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# COMPUTING MODELS OF SOCIAL SECURITY

Ayşe İmrohoroğlu, Selahattin İmrohoroğlu, and Douglas H. Joines[1]

## 10.1 Introduction

In the United States and most other developed countries, the public pension system and associated benefit payments to the retired and their families (including disability, medical, and survivor benefits) constitute the largest item in the government budget. Partly because of their scale, these payments have during the last quarter century become the object of intense study by economists.

Most of the issues concerning the effect of unfunded social security programmes on the economy have been analysed qualitatively using standard models such as the two- or three-period overlapping generations model, and some of the empirical predictions have been tested. More recently, some of these questions as well as other issues in fiscal policy have been analysed quantitatively using larger overlapping generations models. The starting point for this literature is Auerbach and Kotlikoff (1987) and a series of papers that preceded that book. Auerbach and Kotlikoff use a non-stochastic, 55-period overlapping generations model to analyse the effects of unfunded social security on both labour supply and the capital stock. Subsequent work modifies the Auerbach–Kotlikoff model by adding borrowing constraints, various sources of uncertainty, and other features. In particular, incorporating two sources of uncertainty into a model of social security seems to be important. First, an uncertain lifespan is essential for many interesting questions concerning social security, which provides partial insurance against this risk in the absence of private annuity markets. Second, introducing earnings uncertainty is desirable for at least two reasons: earnings uncertainty interacts with borrowing constraints and yields within-cohort heterogeneity which can address questions about the distribution of consumption and wealth; and an unfunded social security system with little or no linkage between benefits and contributions provides some insurance for earnings uncertainty.

## 10.2 A model of social security with heterogeneous agents

This section describes the İmrohoroğlu *et al.* (1995) set-up, which is related to several recent large-scale general equilibrium, overlapping generations models.[2]

[1] The authors' correspondence address is Department of Finance and Business Economics, Marshall School of Business, University of Southern California, Los Angeles, CA 90089-1421, USA.

[2] Among others, important quantitative work using overlapping generations models includes Hubbard and Judd (1987), Ríos-Rull (1996), Huggett and Ventura (1998), Cooley and Soares (1996; 1998), İmrohoroğlu *et al.* (1998a), Rust and Phelan (1997), Storesletten *et al.* (1997), İmrohoroğlu (1998) and Conesa and Krueger (1998).

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## 10.2 A model of social security with heterogeneous agents

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### 10.2.1 *Demographics*

The economy is populated by overlapping generations of long but finite-lived individuals with total measure one. Individuals face random survival from age $j-1$ to $j$, as represented by the conditional survival probabilities $\psi_j \in (0, 1)$. Some consumers may survive through the maximum possible lifespan, $J$. Each period the number of newborns grows relative to the last cohort by a constant proportion $n$. To obtain a stationary population, cohort shares $\{\mu_j\}_{j=1}^{J}$ are calculated by $\mu_j = \psi_j \mu_{j-1}/(1+n)$, $\sum_{j=1}^{J} \mu_j = 1$.[3] Aggregate quantities in the economy are weighted averages of individual quantities where individual measures as well as the cohort measures serve as weights.

### 10.2.2 *Budget constraints*

Each period individuals who are below a mandatory retirement age $j_R$ face a stochastic employment opportunity. Let $s \in S = \{e, u\}$ denote the employment opportunities state and assume that it follows a first-order Markov process. The transition function for the individual earnings state is given by the $2 \times 2$ matrix $\Pi(s', s) = [\pi_{ij}]$, $i, j = e, u$, where $\pi_{ij} = \text{Prob}\{s_{t+1} = j \mid s_t = i\}$. If $s = e$, the individual is employed and earns $w\varepsilon_j$ where $w$ is the wage rate per efficiency unit of labour, the labour supply is unity, and $\varepsilon_j$ is an age-indexed efficiency of labour. If $s = u$, the agent is unemployed and receives unemployment insurance benefits in the amount $\phi w \varepsilon_j$, where $\phi$ is the replacement ratio. During retirement the individual receives a pension $b$ and decumulates assets. The social security benefits are calculated to be a fraction, $\theta$, of some base income, taken to be the average lifetime employed income. That is

$$b_j = \begin{cases} 0 & j = 1, 2, \ldots, j_R - 1 \\ \theta \dfrac{\sum_{i=1}^{j_R-1} w\varepsilon_i}{j_R - 1} & j = j_R, j_R + 1, \ldots, J \end{cases} \tag{10.1}$$

Note that an agent's social security benefit is independent of the agent's employment history. The after-tax income of an individual is given by

$$q_j = \begin{cases} (1 - \tau_s - \tau_u) w\varepsilon_j & j \in [1, j_R), s = e \\ \phi w \varepsilon_j & j \in [1, j_R), s = u \\ b & j \in [j_R, J] \end{cases} \tag{10.2}$$

where $\tau_s$ and $\tau_u$ are social security and unemployment insurance payroll tax rates, respectively.

The infinitely-lived government administers the unemployment insurance and social security schemes. Given unemployment insurance and social security benefits, the government chooses the unemployment insurance and the social security tax rates so that each of these schemes is self-financing.

In this economy, there are no private markets for insurance against the risk of unemployment or living longer than expected. Unfunded social security provides partial

[3] The cohort shares are assumed to be time-invariant in order to restrict the computations to steady states. In this class of general equilibrium, heterogeneous-agent, large-scale overlapping generations models, computing transitions is not a simple task.



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insurance against the latter risk, but the former can only be partially insured against by private saving. We assume that agents may not have negative assets at any age. Hence, the restriction on the amount of assets carried over from age $j$ to $j+1$, $a_j$, is that

$$a_j \geq 0 \tag{10.3}$$

Since there is no altruistic bequest motive and death is certain after age $J$, individuals who survive to age $J$ liquidate all their assets at that age so that $a_J = 0$. However, uncertain survival until age $J$ means that there are accidental bequests.

Consumption and asset accumulation at age $j$, denoted by $c_j$ and $a_j - a_{j-1}$, respectively, follow

$$c_j + a_j = (1+r)a_{j-1} + q_j + \xi \tag{10.4}$$

where $r$ is the return on physical capital net of depreciation and $\xi$ is a lumpsum transfer of accidental bequests.[4]

### 10.2.3 *Preferences*

Each individual maximizes the expected, discounted lifetime utility

$$E_0 \sum_{j=1}^{J} \beta^{j-1} \left[ \prod_{k=1}^{j} \psi_k \right] u(c_j) \tag{10.5}$$

where $\beta$ is the subjective discount factor. The period utility function is assumed to take the form

$$u(c_j) = \frac{c_j^{1-\gamma}}{1-\gamma} \tag{10.6}$$

where $\gamma$ is the coefficient of relative risk aversion.

### 10.2.4 *Technology*

The production technology of the economy is given by a constant returns to scale Cobb–Douglas function

$$Q = BK^{1-\alpha}N^{\alpha} \tag{10.7}$$

where $B > 0$, $\alpha \in (0, 1)$ is labour's share of output, and $K$ and $N$ are aggregate capital and labour inputs, respectively. The aggregate capital stock is assumed to depreciate at the rate $\delta$.

The profit-maximizing behaviour of the firm gives rise to first-order conditions which determine the net real return to capital and the real wage

$$r = (1-\alpha)B\left[\frac{K}{N}\right]^{-\alpha} - \delta \tag{10.8}$$

$$w = \alpha B\left[\frac{K}{N}\right]^{1-\alpha}$$

[4]The particular assumption for the redistribution of accidental bequests may have an impact on the quantitative results. See İmrohoroğlu *et al.* (1995) and İmrohoroğlu (1998).

### 10.2.5 *Decision rules*

Let $D = \{d_1, d_2, \ldots, d_m\}$ denote the discrete grid of points on which asset holdings will be required to fall. For any beginning-of-period asset holding and employment state $(a, s) \in D \times S$, define the constraint set of an age-$j$ agent $\Omega_j(a, s) \in R^2_+$ as all pairs $(c_j, a_j)$ such that equations (10.3) and (10.4) hold. Let $V_j(a, s)$ be the (maximized) value of the objective function of an age-$j$ agent with beginning-of-period asset holdings and employment state $(a, s)$. $V_j(a, s)$ is given as the solution to the dynamic program

$$V_j(a,s) = \max_{(c,a')\in\Omega_j(a,s)} \left\{u(c) + \beta\psi_{j+1}E_{s'}V_{j+1}(a',s')\right\}, \qquad j = 1, 2, \ldots, J \tag{10.9}$$

where a prime on a variable indicates its value for the next age and the notation $E_{s'}$ means that the expectation is over the distribution of $s'$.

The optimization problem faced by an individual in this economy is one of finite-state, finite-horizon dynamic programming.[5] The value functions and the decision rules for each age $j = 1, 2, \ldots, J$ can be found by a single recursion working backwards from the last period of life. Using the budget constraint (10.4) to substitute for $c_j$ in Bellman's equation (10.9), the problem reduces to choosing the decision variable $a_j$. We assume that $a_j \in D \equiv \{d_1, d_2, \ldots, d_m\}$. For individuals at age $j_R$ or older, namely the retired, the state space is an $m \times 1$ vector $X = \{x = a\colon\ a \in D\}$. For individuals who are subject to idiosyncratic employment risk, at age $j_R - 1$ or younger, the state space is an $m \times 2$ matrix $\tilde{X} = \{\tilde{x} = (a, s)\colon\ a \in D, s \in S\}$. The control space for individuals of all ages is the $m \times 1$ vector $D$. For $j = j_R, j_R + 1, \ldots, J$, the decision rules take the form of an $m \times 1$ vector of asset holdings that solves the above problem. For $j = 1, 2, \ldots, j_R - 1$, the decision rules are $m \times 2$ matrices, one such matrix for each $j$, showing the utility maximizing asset holding for each level of beginning-of-period assets and employment state realization.

Since death is certain beyond age $J$ ($\psi_{J+1} = 0$) the value function at $J + 1$ is identically zero. Hence, the solution to

$$V_J(x_J) = \max_{\{c_J, a_J\}} u(c_J)$$

subject to

$$c_J = (1 + r)a_{J-1} + q_J + \xi$$

is an $m \times 1$ vector decision rule for age-$J$ individuals, $A_J$. Note that this is a vector of zeros since there is no bequest motive and death is certain after $J$. The value function at age $J$, $V_J$, is an $m \times 1$ vector whose entries correspond to the value of the utility function at $(1 + r)a_{J-1} + b + \xi$ with $a_{J-1}$ taking on the values $d_1, d_2, \ldots, d_m$. This value function $V_J$ is passed on to the next step where the age-$(J - 1)$ decision rule and value function are calculated. The age-$(J - 1)$ decision rule is found by obtaining

$$V_{J-1}(x_{J-1}) = \max_{\{c_{J-1}, a_{J-1}\}} \{u(c_{J-1}) + \beta\psi_J V_J(x_J)\}$$

[5] See Sargent (1987) and Stokey *et al.* (1989) for a description of dynamic programming as a tool for solving a large class of general equilibrium models.

subject to

$$c_{J-1} + a_{J-1} = (1+r)a_{J-2} + b + \xi, \quad c_{J-1} \geq 0, \;\; a_{J-1} \geq 0$$

The decision rule is found as follows.[6] For $a_{J-2} = d_1$, the value of $a_{J-1} \in D$ that solves the above problem is obtained by evaluating the objective function at each point on the grid $D$. This value is reported as the first element of the $m \times 1$ decision rule $A_{J-1}$. By repeating this procedure for all possible initial asset levels $a_{J-2} \in D$ the entire vector $A_{J-1}$ is filled. Simultaneously, the age-$(J-1)$ value function $V_{J-1}$ is found as an $m \times 1$ vector with entries corresponding to the right-hand side of the above objective function evaluated at the decision rule $A_{J-1}$.

Working the backward recursion, we come to age $j_R - 1$, the age immediately before the mandatory retirement age of $j_R$. The problem to solve is

$$V_{j_R-1}(\tilde{x}_{j_R-1}) = \max_{\{c_{j_R-1}, a_{j_R-1}\}} \{u(c_{j_R-1}) + \beta\psi_{j_R} V_{j_R}(\tilde{x}_{j_R})\}$$

subject to

$$c_{j_R-1} + a_{j_R-1} = (1+r)a_{j_R-2} + q_{j_R-1} + \xi, \quad c_{j_R-1} \geq 0, \;\; a_{j_R-1} \geq 0$$

When the individual is at age $j_R - 1$ or younger, disposable income is no longer independent of the idiosyncratic employment risk. In fact, for $j = 1, 2, \ldots, j_R - 1$, disposable income can take one of two values, $(1 - \tau_r - \tau_u)w\varepsilon_j$ or $\phi w \varepsilon_j$, depending on the realization of $s$. The decision rule for age $j_R - 1$ (and also for younger individuals) is an $m \times 2$ matrix describing the utility maximizing levels of asset holdings for each point in the state space $\tilde{X} = D \times S$. Consequently, the value function $V_{j_R-1}$ is also an $m \times 2$ matrix.

For $j = 1, 2, \ldots, j_R - 2$, the optimality equation is given by

$$V_j(\tilde{x}_j) = \max_{\{c_j, a_j\}} \left\{ u(c_j) + \beta\psi_{j+1} \sum_{s'} \Pi(s', s) V_{j+1}(\tilde{x}_{j+1}) \right\}$$

subject to

$$c_j + a_j = (1+r)a_{j-1} + q_j + \xi, \quad c_j \geq 0, \;\; a_j \geq 0$$

For $a_{j-1} = d_1$ and $s = e$, we search over $a_j \in D$ that solves the above problem and report that value as the $1 \times 1$ element of the $m \times 2$ decision rule $A_j$. Then we search over $a_j \in D$ for given $a_{j-1} = d_1$ and $s = u$, and report the optimal value as the $1 \times 2$ element of the decision rule for age $j$. This process is repeated until all elements of the decision rule $A_j$ are computed. This completes the computation of the decision rules $A_j$ and value functions $V_j$ for all ages; two $(j_R - 1)$ matrices each $m \times 2$ and two $(J - j_R + 1)$ vectors each $m \times 1$.

[6] Because of the concavity of the value function, it is not necessary to evaluate the second term on the right-hand side of equation (10.9) at every grid point. One useful approach is to start with a coarse grid over the entire decision space and then use successively finer grids in the neighbourhood of the optimum. An alternative approach is to compute the value function using a coarse grid on the state space and use linear interpolations to evaluate the value function for in-between grid points.

### 10.2.6 *Age-dependent distributions of agents*

To obtain the distribution of age-$j$ agents, $\lambda_j(a, s)$, into beginning-of-period asset holding levels and employment categories, we start from a given initial wealth distribution $\lambda_1$. We assume that newborns have zero asset holdings, so $\lambda_1$ is taken to be an $m \times 2$ matrix with zeros everywhere except the first row, which is equal to $(u_1, u_2)$, the expected employment and unemployment rates, respectively. The distribution of agents at the end of age 1, or equivalently, at the beginning of age 2, is found by

$$\lambda_2(a', s') = \sum_{s} \sum_{a:a' \in A_1(a,s)} \Pi(s', s)\lambda_1(a, s)$$

Starting from the initial wealth distribution $\lambda_1$, some individuals will be employed and some of them will be unemployed at age 1. Depending on the realization of the employment status, individuals will make asset holding decisions which are already calculated. Therefore, at the beginning of age 2, they will go to (possibly) different points in the state-space matrix $(a, s)$. Each entry in the $m \times 2$ matrix $\lambda_2$ gives the fraction of 2-year-old agents at that particular combination of asset holdings (chosen at the end of the age-1 optimization problem) and period-2 employment status. Note that, for each $j$, each element of $\lambda_j$ is non-negative, and the sum of all entries equals 1.

In general, given $J$ decision rules $A_j$ and an initial wealth distribution $\lambda_1$, the age-dependent distributions are computed from the forward recursion

$$\lambda_j(a', s') = \sum_{s} \sum_{a:a' \in A_j(a,s)} \Pi(s', s)\lambda_{j-1}(a, s) \tag{10.10}$$

Note that for $j = j_R, j_R + 1, \ldots, J$, $\lambda_j$ is $m \times 1$ since the retired individuals are not subject to idiosyncratic employment risk.

Using these age-dependent distributions we can compute age profiles for consumption, assets, and income. We also compute aggregate values for these variables.

Alternatively, one could simulate the histories of a large number of agents using Monte Carlo methods and calculate the summary statistics from these simulations. This approach starts with an initial distribution of asset holdings and randomly draws the survival probabilities and the realization of the employment state for a single agent. Given these realizations, and the optimal decision rules, the next period's asset holdings are computed, which become the following period's state variables. This procedure is recursively followed forward until the agent dies, which is no later than age $J$. This procedure is repeated for a large number of agents, and averages are computed, until convergence of the calibrated cohort shares and the unemployment rate.[7]

### 10.2.7 *Stationary equilibrium*

A *stationary equilibrium* for a given set of policy arrangements $\{\theta, \phi, \tau_s, \tau_u\}$ is a collection of value functions $V_j(a, s)$, individual policy rules $A_j : D \times S \to R_+$, $A_j : D \times S \to D$, age-dependent (but time-invariant) measures of agent types $\lambda_j(a, s)$

[7] İmrohoroğlu *et al.* (1998b) replicate 220 000 agent histories to match the cohort shares to within 0.00001.



for each age $j = 1, 2, \ldots, J$, re
transfer $\xi$ such that the followi

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2. Relative prices $\{w, r\}$ solv equation (10.8).
3. Given relative prices $\{w, r\}$, fer $\xi$, the individual policy program (10.9).
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for each age $j = 1, 2, \ldots, J$, relative prices of labour and capital $\{w, r\}$, and a lump-sum transfer $\xi$ such that the following hold:

1. Individual and aggregate behaviour are consistent:

$$K = \sum_j \sum_a \sum_s \mu_j \lambda_j(a, s) a_{j-1} \quad \text{and} \quad N = \sum_{j=1}^{j_R - 1} \sum_a \mu_j \lambda_j(a, s = e)\varepsilon_j \quad (10.11)$$

2. Relative prices $\{w, r\}$ solve the firm's profit maximization problem by satisfying equation (10.8).
3. Given relative prices $\{w, r\}$, government policy $\{\theta, \phi, \tau_s, \tau_u\}$, and a lump-sum transfer $\xi$, the individual policy rules $C_j(a, s)$, $A_j(a, s)$ solve the individuals' dynamic program (10.9).
4. The commodity market clears:

$$\sum_j \sum_a \sum_s \mu_j \lambda_j(a, s)\{C_j(a, s) + [A_j(a, s) - (1 - \delta)A_{j-1}(a, s)]\} = Q \quad (10.12)$$

   where the initial wealth distribution of agents, $A_0$, is taken as given.
5. The collection of age-dependent, time-invariant measures $\lambda_j(a, s)$ for $j = 2, 3, \ldots, J$, satisfies

$$\lambda_j(a', s') = \sum_s \sum_{a:a' = A_j(a,s)} \Pi(s', s)\lambda_{j-1}(a, s)$$

   where the initial measure of agents at birth, $\lambda_1$, is taken as given.
6. The social security system is self-financing:

$$\tau_s = \frac{\sum_{j=j_R}^{J} \sum_a \mu_j \lambda_j(a, s) b}{\sum_{j=1}^{j_R - 1} \sum_a \mu_j \lambda_j(a, s = e) w \varepsilon_j}$$

7. The unemployment insurance benefits scheme is self-financing:

$$\tau_u = \frac{\sum_{j=1}^{j_R - 1} \sum_a \mu_j \lambda_j(a, s = u) \phi w \varepsilon_j}{\sum_{j=1}^{j_R - 1} \sum_a \mu_j \lambda_j(a, s = e) w \varepsilon_j} = \phi \frac{u_2}{u_1}$$

8. The lump-sum distribution of accidental bequests is determined by

$$\xi = \sum_j \sum_a \sum_s \mu_j \lambda_j(a, s)(1 - \psi_{j+1})A_j(a, s)$$

### 10.2.8 *Measures of utility and welfare benefits*

In order to compare alternative social security arrangements, we need a measure of "average steady-state utility". Given a policy arrangement $\Gamma = \{\theta, \phi, \tau_s, \tau_u\}$, we calculate

$$W(\Gamma) = \sum_{j=1}^{J} \sum_{y} \sum_{s} \beta^{j-1} \left\{ \prod_{k=1}^{j} \psi_k \lambda_j(a, s) u(C_j(a, s)) \right\} \tag{10.13}$$

as our measure of utility. $W(\Gamma)$ is the expected discounted utility a newborn individual derives from the consumption policy functions $\{C_j(a, s)\}$ under a given social security arrangement.

Second, we need a measure to quantify the welfare benefits (or costs) of alternative social security arrangements. As our reference economy, we take the benchmark equilibrium under a zero social security replacement rate. Our measure of welfare benefit (or cost) is calculated as the consumption supplement in each period of life required to make a newborn individual indifferent between being born into an economy with a given social security replacement rate and an economy with no social security. Let $W_0 = W(\Gamma_0)$ and $W_1 = W(\Gamma_1)$ denote the utility under policy arrangements $\Gamma_0 = \{\theta_0 = 0, \phi, \tau_{s0} = 0, \tau_u\}$ and $\Gamma_1 = \{\theta_1 > 0, \phi, \tau_{s1} > 0, \tau_u\}$, respectively. Our measure of welfare benefits is $\kappa = \ell / Q_0$ where $\ell$ is a lump-sum compensation required to make a newborn indifferent between policy arrangements $\Gamma_0$ with compensation $\ell$ in each period of life, and an alternative policy arrangement $\Gamma_1$ without compensation, and $Q_0$ is real gross national product under arrangement $\Gamma_0$.

Note that steady-state equilibria calculated in this class of models do not, in general, result in allocations that are Pareto optimal for a variety of reasons such as the presence of liquidity constraints and dynamic inefficiency associated with overlapping generations models. In order to quantify the extent to which these equilibria suffer from these problems, it might be desirable to characterize the following first-best solution. Consider the problem faced by a social planner whose task is to allocate the economy's output among investments in physical capital and consumption of the 65 generations alive in any period. The planner is restricted to choose among steady states, and the objective is to maximize the expected lifetime utility of an individual born into the chosen steady state. In a steady state, investment is equal to $(\delta + n)K$. The planner's problem is thus to choose a capital stock $K$ and a consumption profile $\{c_j\}_{j=1}^{J}$ to maximize the objective function (10.13) subject to the constraint

$$f(K, N) = (\delta + n)K + \sum_{j=1}^{J} \mu_j c_j$$

The first-order condition associated with $K$ is that the marginal product of capital equals $\delta + n$. This condition requires that the planner choose the golden rule capital stock, thus maximizing aggregate consumption. The remaining optimality conditions concern the allocation of aggregate consumption among the $J$ living generations, or alternatively (because the planner is restricted to choose among steady states), over the $J$ periods

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### 10.2.9 *Calibration*

In order to obtain numerical s
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### 10.2.10 *Computing a statio*

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dy states), over the $J$ periods



of an individual's life. Given the form of the utility function in equation (10.5), these conditions give rise to expressions of the form

$$\left(\frac{c_{j+1}}{c_j}\right)^{-\gamma} = \beta(1+n)$$

Note that the general shape of the consumption profile implied by these expressions does not depend on the level of aggregate consumption. If individuals were not subject to liquidity constraints, they would allocate consumption over the life cycle according to

$$E_0\left(\frac{c_{j+1}}{c_j}\right)^{-\gamma} = \beta(1+r)\psi_{j+1}$$

The consumption path implied by this condition differs from that chosen by the social planner for two reasons. First, the planner pools the mortality risks represented by $\psi_j$s, whereas individuals in our model are unable to do so due to the absence of annuity markets. As a result, the age–consumption profile chosen by individuals tends to be less steep than that chosen by the planner. Second, the planner's optimality conditions involve the population growth rate (which equals the economy's growth rate in the absence of productivity growth), whereas the individual's involve the market interest rate. These rates will differ unless the economy is at the golden rule capital stock. In addition, an individual subject to binding liquidity constraints would not allocate consumption according to the above Euler equations, possibly causing a further divergence between the individual's consumption profile and that chosen by the planner. Social security can affect welfare by altering the steady-state capital stock, and thus aggregate consumption, and by influencing the shape of the age–consumption profile.

### 10.2.9 *Calibration*

In order to obtain numerical solutions to the model, it is necessary to choose particular values for the parameters. The general strategy is to choose parameter values so that the model economy reproduces certain long-run empirical characteristics of the economy being studied. This entails matching model quantities with empirical counterparts that should be constant along a balanced growth path. Examples of such quantities are the growth rates of population and total factor productivity and ratios such as the capital–output and investment–capital ratios. Although these empirical quantities are not literally constant, they generally appear to be mean stationary time series, and the means of these time series generally can be estimated fairly precisely. Cooley and Prescott (1995) provide a general discussion of this strategy for choosing model parameters. The specifics of calibration differ from model to model, and the reader is referred to individual papers for details. See, for example, İmrohoroğlu *et al.* (1998a; 1998c) for a detailed discussion of the calibration of a particular overlapping generations model as it is applied to social security.

### 10.2.10 *Computing a stationary equilibrium*

Let $\epsilon_1$ and $\epsilon_2$ denote the convergence criteria for the aggregate capital stock and unintended bequests, respectively. These criteria are usually obtained through experimentation. A smaller $\epsilon$ increases the number of iterations whereas a larger $\epsilon$ may change the results significantly. Also choose the step sizes $\hat{\alpha}_1$ and $\hat{\alpha}_2$ governing the adjustment to capital and bequests between iterations. Computing an equilibrium requires finding a fixed point in the capital stock, $K$, and the transfer of unintended bequests, $\xi$, and consists of the following steps:

1. Guess $K_0$ and $\xi_0$. Compute the aggregate labour input $N = u_1 \sum_{j=1}^{j_R-1} \mu_j \varepsilon_j$. Use the first-order conditions from the firm's profit maximization problem to obtain the implied values for the relative factor prices $w$ and $r$, and substitute these in the individual's budget constraint.
2. Compute the decision rules for each cohort by completing a backward recursion, and the distribution of agent types for each cohort by completing a forward recursion.
3. Compute the new aggregate capital stock $K_1 = \sum_j \sum_a \sum_s \mu_j \lambda_j(a, s) A_j(a, s)$ and the new lump-sum transfer $\xi_1 = \sum_j \sum_a \sum_s \mu_j \lambda_j(a, s)(1 - \psi_{j+1}) A_j(a, s)$, and check if $\frac{|K_1 - K_0|}{K_0} < \epsilon_1$ and $\frac{|\xi_1 - \xi_0|}{\xi_0} < \epsilon_2$. If not, compute $K_2 = \hat{\alpha}_1 K_0 + (1 - \hat{\alpha}_1) K_1$ and $\xi_2 = \hat{\alpha}_2 \xi_0 + (1 - \hat{\alpha}_2)\xi_1$. Set $K_0 = K_2$ and $\xi_0 = \xi_2$ and go to step 1. For each of the $j_R - 1$ working ages, computing the decision rules involves $d_m \times d_m \times 2$ function evaluations, and for each of $J - j_R + 1$ retired ages, obtaining decision rules requires $d_m \times d_m$ function evaluations.[8]
4. Compute aggregate consumption, investment, and output using the decision rules, distribution of agent types, and the population shares of cohorts, and check whether the commodity market clearing condition given by equation (10.12) is approximately satisfied.[9] If the problem is correctly specified and the code is accurate, *excess demand* is typically less than 0.01% of output when the capital stock converges. If *excess demand* is sufficiently small when the aggregate capital stock converges, then stop. If not, check the code for accuracy or the economic model for internal consistency and start again.

## 10.3 A linear quadratic model of social security

De Nardi *et al.* (1998) demonstrate how a demographic transition can be incorporated in a general equilibrium model with long-lived overlapping generations of individuals facing several sources of uncertainty. The emphasis is on the computation of an equilibrium transition path between steady states which is induced by a demographic transition and the government's fiscal response to it.

[8] The number of grid points varies from one paper to the next. For example, İmrohoroğlu *et al.* (1995) use 601 grid points, whereas İmrohoroğlu *et al.* (1998c) use 4097 grid points. In all cases, the computer code is written in FORTRAN. In the model with 4097 grid points, each iteration takes about 90 seconds on a 200 MHz Pentium Pro. Finding an equilibrium generally requires between five and eight iterations and rarely takes more than ten iterations.

[9] Note that this is merely a check on the internal consistency of the model and the accuracy of the code that performs the computations. When the model is well specified and the decision rules and the distribution of the agent types and the aggregate variables are calculated correctly, the market clearing condition should hold in equilibrium since it is a weighted average of the individuals' budget constraints.



10.3.1 *Demographics*

For any variable $z$, the subscri
calendar time. For example, $N_t$
Time is discrete and indexe
$N_0(s)$ arrives. These are "age C
live through $s, s+1, s+2, \dots$
conditional probability of survi
$t$ people alive at time $s$ moves a

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Iterating on (10.14) gives $N_t(s$
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De Nardi *et al.* assume that at
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$f_t$

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Social Security Administratior
the model at "age" 0 ($t = 0$)
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During the first $t_R + 1$ peri
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10.3.2 *Technology*

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$(a,s)(1-\psi_{j+1}) A_j(a,s)$, and
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btaining decision rules requires

put using the decision rules, dis-
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the individuals' budget constraints.



### 10.3.1 *Demographics*

For any variable $z$, the subscript $t$ denotes age and the index $s$ in parentheses denotes calendar time. For example, $N_t(s)$ denotes the number of age-$t$ people at time $s$.

Time is discrete and indexed by $s$. At each date $s$, a cohort of individuals of measure $N_0(s)$ arrives. These are "age 0" individuals who face random survival. The lucky ones live through $s, s+1, s+2, \ldots, s+T$, for a total of $T+1$ years. Let $\alpha_t(s)$ denote the conditional probability of surviving from age $t$ to age $t+1$ at time $s$. The number age $t$ people alive at time $s$ moves according to

$$N_{t+1}(s+1) = \alpha_t(s) N_t(s) \tag{10.14}$$

Iterating on (10.14) gives $N_t(s) = \alpha_{t-1}(s-1)\,\alpha_{t-2}(s-2)\cdots\alpha_0(s-t)\,N_0(s-t)$. We compute the probability that a person born at $s-t$ survives to age $t$ as

$$\lambda_t(s) \equiv \prod_{h=1}^{t} \alpha_{t-h}(s-h) \tag{10.15}$$

De Nardi *et al.* assume that at time $s$, the number of new individuals grows at the rate $n(s)-1$, so that $N_0(s) = n(s)N_0(s-1)$, which implies $N_0(s) = \prod_{h=1}^{s} n(h) N_0(0)$. Let $\nu(s) = \prod_{h=1}^{s} n(h)$. Then the fraction $f_t(s)$ of age-$t$ people at time $s$ is given by

$$f_t(s) = \frac{\lambda_t(s)\nu(s)}{\sum_{i=0}^{T} \lambda_i(s)\nu(s-i)} \tag{10.16}$$

which will be used as cohort weights to compute aggregate quantities. The entire population at time $s$ is given by $N(s) = \sum_{t=0}^{T} N_t(s)$. The paths $n(s)$ and $\alpha_t(s)$ for $s = 1970, \ldots, 2060 + 3T$ are taken as given and calibrated using the projections of the Social Security Administration for the United States. Note that the people who enter the model at "age" 0 ($t = 0$) are 21 years old. The mandatory retirement age is 65 ($t = t_R + 2$) and old agents may live up to 90 years old ($t = T$).

During the first $t_R + 1$ periods of life, a consumer supplies labour in exchange for wages that he allocates among consumption, taxes, and asset accumulation. During the final $T - t_R$ periods of life, the consumer receives social security benefits. In addition to lifespan risk, agents face different income shocks that they cannot insure. They can smooth consumption by accumulating two risk-free assets: physical capital and government bonds. The government taxes consumption and income from capital and labour, issues and services debt, purchases goods, and pays retirement benefits. There is a constant returns-to-scale Cobb–Douglas aggregate production function and no aggregate uncertainty. As a consequence, factor prices will be time-varying but deterministic.

### 10.3.2 *Technology*

The aggregate technology is described by a constant returns to scale Cobb–Douglas production function. Prices of capital and labour at time $s$, denoted by $r(s-1)$ and $w(s)$, respectively, are determined from the firm's profit maximization problem in a competitive equilibrium:

$$r(s-1) = \tilde{\alpha} A \left[\frac{K(s-1)}{L(s)}\right]^{\tilde{\alpha}-1} \tag{10.17}$$

$$w(s) = (1-\tilde{\alpha}) A \left[\frac{K(s-1)}{L(s)}\right]^{\tilde{\alpha}} \tag{10.18}$$

where $L(s) = \sum_{t=0}^{t_R} \epsilon_t\, \ell_t(s)\, N_t(s)$ is the aggregate labour input in efficiency units, $\epsilon_t$ is a time-invariant and exogenous age-efficiency index, and $\ell_t(s)$ is the labour supply of an agent of age $t$ at time $s$. The aggregate capital input is given by $K(s-1) = \sum_{t=0}^{t_R} k_t(s-1)\, N_t(s)$, where $k_t(s)$ is the physical capital holdings of an agent of age $t$ at time $s$, $\tilde{\alpha} \in (0,1)$ is the income share of capital, and $A$ is total factor productivity.

### 10.3.3 *Government*

An age-$t$ person divides his time-$s$ asset holdings $a_t(s)$ between government bonds and private capital: $a_t(s) = b_t(s) + k_t(s)$, where $b_t(s)$ is government debt.[10] The government's budget constraint at $s$ is:

$$g(s)N(s) + \sum_{t=t_R+1}^{T} S_t(s)N_t(s) + R(s-1)\sum_{t=0}^{T} b_t(s-1)N_t(s)$$

$$= \tau_b R(s-1)Beq(s) + \sum_{t=0}^{T} b_t(s)N_t(s) + \sum_{t=0}^{T} N_t(s)\{\tau_a(s)[R(s-1)-1]a_{t-1}(s-1)$$

$$+ \tau_\ell(s)w(s)\epsilon_t\ell_t(s) + \tau_c(s)c_t(s)\} \tag{10.19}$$

where

$$Beq(s) = \sum_{t=0}^{T} [1-\alpha_t(s)]\, a_t(s-1)N_t(s-1) \tag{10.20}$$

and

$$a_{-1}(s-1) = \frac{Beq(s)(1-\tau_b)}{N_0(s)} \tag{10.21}$$

In equation (10.19), $g(s)$ is the amount of government purchases at time $s$, $S_t(s)$ is the social security benefits received by an age-$t$ individual at time $s$, $R(s-1) = 1 + r(s-1) - \delta$ is the rate of return on asset holdings net of depreciation, $\tau_b$ is the tax on inheritances, $\tau_a(s)$, $\tau_\ell(s)$ and $\tau_c(s)$ are taxes on asset income, labour income, and consumption, respectively. The amount of assets inherited at time $s$ by each new worker is denoted by

[10]We assume that these two assets pay the same return, which implies that individual portfolios are indeterminate. We can compute the aggregate holdings of each asset since the economy's resource constraint yields the amount of aggregate physical capital, and government bond holdings are then computed as a residual after the total asset holdings are computed.

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bonds in the same proportions

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$b_{-1}(s -$

In the benefit formula, $fix$

where $AV$ records the average
For people living during the tr
tions in the initial and final ste

### 10.3.4 *Household's problem*

#### 10.3.4.1 *Budget constraints*

$c_t(s) + a_t(s) = R(s-1)a_{t-}$

$\Upsilon_t = \tau_0(s) + \tau_\ell($

$+ \tau_a(s)[R($

$e_t(s) = e_{t-1}(s-1)$

$S_t(s) = \begin{cases} 0 \\ fixben_t( \end{cases}$

$z_{t+1} = A_{22}z_t + C_2$

$\begin{bmatrix} d_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} U_{d,t} \\ U_\gamma \end{bmatrix} z_t$

In equation (10.23a), $\tau_0(s)$ is
individual. The benefit formul
or benefits that are related to
evolution of the information v
$U_{d,t}$ and $U_\gamma$ are selector vecto
point $\gamma_t$. De Nardi *et al.* set the
process with mean zero: $d_t =$
is adapted to $J_t = (\omega_0^t, x_0)$,
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[11]We distribute bequests as follov
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distribution scheme implies that wi
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it leaves behind.

$a_{-1}(s-1)$, which is assumed to be divided between physical capital and government bonds in the same proportions that these are held in the aggregate portfolio:

$$k_{-1}(s-1) = \frac{\sum_{t=0}^{T}[1-\alpha_t(s)]\,k_t(s)N_t(s)}{N_0(s)} \qquad (10.22)$$
$$b_{-1}(s-1) = \frac{\sum_{t=0}^{T}[1-\alpha_t(s)]\,b_t(s)N_t(s)}{N_0(s)}$$

In the benefit formula, $fixben$, for people living in a steady state, is given by

$$fixben = fixrate \cdot AV$$

where $AV$ records the average earnings of a worker who has survived to retirement age. For people living during the transition, $fixben$ is a linear combination of the contributions in the initial and final steady states.[11]

### 10.3.4 *Household's problem*

10.3.4.1 *Budget constraints* Individuals face the following budget constraints:

$$c_t(s) + a_t(s) = R(s-1)a_{t-1}(s-1) + w(s)\epsilon_t\ell_t(s) + S_t(s) - \Upsilon_t(s) + d_t \qquad (10.23a)$$
$$\Upsilon_t = \tau_0(s) + \tau_\ell(s)\,[w(s)\epsilon_t\ell_t(s) + d_t] + \tau_a(s)\,[R(s-1)-1]\,a_{t-1}(s-1) + \tau_c(s)c_t(s) \qquad (10.23b)$$
$$e_t(s) = e_{t-1}(s-1) + w(s)\epsilon_t\ell_t(s) \qquad (10.23c)$$
$$S_t(s) = \begin{cases} 0 & \text{for } t \le t_R + 1 \\ fixben_t(s) + rrate_t(s)e_{t-1}(s-1) & \text{for } t > t_R + 1 \end{cases} \qquad (10.23d)$$
$$z_{t+1} = A_{22}z_t + C_2\omega_{t+1} \qquad (10.23e)$$
$$\begin{bmatrix} d_t \\ \gamma_t \end{bmatrix} = \begin{bmatrix} U_{d,t} \\ U_\gamma \end{bmatrix} z_t \qquad (10.23f)$$

In equation (10.23a), $\tau_0(s)$ is a lump-sum tax, and $e_t(s)$ the cumulated earnings of an individual. The benefit formula (10.23d) allows for either a lump-sum retirement benefit or benefits that are related to past cumulated earnings. Equation (10.23e) describes the evolution of the information variable $z_t$, where $\omega_{t+1}$ is a martingale difference process. $U_{d,t}$ and $U_\gamma$ are selector vectors that specify the income shock $d_t$ and the stochastic bliss point $\gamma_t$. De Nardi *et al.* set the preference shock to a constant but specify $d_t$ to be random process with mean zero: $d_t = \rho_1 d_{t-1} + \omega_{1t}$. The martingale difference sequence $\omega_{t+1}$ is adapted to $J_t = (\omega_0^t, x_0)$, with $E(\omega_{t+1}|J_t) = 0$, $E(\omega_{t+1}\omega_{t+1}'|J_t) = I$. We assume that the individual income shocks are independent across individuals. The law of large numbers then implies that all uncertainty at the individual level averages out and that the aggregate economy is deterministic.

[11] We distribute bequests as follows. Each agent born at time $s$ begins life with assets $a_{-1}(s-1)$, which we set equal to a per-capita share of total bequests from people who died at the end of period $s-1$. This distribution scheme implies that within a steady state, per-capita initial assets equal per-capita bequests adjusted for population growth. However, during either policy or demographic transitions between steady states, this distribution scheme implies that what a generation receives in bequests no longer equals what it leaves behind.

10.3.4.2 *Preferences* The one-period utility function for an age-$t$ person is given by

$$u\left(c_t(s), \ell_t(s)\right) = -\tfrac{1}{2}\left[(c_t(s) - \gamma_t(s))^2 + (\pi_2 \ell_t(s))^2\right] \qquad (10.24)$$

where $\pi_2$ is a parameter and $\gamma_t(s)$ is a stochastic bliss point. There is a subjective discount factor $\beta$ which is common across individuals and cohorts. The effective discount factor from age $t$ to $t+1$ at time $s$ is the product $\beta\alpha_t(s)$. Let $x_t(s) = \left[a_{t-1}(s-1), e_{t-1}(s-1), z_t'\right]'$ denote the *state* vector of an age $t$ individual at the beginning of period $s$. If an individual dies at the end of age $t-1$, his value function is given by $V_t(x_t(s) \mid \text{dead at } t) = V_{T+1}(x_t(s)) = x_t(s)' P_{T+1} x_t(s)$, where $P_{T+1}$ is a negative semi-definite matrix with parameters that determine the strength of the bequest motive. This formulation of bequest motive is termed "the joy of giving" in the literature.[12]

Our formulation gradually activates the bequest motive, intensifying it with age as the mortality table makes the household think more about the hereafter.

For $t = 0, \ldots, T$, let $V_t(x_t(s))$ be the optimal value function for an age-$t$ person at time $s$. The household's Bellman equations are

$$\begin{aligned} V_t\left(a_{t-1}(s-1), e_{t-1}(s-1), z_t\right) = & \max_{\{c_t(s), a_t(s), \ell_t(s)\}} \{u(c_t(s), \ell_t(s)) \\ & + \beta\alpha_t(s) E_t V_{t+1}\left(a_t(s), e_t(s), z_{t+1}\right) \\ & + \beta[(1-\alpha_t(s)] E_t V_{T+1}\left(a_t(s), e_t(s), z_{t+1}\right)\} \end{aligned}$$

where the maximization is subject to the constraints (10.23).

Using standard linear quadratic control theory, the solution to the above finite-state, finite-horizon dynamic program is obtained as follows. Suppressing the time subscript for ease of exposition, we can express Bellman equations as

$$V_t(x_t) = \max_{u_t, x_{t+1}} \{u_t' Q_t u_t + x_t' R_t x_t + \beta E_t V_{t+1}(x_{t+1})\} \qquad (10.25)$$

where

$$\begin{aligned} E_t V_{t+1}(x_{t+1}) &= \alpha_t(s) E_t[V_{t+1}(x_{t+1}) \mid \text{alive}] \\ &\quad + [1-\alpha_t(s)] E_t[V_{t+1}(x_{t+1}) \mid \text{dead}] \\ V_t(x_t \mid \text{alive}) &= x_t' P_t x_t + \xi_t \\ V_t(x_t \mid \text{dead}) &= x_t' P_{T+1} x_t \\ x_t' P_{T+1} x_t &= -JG\left((1-\tau_b)a_{t-1} - JB\right)^2 \end{aligned} \qquad (10.26)$$

The last equation describes the bequest motive. $JG$ is a parameter governing the intensity of the bequest motive and $JB$ is an inheritance bliss point. Since the individual's survival probability declines over the life cycle, equation (10.26) reveals the higher weight attached to death and hence bequests as the individual ages.

[12] An altruistic bequest motive helps the model to produce an empirically plausible capital–output ratio. Also, the presence of a bequest motive makes private saving and hence the aggregate capital stock more resilient to changes in the environment. Fuster (1997) emphasizes the importance of this feature of her model in yielding results that are different from those in Auerbach and Kotlikoff (1987).

The matrix Riccati equatio

$F_t = (Q_t + \beta\alpha_t(s)I$

$\times (\beta\alpha_t(s)B_t'P_t$

$P_t = R_t + F_t'Q_tF_t$ ·

$+ \beta[1-\alpha_t(s)]$

$\xi_t = \beta\alpha_t(s)\left[\text{tr}(P_{t+}\right.$

Reintroducing the time sut dependent decision rules

and the law of motion

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Note that the certainty equivale independent of the noise statist

Given a mean and covarian first two moments of the state \

$\mu_{t+1}(s+1$

$\Sigma_{t+1}(s+1$

The mean and standard deviatic investment, output, and physic: averages of the above moments

10.3.5 *Resource constraint*

The national income identity at

$g(s)N(s) + \sum_{t=0}^{T} c_t(s)N_t(s)$ -

10.3.6 *Time variation in dem*

De Nardi *et al.* (1998) incorpoı sition in the demographic struc constant pre-1975 values of the

[13] Huang *et al.* (1997) depart from Hansen and Sargent (1995), the lin influence the decision rules.

The matrix Riccati equations for $P_t$, $F_t$ and $\xi_t$ are:

$$
\begin{aligned}
F_t &= \left(Q_t + \beta\alpha_t(s)B_t'P_{t+1}B_t + \beta[1-\alpha_t(s)]B_t'P_{T+1}B_t\right)^{-1} \\
&\quad \times \left(\beta\alpha_t(s)B_t'P_{t+1}A_t + \beta[1-\alpha_t(s)]B_t'P_{T+1}A_t\right) \\
P_t &= R_t + F_t'Q_tF_t + \beta\alpha_t(s)[A_t - B_tF_t]'P_{t+1}[A_t - B_tF_t] \\
&\quad + \beta[1-\alpha_t(s)][A_t - B_tF_t]'P_{T+1}[A_t - B_tF_t] \\
\xi_t &= \beta\alpha_t(s)\left[\operatorname{tr}(P_{t+1}C'C) + \xi_{t+1}\right] + \beta[1-\alpha_t(s)]\left[\operatorname{tr}(P_{T+1}C'C)\right]
\end{aligned}
$$

Reintroducing the time subscript, the above recursions produce the time- and age-dependent decision rules

$$u_t(s) = -F_t(s)x_t(s)$$

and the law of motion

$$x_{t+1}(s+1) = A_t(s)x_t + C_t(s)\omega_{t+1}$$

Note that the certainty equivalence specification of preferences makes the decision rules independent of the noise statistics, $\{C_t(s)\}$.[13]

Given a mean and covariance matrix for the initial state vector, $(\mu_0(s), \Sigma_0(s))$, the first two moments of the state vector follow the law of motion

$$
\begin{aligned}
\mu_{t+1}(s+1) &= A_t(s)\mu_t(s) \\
\Sigma_{t+1}(s+1) &= A_t(s)\Sigma_t(s)A_t(s)' + C_t(s)C_t(s)'
\end{aligned}
$$

The mean and standard deviation of aggregate quantities such as aggregate consumption, investment, output, and physical capital stock can then be easily computed as weighted averages of the above moments of the distribution of the state vector.

### 10.3.5 *Resource constraint*

The national income identity at time $s$ in this economy is given by

$$g(s)N(s) + \sum_{t=0}^{T} c_t(s)N_t(s) + K(s) = R(s-1)K(s-1) + w(s)\sum_{t=0}^{t_R} \epsilon_t\ell_t(s)\,N_t(s)$$

### 10.3.6 *Time variation in demographics*

De Nardi *et al.* (1998) incorporate the ageing of the population in their model as a transition in the demographic structure of the model. An initial steady state, associated with constant pre-1975 values of the demographic parameters $\{\alpha_t, n\}$, is specified. Then, the

[13] Huang *et al.* (1997) depart from certainty equivalence by employing nonexpected utility. Following Hansen and Sargent (1995), the linearity of decision rules is preserved although the noise statistics influence the decision rules.

projected mortality tables from the Social Security Administration (SSA) are used for the years between 1975 and 2060, so that

$$\alpha_t(s) = \begin{cases} \alpha_t^0 & \text{for } s \leq 1974 \\ \hat{\alpha}_t(s) & \text{for } 1975 \leq s \leq 2060 \\ \alpha_t^1 & \text{for } s > 2060 \end{cases}$$

where $\alpha_t^0 = \alpha_t(1970)$ from the mortality table, $\alpha_t^1 = \alpha_t(2060 + t)$, and the SSA numbers for the cohort to be born in 2060; the $\hat{\alpha}_t(s)$ are taken from the SSA.[14] The path for the growth rate of newborns is calibrated in order to match the SSA's forecasts of the dependency ratio, which is projected to increase from 18% in 1974 to 50% in 2060.

De Nardi *et al.* assume that individuals in the economy suddenly realize in 1975 that the mortality tables have changed and that they start using the new tables. The mortality tables are assumed to reach a steady state in 2060 in line with the SSA projections. The demographic structure changes for another $T+1$ years, until it reaches a new steady state in $2060+(T+1)$. The demographic transition requires the government to make fiscal adjustments and causes the individuals to recompute their decision rules in light of all the surprise changes in their environment. In steps, the government increases one tax rate (either $\tau_\ell$ or $\tau_c$) during a policy transition period, leaving all other tax rates constant. These tax changes are scheduled and announced as follows. In 1975 the government announces that, starting in year 2000, it will increase the tax on labour income (in experiments 1, 3, 5, and 6) or on consumption (in experiments 2 and 4) every 10 years in order to reach the terminal steady state with the desired ratio of debt to gross domestic product (GDP). Starting in 2060, that tax rate is held constant at its new steady-state level, but the wage rate and interest rate continue to vary for another $2(T+1)$ periods, after which time they are held fixed.

### 10.3.7 *Computing an equilibrium transition path*

1. Compute the initial steady-state equilibrium. Use a backward recursion to compute the agents' value functions and policy functions, taking as given government policy, bequests, and prices. Iterate until convergence on the following four-dimensional fixed-point problem with arguments given by:
   (a) the social security pension, in order to match the desired replacement rate;
   (b) bequests, so that planned bequests coincide with received ones;
   (c) the labour income or consumption tax to satisfy the government budget constraint;
   (d) factor prices, to match the firms' first-order conditions.[15]
2. Compute the final steady-state equilibrium. In addition to following the above procedure for the initial steady state, there is an additional do-loop layer in which iterations are performed on the government debt level to match the debt-to-GDP ratio to a prescribed value such as that in the initial steady state.

[14] The life tables are taken from Bell *et al.* (1992).

[15] In practice, the wage rate is a function of the real interest rate through the Cobb–Douglas production function. Therefore, the last component of the steady-state fixed point is just the real interest rate.



3. Compute the equilibrium tra
path of factor prices, bequest
sition dynamics by solving
functions, and then:
   (a) iterate until convergence
   final debt-to-GDP ratio;
   (b) iterate until convergence
   first-order conditions.

Although the model econon
only asymptotically (because pı
and Kotlikoff (1987) and assum

### 10.4 Conclusions

This chapter presents two versio
markets and describes how this
pension systems such as the un
United States and many other de
from the Arrow–Debreu world
presence of exogenously given b
that private annuity markets are
against uncertain lifespans.

The two versions of the mod
respects. The first version assur
ond version relaxes this assump
motive, whereas the first version
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and Ventura (1998) and Fuster (
model with preferences similar
here. In addition, Fuster's mode
described here.

The chapter presents the nur
equilibria for each version of th
ter also describes the solution a
states.

[16] To compute a steady-state equili
method to find the root of a system of
between steady states, they use a relax
value problems. See Press *et al.* (1986
secant and relaxation algorithms.

3. Compute the equilibrium transition path between the steady states. For a given time path of factor prices, bequests, and government policy parameters, compute the transition dynamics by solving backward the sequence of value functions and policy functions, and then:
   (a) iterate until convergence on a parameterized path for the tax rate to match the final debt-to-GDP ratio;
   (b) iterate until convergence on the time path of factor prices to match the firms' first-order conditions.

Although the model economy would converge to the final steady-state equilibrium only asymptotically (because prices are endogenous) De Nardi *et al.* follow Auerbach and Kotlikoff (1987) and assume that convergence obtains in $3T$ periods.[16]

## 10.4 Conclusions

This chapter presents two versions of an overlapping generations model with incomplete markets and describes how this model can be used to analyse issues related to public pension systems such as the unfunded social security system currently in place in the United States and many other developed countries. The first version of the model departs from the Arrow–Debreu world of complete contingent claims markets by assuming the presence of exogenously given borrowing constraints. Both versions of the model assume that private annuity markets are missing, thereby limiting the ability of agents to insure against uncertain lifespans.

The two versions of the model differ in their preference structures as well as in other respects. The first version assumes that labour is supplied inelastically, whereas the second version relaxes this assumption. The second version incorporates a form of bequest motive, whereas the first version is populated by pure life-cycle consumers. The first version allows for a richer set of within-cohort heterogeneity, whereas the second version (essentially) assumes that intra-cohort heterogeneity is normally distributed. Huggett and Ventura (1998) and Fuster (1997) have incorporated a variable labour supply into a model with preferences similar to those used in the first version of the model presented here. In addition, Fuster's model includes a bequest motive that is different from the one described here.

The chapter presents the numerical solution algorithms used to compute steady-state equilibria for each version of the model. For the second version of the model, the chapter also describes the solution algorithm for computing transition paths between steady states.

[16]To compute a steady-state equilibrium, De Nardi *et al.* (1998) use a secant algorithm which is a method to find the root of a system of nonlinear equations. In computing an equilibrium transition path between steady states, they use a relaxation algorithm which is a method for solving two-point boundary value problems. See Press *et al.* (1986) and references contained therein for a detailed description of the secant and relaxation algorithms.