Review of Economic Dynamics **3,** 771–800 (2000)
doi:10.1006/redy.1999.0090, available online at http://www.idealibrary.com on IDEAL®

Why Do Some Households Save So Little? A Rational Explanation[1]

Siu Fai Leung

Department of Economics, Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong
E-mail: sfleung@ust.hk

Received February 18, 1997; revised November 17, 1999

This paper offers an explanation for the puzzle of low wealth holdings among a significant fraction of the elderly. Instead of invoking irrational, nonrational, or nonoptimal behavior to resolve the puzzle, it is shown that widespread low wealth holdings are consistent with a rational life-cycle model of saving with uncertain lifetime and borrowing constraint. When there is uncertainty about the length of life, it is optimal for some individuals to save little and exhaust their wealth early. The characteristics of these individuals are derived. The simulation results support that the model can account for low wealth holdings as well as early terminal wealth depletion. The analysis also rejects the common perception that uncertain lifetime reduces dissaving. *Journal of Economic Literature* Classification Numbers: D11, D91, E21, I12, J14.

Key Words: life-cycle; uncertain lifetime; saving; wealth; dissaving; elderly; retirement; rational behavior.

I. INTRODUCTION

A prominent finding in various national surveys is that a significant fraction of the elderly in the United States, generally at least 20%, have little or no liquid wealth.[2] Despite all the differences in the design and the year of study of the surveys, the finding of a substantial number of low-wealth elderly is very robust. The stylized fact has puzzled many economists. Why do so many people save so little? Unless lifetime earnings

[1] I thank John Boyd III, Gregory Chow, Thomas Cooley, George Chressanthis, James Davies, Issac Ehrlich, Eric Engen, Bruce Jacobs, James Kahn, Laurence Kotlikoff, Thad Mirer, Nancy Stokey, Danyang Xie, and a referee for helpful comments and suggestions.

[2] For evidence based on earlier surveys, see Diamond (1977) and Diamond and Hausman (1984). For more recent surveys, see Eller (1994), Poterba *et al*. (1994), Gustman and Juster (1995), Mitchell and Moore (1997), and the references therein.

1094-2025/00 $35.00

are particularly low or discount rates are unusually high, widespread low wealth accumulation does not seem to be consistent with optimal or rational behavior.

The difficulties in resolving the puzzle within the conventional economic framework have affected economic research both empirically and theoretically. On the empirical side, some economists choose to eliminate all the low-wealth individuals from the sample in their data analyses because they believe that the standard life-cycle theory of saving is not applicable to those individuals (e.g., King and Dicks-Mireaux, 1982; Diamond and Hausman, 1984; Bernheim, 1991). On the theoretical front, a number of economists contend that the rational, forward-looking, and optimizing approach is not suitable for studying saving behavior at all because the decision problem is too complicated and the assumptions are simply unrealistic (e.g., Bernheim and Scholz, 1993; Thaler, 1994). They abandon the conventional economic rational approach and develop either quasi-rational or nonrational models to solve the puzzle.

There are very few systematic attempts to reconcile widespread low wealth holdings with a rational model of saving. In a recent contribution, Hubbard *et al.* (1995) propose that a standard life-cycle model of precautionary saving with asset-based, means-tested social insurance can account for low wealth holdings among the aged as well as the nonaged. Welfare programs discourage saving especially for low-lifetime-income households because the transfer payments provide a safety net in times of adversity and the eligibility tests impose an implicit 100% tax on both earnings and assets.

In this paper, I offer a rational theory of low wealth accumulation that does not rely on social insurance. While asset-based, means-tested welfare programs may play a significant role in reducing saving, there are at least two reasons to suspect that other important forces are also at work. First, the definition of wealth in the model of Hubbard *et al.* is debatable. Many welfare programs like Aid to Families with Dependent Children (AFDC), Supplemental Security Income (SSI), and food stamps do not include home equity (as well as certain types of assets) in the asset tests, whereas there is no distinction between housing wealth and nonhousing wealth (or any type of wealth) in their model. Hence, all of the wealth is counted in the asset test in their model, but in reality only part of the wealth is included. Thus, the asset-tested welfare programs actually overtax the wealth in their model, producing a larger percentage of families with wealth less than income. This upward bias suggests that their model may actually account for only part of the low-wealth families. Second, the social insurance argument solves only part of the puzzle because it is mainly applicable to low-lifetime-income households. As Hubbard *et al.* themselves acknowledge, welfare programs have a much weaker impact on higher-lifetime-income households than lower-lifetime-income households because welfare benefits and the uninsured risks of medical expenses are a much smaller fraction of the lifetime resources of the former group. Hence, low wealth holdings among a substantial fraction of higher-lifetime-income households remains a puzzle.[3] Because of these considerations, it is worthwhile to search for additional determinants of widespread low wealth holdings.

Adhering to the conventional economic rational framework, this paper aims to show that, without social insurance, uncertain lifetime and borrowing constraint can also contribute to low wealth holdings. In the absence of bequest motives and life annuities, I demonstrate that it is optimal and rational for some individuals to save little and entirely decumulate their assets well before the maximum lifetime. When there is uncertainty about the length of life and borrowing against future earnings is not permitted, the force of mortality increases impatience to such an extent that some individuals will choose to consume most of their income and save little because the assets are useless if they die sooner than expected.

It has long been recognized that lifetime uncertainty can reduce saving (e.g., Rae, 1834; Fisher, 1930). A novel contribution of this paper is to explain *why* and *how* uncertain lifetime reduces saving via terminal wealth depletion. By proving analytically that wealth must be exhausted at some instant before the maximum lifetime and that there is entirely no asset accumulation after that instant, I can then utilize the terminal wealth depletion time to explain why some individuals, especially the elderly, hold little or no wealth. The earlier the wealth is terminally exhausted, the longer will be the duration of low and zero wealth. The depletion is terminal, not temporary. I provide some simulation results to demonstrate that terminal wealth depletion can occur very early. In some cases it happens in just a few years after retirement. My model can therefore account for widespread low wealth holdings especially among the elderly.

In addition to solving the puzzle, my theory also provides an answer for the heretofore unexplained finding of terminal wealth depletion in the

[3] For evidence that a significant fraction of higher-lifetime-income households also hold very little wealth, see Diamond (1977) and Diamond and Hausman (1984). They demonstrate that the low-wealth phenomenon is not confined to the bottom of the income distribution. In a different but related study, Masson (1988) shows that income or permanent income explains only a minor part of the wealth distribution. The fact that variations in wealth at retirement are much larger than variations in lifetime income suggest that low wealth accumulation is not just a reflection of low lifetime income; other factors have also contributed to the outcome. In fact, the data in Hubbard *et al.* (1995, Table I) also indicate that low-wealth households are not always low-lifetime-income households, e.g., 32.3 (13.0)% of households headed by high school (college) graduates aged 60 or above had median nonhousing wealth less than one-half of income.

empirical work of Hurd (1989) and the simulation studies of Mirer (1992, 1994). In an empirical analysis of the Longitudinal Retirement History Survey, Hurd (1989) finds a high wealth decumulation rate among the retired elderly. His estimates imply that some people will completely exhaust their wealth in only a few years after retirement and will live entirely on their income thereafter. Similarly, Mirer (1992, 1994) discovers a notable terminal wealth depletion time in his simulation work. Despite their attempts to explain the results, both Hurd and Mirer have not been able to demonstrate theoretically why and how terminal wealth depletion takes place. My theory fits right into these new and unique findings.

Uncertainty about the length of life exerts two opposite effects on saving. It raises precautionary saving because life may be longer than expected, but it may also discourage saving because unused assets are wasted (especially for those who have no bequest motives) if life is shorter than expected. Because of these two offsetting forces, the net impact of uncertain lifetime on saving, in the context of a standard life-cycle model, is theoretically ambiguous. Despite the ambiguity, there is a common perception among many economists, based primarily on the widely cited work of Davies (1981), that uncertain lifetime reduces dissaving (or increases saving). In an attempt to account for the observed slow dissaving among the elderly, Davies proposes a theory to show that, with the most plausible parameters, uncertain lifetime will likely have a negative impact on consumption (relative to what it would be if lifetime was certain). He offers illustrative computations to show that the negative impact indeed prevails and the magnitude is large enough to explain much of the lack of decumulation by the elderly. Thus, Davies' results suggest that uncertain lifetime reduces decumulation, which is opposite to the thesis of this paper that uncertain lifetime raises decumulation. To resolve this inconsistency, I conduct a simulation study similar to Davies', adding an explicit terminal wealth depletion time into the setup that is absent in Davies' computational procedure. In contrast to Davies' results, I find that in almost all cases, consumption under uncertain lifetime is higher than consumption under certain lifetime, which means that uncertain lifetime hastens dissaving. Thus, my simulation results do not support the claim that uncertain lifetime is responsible for the low rate of dissaving among the retired. To the contrary, uncertain lifetime accounts for low wealth holdings and rapid asset decumulation. These findings signify the importance of incorporating an explicit terminal wealth depletion time into simulation work, call into question the common perception that uncertain lifetime reduces dissaving, and suggest that one has to look for other factors to explain the (tentative) observed slow dissaving among the aged.

Finally, I discuss why only a significant fraction, but not all, of the elderly hold little or no wealth. The analysis produces a list of probable characteristics of those elderly people who have little or no wealth. They are most likely to be individuals who have no bequest motives, no actuarially fair life annuities, low initial wealth, low risk aversion, high discount rates, low interest rates, high income, or poor health.

The rest of the paper is organized as follows. Section II presents the model and investigates what types of individuals would be more likely to deplete their wealth early.[4] Section III demonstrates the importance of incorporating terminal wealth depletion into the design of numerical analysis. Section IV offers some simulation results to substantiate the theory and reviews other supportive empirical evidence in the literature. Section V compares the findings with Davies' results. Section VI explores why only a significant fraction of the elderly hold little or no wealth. Section VII concludes the paper.

II. A THEORY OF TERMINAL WEALTH DEPLETION

I employ a standard pure life-cycle model of saving and add two features into the model: uncertain lifetime and borrowing constraint. As explained in Yaari (1965), it is necessary to include a borrowing constraint in this type of model to avoid the possibility of dying with debts. For reasons given in Leung (1995), I adopt a continuous-time model instead of a discrete-time formulation.

Consider an individual whose lifetime T is a continuous random variable distributed on $[0, \bar{T}]$, where $\bar{T}$ (a finite positive number) is the maximum possible lifetime. The individual has survival probability $\Omega(t)$ at each time t, where $\Omega(0) = 1$ and $\Omega(\bar{T}) = 0$. Let $\pi(t)$ and $\pi_t(t)$ denote the probability density function of T and the hazard rate of death (or force of mortality, mortality hazard) at time t, respectively. Then $\pi_t(t) = \pi(t)/\Omega(t)$ and $\Omega(t) = P(T > t) = e^{-\int_0^t \pi_x(x)\,dx}$. It is assumed that $\pi(t) > 0$ for $t \in (0, \bar{T})$ and $\lim_{t \to \bar{T}} \pi_t(t) = \infty$.

[4] While the theory presented in Section II bears some resemblance to that in my earlier paper (Leung, 1994), there are some major differences. The sole objective of Leung (1994) is to derive a new characterization (terminal wealth depletion) of the model of Yaari (1965) under very general conditions. It focuses primarily on technical issues. In contrast, the present paper focuses on applications and the scope is much broader. The theoretical analysis here is simpler and more complete in at least three ways. First, a new mathematical proof of terminal wealth depletion is offered. The proof is considerably simpler and more intuitive than that in Leung (1994). The simplicity allows a much more transparent economic interpretation of the theoretical results. Second, comparative statics results, which are absent in Leung (1994), are derived. These results help reveal the properties of the model. Third, the subtle relationship between theory and numerical analysis is examined in detail. The investigation offers an important guideline for the design of numerical analysis on this type of life-cycle models.

The individual has a twice continuously differentiable utility function $g(c)$ and discounts the future at a fixed rate α, where $g'(c) > 0$, $g''(c) < 0$, and $\alpha \geq 0$. At each time $t \in [0, \overline{T}]$, the individual possesses wealth (assets, accumulated savings) $S(t)$, consumes $c(t)$, and receives two sources of income: interest income $jS(t)$ and noninterest income $m(t)$ (e.g., wage income), where j (a constant) denotes the interest rate. Assuming that there are no bequest motives and life annuities are not available, the individual's objective is to

$$\max_{c(t)} \int_0^{\overline{T}} \Omega(t) e^{-\alpha t} g(c(t))\, dt \tag{1}$$

subject to

$$c(t) \geq 0, \tag{2}$$

$$S(t) \geq 0, \tag{3}$$

$$S'(t) = jS(t) + m(t) - c(t), \tag{4}$$

$$S(0) = S_0, \tag{5}$$

and

$$S(\overline{T}) = 0. \tag{6}$$

The inequality in (3) has been known as the wealth, liquidity, or borrowing constraint.[5] As in Leung (1994), it is assumed that $\pi(t)$ is continuous, $m(t)$ is continuously differentiable, $c(t)$ is piecewise continuous, and $S(t)$ is piecewise continuously differentiable. Let the Hamiltonian for this optimal control problem be

$$\begin{aligned} \mathscr{H}(c(t), S(t)) = \Omega(t) e^{-\alpha t} g(c(t)) + \lambda(t)[jS(t) + m(t) - c(t)] \\ + \eta(t) c(t) + \mu(t) S(t), \end{aligned} \tag{7}$$

where $\lambda(t)$, $\eta(t)$, and $\mu(t)$ are multipliers. Let $c^{\#}(t)$ denote the optimal solution to the control problem and $S^{\#}(t)$ be the associated wealth path defined by

$$S^{\#}(t) = e^{jt}\left[S(0) + \int_0^t e^{-jz} m(z)\, dz - \int_0^t e^{-jz} c^{\#}(z)\, dz\right]. \tag{8}$$

[5] The borrowing constraint is not as restrictive as it first appears to be. Debts in the form of collateralized loans (such as home mortgages) are permitted, as long as they are not allowed to exceed the total current value of traded assets. The purpose of the constraint is to prohibit the individual from borrowing against his future income.

Then $c^{\#}(t)$ and $S^{\#}(t)$ satisfy the necessary optimality conditions (9)–(12),

$$\frac{\partial \mathscr{H}\left(c^{\#}(t), S^{\#}(t)\right)}{\partial c(t)} = \Omega(t)e^{-\alpha t}g'\left(c^{\#}(t)\right) - \lambda(t) + \eta(t) = 0, \quad (9)$$

$$\frac{\partial \mathscr{H}\left(c^{\#}(t), S^{\#}(t)\right)}{\partial S(t)} = -\lambda'(t) = j\lambda(t) + \mu(t), \quad (10)$$

$$\eta(t) \geq 0, \qquad \eta(t)c^{\#}(t) = 0, \quad (11)$$

and

$$\mu(t) \geq 0, \qquad \mu(t)S^{\#}(t) = 0. \quad (12)$$

As $g'(c) > 0$ and $\Omega(t) > 0$ on $[0, \overline{T})$, (9) implies that $\lambda(t) > 0$ for all $t \in [0, \overline{T})$. Solving (10), one obtains $\lambda(t) = \lambda(z)e^{-j(t-z)} - \int_z^t e^{-j(t-x)}\mu(x)\,dx$ for any z and t such that $0 \leq z < t \leq \overline{T}$. Substituting this into (9),

$$\Omega(t)e^{-\alpha t}g'\left(c^{\#}(t)\right) + \eta(t) = \lambda(z)e^{-j(t-z)} - \int_z^t e^{-j(t-x)}\mu(x)\,dx \quad (13)$$

for $0 \leq z < t \leq \overline{T}$. Denote $dc^{\#}(t)/dt$ by $c^{\#\prime}(t)$. For $t \in [0, \overline{T})$, if $\eta(t) = 0$ and $\mu(t) = 0$, then (10) and (13) imply that

$$c^{\#\prime}(t) = \frac{\left[\pi_t(t) + \alpha - j\right]g'\left(c^{\#}(t)\right)}{g''\left(c^{\#}(t)\right)}. \quad (14)$$

An important property of this model is the existence of terminal wealth depletion.

PROPOSITION 1. *If $m(\overline{T}) > 0$, then there exists a $t^* \in [0, T)$ such that $S^{\#}(t) = 0$ and $c^{\#}(t) = m(t)$ for all $t \in [t^*, \overline{T}]$.*

Proof. Since $m(\overline{T}) > 0$, (3), (4), and (6) imply that $c^{\#}(\overline{T}) > 0$, hence $g'(c^{\#}(t)) < \infty$ for all $t \in [0, \overline{T}]$. As $\lim_{t \to \overline{T}} \pi_t(t) = \infty$, one can choose a $\tau \in [0, \overline{T})$ such that $c^{\#\prime}(t) < m'(t)$ for all $t > \tau$. Since $S^{\#\prime\prime}(t) = jS^{\#\prime}(t) + m'(t) - c^{\#\prime}(t)$, this implies that $S^{\#\prime}(t) \leq 0$ for $t \in (\tau, \overline{T}]$ because if $S^{\#\prime}(t) > 0$ for some $t \in (\tau, \overline{T}]$, then $S^{\#\prime\prime}(t) > 0$, which means that $S^{\#}(\overline{T})$ can never reach zero (as $S^{\#}(t)$ is increasing at an increasing rate), contradicting (6). Now suppose the proposition is not true; then $S^{\#\prime}(t) \leq 0$ on $(\tau, \overline{T}]$ and $S^{\#}(\overline{T}) = 0$ imply that there must exist an ω such that $\tau \leq \omega < \overline{T}$ and $S^{\#}(t) > 0$ for all $t \in (\omega, \overline{T}]$, so $\mu(t) = 0$ for $t \in (\omega, \overline{T}]$ and $\int_\omega^{\overline{T}} e^{-j(\overline{T}-x)}\mu(x)\,dx = 0$. Using these results as well as $\eta(\overline{T}) = 0$ and $\Omega(\overline{T}) = 0$ to evaluate (13) at $z = \omega$ and $t = \overline{T}$, one obtains $\lambda(\omega) = 0$, contra-

dicting the fact that $\lambda(t) > 0$ on $[0, \bar{T})$. Thus, there must exist a $t^* \in [0, \bar{T})$ such that $S^{\#}(t) = 0$ for all $t \in [t^*, \bar{T}]$, hence $c^{\#}(t) = m(t)$ on $[t^*, \bar{T}]$.
Q.E.D.

Proposition 1 states that there must be a terminal wealth depletion time t^* before $\bar{T}$ if $m(\bar{T}) > 0$.[6] The entire wealth is exhausted at t^* and the depletion is *terminal* (as opposed to temporary) because it remains zero throughout the interval $[t^*, \bar{T}]$. Consumption after t^* will be exactly equal to income. The condition $m(\bar{T}) > 0$ is likely to be satisfied in practice because of the provision of social security benefits and the existence of private pensions. Hence, the model demonstrates how social security can completely displace private savings. In addition, Proposition 1 provides a rigorous analytical account for the heretofore unexplained finding of terminal wealth depletion in the empirical work of Hurd (1989) and the simulation studies of Mirer (1992, 1994) because the condition $m(\bar{T}) > 0$ is satisfied in these studies.[7]

While this model is a special case of the one studied in Leung (1994), the proof of Proposition 1 is new. The proof is considerably simpler than that in Leung (1994) mainly because of the assumption that j is a constant, which allows the sign of $S^{\#\prime}(t)$ to be deduced from $S^{\#\prime\prime}(t)$. A simple and intuitive interpretation which highlights the rationale behind the proof can be given as follows. Suppose (2) and (3) are never binding; then (9) and (10) give the usual optimality condition $\Omega(t)e^{-\alpha t}g'(c^{\#}(t)) = \lambda(t) = \lambda(0)e^{-jt}$; i.e., the expected discounted marginal utility of consumption, $\Omega(t)e^{-\alpha t}g'(c^{\#}(t))$, is equal to the marginal value of wealth $\lambda(t)$. If $m(\bar{T}) > 0$, then $g'(c^{\#}(t)) < \infty$ for all $t \in [0, \bar{T}]$. Since $\Omega(\bar{T}) = 0$, this implies that $\Omega(t)e^{-\alpha t}g'(c^{\#}(t))$ will eventually decline to zero as t approaches $\bar{T}$, while $\lambda(t)$ will fall to its minimum $\lambda(0)e^{-j\bar{T}} > 0$. Hence, there exists a time τ such that $\Omega(t)e^{-\alpha t}g'(c^{\#}(t)) < \lambda(0)e^{-jt}$, $t \in [\tau, \bar{T}]$. In other words, the equality $\Omega(t)e^{-\alpha t}g'(c^{\#}(t)) = \lambda(0)e^{-jt}$ cannot hold for all t, which means that (2) or (3), or both, must bind on $[t^*, \bar{T}]$ for some $t^* \in [0, \bar{T})$. Since $m(\bar{T}) > 0$, (2) cannot bind in the neighborhood of $\bar{T}$ because this will violate (6). Thus, (3) must bind in the neighborhood of $\bar{T}$, in which case $\int_{t^*}^{\bar{T}} e^{-j(t-x)}\mu(x)\,dx > 0$, so that $\lambda(t)$ can reach zero as t approaches $\bar{T}$ (see (13)).

The above interpretation shows that the key factor that causes wealth depletion is uncertain lifetime because it eventually drives the effective discount rate for the future to infinity ($\Omega(t)$ in (9) and $\Omega(\bar{T}) = 0$). As

[6] The case $m(\bar{T}) = 0$ is discussed in Leung (1997).

[7] Mirer (1992, 1994) justifies his simulation algorithm by appealing to Propositions 1 and 2 of Mariger (1987). However, Mariger's propositions only deal with finding cutoff dates and do not really prove the existence of a terminal wealth depletion time. More importantly, there is no lifetime uncertainty in Mariger's model.

emphasized in Rae (1834) and Fisher (1930), uncertain lifetime increases impatience and tends to reduce saving. What is new here is the analytical and unambiguous result that the uncertainty of lifetime will eventually lead to terminal wealth depletion before the maximum lifetime.

Although uncertain lifetime drives terminal wealth depletion, there are many factors that determine when it will take place. As explained in Leung (1997), one can investigate the determinants of t^* by the following procedure. Assume that $\eta(t) = \mu(t) = 0$ for $t \in [0, t^*]$; then (9) and (10) imply that $\Omega(t)e^{-\alpha t}g'(c^{\#}(t)) = \lambda(t) = \lambda(0)e^{-jt}$, hence $\Omega(t)e^{-\alpha t}g'(c^{\#}(t)) = \Omega(t^*)e^{-\alpha t^*}g'(c^{\#}(t^*))e^{j(t^*-t)}$. Since $c^{\#}(t^*) = m(t^*)$, therefore

$$c^{\#}(t) = (g')^{-1}\left(\frac{g'(m(t^*))\Omega(t^*)e^{(j-\alpha)t^*}}{\Omega(t)e^{(j-\alpha)t}}\right) \tag{15}$$

for $t \in [0, t^*]$, where $(g')^{-1}$ denotes the inverse of g'. Substituting (15) into the lifetime budget constraint $S(0) = \int_0^{\bar{T}} e^{-jt}[c^{\#}(t) - m(t)]\,dt$, one obtains

$$S(0) = \int_0^{t^*} e^{-jt}\left[(g')^{-1}\left(\frac{g'(m(t^*))\Omega(t^*)e^{(j-\alpha)t^*}}{\Omega(t)e^{(j-\alpha)t}}\right) - m(t)\right]dt. \tag{16}$$

Hence, t^* is determined implicitly by (16). Although (16) does not offer an explicit solution for t^*, it can be employed to derive the following sensitivity results.

PROPOSITION 2. *Let* $\Delta(t) = -\pi_t(t) - \alpha + j - m'(t)g''(m(t))/g'(m(t))$ *and assume that* $\Delta(t^*) < 0$. *Then*

(i) $\partial t^*/\partial S(0) > 0$, $\partial t^*/\partial \alpha < 0$, $\partial t^*/\partial j > 0$.

(ii) *Let* $\Omega(t) = e^{-\int_0^t \phi\pi_x(x)\,dx}$, *where* ϕ *is a shift parameter for the mortality hazard* $\pi_x(x)$; *then* $\partial t^*/\partial\phi < 0$.

(iii) *Suppose* $g(c) = c^{1-\gamma}/(1-\gamma)$, $\gamma > 0$ ($g(c) = \log c$ *if* $\gamma = 1$). *If* $\Omega(t)e^{(j-\alpha)t} > \Omega(t^*)e^{(j-\alpha)t^*}$ *for all* $t \in [0, t^*)$, *then* $\partial t^*/\partial\gamma > 0$.

(iv) *Suppose* $g(c) = c^{1-\gamma}/(1-\gamma)$, $\gamma > 0$ ($g(c) = \log c$ *if* $\gamma = 1$), *and a shift parameter* ξ *is introduced into* $m(t)$, *i.e.*, $m(t) = \xi m_0(t)$. *Then* $\partial t^*/\partial\xi \le 0$.

Proof. Let $\Psi(t^*) = \int_0^{t^*} e^{-jz}[g'(c^{\#}(z))/g''(c^{\#}(z))]\,dz$. (i) See Leung (1997) for the proofs of $\partial t^*/\partial S(0) > 0$ and $\partial t^*/\partial\alpha < 0$. To prove $\partial t^*/\partial j > 0$, differentiate (16) with respect to j, $\Delta(t^*)\Psi(t^*)[\partial t^*/\partial j] = \int_0^{t^*} e^{-jt}t[c^{\#}(t) - m(t)]dt - \int_0^{t^*} e^{-jt}(t^* - t)[g'(c^{\#}(t))/g''(c^{\#}(t))]\,dt$. Using integration by parts, one can verify that $\int_0^{t^*} e^{-jt}t[c^{\#}(t) - m(t)]\,dt =$

$\int_0^{t^*} e^{-jt}S^{\#}(t)\,dt > 0$. Since $\Delta(t^*) < 0$, $\Psi(t^*) < 0$, and $\int_0^{t^*} e^{-jt}(t^* - t)[g'(c^{\#}(t))/g''(c^{\#}(t))]\,dt < 0$, $\partial t^*/\partial j > 0$ follows. (ii) Differentiating (16) with respect to ϕ, $\Delta(t^*)\Psi(t^*)[\partial t^*/\partial \phi] = \int_0^{t^*} e^{-jt}[\int_z^{t^*} \pi_x(x)\,dx][g'(c^{\#}(z))/g''(c^{\#}(z))]\,dz$, hence $\partial t^*/\partial \phi < 0$. (iii) See Leung (1997). (iv) As $\partial m(t)/\partial \xi = m(t)/\xi$ and $c^{\#}(t) = m(t^*)\{\Omega(t)e^{(j-\alpha)t}/[\Omega(t^*)e^{(j-\alpha)t^*}]\}^{1/\gamma}$, differentiating (16) with respect to ξ yields $\Delta(t^*)\Psi(t^*)[\partial t^*/\partial \xi] = 1/\xi)\{\int_0^{t^*} e^{-jz}m(z)\,dz - m(t^*)\int_0^{t^*} e^{-jz}[\Omega(z)e^{(j-\alpha)z}/\Omega(t^*)e^{(j-\alpha)t^*}]^{1/\gamma}\,dz\} = 1/\xi)\int_0^{t^*} e^{-jz}[m(z) - c^{\#}(t)]\,dz = -S(0)/\xi$; therefore $\partial t^*/\partial \xi \leq 0$. Q.E.D.

The comparative statics results are intuitively reasonable. Increases in the initial wealth delay the terminal wealth depletion time. The more impatient the individual, the sooner the wealth will be exhausted. When the interest rate is low, the returns to assets are small, so the wealth is exhausted earlier. An individual with a higher mortality hazard will run down his assets earlier because he may not live long enough to enjoy his wealth. More risk averse individuals hold on to their wealth for a longer period of time. Other things being equal, people with higher income deplete their wealth earlier. Hence, in contrast to the social insurance argument of Hubbard *et al.* (1995), my model can explain why even some higher-lifetime-income households may hold little wealth.

III. THEORY AND NUMERICAL ANALYSIS

Apart from theoretical significance, the findings in Section II also have important implications for numerical analysis. The literature on life-cycle saving is filled with studies employing numerical analysis to examine either the time paths of $c(t)$ and $S(t)$ (as tractable characterizations of the trajectories are in general difficult to obtain) or the quantitative properties of the model (as in calibration exercises). By specifying functional forms and assigning parameter values for the model, these studies compute numerically the optimal solutions and other functions or parameters of interest.

A referee of this article raised an important issue concerning the relevance of the preceding theoretical results on the proper design of the numerical analysis: Should special steps be taken to explicitly incorporate terminal wealth depletion into the design of the numerical analysis? The referee asserts that it is not necessary to take special steps to incorporate terminal wealth depletion into the design of the numerical analysis because if such depletion is optimal, it will emerge naturally as a result of an appropriate analysis. According to the referee, the solution $c(t)$ that satisfies (3) and obeys (14) from $t = 0$ to $t = t^*$, where $S(t^*) = 0$ occurs, must also be such that $c(t^*) = m(t^*)$ if $t^* < \overline{T}$. Knoweldge of (i) S_0, (ii)

Eq. (14), and (iii) the time path of $m(t)$ is all one needs to compute the optimal consumption path in the numerical analysis. While this is a natural and reasonable claim, it misses a subtle feature of the model. In this section, I provide an example to disprove the referee's claim and offer an explanation for the necessity of incorporating terminal wealth depletion into the design of the numerical analysis.

To enhance exposition, I employ a simple example that can be solved analytically. Although the analytical solvability of the example may render the purported numerical analysis somewhat redundant, it actually has the distinct advantage that all the arguments involved can be precisely proved or disproved. Consider an individual with a logarithmic utility and a uniformly distributed lifetime. More specifically, let $g(c(t)) = \log(c(t))$, $\alpha = j = 0$, $\Omega(t) = (\bar{T} - t)/\bar{T}$, $m(t) = m$ (a constant) > 0, and $S_0 > 0$. Since $\pi_t(t) = 1/(\bar{T} - t)$, (14) becomes

$$c^{\#\prime}(t) = -\frac{c^{\#}(t)}{\bar{T} - t}. \tag{17}$$

Suppose one follows the referee's suggestion to solve for $c^{\#}(t)$ numerically. The numerical analysis will typically involve three steps: (i) Pick a positive $c(0)$. (ii) Given S_0 and $c(0)$, solve numerically the differential equations (17) and (4) for $c(t)$ and $S(t)$, $t \in (0, \bar{T}]$. (iii) Check whether $S(\bar{T}) = 0$ is satisfied. If yes, $c(t)$ and $S(t)$ are optimal. If no, return to step (i).[8] Clearly, this algorithm does not take any special steps to incorporate terminal wealth depletion. To evaluate whether the numerical analysis is effective, assume that it has located a $c(0)$ that satisfies $S(\bar{T}) = 0$. From a theoretical point of view, what the algorithm essentially does is to solve for $c(t)$ and $S(t)$ from (4), (6), and (17), given the initial condition S_0. Since this control problem is analytically solvable, it is possible to examine the analytical solution to ascertain whether the numerical analysis can produce the correct optimal solution.

To derive the analytical solution, first solve the first-order differential equation (17) to get

$$c^{\#}(t) = \frac{(\bar{T} - t)c^{\#}(0)}{\bar{T}}. \tag{18}$$

To solve for the starting point $c^{\#}(0)$, plug (18) into (6) (i.e., the lifetime budget constraint $S(\bar{T}) = S_0 + \int_0^{\bar{T}} m(t)\,dt - \int_0^{\bar{T}} c(t)\,dt = 0$); then one ob-

[8] This is essentially the shooting algorithm for solving finite-horizon optimal control problems as described in Judd (1998, pp. 353–354, Algorithm 10.2).

tains

$$c^{\#}(0) = \frac{2\left(S_0 + m\overline{T}\right)}{\overline{T}}. \tag{19}$$

It follows from (18) and (19) that

$$c^{\#}(t) = \frac{2(\overline{T} - t)\left(S_0 + m\overline{T}\right)}{\overline{T}^2}. \tag{20}$$

Substituting (20) into the wealth path $S^{\#}(t) = S_0 + \int_0^t m(t)\,dt - \int_0^t c^{\#}(t)\,dt$ yields

$$S^{\#}(t) = \frac{(\overline{T} - t)\left[S_0\overline{T} - t\left(S_0 + m\overline{T}\right)\right]}{\overline{T}^2}. \tag{21}$$

Hence, the algorithm will generate (20) and (21) as the optimal solution. One can readily see from (21) that $S^{\#}(t) > 0$ for $t \in [0, S_0\overline{T}/(S_0 + m\overline{T}))$ and $S^{\#}(t) < 0$ for $t \in (S_0\overline{T}/(S_0 + m\overline{T}), \overline{T})$. Thus, following this algorithm, wealth depletion will emerge at

$$t^* = \frac{S_0\overline{T}}{S_0 + m\overline{T}} \tag{22}$$

because $S^{\#}(t^*) = 0$. However, substituting (22) into (20) gives $c^{\#}(t^*) = 2m > m$. The result $c^{\#}(t^*) \neq m$ proves that the referee's claim is incorrect. Although (21) and (20) satisfy (3) and obey (14) for $t \in [0, t^*)$ and $S^{\#}(t^*) = 0$ is fulfilled, $c^{\#}(t^*) \neq m(t^*)$. Without explicitly incorporating the terminal wealth depletion time into the setup, the referee's solution procedure cannot guarantee that, when the wealth is depleted at $t^* < \overline{T}$, consumption $c^{\#}(t^*)$ must be equal to income $m(t^*)$. Knowledge of S_0, Eq. (14), and the time path of $m(t)$ is not sufficient to compute the optimal consumption path.

There is another problem with the algorithm. Equation (21) shows that $S^{\#}(t) < 0$ for all $t \in (t^*, \overline{T})$, hence (3) is violated. In order to remove this problem, one may be tempted to impose the constraint (3) on step (ii) of the algorithm. However, such an amended algorithm will not be able to produce a solution for this control problem because (19) and (21) are the *only* solution that satisfies (4), (6), and (17). Put differently, there does not exist a solution $(c(t), S(t))$ that *simultaneously* satisfies (3), (4), (6), and (17). The conundrum arises because the algorithm does not take into account terminal wealth depletion.

To find out the source of the problem, notice that (14) (or (17) in the example) implicitly assumes that the multiplier $\mu(t)$ equals 0. If $\mu(t) > 0$ for some $t \in (0, \overline{T})$, then (10) and (13) imply that $\Omega(t)e^{-\alpha t}\{g''(c^{\#}(t))c^{\#\prime}(t) - [\pi_t(t) + \alpha - j]g'(c^{\#}(t))\} = -\mu(t)$.[9] Multiplying both sides by $S^{\#}(t)$ and using (12), one gets

$$\left\{g''\left(c^{\#}(t)\right)c^{\#\prime}(t) - \left[\pi_t(t) + \alpha - j\right]g'\left(c^{\#}(t)\right)\right\}S^{\#}(t) = 0. \quad (23)$$

Therefore, the general solution $c^{\#}(t)$ should obey (23) instead of (14). If $S^{\#}(t) > 0$, then (14) holds. By restricting the solution to obey (14), the algorithm excludes the possibility that $g''(c^{\#}(t))c^{\#\prime}(t) - [\pi_t(t) + \alpha - j]g'(c^{\#}(t)) \neq 0$. Such exclusion is too restrictive because $g''(c^{\#}(t))c^{\#\prime}(t)$ can diverge from $[\pi_t(t) + \alpha - j]g'(c^{\#}(t))$ when $S^{\#}(t) = 0$. As shown in Proposition 1, $c^{\#}(t) = m(t)$ for $t \in (t^*, \overline{T})$, hence $c^{\#\prime}(t) = m'(t)$. Therefore, (14) holds if $g''(m(t))m'(t) = [\pi_t(t) + \alpha - j]g'(m(t))$. Obviously, there is no a priori reason for $m(t)$ to satisfy this equality for all $t \in (t^*, \overline{T})$. Using the example as an illustration, $m'(t) = 0$ for $t \in (t^*, \overline{T})$ (as m is a constant), hence $g''(m(t))m'(t) = 0$ while $[\pi_t(t) + \alpha - j]g'(m(t)) > 0$. Therefore, the algorithm can never produce the right solution for the example because the optimal solution does not obey (14) for $t \in (t^*, \overline{T})$.[10]

An easy and efficient way to resolve the conundrum is to incorporate t^* directly into the numerical analysis. The incidence of terminal wealth depletion in effect shortens the horizon of the optimal control problem from $\overline{T}$ to t^*. As the endpoint of the control problem changes from a known $\overline{T}$ to an unknown t^*, it is natural to include t^* explicitly in the algorithm in order to solve for $c(t)$ and $S(t)$ in an efficient and cost-effective way. Using the previous algorithm as an example, one can modify the steps as follows: (i) Pick a positive t^* ($t^* \leq \overline{T}$) and a positive $c(0)$. (ii) Given S_0 and $c(0)$, solve numerically the differential equations (17) and (4) for $c(t)$ and $S(t)$, $t \in (0, t^*]$. (iii) Check whether (2), (3), and $S(t^*) = 0$ are satisfied. If yes, $c(t)$ and $S(t)$ are optimal. If no, return to step (i).[11] In this algorithm, the terminal wealth depletion time t^*, which replaces the $\overline{T}$ in the previous algorithm, is treated explicitly as a parameter to be determined. The objective of the algorithm is to solve for $c(t)$, $S(t)$, *and* t^*

[9] Notice that $\eta(t) = 0$ because of (11) and the fact that $c^{\#}(t) > 0$ for $t \in [0, \overline{T}]$ (see the proof of Proposition 1).

[10] It follows that the shooting algorithm (Judd, 1998, pp. 353–354) cannot be applied to this type of model without further modifications.

[11] For the above example, the optimal solution, which can be derived analytically, is given by $c^{\#}(t) = m^2(\overline{T} - t)/(S_0 + m\overline{T} - \sqrt{S_0^2 + 2m\overline{T}S_0})$, $S^{\#}(t) = S_0 + mt + m^2[(\overline{T} - t)^2 - \overline{T}^2]/[2(S_0 + m\overline{T} - \sqrt{S_0^2 + 2m\overline{T}S_0})]$, and $t^* = (\sqrt{S_0^2 + 2m\overline{T}S_0} - S_0)/m$, $t \in [0, t^*]$.

simultaneously. While other algorithms are possible, this one is probably the simplest one. If it is ever possible to design a numerical analysis without explicitly incorporating terminal wealth depletion, the required algorithm will be computationally demanding. The benefits of such a complicated algorithm will certainly outweigh the costs especially when it is compared with the simple algorithm given here. Thus, on both theoretical and practical grounds, it is necessary and advantageous to incorporate terminal wealth depletion into the design of the numerical analysis.

IV. EVIDENCE

To investigate practically how early the wealth is depleted, I conduct a simulation study using the conventional CRRA utility function, i.e., $g(c) = c^{1-\gamma}/(1-\gamma)$, $\gamma > 0$, ($g(c) = \log(c)$ if $\gamma = 1$). To highlight the role of uncertain lifetime, I focus on the elderly and investigate the model over the time interval $[65, \overline{T}]$. For simplicity, I assume that $m(t) = M$ for $t \geq 65$, where M is a positive constant. The assumption of a constant stream of income after retirement is a good approximation; see, e.g., the evidence in Diamond and Hausman (1984) and Hurd (1989). Under these assumptions, one can verify that (16) becomes

$$\frac{S(65)}{M} = \int_{65}^{t^*} e^{-j(t-65)} \left\{ \left[\frac{\Omega(t) e^{(j-\alpha)t}}{\Omega(t^*) e^{(j-\alpha)t^*}} \right]^{1/\gamma} - 1 \right\} dt. \tag{24}$$

Following the convention in the literature, I set j at 0.03. As there is no clear consensus on the values of γ and α, several values are examined (ranging from 0.1 to 4 for γ and 0.01 to 0.1 for α) and the choices are in line with either existing empirical findings or previous simulation studies.[12] Assuming that mortality follows the Gompertz Law (Leung, 1994), I set

$$\Omega(t) = e^{-0.00093\phi(e^{0.087t}-1)}, \qquad \phi = 1, 2. \tag{25}$$

Two values of ϕ are used to study how t^* varies with healthiness. An individual with $\phi = 2$ has twice the hazard rate of death than an individual with $\phi = 1$. There is no need to specify $\overline{T}$ in the simulations because the domain of the Gompertz distribution is $[0, \infty)$, hence no finite $\overline{T}$ appears in (25).

[12] Notice that small values of γ and large values of α are not implausible. For example, Lawrance (1991) finds estimates of γ as low as 0.625 and estimates of α exceeding 0.1. Some of the estimates of γ in Hansen and Singleton (1983) are about 0.16, which is even smaller than Lawrance's estimates.

One may be concerned about whether Proposition 1 is applicable to this simulation setup because the Gompertz Law (25) does not satisfy the conditions $\Omega(\overline{T}) = 0$ and $\lim_{t \to \overline{T}} \pi_t(t) = \infty$ with a finite $\overline{T}$. To answer this query, multiply (25) by $e^{-\varepsilon t/[\overline{T}(\overline{T}-t)]}$ to obtain $\tilde{\Omega}(t) = e^{-\varepsilon t/[\overline{T}(\overline{T}-t)]}\Omega(t)$, where ε and $\overline{T}$ are positive numbers. Clearly, $\tilde{\Omega}(0) = 1$, $\tilde{\Omega}(\overline{T}) = 0$, and $\tilde{\Omega}'(t) \leq 0$ for $t \in [0, \overline{T}]$, hence $\tilde{\Omega}(t)$ is a well-defined survival function. Let the mortality hazard be $\tilde{\pi}_t(t) = -\tilde{\Omega}'(t)/\tilde{\Omega}(t)$; then $\lim_{t \to \overline{T}} \tilde{\pi}_t(t) = \lim_{t \to \overline{T}}[0.00008091e^{0.087t} + \varepsilon/(\overline{T} - t)^2] = \infty$. Thus, $\tilde{\Omega}(t)$ and $\tilde{\pi}_t(t)$ satisfy the pertinent conditions of Proposition 1. Hence, for the sake of theoretical nicety, one can replace $\Omega(t)$ with $\tilde{\Omega}(t)$ in the simulation study. Nevertheless, such a replacement is unnecessary from a practical point of view because $\tilde{\Omega}(t)$ can be made arbitrarily close to $\Omega(t)$ by choosing a small value for ε. For example, suppose $\overline{T} = 130$; then by setting $\varepsilon = 0.0001$, the numerical values of $\Omega(t)$ and $\tilde{\Omega}(t)$ will be so close to each other that they produce practically the same t^* in the simulations.[13] Therefore, it makes no difference whether $\Omega(t)$ or $\tilde{\Omega}(t)$ is used in the simulation study. In other words, one can always justify $\Omega(t)$ by using $\tilde{\Omega}(t)$, so the choice of the familiar Gompertz Law should not be a concern in this regard.[14]

Under this setup, t^* can be solved numerically from (24). The solution t^* is unique because $\Delta(t) = -0.00008091e^{0.087t} - \alpha + 0.03$ for all $t \in [65, \overline{T}]$, hence $\Delta'(t) < 0$ and the condition in Leung (1997, Proposition 4) is satisfied. Table I reports the simulation results for three different values of $S(65)/M$.[15] Several observations are in order.

First, it is clear that terminal wealth depletion can occur very early. In some cases it happens in only a few years after retirement. In the extreme case where $\gamma = 0.1$ and $\alpha = 0.1$, even a relatively wealthy individual with $S(65)/M = 10$ will deplete his assets at age 68. Second, the magnitude of t^* varies appreciably with the values of γ, α, ϕ, and $S(65)/M$. Consistent with the predictions in Proposition 2, t^* increases with γ and $S(65)/M$,

[13] This indicates that the conditions $\Omega(\overline{T}) = 0$ and $\lim_{t \to \overline{T}} \pi_t(t) = \infty$ are sufficient but not necessary for Proposition 1 to hold. As long as $\Omega(\overline{T})$ is sufficiently small and $\lim_{t \to \overline{T}} \pi_t(t)$ is sufficiently large, terminal wealth depletion will take place.

[14] This result is not limited to the Gompertz Law only. In general, for any reasonable survival function for human mortality $\Omega(t)$ defined on $[0, \infty)$, one can find a modified $\tilde{\Omega}(t)$ defined on $[0, \overline{T}]$ for an appropriately chosen $\overline{T}$ $(0 < \overline{T} < \infty)$ such that $\tilde{\Omega}(0) = 1$, $\tilde{\Omega}(\overline{T}) = 0$, $\lim_{t \to \overline{T}} -\tilde{\Omega}'(t)/\tilde{\Omega}(t) = \infty$, and $\tilde{\Omega}(t)$ can be made arbitrarily close to $\Omega(t)$ for $t \in [0, \overline{T})$ (e.g., $\tilde{\Omega}(t) = e^{-\varepsilon t/[\overline{T}(\overline{T}-t)]}\Omega(t)$).

[15] Empirical evidence suggests that the average wealth to income ratio at retirement, $S(65)/M$, is about 5; see, for example, Diamond and Hausman (1984) and Hurd (1989). The median wealth to income ratio is likely to be smaller than 5 because the distribution of wealth is more skewed than that of income.

TABLE I
Terminal Wealth Depletion Time t^*

		t^*					
		$\phi = 1$ $\frac{S(65)}{M}$			$\phi = 2$ $\frac{S(65)}{M}$		
γ	α	1	5	10	1	5	10
4	0.10	74	82	87	73	80	84
	0.05	77	86	90	75	82	86
	0.03	79	87	92	75	83	86
	0.01	81	89	93	76	84	87
1	0.10	70	74	76	69	73	75
	0.05	71	77	80	70	74	77
	0.03	73	79	82	71	75	78
	0.01	75	81	84	72	77	79
0.5	0.10	68	71	73	68	70	72
	0.05	70	73	76	69	72	74
	0.03	71	75	78	69	73	75
	0.01	73	78	80	70	74	76
0.1	0.10	66	67	68	66	67	68
	0.05	67	69	70	67	68	69
	0.03	68	70	71	67	68	69
	0.01	69	72	73	68	69	70

and decreases with α and ϕ. Third, one cannot infer from t^* *alone* the characteristics of the individual. For example, an individual whose t^* is 70 can be relatively rich ($S(65)/M = 10$, $\gamma = 0.1$, $\alpha = 0.05$, $\phi = 1$) or poor ($S(65)/M = 1$, $\gamma = 1$, $\alpha = 0.1$, $\phi = 1$). Hence, the claim of Hurd (1989) that low initial wealth is the cause of all the early terminal wealth depletion in his sample may be overly simplistic. Other factors such as poor health, low risk aversion, low interest rate, or high income could also be responsible. In sum, the simulation results support that the model can account for low wealth holdings and early terminal wealth depletion.

The above simulation exercise concentrates on the saving behavior between age 65 and $\overline{T}$. An alternative is to start the simulation at a younger age, say 18, and trace the age-wealth profile between age 18 and $\overline{T}$ to examine whether the model predicts many people retiring with little or no assets. While this is apparently a more complete investigation, it is not really necessary to adopt such an approach to illustrate the point underscored in this paper. First, simulating the age-wealth profile between age 18 and $\overline{T}$ requires specific assumptions on the age-earnings profile between age 18 and 65. As exemplified in Hubbard *et al*. (1995), one can specify at least three different age-earnings profiles to capture differences in permanent income through educational attainment. Once the parameter values for (γ, α, ϕ) are chosen, the value of $S(65)/M$ will solely be determined by the age-earnings profile and $S(18)$. Thus, one can simulate the age-wealth profile for either different age-earnings profiles or different values of $S(65)/M$; the two are closely related because the choice of the age-earnings profile determines $S(65)/M$. I have attempted both and I do not report the results for different age-earnings profiles here because the borrowing constraint tends to bind several times or over a lengthy period of time within the interval $[18, 65]$ (especially for people with a low age-earnings profile), making the simulations unnecessarily complicated while adding very little in terms of understanding the essence of the problem. Second, it is not surprising to find that people with a low age-earnings profile reach retirement with little assets. One of the main objectives of this paper, however, is to claim and verify the more subtle point that even people with higher age-earnings profiles, and hence higher wealth to income ratios at retirement, may also exhaust their wealth rather rapidly after retirement. For this purpose it is sufficient to simulate the age-wealth profile between age 65 and $\bar{T}$ using higher values of $S(65)/M$ such as 5 and 10.

Besides the simulation results, there is other supportive empirical evidence in the literature. In a nonparametric study using ten years of panel data from the Longitudinal Retirement History Survey, Hurd (1987) finds that the elderly in the sample generally decumulated real wealth. Excluding home equity, the estimated average rate of real wealth change is about -3.2% per annum, which suggests that a household will decumulate about half of its wealth in 20 years.[16] In a subsequent study, Hurd (1989) uses the same data set but estimates parametrically a structural model of life-cycle saving for the retired singles. The estimates imply that many retired singles will decumulate their entire wealth in less than 20 years. For instance, one set of estimates shows that the mean and the median times (after age 65) to terminal depletion of wealth are 15.9 and 16.4 years, respectively (Hurd, 1989, p. 802). Additional evidence of dissaving is reported in recent studies by Attanasio and Hoynes (1995) and Hurd (1999).

[16] If housing wealth is included, then the estimated average rate of real wealth change is about -1.5% per annum, which means that about one quarter of wealth will be decumulated in 20 years. The role of housing wealth in testing life-cycle models of saving is discussed in detail in Hurd (1987, 1989). In a sensitivity analysis, Kuehlwein (1995) confirms the dissaving finding of Hurd (1987) in many specifications, but raises questions about dissaving among couples and the method of elderly dissaving. More research is needed to assess the robustness of their procedures and findings.

These findings support my theory in two ways. First, the emergence of terminal wealth depletion in Hurd (1989) is exactly predicted by my theory. Second, the elderly, especially the singles, do decumulate their assets in a significant way. The dissaving finding of Attanasio and Hoynes (1995) and Hurd (1987, 1999) stands in sharp contrast to the prevailing literature, which is largely based on cross-section data, that shows little evidence of dissaving. Although some of the estimated dissaving rates (e.g., 3.2% per annum) may not appear to be high enough to effect terminal wealth depletion within a short period of time, this should not be construed as evidence against my theory because the dissaving rates reported in the literature are typically mean or median values. What is more relevant to the present study is neither the mean nor the median rate of wealth decumulation, but the rates of wealth decumulation among the bottom two deciles of the distribution. As the focus of this paper is on the low-wealth elderly, empirical estimates of the average dissaving rate are not the most pertinent gauges for the validity of my theory. If the average elderly household decumulates its wealth at a rate of 3.2% per annum, then the bottom two deciles of elderly households must dissave more rapidly than 3.2% per annum. These bottom two deciles of elderly households will most likely account for a significant portion of the low-wealth elderly observed in the data. Unfortunately, none of these studies reports the distribution of the rate of wealth decumulation, so one cannot ascertain the dissaving rates among the bottom deciles. Nevertheless, given the high variability of the wealth distribution, there are good reasons to believe that the dissaving rates among the bottom two deciles of elderly households will be high enough to generate among them low wealth holdings and eventual wealth depletion within a reasonable period of time.

Another set of findings that lends credence to my theory is Sheiner and Weil (1992) and Jones (1996). In the above theoretical analysis, home equity is treated no different from the other financial assets. Despite popular beliefs that elderly households do not decumulate their housing wealth, Sheiner and Weil find that households do significantly reduce their home equities as they age: about 58% of households will not leave behind a house when the last member dies. In addition, most households do not keep the money received from selling the houses. Sheiner and Weil strongly argue that previous studies have underestimated the decumulation rate of housing wealth. Similar findings of substantial home equity reduction by the elderly using both U.S. and Canadian data are reported in the preliminary study of Jones (1996).

A separate and growing body of empirical work on the relationship between mortality and wealth also provides evidence in favor of my theory. Jianakoplos *et al.* (1989) find that the mortality rates of the elderly in the bottom two deciles of the wealth distribution are three times higher than those of the elderly in the top decile. Using a different data set, Attanasio and Hoynes (1995) obtain a similar result that the mortality rate of the elderly in the lowest wealth quartile is three times higher than that in the top wealth quartile. Menchik (1993) finds a strong inverse relationship between mortality and wealth even after controlling for other variables such as permanent income. These findings on the negative relationship between mortality and wealth are all consistent with Proposition 2(ii).[17]

V. A COMPARISON WITH DAVIES (1981)

The results in the previous sections appear to be at odds with Davies (1981). In a widely cited study, Davies proposes that uncertain lifetime can account for much of the observed slow dissaving among the elderly. By calibrating a life-cycle model of saving with income and survival data, he compares numerically consumption under certain lifetime with consumption under uncertain lifetime using different sets of parameter values. In a majority of the cases investigated, including those that are based on what Davies regards as "best guess" parameter values, he finds that consumption under certain lifetime is notably higher than consumption under uncertain lifetime. According to Davies, the results suggest that uncertain lifetime reduces dissaving. Such a finding is just opposite to the theme of this paper that uncertain lifetime increases dissaving. Because of the importance and influence of Davies' results, this issue warrants further investigation.

A critical problem in Davies' investigation is that it does not incorporate terminal wealth depletion into the design of the numerical analysis, despite the fact that his pension model is a special case of the model analyzed in Section II. As shown in Sections II and III, there must be terminal wealth depletion before the maximum lifetime and the depletion time should be explicitly incorporated into the design of the numerical analysis. To study the impact of incorporating terminal wealth depletion, I conduct a numerical analysis similar to Davies' and investigate whether uncertain lifetime tends to reduce consumption. As in the previous section, I study the model over the period $[65, \bar{T}]$ and assume that $m(t) = M$ for $t \in [65, \bar{T}]$, where M is a positive constant.

Let $\hat{c}(t)$ denote the optimal consumption in a standard life-cycle model of saving under certain lifetime; then one can follow the derivation of Eq.

[17] Notice that the model here only considers the effect of mortality (health) on wealth. Of course, the reverse causation is also possible (e.g., lower wealth contributing to higher mortality).

(19) in Davies (1981, p. 571) to show that

$$\frac{\hat{c}(t)}{M} = \frac{[S(t)/M] + [1 - e^{-j(E(T\mid t)-t)}]/j}{[1 - e^{-(j-\delta)(E(T\mid t)-t)}]/(j-\delta)}, \tag{26}$$

where $\delta = (j - \alpha)/\gamma$ and $E(T|t) = t + \int_t^{\overline{T}} [\Omega(x)/\Omega(t)]\,dx$ is the expected lifetime (life expectancy) at time t given that the individual is alive at t.[18] In the simulations, I arbitrarily set $\overline{T} = 120$. As in the previous section, I assume $j = 0.03$ and $\Omega(t)$ follows (25) with ϕ set at 1.

Under uncertain lifetime, (15) implies that the ratio of consumption to income is given by

$$\frac{c^{\#}(t)}{M} = \left[\frac{\Omega(t)e^{(j-\alpha)t}}{\Omega(t^*)e^{(j-\alpha)t^*}}\right]^{1/\gamma}, \qquad 65 \le t \le t^*. \tag{27}$$

To compare $c^{\#}(t)$ and $\hat{c}(t)$, one needs to make some assumptions on the evolution of $S(t)$. Following Davies, the ratio $c^{\#}(t)/\hat{c}(t)$ is calculated in the following way. For each set of parameter values specified for $\{\gamma, \alpha, S(65)/M\}$, I use (27) to calculate $c^{\#}(t)/M$, $t = 65, 66, \ldots, t^*$, where t^* is taken from Table I. Using these values of $c^{\#}(t)/M$, I obtain the corresponding ratio of wealth to income, $S^{\#}(t)/M$, by means of the formula (see (8))

$$\frac{S^{\#}(t)}{M} = \frac{S(65)}{M}e^{j(t-65)} + \int_{65}^{t} e^{j(t-z)}\left[1 - \frac{c^{\#}(z)}{M}\right]dz \tag{28}$$

for $t = 65, 66, \ldots, t^*$. Replacing $S(t)/M$ in (26) with $S^{\#}(t)/M$, I obtain the value of $\hat{c}(t)/M$ for each t. Dividing (27) by (26) gives the desired ratio $c^{\#}(t)/\hat{c}(t)$ for $t = 65, 66, \ldots, t^*$. This simulation design and procedure are similar to those of Davies (1981, Sect. IV) except that it explicitly takes into account the terminal wealth depletion time t^* and expoits the constant income stream M in the formulation of the ratios $c^{\#}(t)/M$ and $\hat{c}(t)/M$. It is easy to see that the procedure only requires an initial value for the ratio $S(65)/M$; neither $S(65)$ nor M needs to be specified.

Table II gives some of the simulation results. For brevity, only the case $S(65)/M = 5$ (with all four values of α) as well as the case $S(65)/M = 10$ and $\alpha = 0.03$ is presented.[19] Contrary to the majority of Davies' results,

[18] As will be shown below, the advantage in considering the ratio $\hat{c}(t)/M$ instead of $\hat{c}(t)$ in (26) is that it is not necessary to assume any specific values for $S(t)$ and M in the computations; only the ratio $S(t)/M$ needs to be specified.

[19] Results for the other cases are available on request. Notice that Table II reports the ratios up to age $t^* - 1$ for the case $S(65)/M = 5$. For $S(65)/M = 10$ and $\alpha = 0.03$, I only report the ratios (if applicable) up to age $t^* - 1$ of the case $S(65)/M = 5$ and $\alpha = 0.01$, as the numbers displayed are sufficient to convey the general pattern of the results.

TABLE II
Ratio of Consumptions $c^{\#}(t)/\hat{c}(t)$

	α				
	0.1	0.05	0.03	0.01	0.03
t	$\frac{S(65)}{M}=5$				$\frac{S(65)}{M}=10$
	$\gamma = 4$				
65	1.059	1.073	1.038	1.039	1.027
66	1.067	1.083	1.044	1.045	1.036
67	1.070	1.089	1.048	1.050	1.042
68	1.073	1.097	1.051	1.055	1.050
69	1.076	1.105	1.056	1.060	1.058
70	1.078	1.113	1.060	1.066	1.068
71	1.080	1.123	1.065	1.073	1.079
72	1.082	1.134	1.070	1.081	1.093
73	1.083	1.145	1.076	1.089	1.108
74	1.083	1.158	1.082	1.099	1.126
75	1.081	1.172	1.088	1.110	1.147
76	1.077	1.187	1.095	1.122	1.172
77	1.070	1.203	1.101	1.135	1.201
78	1.060	1.221	1.108	1.149	1.236
79	1.046	1.239	1.114	1.165	1.278
80	1.026	1.257	1.119	1.182	1.329
81	1.000	1.276	1.122	1.201	1.391
82		1.293	1.124	1.221	1.468
83		1.307	1.122	1.241	1.565
84		1.317	1.115	1.261	1.690
85		1.319	1.102	1.280	1.854
86			1.081	1.296	2.079
87				1.306	2.400
88				1.306	2.890
	$\gamma = 1$				
65	1.171	1.281	1.331	1.322	1.346
66	1.176	1.297	1.352	1.339	1.375
67	1.165	1.303	1.365	1.351	1.397
68	1.148	1.308	1.379	1.363	1.421
69	1.123	1.311	1.392	1.375	1.448
70	1.090	1.309	1.404	1.387	1.477
71	1.047	1.304	1.415	1.398	1.508
72	0.993	1.292	1.424	1.409	1.543
73	0.927	1.273	1.429	1.417	1.581
74		1.245	1.429	1.423	1.623
75		1.205	1.422	1.425	1.668
76		1.152	1.406	1.422	1.715
77			1.377	1.411	1.764
78			1.333	1.389	1.812
79				1.355	1.854
80				1.305	1.884

TABLE II—*Continued*

			$\gamma = 0.5$		
65	1.059	1.278	1.478	1.719	1.669
66	1.020	1.264	1.484	1.744	1.716
67	0.953	1.230	1.474	1.761	1.749
68	0.871	1.187	1.458	1.776	1.782
69	0.777	1.132	1.434	1.789	1.817
70	0.673	1.065	1.399	1.798	1.851
71		0.986	1.352	1.801	1.884
72		0.896	1.291	1.797	1.913
73			1.215	1.783	1.935
74			1.124	1.755	1.945
75				1.709	1.935
76				1.641	1.897
77				1.548	1.819
			$\gamma = 0.1$		
65	0.539	1.658	3.009	9.310	3.381
66	0.276	1.531	2.845	8.773	3.342
67		1.259	2.531	8.069	3.126
68		0.934	2.133	7.261	2.797
69			1.687	6.349	2.359
70				5.356	1.849
71				4.322	

Table II shows that, with few exceptions, $c^{\#}(t)$ is notably higher than $\hat{c}(t)$ for all t before t^*.[20] Thus, the rate of dissaving is generally higher under uncertain lifetime than that under certain lifetime. For example, when $\gamma = 4$, $\alpha = 0.01$ (which are closest to Davies' "best guess" parameter values), and $S(65)/M = 5$, $c^{\#}(t)$ stays above $\hat{c}(t)$ for all $t \in [65, 88]$. Even when t^* is as large as 92 (when $\gamma = 4$, $\alpha = 0.03$, $S(65)/M = 10$), $c^{\#}(t)$ is always greater than $\hat{c}(t)$ and the ratio $c^{\#}(t)/\hat{c}(t)$ increases monotonically with t. In some extreme cases (e.g., $\gamma = 0.1$, $\alpha = 0.01$, $S(65)/M = 5$), $c^{\#}(t)$ is greater than $\hat{c}(t)$ by about 400 to 900%. The ratios $c^{\#}(t)/\hat{c}(t)$ in Table II are almost always substantially larger than those in Davies. When there is a terminal wealth depletion time, consumption under uncertain lifetime has to be higher than consumption under certain lifetime so that the

[20] Table 3 of Davies (1981) shows that, out of the 168 ratios of $c^{\#}(t)/\hat{c}(t)$ calculated, 42 are greater than 1, which means that consumption under uncertain lifetime is higher than consumption under certain lifetime in only 25% of the cases investigated.

wealth can be run down at a faster rate. This demonstrates the importance of incorporating an explicit wealth depletion time into the simulations.[21]

It is difficult to claim that the discrepancies between Davies' results and the findings presented in Table II can be *entirely* attributed to the incorporation of terminal wealth depletion because there are two major differences in the calibrations of the model. First, Davies employs a monotonically declining post-retirement income $m(t)$ while I employ a constant $m(t)$ in the simulations. As his calibrations are based on Canadian data, his choice of a falling $m(t)$ may be appropriate for the time and country studied. On the other hand, my calibrations are based on U.S. data, so my choice of a constant $m(t)$ may be more relevant to the United States in recent decades. Second, I calibrate the survival function $\Omega(t)$ by means of the conventional Gompertz Law, whereas Davies uses a peculiar inverse logistic curve for $\Omega(t)$. Unless there are typographical errors in the article, Davies' $\Omega(t)$ does not match actual survival data.[22] Because of Davies' unusual $\Omega(t)$, it is difficult to make a reliable and thorough comparison between his results and my findings. Furthermore, it does not seem appropriate to adopt Davies' peculiar $\Omega(t)$ in my calibrations to check the robustness of my findings to different choices of $\Omega(t)$. As a result, it is premature to conclude that the incorporation of terminal wealth depletion is the only factor behind the discrepancies between Davies' results and my findings.

Despite the above caveats, Table II at least illustrates that uncertain lifetime cannot account for the observed slow decumulation among the

[21] To verify this claim, I run the simulations without adding an explicit terminal wealth depletion time into the model. I find that in many cases, consumption under uncertain lifetime is no longer higher than consumption under certain lifetime, which is consistent with Davies' results. Due to their length, these results are not reported here, but they are available on request.

[22] The reasons are as follows. First, the values of $\Omega(t)$ calculated from the formula and the parameter values given in footnote 17 of Davies (1981, p. 572) are substantially lower than the actual survival data displayed in his Fig. 2 (Davies, 1981, p. 573). For instance, $\Omega(20) = 0.106$, $\Omega(30) = 0.026$, and $\Omega(40) = 0.006$, which are much lower than those portrayed in his Fig. 2. Second, using Davies' symbols, the mortality hazard calculated from the inverse logistic curve is given by $\pi_t(t) = (b + a(b + c)e^{ct})/(1 + ae^{ct})$. Given Davies' parameter values for a, b, and c, $\pi_t(t)$ is virtually a constant, e.g., $\pi_{20}(20) = 0.137$, $\pi_{30}(30) = 0.143$, and $\pi_t(t) \cong 0.144$ for all $t \geq 40$. This mortality hazard is much higher than that observed in actual life tables for t between 20 and 80, and much lower for t greater than 90. Third, $\lim_{t \to \infty} \pi_t(t) = b + c = 0.144396$, hence Davies' $\pi_t(t)$ does not satisfy the condition $\lim_{t \to \bar{T}} \pi_t(t) = \infty$. Fourth, Davies' $\pi_t(t)$ is already very close to the limiting value 0.144396 when $t = 30$. All these features of Davies' $\Omega(t)$ cannot be supported by actual survival data.

elderly. Instead it shows the contrary that uncertain lifetime raises the rate of dissaving beyond what it would be if lifetime was certain. This result is consistent with the finding in the previous sections that uncertain lifetime reduces the desire to provide for the future. To explain the slow decumulation among the retired, one may have to turn to other factors that mitigate impatience, e.g., self-control, habit, bequest motive (Fisher, 1930).[23]

An immediate question arises from the above analysis. Is the theory of terminal wealth depletion plainly rejected by the observed slow dissaving among the elderly? As shown in Table I, some of the terminal wealth depletion times are so small that they imply a very high asset decumulation rate. For example, $t^* = 70$ in the case where $\gamma = 1$, $\alpha = 0.1$, $S(65)/M = 1$, and $\phi = 1$, which implies that the annual dissaving rate after retirement is about 20%. How can such a high dissaving rate be reconciled with the observed low dissaving rate that Davies and others attempt to explain? Two answers can be offered. First, the observed dissaving rate refers to the mean of the distribution of dissaving rates. The present model, however, is designed to explain the lower tail of the distribution of dissaving rates because it addresses why a special group of elderly people has little or no wealth. Hence, a high dissaving rate (say 20% per annum) for one particular group of elderly people can be consistent with a low average dissaving rate (say 4% per annum) for the *entire* group of elderly people. Second, the observed slow decumulation among the elderly may not be a genuine phenomenon. There is increasing evidence that previous studies have underestimated the asset decumulation rates because they utilize cross-section (instead of longitudinal) data and they fail to control for differential mortality effects (see, e.g., Modigliani, 1986; Hurd, 1987; Jianakoplos *et al.*, 1989; Attanasio and Hoynes, 1995). Thus, the observation is still tentative.

[23] To be complete, it should be pointed out that Davies does find something like terminal wealth depletion in a few cases (3 out of the 24 cases studied) in his Table 3 (e.g., the case $\gamma = 0.5$, $r = 0.03$, and $\rho = r$), although it is not clear whether his numerical analysis is conducted in an appropriate way (as discussed in Section III above) and whether the binding of the wealth constraint is temporary or terminal. In those cases, the results are similar to mine: consumption under uncertain lifetime is higher than that under certain lifetime. In fact, in his theoretical analysis, Davies (1981, pp. 571–572) speculates that consumption under uncertain lifetime will tend to be higher than consumption under certain lifetime if wealth is exhausted before the maximum lifetime. As he is not aware that wealth must be exhausted before the maximum lifetime and there are only a few cases of terminal wealth depletion in his simulation, he draws his conclusions based on the other cases.

VI. BEQUEST MOTIVE AND LIFE ANNUITY

The analysis so far has only explained why it is optimal for some elderly people to hold little or no wealth. To give a complete account for the stylized fact, one has to explain why only a significant fraction, but not all, of the elderly have little or no wealth. An immediate answer, which follows from Proposition 2, is that most elderly people may have high risk aversion, low discount rates, good health, high initial wealth, high interest rates, or low income. To add to this list of factors, this section shows that there are two more reasons for not depleting one's wealth early.

Suppose the individual has a bequest motive; then the decision problem in Section II becomes

$$\max_{c(t)} \int_0^{\overline{T}} \left[\Omega(t) e^{-\alpha t} g(c(t)) + \pi(t)\beta(t)\varphi(S(t)) \right] dt \qquad (29)$$

subject to (2), (4), and (5), where $\beta(t)$ is a subjective weighing function for bequest and $\varphi(S(t))$ is the utility derived from leaving a bequest of $S(t)$ (see Yaari, 1965). The Hamiltonian for this optimal control problem is given by

$$\mathscr{H}^{\mathscr{A}}(c(t), S(t)) = \Omega(t) e^{-\alpha t} g(c(t)) + \pi(t)\beta(t)\varphi(S(t)) + \lambda(t)[jS(t) + m(t) - c(t)] + \eta(t)c(t). \qquad (30)$$

To ease notational burden, I use the same notations $(c^{\#}(t), S^{\#}(t))$ to denote the optimal solution to this control problem. The necessary optimality conditions are

$$\frac{\partial \mathscr{H}^{\mathscr{A}}(c^{\#}(t), S^{\#}(t))}{\partial c(t)} = \Omega(t) e^{-\alpha t} g'(c^{\#}(t)) - \lambda(t) + \eta(t) = 0, \qquad (31)$$

$$\frac{\partial \mathscr{H}^{\mathscr{A}}(c^{\#}(t), S^{\#}(t))}{\partial S(t)} = -\lambda'(t) = \pi(t)\beta(t)\varphi'(S^{\#}(t)) + j\lambda(t), \qquad (32)$$

as well as $\eta(t) \geq 0$ and $\eta(t)c^{\#}(t) = 0$. Solving (32),

$$\lambda(t) = \lambda(0) e^{-jt} - \int_0^t e^{-j(t-z)} \pi(z)\beta(z)\varphi'(S^{\#}(z))\, dz. \qquad (33)$$

By inspecting (31)–(33), one can see that it is no longer necessary for the wealth to be depleted before $\overline{T}$ because $\lambda(\overline{T}) = 0$ is now feasible. For example, let $g(c) = c^{1-\gamma}/(1-\gamma)$ $(\gamma > 0)$, $\varphi(S) = S$, $\alpha = 0$, $\beta(t) = 1$,

$j = 0$, $\Omega(t) = 1 - t$, $\overline{T} = 1$, and assume that $S(0) \geq 1$. One can verify that $c^{\#}(t) = [(1-t)/(2-t)]^{1/\gamma}$, $t \in [0,1]$. Since $\int_0^t c^{\#}(z)\,dz < 1$ for all $t \in [0,1]$, $S^{\#}(t) = S(0) + \int_0^t m(z)\,dz - \int_0^t c^{\#}(z)\,dz > 0$ for all $t \in [0,1]$. Hence, the inequality $S^{\#}(t) > 0$ holds throughout $[0, \overline{T})$ in this example. In general, it is easy to verify that, whenever utility depends on wealth (be it from bequest, status, or other motives), then wealth depletion may never occur. As wealth holdings generate utility, it is intuitively reasonable that individuals will maintain positive wealth throughout the lifetime.

Now suppose that the individual has no bequest motives but actuarially fair life annuities are available. As in Yaari (1965), the individual chooses $c(t)$ to maximize $\int_0^{\overline{T}} \Omega(t)e^{-\alpha t} g(c(t))\,dt$ subject to (2) as well as the lifetime budget constraint

$$S(0) = \int_0^{\overline{T}} e^{-\int_0^t r(x)\,dx}[c(t) - m(t)]\,dt, \tag{34}$$

where $r(t)$ is the interest rate on actuarial notes. Let $Y = \int_0^{\overline{T}} e^{-\int_0^t r(x)\,dx} m(t)\,dt$ and $W(t) = \int_0^t e^{-\int_0^z r(x)\,dx} c(z)\,dz$; then (34) is equivalent to the following set of constraints: $W(0) = 0$, $W(\overline{T}) = S(0) + Y$, and $W'(t) = c(t)e^{-\int_0^t r(x)\,dx}$.

Let $\psi(t)$ be the multiplier for $W'(t)$; then the Hamiltonian for this optimal control problem is given by

$$\mathscr{H}^{\mathscr{B}}(c(t), S(t)) = \Omega(t)e^{-\alpha t} g(c(t)) + \eta(t)c(t) + \psi(t)c(t)e^{-\int_0^t r(x)\,dx}. \tag{35}$$

Again let $(c^{\#}(t), S^{\#}(t))$ denote the optimal solution to this control problem; the necessary conditions are given by

$$\frac{\partial \mathscr{H}^{\mathscr{B}}(c^{\#}(t), S^{\#}(t))}{\partial c(t)} = \Omega(t)e^{-\alpha t} g'(c^{\#}(t)) + \eta(t) + \psi(t)e^{-\int_0^t r(x)\,dx} = 0, \tag{36}$$

$$\frac{\partial \mathscr{H}^{\mathscr{B}}(c^{\#}(t), S^{\#}(t))}{\partial W(t)} = -\psi'(t) = 0, \tag{37}$$

as well as $\eta(t) \geq 0$ and $\eta(t)c^{\#}(t) = 0$. Assume that the interest rate on actuarial notes is fair, i.e., $r(t) = j + \pi_t(t)$; then $e^{-\int_0^t r(x)\,dx} = e^{-\int_0^t [j + \pi_t(t)]\,dx}$

$= \Omega(t)e^{-jt}$. Thus, (36) becomes $\Omega(t)[e^{-\alpha t}g'(c^{\#}(t)) + \psi(t)e^{-jt}] + \eta(t) = 0$. As $\Omega(t) > 0$ for $t \in [0, \bar{T})$, it follows that

$$e^{-\alpha t}g'\big(c^{\#}(t)\big) + \psi(t)e^{-jt} = 0 \tag{38}$$

whenever $c^{\#}(t) > 0$. The survival probability $\Omega(t)$ does not appear in (38), so the arguments leading to Proposition 1 do not apply here. Thus, there is no terminal wealth depletion in this model. The presence of life annuities essentially removes the effect of uncertain lifetime from the saving decision. Based on these analyses, one can therefore add bequest motive and life annuity to the list of factors that may explain why a majority of the elderly do not deplete their wealth early.

Although the annuity markets are quite well developed, few people purchase life annuities (Friedman and Warshawsky, 1988). However, the elderly may insure against uncertain lifetime by means of risk pooling within the family (Kotlikoff and Spivak, 1981). If family risk pooling operates as efficient as a perfect life annuity market, then these households do not need to deplete their wealth. Even if family risk pooling is less than perfect, it may be strong enough to put off the terminal wealth depletion time.

VII. CONCLUSION

This paper develops a rational theory of terminal wealth depletion to explain the puzzle of widespread low wealth holdings especially among the elderly. Previous studies of saving have only vaguely discussed, but never proved analytically and unambiguously, how low wealth accumulation or terminal wealth depletion can take place. By incorporating uncertain lifetime and borrowing constraint into the standard life-cycle model of saving, I demonstrate rigorously why and how terminal wealth depletion takes place and employ the comparative statics results to infer the probable characteristics of those individuals who save little and exhaust their wealth early. In addition to solving the puzzle, the analysis offers an analytical account for the unique finding of terminal wealth depletion in several recent studies, reveals the subtle relationship between theory and numerical analysis, and rejects the common perception that uncertain lifetime reduces dissaving.

A simple alternative explanation for the puzzle of widespread low wealth holdings among the elderly is that these households enter into retirement with little assets. Besides the low-lifetime-income argument, this explanation begs the question of why these households choose to have little assets when they begin to retire. My theory can help fill this void because for some households the asset decumulation process, which leads to terminal wealth depletion, may have started before they reach retirement. Although the simulations presented in the paper begin at age 65, it does not imply that in reality households decumulate only after they reach 65. The choice of age 65 is made simply for illustrative purposes; it is entirely possible, and in fact likely, that some households start dissaving earlier.

Although the analysis emphasizes the role of uncertain lifetime in depressing saving and depleting wealth, mortality risk is not necessarily the most important factor that determines when early terminal wealth depletion will take place. As illustrated in the comparative statics analysis, a young and healthy individual may deplete his wealth early depending on the interplay with other factors such as risk aversion, discount rate, interest rate, income, and initial wealth. From an analytical point of view, the main role of uncertain lifetime is to provide a *mechanism* by which terminal wealth depletion can be proved and characterized. For instance, many scholars have argued that the provision of social security pension is a major cause of low private savings. This paper delivers a rigorous theory to demonstrate why and how social security can displace private savings.

As reviewed in Jappelli (1990), there is a good deal of evidence that a significant fraction of households (especially the younger ones) are liquidity constrained because they cannot borrow against their future earnings. The model can therefore provide a complete story for low wealth holdings among a significant fraction of the aged as well as the nonaged. When people are young, the borrowing constraint is binding because earnings are low. As they get older, increases in earnings ease the borrowing constraint. In the meantime, uncertain lifetime begins to exert its effect on saving and eventually leads to terminal wealth depletion.

The present analysis complements those of Hubbard *et al.* (1995) and Feldstein (1995). Uncertain lifetime, low lifetime income, borrowing constraint, social insurance programs, and college scholarship rules together solve various pieces of the puzzle of widespread low wealth holdings. None of these factors alone can satisfactorily resolve the puzzle because there are many different motives for saving and there are various government policies and programs that influence the saving decision. The theory developed in this paper offers a general framework to enrich the analysis of saving behavior. For example, the model can be applied to study the impact of social insurance programs on the terminal wealth depletion time.

The theory proposed in this paper is testable because it predicts what types of people are more likely to save little and deplete their wealth early. In addition to an empirical test of the theory, an extensive simulation analysis along the lines of Hubbard *et al.* (1995) would be useful to determine how well the theory matches the data.

REFERENCES

Attanasio, Orazio, and Hoynes, Hilary (1995). "Differential Mortality and Wealth Accumulation," NBER working paper 5126.

Bernheim, B. Douglas (1991). "How Strong Are Bequest Motives? Evidence Based on Estimates of the Demand for Life Insurance and Annuities," *Journal of Political Economy* **99**, 899–927.

Bernheim, B. Douglas, and Scholz, John (1993). "Private Saving and Public Policy," *in Tax Policy and the Economy* (James Poterba, Ed.), Vol. 7, Cambridge, MA: MIT Press.

Davies, James B. (1981). "Uncertain Lifetime, Consumption, and Dissaving in Retirement," *Journal of Political Economy* **89**, 561–577.

Diamond, Peter (1977). "A Framework for Social Security Analysis," *Journal of Public Economics* **8**, 275–298.

Diamond, Peter, and Hausman, Jerry (1984). "Individual Retirement and Savings Behavior," *Journal of Public Economics* **23**, 81–114.

Eller, T. J. (1994). *Household Wealth and Asset Ownership*: *1991*. U.S. Bureau of the Census, Current Population Reports, P70-34, Washington, DC: U.S. Government Printing Office.

Feldstein, Martin (1995). "College Scholarship Rules and Private Saving," *American Economic Review* **85**, 552–566.

Fisher, Irving (1930). *The Theory of Interest*. New York: Macmillan.

Friedman, Benjamin, and Warshawsky, Mark (1988). "Annuity Prices and Saving Behavior in the United States," *in Pensions in the U.S. Economy* (Zvi Bodie, John Shoven, and David Wise, Eds.), Chicago: NBER and Univ. of Chicago Press.

Gustman, Alan, and Juster, Thomas (1995). "Income and Wealth of Older American Households: Modeling Issues for Public Policy Analysis," NBER working paper 4996.

Hansen, Lars, and Singleton, Kenneth (1983). "Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns," *Journal of Political Economy* **91**, 249–265.

Hubbard, R. Glenn, Skinner, Jonathan, and Zeldes, Stephen (1995). "Precautionary Saving and Social Insurance," *Journal of Political Economy* **103**, 360–399.

Hurd, Michael (1987). "Savings of the Elderly and Desired Bequests," *American Economic Review* **77**, 298–312.

Hurd, Michael (1989). "Mortality Risk and Bequests," *Econometrica* **57**, 779–813.

Hurd, Michael (1999). "Mortality Risk and Consumption by Couples," NBER working paper 7048.

Jianakoplos, Nancy, Menchik, Paul, and Irvine, Owen (1989). "Using Panel Data to Assess the Bias in Cross-sectional Inferences of Life-cycle Changes in the Level and Composition of Household Wealth," *in Measurement of Savings*, *Investment and Wealth* (R. Lipsey and H. Tice, Eds.), Chicago: NBER and Univ. of Chicago Press.

Jappelli, Tullio (1990). "Who is Credit Constrained in the U.S. Economy?" *Quarterly Journal of Economics* **105**, 219–234.

Jones, Lawrence (1996). "Housing Tenure Transition and Dissaving by the Elderly," *Canadian Journal of Economics* **29**, S505–S509.

Judd, Kenneth (1998). *Numerical Methods in Economics* Cambridge, MA: MIT Press.

King, M. A., and Dicks-Mireaux, L.-D. L. (1982). "Asset Holdings and the Life-cycle," *Economic Journal* **92**, 247–267.

Kotlikoff, Laurence, and Spivak, Avia (1981). "The Family as an Incomplete Annuities Market," *Journal of Political Economy* **89**, 372–391.

Kuehlwein, Michael (1995). "A Close Look at Dissaving in the Longitudinal Retirement History Survey," *Review of Income and Wealth* **41**, 161–176.

Lawrance, Emily (1991). "Poverty and the Rate of Time Preference: Evidence from Panel Data," *Journal of Political Economy* **99**, 54–77.

Leung, Siu Fai (1994). "Uncertain Lifetime, the Theory of the Consumer, and the Life Cycle Hypothesis," *Econometrica* **62**, 1233–1239.

Leung, Siu Fai (1995). "A Distinction between Continuous-time and Discrete-time Models of Uncertain Lifetime," *Economics Letters* **47**, 291–296.

Leung, Siu Fai (1997). "Life-cycle Saving under Uncertain Lifetime: Characterization, Sensitivity Analysis, and Application to Social Security," working paper.

Masson, Andre (1988). "Permanent Income, Age and the Distribution of Wealth," *Annales d'Economie et de Statistique* **9**, 227–256.

Mariger, Randall (1987). "A Life-cycle Consumption Model with Liquidity Constraints: Theory and Empirical Results," *Econometrica* **55**, 533–557.

Menchik, Paul (1993). "Economic Status as a Determinant of Mortality Among Black and White Older Males—Does Poverty Kill?" *Population Studies* **47**, 427–436.

Mirer, Thad (1992). "Life-cycle Valuation of Social Security and Pension Wealth," *Journal of Public Economics* **48**, 377–384.

Mirer, Thad (1994). "The Dissaving of Annuity Wealth and Marketable Wealth in Retirement," *Review of Income and Wealth* **40**, 87–97.

Mitchell, Olivia, and Moore, James (1997). "Retirement Wealth Accumulation and Decumulation: New Developments and Outstanding Opportunities," NBER working paper 6178.

Modigliani, Franco (1986). "Life Cycle, Individual Thrift, and the Wealth of Nations," *American Economic Review* **76**, 297–313.

Poterba, James, Venti, Steven, and Wise, David (1994). "Targeted Retirement Saving and the Net Worth of Elderly Americans," *American Economic Review Papers and Proceedings* **84**, 180–185.

Rae, John (1834). *The Sociological Theory of Capital* (reprint, *The New Principles of Political Economy*, 1834), New York: Macmillan, 1905.

Sheiner, Louise, and Weil, David (1992). "The Housing Wealth of the Aged," NBER working paper 4115.

Thaler, Richard (1994). "Psychology and Savings Policies," *American Economic Review Papers and Proceedings* **84**, 186–192.

Yaari, Menahem (1965). "Uncertain Lifetime, Life Insurance, and the Theory of the Consumer," *Review of Economic Studies* **32**, 137–150.