6
  - N: 100
Long Term Rental Revenues

Distribution of Revenue from Long Term Rentals Luxury 2, all locations

- Actual
  - Mean: 28206.9
  - Median: 28154.2
  - Minimum: 6320.9
  - Maximum: 50427.3
  - Std dev: 10631.9
  - N: 40

- Simulated
  - Mean: 28621.0
  - Median: 28126.8
  - Minimum: 7880.7
  - Maximum: 65005.2
  - Std dev: 9605.9
  - N: 100
Short Term Rental Revenues

Distribution of Revenue from Short Term Rentals Luxury 2, all locations

- Actual
  - Mean: 6106.0
  - Median: 3246.2
  - Minimum: 0.0
  - Maximum: 24102.7
  - Std dev: 7336.7
  - N: 40

- Simulated
  - Mean: 7446.1
  - Median: 5208.4
  - Minimum: 0.0
  - Maximum: 33050.7
  - Std dev: 7587.5
  - N: 100
Total Profits

![Distribution of Total Profits Luxury 2, all locations](image)

**Actual**
- Mean: 22244.3
- Median: 21868.1
- Minimum: 10942.0
- Maximum: 37184.1
- Std dev: 6223.7
- N: 40

**Simulated**
- Mean: 22212.7
- Median: 22403.1
- Minimum: 3979.2
- Maximum: 51391.9
- Std dev: 9608.2
- N: 100
Internal Rates of Return

Distribution of Internal Rate of Return (%) Luxury 2, all locations

Actual
Mean 49.2
Median 48.7
Minimum 28.6
Maximum 79.5
Std dev 10.4
N 40

Simulated
Mean 47.1
Median 48.4
Minimum 18.4
Maximum 60.5
Std dev 6.9
N 100
1. While it is possible to evaluate *specific hypothetical alternatives* to the company’s *status quo* operating policy using simulation methods similar to the previous section, there are more efficient methods available for searching over *all possible policies* to find the *optimal replacement policy*. 
1. While it is possible to evaluate *specific hypothetical alternatives* to the company’s *status quo* operating policy using simulation methods similar to the previous section, there are more efficient methods available for searching over *all possible policies* to find the *optimal replacement policy*. 

2. Mathematically, the optimal replacement problem is equivalent to a specific type of *optimal stopping problem* known as a *regenerative optimal stopping problem* (see Rust, 1987).
Optimal Stopping Theory

1. While it is possible to evaluate specific hypothetical alternatives to the company’s status quo operating policy using simulation methods similar to the previous section, there are more efficient methods available for searching over all possible policies to find the optimal replacement policy.

2. Mathematically, the optimal replacement problem is equivalent to a specific type of optimal stopping problem known as a regenerative optimal stopping problem (see Rust, 1987).

3. The term “regenerative” is used, since the decision to replace a vehicle does not stop or end the decision process, but rather results in a “regeneration” or “rebirth”, i.e. a replacement of an old vehicle by a brand new one.
1. We use the method of *dynamic programming* to formulate and solve the optimal stopping problem.
1. We use the method of *dynamic programming* to formulate and solve the optimal stopping problem.

2. We show that the optimal strategy takes the form of a *threshold rule*, i.e. the optimal time to replace a car occurs when its odometer value $o$ exceeds a threshold value $\bar{o}(d, r, \tau)$ that depends on the current rental state $r$, the duration in that state $d$, and the car type $\tau$. 
1. We use the method of *dynamic programming* to formulate and solve the optimal stopping problem.

2. We show that the optimal strategy takes the form of a *threshold rule*, i.e. the optimal time to replace a car occurs when its odometer value $o$ exceeds a threshold value $\bar{o}(d, r, \tau)$ that depends on the current rental state $r$, the duration in that state $d$, and the car type $\tau$.

3. Using numerical methods, we solve the dynamic programming problem and calculate the optimal stopping thresholds $\bar{o}(d, r, \tau)$. 

---

**Dynamic Programming**

1. Introduction
2. Data
3. An Econometric Model
4. Simulation Results
5. Optimal Replacement Policy
   - Optimal Stopping Theory
   - Valuing Alternative Policies
   - Bellman Equation
   - Valuing Suboptimal Policies
   - Expected Value Functions
   - Expected Revenue Functions
   - Optimal Stopping Threshold
6. Numerical Results
7. Extensions
8. Conclusions
1. We use the method of *dynamic programming* to formulate and solve the optimal stopping problem.

2. We show that the optimal strategy takes the form of a *threshold rule*, i.e. the optimal time to replace a car occurs when its odometer value $o$ exceeds a threshold value $\bar{o}(d, r, \tau)$ that depends on the current rental state $r$, the duration in that state $d$, and the car type $\tau$.

3. Using numerical methods, we solve the dynamic programming problem and calculate the optimal stopping thresholds $\bar{o}(d, r, \tau)$.

4. We also compute the optimal *value functions* $V(r, d, o, \tau)$. This function provides the expected discounted profits (over an infinite horizon) under the optimal replacement policy for a vehicle that is in state $(r, d, o)$. 

---

*Dynamic Programming*
1. It is also possible to compute the value of any alternative operating strategy $\mu$, which can include *mixed* or probabilistic operating strategies where the decision to replace a car is given by a conditional probability distribution $\mu(r, d, o, \tau)$. 
1. It is also possible to compute the value of any alternative operating strategy $\mu$, which can include mixed or probabilistic operating strategies where the decision to replace a car is given by a conditional probability distribution $\mu(r, d, o, \tau)$.

2. Let $V_\mu(r, d, o, \tau)$ denote the expected discounted profits (again over an infinite horizon) under the alternative replacement policy $\mu$. 
1. It is also possible to compute the value of any alternative operating strategy \( \mu \), which can include mixed or probabilistic operating strategies where the decision to replace a car is given by a conditional probability distribution \( \mu(r, d, o, \tau) \).

2. Let \( V_\mu(r, d, o, \tau) \) denote the expected discounted profits (again over an infinite horizon) under the alternative replacement policy \( \mu \).

3. We will calculate both \( V \) and \( V_\mu \) where \( \mu \) is an approximation to the company’s status quo operating policy.
1. It is also possible to compute the value of any alternative operating strategy $\mu$, which can include mixed or probabilistic operating strategies where the decision to replace a car is given by a conditional probability distribution $\mu(r, d, o, \tau)$.

2. Let $V_{\mu}(r, d, o, \tau)$ denote the expected discounted profits (again over an infinite horizon) under the alternative replacement policy $\mu$.

3. We will calculate both $V$ and $V_{\mu}$ where $\mu$ is an approximation to the company’s status quo operating policy.

4. The difference $V(r, d, o, \tau) - V_{\mu}(r, d, o, \tau)$ will represent our estimate of the gain in profits from adopting an optimal replacement policy.
1. It is also possible to compute the value of any alternative operating strategy $\mu$, which can include mixed or probabilistic operating strategies where the decision to replace a car is given by a conditional probability distribution $\mu(r, d, o, \tau)$.

2. Let $V_\mu(r, d, o, \tau)$ denote the expected discounted profits (again over an infinite horizon) under the alternative replacement policy $\mu$.

3. We will calculate both $V$ and $V_\mu$ where $\mu$ is an approximation to the company’s status quo operating policy.

4. The difference $V(r, d, o, \tau) - V_\mu(r, d, o, \tau)$ will represent our estimate of the gain in profits from adopting an optimal replacement policy.

5. *We will show that the optimal policy entails keeping cars significantly longer than the company currently keeps them.*
1. $V$ given by

$$V(r, d, o) = \max \left[ EP(o) - \overline{P} + \beta EV(r_0, 0, 0), \right.$$

$$\left. ER(r, d, o) - EM + \beta EV(r, d, o) \right]$$

(11)
1. $V$ given by

$$V(r, d, o) = \max \left[ EP(o) - \bar{P} + \beta EV(r_0, 0, 0), \right.$$  
$$\left. ER(r, d, o) - EM + \beta EV(r, d, o) \right]$$  

(11)

2. There are two $EV$ functions in the Bellman equation, $EV(r, d, o)$ and $EV(r_0, 0, 0)$. 
Bellman Equation

1. $V$ given by

\[ V(r, d, o) = \max \left[ EP(o) - \bar{P} + \beta EV(r_0, 0, 0), \right. \]

\[ \left. ER(r, d, o) - EM + \beta EV(r, d, o) \right] \tag{11} \]

2. There are two $EV$ functions in the Bellman equation, $EV(r, d, o)$ and $EV(r_0, 0, 0)$. 

3. $EV(r, d, o)$ is the expected value of an existing car which has an odometer value of $o$ and has been in rental state $r$ for a duration of $d$ days.
Bellman Equation

1. $V$ given by

$$V(r, d, o) = \max \left[ EP(o) - \overline{F} + \beta EV(r_0, 0, 0), \right.$$

$$\left. ER(r, d, o) - EM + \beta EV(r, d, o) \right]$$  \hspace{1cm} (11)

2. There are two $EV$ functions in the Bellman equation, $EV(r, d, o)$ and $EV(r_0, 0, 0)$.

3. $EV(r, d, o)$ is the expected value of an existing car which has an odometer value of $o$ and has been in rental state $r$ for a duration of $d$ days.

4. $EV(r_0, 0, 0)$ is the expected value of a new car just after it has been purchased when it is on the lot waiting for its first rental.
1. The Bellman equation (11) actually applies only when \( r \in \{3, 4\} \) or \( d = 0 \) if \( r \in \{1, 2\} \), since we assume that the company will not interrupt an ongoing rental contract to sell a vehicle.
1. The Bellman equation (11) actually applies only when \((r \in \{3, 4\})\) or \((d = 0\) if \(r \in \{1, 2\})\), since we assume that the company will not interrupt an ongoing rental contract to sell a vehicle.

2. For cars in the midst of a rental spell, \((r = 1\) or \(r = 2\) and \(d > 0\)), we have

\[
V(r, d, o) = ER(r, d, o) - EM + \beta EV(r, d, o). 
\]

where \(ER\) are expected rental revenues, and \(EM\) is the expected maintenance cost.
1. The Bellman equation (11) actually applies only when \( r \in \{3, 4\} \) or \( d = 0 \) if \( r \in \{1, 2\} \), since we assume that the company will not interrupt an ongoing rental contract to sell a vehicle.

2. For cars in the midst of a rental spell, \( (r = 1 \text{ or } r = 2 \text{ and } d > 0) \), we have

\[
V(r, d, o) = ER(r, d, o) - EM + \beta EV(r, d, o).
\]

where \( ER \) are expected rental revenues, and \( EM \) is the expected maintenance cost.

3. There is a similar equation for computing the value of a mixed replacement policy \( \mu \)

\[
V_\mu(d, r, o) = \mu(d, r, o)[EP(o) - \bar{P} + \beta EV_\mu(0, r_0, 0)] + \\
(1 - \mu(d, r, o))[ER(d, r, o) - EM + \beta EV_\mu(d, r, o)],
\]
1. For lot spells

\[ EV(r, d, o) = h(d, r) \left[ V(1, 1, o)\pi(1|r, d, o) + V(2, 1, o)[1 - \pi(1|r, d, o)]\right] + [1 - h(d, r)]V(r, \min(d + 1, 31), o). \]
Expected Value Functions

1. For lot spells

\[ EV(r, d, o) = h(d, r) \left[ V(1, 1, o)\pi(1|r, d, o) + V(2, 1, o)[1 - \pi(1|r, d, o)] \right] + \]
\[ [1 - h(d, r)]V(r, \min(d + 1, 31), o). \]

(14)

2. For rental spells

\[ EV(r, d, o) = [1 - h(d, r)]V(l(r), d + 1, o) + \]
\[ h(d, r) \int_{o'} \left[ \sum_{r'} V(r', 0, o')\pi(r'|r, d, o) \right] f(o'|r, d, o)do' \]

(15)

where \( l(r) \) is the lot state following rental state \( r \), i.e. \( l(1) = 3 \) and \( l(2) = 4 \).
1. For short term contracts, $ER(r, d, o)$ is given by

$$ER(2, d, o) = h(d, 2)EDR(2)d,$$

where $EDR(2)$ is the expected daily rental rate for a short term contract.
Expected Revenue Functions

1. For short term contracts, \( ER(r, d, o) \) is given by

\[
ER(2, d, o) = h(d, 2) EDR(2)d, 
\]

where \( EDR(2) \) is the expected daily rental rate for a short term contract.

2. For long term contracts, it is given by

\[
ER(1, d, o) = \begin{cases} 
  h(d, 1) EDR(1)d & \text{if } d \geq \bar{d} \\
  h(d, 1)[EDR(1) + \rho]d & \text{if } d < \bar{d} 
\end{cases} 
\]
1. While it is possible that more complicated types of optimal replacement rules could arise, the optimal replacement policy will generally take the form of an *optimal stopping threshold* $\overline{\sigma}(r, d)$. 
1. While it is possible that more complicated types of optimal replacement rules could arise, the optimal replacement policy will generally take the form of an \textit{optimal stopping threshold} \( \sigma(r, d) \).

2. This is simply a rule that says that it is optimal to keep the current car if its odometer \( o \) satisfies \( o < \sigma(r, d) \) and to replace the car otherwise.
1. While it is possible that more complicated types of optimal replacement rules could arise, the optimal replacement policy will generally takes the form of an *optimal stopping threshold* $\bar{o}(r, d)$.

2. This is simply a rule that says that it is optimal to keep the current car if its odometer $o$ satisfies $o < \bar{o}(r, d)$ and to replace the car otherwise.

3. The optimal stopping threshold is the value of $o$ where the firm is indifferent between keeping the car and replacing it. That is, it is the solution to the equation

$$EP(\bar{o}(r, d)) - \bar{P} + \beta EV(r_0, 0, 0) = ER(r, d, \bar{o}(r, d)) - EM + \beta EV(r, d, \bar{o}(r, d)),$$

(18)
1. As we noted above, if we solve the regenerative optimal stopping problem under the assumption that the only aging effects are 1) the depreciation in vehicle resale values, and 2) the “rental contract composition effect”, then the optimal stopping thresholds is $\bar{o}(r, d) = \infty$, i.e. it is never optimal to sell an existing vehicle.
1. As we noted above, if we solve the regenerative optimal stopping problem under the assumption that the only aging effects are 1) the depreciation in vehicle resale values, and 2) the “rental contract composition effect”, then the optimal stopping thresholds is $\bar{o}(r, d) = \infty$, i.e. it is never optimal to sell an existing vehicle.

2. This follows from the assumption that average daily maintenance costs $EM$ do not increase as a function of odometer value, and that rental rates do not decrease as a function of odometer values.
No Extrapolation Case

1. As we noted above, if we solve the regenerative optimal stopping problem under the assumption that the only aging effects are 1) the depreciation in vehicle resale values, and 2) the “rental contract composition effect”, then the optimal stopping thresholds is $\overline{\sigma}(r, d) = \infty$, i.e. it is *never optimal to sell an existing vehicle*.

2. This follows from the assumption that average daily maintenance costs $EM$ do not increase as a function of odometer value, and that rental rates do not decrease as a function of odometer values.

3. While there is substantial empirical justification for these assumptions *over the range of our observations*, it is questionable that these assumptions will continue to be valid as a vehicle’s odometer and age increases indefinitely, far beyond the range for which we have any observations.
1. We calculated the optimal replacement policy under extremely pessimistic assumptions about increases in maintenance costs and decreases in rental rates beyond the range of our data.
The Pessimistic Case

1. We calculated the optimal replacement policy under extremely pessimistic assumptions about increases in maintenance costs and decreases in rental rates beyond the range of our data.

2. That is, we will assume that beyond the range of our observations, maintenance costs increase at a very rapid rate as odometer increases, and that to induce customers to rent older vehicles, daily rental rates must be steeply discounted.
The Pessimistic Case

1. We calculated the optimal replacement policy under extremely pessimistic assumptions about increases in maintenance costs and decreases in rental rates beyond the range of our data.

2. That is, we will assume that beyond the range of our observations, maintenance costs increase at a very rapid rate as odometer increases, and that to induce customers to rent older vehicles, daily rental rates must be steeply discounted.

3. Specifically, after a vehicle hits 130,000 kilometers, we assume that maintenance costs increase rapidly and rental rates must be decreased rapidly to induce customers to rent older cars.
Multiplication Factors

Assumed Multiplication Factor for Daily Maintenance Costs

Assumed Multiplication Factor for Daily Rental Rates
6.1 Results for Compact
Optimal Replacement Strategy: Compact, all locations

- Replacement Threshold, 1st day of long term rental: 156.2
- Replacement Threshold, 1st day of short term rental: 201.6
Optimal Values: Compact

Value Function for Compact, all locations (first day of rental spell)

- Long term rental
- Short term rental
- Value of Selling

Expected discounted Profit per car vs. Odometer (Thousands of kilometers)
Optimal Values: Compact

Value Function for Compact, all locations (ov=47.15)

- Long term rental
- Short term rental
- Lot spell, previous ltr
- Lot spell, previous str

Expected discounted Profit per car vs. Duration of spell (days)
Expected Rental Revenues given Rental Duration for Compact, all locations

- Long term rental
- Short term rental

Expected daily profit per car vs. Duration of rental spell (days)
6.2 Results for Luxury
Optimal Replacement Strategy: Luxury 2, all locations

Replacement Threshold, 1st day of long term rental: 175.3
Replacement Threshold, 1st day of short term rental: 192.1
Optimal Values: Luxury

Value Function for Luxury 2, all locations (first day of rental spell)

- Long term rental
- Short term rental
- Value of Selling

Expected discounted Profit per car vs. Odometer (Thousands of kilometers)
Optimal Values: Luxury

Value Function for Luxury 2, all locations (ov=47.15)

Expected discounted Profit per car

Duration of spell (days)

- Long term rental
- Short term rental
- Lot spell, previous ltr
- Lot spell, previous str
Expected Rental Revenues given Rental Duration for Luxury 2, all locations

Duration of rental spell (days)

Expected daily profit per car

- Long term rental
- Short term rental

Expected Revenue: Luxury
6.3 Results for RV
Optimal Replacement Strategy: RV, all locations

- Replacement Threshold, 1st day of long term rental: 170.6
- Replacement Threshold, 1st day of short term rental: 202.5
Optimal Values: RV

Value Function for RV, all locations (first day of rental spell)

Expected discounted Profit per car

Odometer (Thousands of kilometers)

Long term rental
Short term rental
Value of Selling
Value Function for RV, all locations (ov=47.15)

Expected discounted Profit per car vs. Duration of spell (days)

- Blue line: Long term rental
- Red dashed line: Short term rental
- Green dashed line: Lot spell, previous ltr
- Redotted line: Lot spell, previous str

Optimal Values: RV
Expected Rental Revenues given Rental Duration for RV, all locations

- Long term rental
- Short term rental
6.4 Profit Comparisons
# Profit Comparisons

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Compact</th>
<th>Luxury</th>
<th>RV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{P}$</td>
<td>9668</td>
<td>23389</td>
<td>18774</td>
</tr>
</tbody>
</table>

**Expected Discounted Values Under Optimal Replacement Policy**

| $V(0,0,r_0)$ | 268963 | 374913 | 327057 |
| $(1 - \beta)V(0,0,r_0)$ | 22.11  | 30.81  | 26.88  |
| $V(0,0,r_0)/\overline{P}$ | 27.8   | 16.0   | 17.4   |

**Expected Discounted Values Under Status Quo Replacement Policy**

| $V_\mu(0,0,r_0)$ | 196589 | 318247 | 136792 |
| $(1 - \beta)V_\mu(0,0,r_0)$ | 16.16  | 26.16  | 11.24  |
| $V(0,0,r_0)_\mu/\overline{P}$ | 20.3   | 13.6   | 7.3    |

**Ratio of Expected Values: Optimal Policy versus Status Quo**

| $V(0,0,r_0)/V_\mu(0,0,r_0)$ | 1.37   | 1.18   | 2.39   |
6.5 Assessing Robustness
1. To assess the robustness of our conclusions, we solved for the optimal replacement policy under even more pessimistic assumptions about maintenance costs and the rental discounts that would be required to induce customers to rent older cars.
1. To assess the robustness of our conclusions, we solved for the optimal replacement policy under even more pessimistic assumptions about maintenance costs and the rental discounts that would be required to induce customers to rent older cars.

2. Under this more pessimistic scenario, maintenance costs start to accelerate far earlier, at \textit{60,000 kilometers}. 
Even More Pessimistic Case

1. To assess the robustness of our conclusions, we solved for the optimal replacement policy under even more pessimistic assumptions about maintenance costs and the rental discounts that would be required to induce customers to rent older cars.

2. Under this more pessimistic scenario, maintenance costs start to accelerate far earlier, at \textit{60,000 kilometers}.

3. We assume that rental rates start decreasing at a linear after 60,000 kilometers until they hit zero at 210,000 kilometers.
1. To assess the robustness of our conclusions, we solved for the optimal replacement policy under even more pessimistic assumptions about maintenance costs and the rental discounts that would be required to induce customers to rent older cars.

2. Under this more pessimistic scenario, maintenance costs start to accelerate far earlier, at **60,000 kilometers**.

3. We assume that rental rates start decreasing at a linear after 60,000 kilometers until they hit zero at 210,000 kilometers.

4. **Even under this even more pessimistic scenario, the optimal replacement policy still entails keeping cars about twice as long (in terms of age or odometer value) as the company currently keeps them.**
Assumed Multiplication Factor for Daily Maintenance Costs

- Scenario 1
- Scenario 2

Multiplication Factor

Odometer (000 kilometers)
Rental Factors

Assumed Multiplication Factor for Daily Rental Rates

Scenario 1

Scenario 2
### Profit Comparison

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Compact</th>
<th>Luxury</th>
<th>RV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V(0, 0, r_0)$</td>
<td>245680</td>
<td>337853</td>
<td>275614</td>
</tr>
<tr>
<td>$(1 - \beta)V(0, 0, r_0)$</td>
<td>20.19</td>
<td>27.77</td>
<td>22.65</td>
</tr>
<tr>
<td>$V(0, 0, r_0)/\bar{P}$</td>
<td>25.4</td>
<td>14.4</td>
<td>14.7</td>
</tr>
</tbody>
</table>

**Ratio of Expected Values: Optimal Policy versus Status Quo**

| $V(0, 0, r_0)/V_\mu(0, 0, r_0)$ | 1.25 | 1.06 | 2.01 |
Our analysis has been focused mainly on the narrow question of the timing of replacement decisions,
Unanswered Questions

- Our analysis has been focused mainly on the narrow question of the timing of replacement decisions,
- We believe we have provided convincing evidence that via modest changes in the company’s operating strategy, it can significantly increase discounted profits.
Our analysis has been focused mainly on the narrow question of the timing of replacement decisions,

We believe we have provided convincing evidence that via modest changes in the company’s operating strategy, it can significantly increase discounted profits.

However our analysis leaves a number of unanswered questions:
Unanswered Questions

- Our analysis has been focused mainly on the narrow question of the timing of replacement decisions,

- We believe we have provided convincing evidence that via modest changes in the company’s operating strategy, it can significantly increase discounted profits.

- However our analysis leaves a number of unanswered questions:

  1. Given how successful this company is at what it does, how could it fail to recognize the benefits from keeping its vehicles longer?
Unanswered Questions

- Our analysis has been focused mainly on the narrow question of the timing of replacement decisions,
- We believe we have provided convincing evidence that via modest changes in the company’s operating strategy, it can significantly increase discounted profits.
- However our analysis leaves a number of unanswered questions:
  1. Given how successful this company is at what it does, how could it fail to recognize the benefits from keeping its vehicles longer?
  2. Are there any overlooked considerations, constraints, or regulations that might explain why the company decides to replace its rental vehicles “too frequently”?
Our analysis of sales prices revealed very large variations in the price received for apparently “observationally equivalent” vehicles.
Unanswered Questions, 2

Our analysis of sales prices revealed very large variations in the price received for apparently “observationally equivalent” vehicles.

1. Why would the company “precommit” to selling a vehicle on a particular date for the best price offered on that date, even if the best price seems below the fair market value for the vehicle?
Our analysis of the relative profitability of long and short term rental contracts revealed that for some vehicles, such as the compact car, short term contracts are significantly more profitable than long term contracts.
Our analysis of the relative profitability of long and short term rental contracts revealed that for some vehicles, such as the compact car, short term contracts are significantly more profitable than long term contracts.

1. Why doesn’t the company adjust the rental rates to equalize the relative profitability of long and short term contracts?
Our analysis of the relative profitability of long and short term rental contracts revealed that for some vehicles, such as the compact car, short term contracts are significantly more profitable than long term contracts.

1. Why doesn’t the company adjust the rental rates to equalize the relative profitability of long and short term contracts?

Our analysis also revealed big differences in the overall profitability of different vehicles. In particular, the stream of discounted profits from rental of the RV or luxury car types are 20 and 40% higher, respectively.
Our analysis of the relative profitability of long and short term rental contracts revealed that for some vehicles, such as the compact car, short term contracts are significantly more profitable than long term contracts.

1. Why doesn’t the company adjust the rental rates to equalize the relative profitability of long and short term contracts?

Our analysis also revealed big differences in the overall profitability of different vehicles. In particular, the stream of discounted profits from rental of the RV or luxury car types are 20 and 40% higher, respectively.

1. If these vehicles are so much more profitable, why not allocate more lot space on the margin to luxury and RVs, or alternatively, increase rental rates on compact cars to increase their relatively profitability?
Economists are accustomed to “marginal arguments” for optimal decision making.
Economists are accustomed to “marginal arguments” for optimal decision making.

The rental company must select a “portfolio” of vehicles for the lots in each of its rental locations.
Economists are accustomed to “marginal arguments” for optimal decision making.

The rental company must select a “portfolio” of vehicles for the lots in each of its rental locations.

Similar to standard portfolio analysis in finance, at an optimal allocation the company should be getting roughly the same expected “risk adjusted return” from an investment of $X in car type $\tau_1$ as it does for an equivalent investment in car type $\tau_2$. 
Vehicle Portfolio Management

- Economists are accustomed to “marginal arguments” for optimal decision making.

- The rental company must select a “portfolio” of vehicles for the lots in each of its rental locations.

- Similar to standard portfolio analysis in finance, at an optimal allocation the company should be getting roughly the same expected “risk adjusted return” from an investment of $X$ in car type $\tau_1$ as it does for an equivalent investment in car type $\tau_2$.

- Otherwise if there is one type of car that has a higher return per dollar invested, then the firm would be better off investing the marginal dollar in the car type that yields the highest possible returns.
Economists are accustomed to “marginal arguments” for optimal decision making.

The rental company must select a “portfolio” of vehicles for the lots in each of its rental locations.

Similar to standard portfolio analysis in finance, at an optimal allocation the company should be getting roughly the same expected “risk adjusted return” from an investment of $X in car type $\tau_1$ as it does for an equivalent investment in car type $\tau_2$.

Otherwise if there is one type of car that has a higher return per dollar invested, then the firm would be better off investing the marginal dollar in the car type that yields the highest possible returns.

Our analysis has revealed that of the three car types we have analyzed, the compact has the highest rate of return even though it has the lowest discounted value of profits per car.
It is not completely obvious that the correct way to think about the firm's allocation problem as choosing to invest in the car with the high marginal return, or to allocate cars to a fixed level of lot space to maximize the overall value of discounted profits.
Maximize Value or Return?

- It is not completely obvious that the correct way to think about the firm’s allocation problem as choosing to invest in the car with the high marginal return, or to allocate cars to a fixed level of lot space to maximize the overall value of discounted profits.

- These two criterion for the portfolio management problem seem to result in different allocations, at least on the margin.
Maximize Value or Return?

- It is not completely obvious that the correct way to think about the firm’s allocation problem as choosing to invest in the car with the high marginal return, or to allocate cars to a fixed level of lot space to maximize the overall value of discounted profits.

- These two criterion for the portfolio management problem seem to result in different allocations, at least on the margin.

- That is, if the company wants to get the highest return on its investment, it would appear it should allocate more of its vehicle “portfolio” to compacts and less to luxury or RVs.
Maximize Value or Return?

- It is not completely obvious that the correct way to think about the firm's allocation problem as choosing to invest in the car with the high marginal return, or to allocate cars to a fixed level of lot space to maximize the overall value of discounted profits.

- These two criterion for the portfolio management problem seem to result in different allocations, at least on the margin.

- That is, if the company wants to get the highest return on its investment, it would appear it should allocate more of its vehicle “portfolio” to compacts and less to luxury or RVs.

- However if it is interested in maximizing the expected present value of profits, then it would appear that it should allocate more of its vehicle portfolio to the luxury and RV car types.
Other Portfolio Consideration

- There could be complimentarities between cars of different types, and the firm should try to cater to its customers’ preferences.
There could be complimentarities between cars of different types, and the firm should try to cater to its customers’ preferences.

Clearly some customers will want to rent compacts, others will prefer RVs and others will prefer to have luxury vehicles.
There could be complementarities between cars of different types, and the firm should try to cater to its customers’ preferences.

Clearly some customers will want to rent compacts, others will prefer RVs and others will prefer to have luxury vehicles.

If the company happens to be “stocked out” of a particular customer’s most preferred type of vehicle, having a portfolio with sufficiently close substitutes may enable the company to keep that customer, as opposed to the customer walking down to the next rental company window to see if a competitor has their preferred vehicle in stock and ready to rent.
Data Requirements

- Our data do not include information on customers, their arrival rates to various rental locations and driving/return patterns (i.e. the probability that a car rented at location A will actually be returned to location B).
Data Requirements

- Our data do not include information on customers, their arrival rates to various rental locations and driving/return patterns (i.e. the probability that a car rented at location A will actually be returned to location B).

- Without more data on customer choices, and data on the company’s competitors, it is difficult for us to formulate a more comprehensive model of the overall operations of this company.
Our data do not include information on customers, their arrival rates to various rental locations and driving/return patterns (i.e. the probability that a car rented at location A will actually be returned to location B).

Without more data on customer choices, and data on the company’s competitors, it is difficult for us to formulate a more comprehensive model of the overall operations of this company.

However we believe the analysis we have conducted in this paper constitutes a fundamental “building block” toward a more complete analysis of this optimal (i.e. profit maximizing) operation of this company.
Data Requirements

- Our data do not include information on customers, their arrival rates to various rental locations and driving/return patterns (i.e. the probability that a car rented at location A will actually be returned to location B).

- Without more data on customer choices, and data on the company’s competitors, it is difficult for us to formulate a more comprehensive model of the overall operations of this company.

- However we believe the analysis we have conducted in this paper constitutes a fundamental “building block” toward a more complete analysis of this optimal (i.e. profit maximizing) operation of this company.

- Whatever portfolio allocation of rental vehicles, and rental rates the company chooses, it will want to adopt a vehicle replacement policy that is optimal conditional on its vehicle portfolio and rental rate structure.
Let $M_i$ be the maximum number of cars that the firm has available in location $i$, $i = 1, \ldots, N$. 

Rental Rate Structures
Let $M_i$ be the maximum number of cars that the firm has available in location $i$, $i = 1, \ldots, N$.

Suppose there are $J$ possible car types (i.e. individual makes and models of cars), and the firm has adopted a rental rate structure $\mathcal{R}$. 
Let $M_i$ be the maximum number of cars that the firm has available in location $i$, $i = 1, \ldots, N$.

Suppose there are $J$ possible car types (i.e. individual makes and models of cars), and the firm has adopted a rental rate structure $\mathcal{R}$.

Initially we adopt the simplication that a rental rate plan for car type $j$ at location $i$ consists of two numbers $\{(R_{ij}^l, R_{ij}^s)\}$ representing flat daily rental rates for long and short term rentals for each car type $j$ at rental location $i$. 
Rental Rate Structures

- Let $M_i$ be the maximum number of cars that the firm has available in location $i$, $i = 1, \ldots, N$.

- Suppose there are $J$ possible car types (i.e. individual makes and models of cars), and the firm has adopted a rental rate structure $\mathcal{R}$.

- Initially we adopt the simplication that a rental rate plan for car type $j$ at location $i$ consists of two numbers $\{(R_{ij}^l, R_{ij}^s)\}$ representing flat daily rental rates for long and short term rentals for each car type $j$ at rental location $i$.

- Thus a rental rate structure consists of the complete array of all rental prices at all rental locations, $\mathcal{R} = \{(R_{ij}^l, R_{ij}^s), j = 1, \ldots, J, i = 1, \ldots, N\}$.
Rental Rate Structures

- Let $M_i$ be the maximum number of cars that the firm has available in location $i, i = 1, \ldots, N$.

- Suppose there are $J$ possible car types (i.e. individual makes and models of cars), and the firm has adopted a rental rate structure $\mathcal{R}$.

- Initially we adopt the simplification that a rental rate plan for car type $j$ at location $i$ consists of two numbers $\{(R_{ij}^l, R_{ij}^s)\}$ representing flat daily rental rates for long and short term rentals for each car type $j$ at rental location $i$.

- Thus a rental rate structure consists of the complete array of all rental prices at all rental locations, $\mathcal{R} = \{(R_{ij}^l, R_{ij}^s), j = 1, \ldots, J, i = 1, \ldots, N\}$.

- Rental rate structures are more complicated if we allow contracts with odometer-based discounts, and usage-based rental schemes.
Let $V_{ij}(\mathcal{R})$ denote the expected discounted value of profits from a car of type $j$ in rental location $i$ under the assumption that the firm follows an optimal replacement strategy for each car type $j$ at each location $i$ under rental rate structure $\mathcal{R}$. 
Let $V_{ij}(\mathcal{R})$ denote the expected discounted value of profits from a car of type $j$ in rental location $i$ under the assumption that the firm follows an optimal replacement strategy for each car type $j$ at each location $i$ under rental rate structure $\mathcal{R}$.

Let $P_j$ be the new purchase price of car type $j$. 
Let $V_{ij}(\mathcal{R})$ denote the expected discounted value of profits from a car of type $j$ in rental location $i$ under the assumption that the firm follows an optimal replacement strategy for each car type $j$ at each location $i$ under rental rate structure $\mathcal{R}$.

Let $\overline{P}_j$ be the new purchase price of car type $j$.

Then we can formulate the overall *optimal rental operations problem* as the following programming problem:

$$
\max_{\mathcal{R}} \max_{\{N_{ij}\}} \sum_{i=1}^{N} \sum_{j=1}^{J} N_{ij} \left[ V_{ij}(\mathcal{R}) - \overline{P}_j \right] \text{ subject to: } \sum_{j=1}^{J} N_{ij} \leq M_i
$$
Toward a Complete Model

- Let \( V_{ij}(R) \) denote the expected discounted value of profits from a car of type \( j \) in rental location \( i \) under the assumption that the firm follows an optimal replacement strategy for each car type \( j \) at each location \( i \) under rental rate structure \( R \).
- Let \( \overline{P}_j \) be the new purchase price of car type \( j \).
- Then we can formulate the overall optimal rental operations problem as the following programming problem

\[
\max_{R} \{ N_{ij} \} \sum_{i=1}^{N} \sum_{j=1}^{J} N_{ij} [V_{ij}(R) - \overline{P}_j] \quad \text{subject to:} \quad \sum_{j=1}^{J} N_{ij} \leq M_i
\]

- Nested within this problem is the regenerative optimal stopping problem, that we have solved in this paper, that delivers the value function \( V_{ij}(R) \) for all car types at all of the firm’s rental locations.
Future Directions

- With better data on all of the company’s rental locations and customer data, it may be possible to solve this programming problem.
Future Directions

- With better data on all of the company's rental locations and customer data, it may be possible to solve this programming problem.

- We also need to recognize that the optimal choice of a rental rate structure depends on the choices $\mathcal{R}_c$ of the company's competitors, $c \in C$. 
Future Directions

- With better data on all of the company’s rental locations and customer data, it may be possible to solve this programming problem.

- We also need to recognize that the optimal choice of a rental rate structure depends on the choices $R_c$ of the company’s competitors, $c \in C$.

- In this larger competitive game, the firm’s value and the optimal strategy for its vehicle portfolio and rental rate structure will clearly depend on the portfolios and rental rate structures chosen by its competitors.
Future Directions

- With better data on all of the company's rental locations and customer data, it may be possible to solve this programming problem.

- We also need to recognize that the optimal choice of a rental rate structure depends on the choices $R_c$ of the company’s competitors, $c \in C$.

- In this larger competitive game, the firm’s value and the optimal strategy for its vehicle portfolio and rental rate structure will clearly depend on the portfolios and rental rate structures chosen by its competitors.

- Solving for the overall competitive equilibrium problem in the rental market remains a challenging area for future research.
Main Findings

1. Our goal was to test the hypothesis that the rental company is maximizing expected discounted profits.
Main Findings

1. Our goal was to test the hypothesis that the rental company is maximizing expected discounted profits.

2. Even though this firm earns extremely high rates of return on its rental cars, we have found that there are alternative operating strategies that it could adopt that could result in significantly higher profits and rates of return.
Main Findings

1. Our goal was to test the hypothesis that the rental company is maximizing expected discounted profits.

2. Even though this firm earns extremely high rates of return on its rental cars, we have found that there are alternative operating strategies that it could adopt that could result in significantly higher profits and rates of return.

3. The alternative operating strategy is to keep vehicles approximately twice as long as the company currently keeps them, and to offer discounts to customers to induce them to rent older vehicles.
Main Findings

1. Our goal was to test the hypothesis that the rental company is maximizing expected discounted profits.

2. Even though this firm earns extremely high rates of return on its rental cars, we have found that there are alternative operating strategies that it could adopt that could result in significantly higher profits and rates of return.

3. The alternative operating strategy is to keep vehicles approximately twice as long as the company currently keeps them, and to offer discounts to customers to induce them to rent older vehicles.

4. This is more profitable because by keeping cars longer, the company is able to “amortize” the high trading costs that results from the rapid early depreciation in the resale price of vehicles.
1. Our analysis has raised questions about the optimality of other aspects of the firm’s operations, including its strategy for selling vehicles, its rental price structure, and the appropriate composition of its portfolio of rental vehicles.
1. Our analysis has raised questions about the optimality of other aspects of the firm’s operations, including its strategy for selling vehicles, its rental price structure, and the appropriate composition of its portfolio of rental vehicles.

2. However lacking appropriate data on customer choices, arrival rates, and behavior of competing firms, this study cannot provide an answer to these broader questions.
1. Our analysis has raised questions about the optimality of other aspects of the firm’s operations, including its strategy for selling vehicles, its rental price structure, and the appropriate composition of its portfolio of rental vehicles.

2. However lacking appropriate data on customer choices, arrival rates, and behavior of competing firms, this study cannot provide an answer to these broader questions.

3. We also lack data necessary to answer related questions such as marketing/advertising decisions, determination of number of rental locations, optimal maintenance policies, and so forth.
1. We recommend that the company undertake *experiments* to test the predictions made in this analysis.
1. We recommend that the company undertake *experiments* to test the predictions made in this analysis.

2. In particular, the company should select several types of vehicles at several different locations to serve as a *treatment group*. 
1. We recommend that the company undertake *experiments* to test the predictions made in this analysis.

2. In particular, the company should select several types of vehicles at several different locations to serve as a *treatment group*.

3. The *treatment* given to these vehicles would be similar to the more profitable operating strategy suggested in this analysis, i.e. *keeping cars longer but offering customers discounts to compensate for renting older vehicles.*
1. We recommend that the company undertake *experiments* to test the predictions made in this analysis.

2. In particular, the company should select several types of vehicles at several different locations to serve as a *treatment group*.

3. The *treatment* given to these vehicles would be similar to the more profitable operating strategy suggested in this analysis, i.e. *keeping cars longer but offering customers discounts to compensate for renting older vehicles*.

4. We recommend that a *consumer survey* be taken to determine the *minimum discounts* that would be necessary to induce customers to rent older vehicles.
1. We recommend that the company undertake *experiments* to test the predictions made in this analysis.

2. In particular, the company should select several types of vehicles at several different locations to serve as a *treatment group*.

3. The *treatment* given to these vehicles would be similar to the more profitable operating strategy suggested in this analysis, i.e. *keeping cars longer but offering customers discounts to compensate for renting older vehicles*.

4. We recommend that a *consumer survey* be taken to determine the *minimum discounts* that would be necessary to induce customers to rent older vehicles.

5. Once we know the minimum discounts, we can solve for the optimal replacement strategy and estimate the expected increase in profits for cars in the treatment group.
1. From a methodological point of view, we have shown how it is possible to integrate econometric duration models and (regenerative) optimal stopping theory in order to evaluate the profitability of the operating strategy of a firm.
1. From a methodological point of view, we have shown how it is possible to integrate econometric duration models and (regenerative) optimal stopping theory in order to evaluate the profitability of the operating strategy of a firm.

2. We have also shown how this apparatus can be used to test the hypothesis that the firm is a profit maximizer, and we have provided convincing evidence that the firm is not maximizing discounted profits.
1. From a methodological point of view, we have shown how it is possible to integrate econometric duration models and (regenerative) optimal stopping theory in order to evaluate the profitability of the operating strategy of a firm.

2. We have also shown how this apparatus can be used to test the hypothesis that the firm is a profit maximizer, and we have provided convincing evidence that the firm is not maximizing discounted profits.

3. A practical contribution of the paper is to provide both a framework and concrete computer code that enables us to characterize the precise form of a profit maximizing replacement strategy for this firm.
1. From a methodological point of view, we have shown how it is possible to integrate econometric duration models and (regenerative) optimal stopping theory in order to evaluate the profitability of the operating strategy of a firm.

2. We have also shown how this apparatus can be used to test the hypothesis that the firm is a profit maximizer, and we have provided convincing evidence that the firm is not maximizing discounted profits.

3. A practical contribution of the paper is to provide both a framework and concrete computer code that enables us to characterize the precise form of a profit maximizing replacement strategy for this firm.

4. Thus, our work represents an application of theoretic tools that may have a concrete practical benefit to this firm and similar firms in the rental car business.