ECON 671
Answers to Computer/Empirical Problem (of 2nd half of Econ 615)
Due: November 26th

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I. The file `data.dat` contains life histories of 1000 individuals starting at age 20 and continuing until the person either dies, or reaches 80. We have not added any random or non-random "survey attrition" other than people dying. The matrix `data.dat` has 10 columns containing the following information

1. The person ID number (1 to 1000)
2. The person's *employment state* e_{it} at the beginning of each year (1=employed 0=not employed)
3. The person's *work decision* d_{it} with $d_{it} = 1$ denoting to stay employed if $e_{it} = 1$ or to search for work if $e_{it} = 0$, and $d_{it} = 0$ denoting the decision to quit if $e_{it} = 1$ or to stay unemployed (i.e. not search for a job) if $e_{it} = 0$.
4. The person's *age* t, $t \in \{20,\ldots,80\}$
5. The person's *average wage* aw_{it} (to be described further below, but basically a moving average of past annual wage earnings)
6. The person's *earnings* (before tax) y_{it} from employment
7. The person's *benefits* b_{it} (either pension benefits if retired, or unemployment benefits if unemployed, or welfare benefits if not working and not searching for work, to be described further below)
8. An indicator $f_{it} = 1$ if a person who is not working $e_{it} = 0$ and searching for work $d_{it} = 1$ finds a job, otherwise $f_{it} = 0$ if the job search was unsuccessful
9. An indicator $u_{it} = 1$ if a person who is employed and chooses not to quit is involuntarily unemployed.
10. An indicator $r_{it} = 1$ if a person is retired and receiving Social Security/pension benefits

There are 54702 rows in the `data.dat` file, which is less than the 60,000 lines that it would contain if there were no mortality and everyone lived to age 80. This file is available for downloading along with some pieces of *Matlab* code to help you get started on the problem. The rest of this document will tell you more about the model I used to generate the data and what I want you to do for this problem set.

In this problem, individuals are eligible for 3 main types of government benefits: 1) a pension after one reaches the age of eligibility for retirement, 62, 2) unemployment benefits to individuals who lose a job and are searching to find a new job (benefits limited to 1 year), and 3) welfare benefits for individuals who do not work, with no limit on the duration of these benefits.

The welfare benefits are very small, \$5,000 per year (all income and benefit amounts in `data.dat` are in thousands). Unemployment insurance and welfare benefits are not taxable, but income and pension benefits are both subject to a tax rate of $\tau = 0.25$. Unemployment benefits, *ub* are calculated as a fraction of the *average wage*

$$ub_{it} = rr_u aw_{it} \tag{1}$$

where ub_{it} is the unemployment benefits paid to person i at age t, rr_u is the unemployment insurance *replacement rate* and aw_{it} is the person's average wage at age t. Unemployment insurance is not very generous and the replacement rate is assumed to be only $rr_u = 0.2$, i.e. a 20% replacement rate. To be eligible for unemployment benefits, two conditions must be satisfied: 1) you must be employed at the start of the year, $e_{it} = 1$, 2) you must have lost your job due to layoff, $d_{it} = 1$ and $u_{it} = 1$. The benefits are paid for one year to replace the lost wage earnings, but not more than one year. We assume that if a person loses their job in a given year, they will not be successful in finding a new job until the *next year* and provided they search for a job in the next year and get a job offer (the details on this will be explained shortly).

Finally pension benefits are given by a similar formula as unemployment benefits, but also reflect a higher replacement rate $rr_p = 0.4$ and a *delayed retirement credit*

$$pb_{it} = rr_p drc(r_{it}) aw_{it} \tag{2}$$

where r_{it} is the age when the person *first applied for Social Security benefits.* We assume that the earliest age that a person can receive retirement benefits is 62 and so if a person first applies for benefits at age 62, the delayed retirement credit formula is then given by

$$drc(r_{it}) = (1+r)^{(r_{it}-62)} \tag{3}$$

where $r = 0.05$, i.e. we assume a 5% delayed retirement credit. It follows that if $r_{it} = 62$ then $drc(r_{it}) = 1$ and there is no credit because the person did not delay their retirement date, but if they first apply at age 63, then $drc(63) = (1+r) = 1.05$, and so forth. Thus, a forward-looking person has to evaluate the benefits they get from retiring early (say at the early retirement age 62 when they are first eligible to retire) versus delaying retirement to get a higher retirement benefit due to the delayed retirement credit.

The average wage aw_{it} is a moving average of the person's wages over their full working career. Assuming nobody starts working prior to age 20, the law of motion for the average wage can be written in a recursive form as

$$aw_{it+1} = aw_{it}(t-19)/(t-18) + y_{it}/(t-19) \tag{4}$$

where we assume the initial value, aw_{i20}, is initialized to a random value for purposes of the simulations. Then at age $t = 21$ the person's average wage is a 50/50 weighted average of aw_{i20} and whatever earnings the person made in the previous year, y_{i20}.

We assume the wages individuals receive if working are given by the following equation

$$\log(y_{it}) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 \log(y_{it-1}) + \sigma\varepsilon_{it} \tag{5}$$

where $\varepsilon_{it} \sim N(0,1)$ and $\{\varepsilon_{it}\}$ is *IID*. If a person did not work in the previous year and $y_{it-1} = 0$, then since $\log(0) = -\infty$ we use the average wage aw_{it-1} in place of the wage to predict the earnings from employment

$$\log(y_{it}) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 + \alpha_3 \log(aw_{it-1}) + \sigma\varepsilon_{it} \tag{6}$$

There are three probabilities governing 1) mortality, 2) involuntary unemployment, and 3) the probability an unemployed job searcher will be successful in finding a job. Let $p_d(ed, aw, t)$ be the probability that an age t individual with average wage aw and employment decision d will die in the coming year and thus not survive to reach age $t+1$. Let $p_u(aw,t)$ be the probability that a person aged t with average wage aw is involuntarily unemployed. Since a person cannot be involuntarily unemployed unless they are employed $e = 1$ and do not quit their job $d = 1$, it follows that $p_u(aw,t)$ implicitly depends on (e,d) but it is zero if $e = 0$ or $d = 0$ and is only positive when $e = 1$ and $d = 1$, so we assume this when we write $p_u(aw,t)$. Finally, let $p_f(aw,t)$ be the probability that a person aged t with average wage aw who is currently not working will find a job. Again this probability implicitly depends on (e,d) but is clearly equal to 0 if $d = 0$

(i.e. we assume that a person has zero chance of finding a job if they don't choose to search, $d = 1$). Also if $e = 1$, then the person is already employed and we assume that a person has the option to stay in his/her job if employed, so $p_f(aw,t) = 1$ if $e = 1$ and $d = 1$, though we did already consider the possibility of involuntary unemployment via the probability $p_u(aw,t)$ above.

We assume that the person consumes all income or other benefits he/she receives each year so there is no consumption/saving decision. The utility from receiving income y and benefits b for a person whose work state and decision is (e,d) is given by

$$u(y+b,e,d) = \theta_1 \exp\{\theta_2 \log(y+b)\} - l(t,e,d) \tag{7}$$

where

$$l(t,e,d) = \begin{cases} \theta_3 t^2 & \text{if } e = 1 \text{ and } d = 1 \\ \theta_4 t^2 & \text{if } e = 0 \text{ and } d = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

I provide the *Matlab* function `benefits.m` that calculates a person's i's benefits at age t. $b_{it} = b_t(aw,e,d,u,r)$ which specifies the benefits a person aged t with average wage aw can expect if their employment state and decision is (e,d) and depending on whether an employed person is involuntarily unemployed u, oe whether the person is retired r. Let $f(y_{t+1}|y_t,t,aw_t)$ be the conditional lognormal density of earnings at age $t+1$ assuming employment given earnings at age t of y_t and average wage aw_t. Then the expected utility of a person who is currently employed and decides to work $(e,d) = (1,1)$ and is not retired, $r = 0$, is given by

$$\begin{aligned} & Eu_t(aw,y,e,r,d,rd) \equiv E\{u(\tilde{y} + b_t(aw,e,d,\tilde{u},rs_t(rd,r)))\} \\ & = (1 - p_u(aw,t))\left[\int_0^\infty \theta_1 \exp\{\theta_2 \log((b_t(aw,e,d,0,rs_t(rd,r)) + y')(1-\tau))\} f(y'|y,t,aw)dy' - \theta_3 t^2\right] \\ & + p_u(aw,t)\theta_1 \exp\{\theta_2 \log(b_t(aw,e,d,1,rs_t(rd,r)))\} \end{aligned} \tag{9}$$

The expected utility for other employment states and decisions and for retirees is calculated similarly. The function $rs(rd,r)$ defines the person's *retirement state transition function* given by

$$rs_t(rd,r) = \begin{cases} 0 & \text{if } t < 62 \\ (t-61) & \text{if } t \geq 62 \text{ and } r = 0 \text{ and } rd = 1 \\ 0 & \text{if } t \geq 62 \text{ and } r = 0 \text{ and } rd = 0 \\ r & \text{if } r > 0 \end{cases} \tag{10}$$

Thus, the retirement state equation tells us that a person cannot retire until they reach the early retirement age, $t = 62$, so their retirement state is $r = 0$ until then, but once they reach 62 they can retire at age 62 and their retirement state becomes $r = 1$ (and remains their for the rest of their life, this corresponds to making the retirement decision $rd = 1$ at $t = 62$), or if they do not retire at 62 and wait to retire at a later age, their retirement state stays at $r = 0$ (indicating that they are not yet retired) but at the first age they apply for benefits, $rd = 1$, then their retirement state becomes $r = t - 61$, meaning that if they first retire at age 63, their retirement state is locked in at $r = 2$, if they first retire at age 64, their retirement state is locked in at $r = 3$ and so on. This retirement state "remembers" the age they first started to collect retirement benefits due to the delayed retirement credit, since the benefit they get will be given by $aw(1+g)^{(r-1)}$ for every remaining year in their life until they die.

I provide the Matlab program `uf.m` that calculates the expected utility for all possible values of (t, aw, y, e, r, d, rd), and you can see I used *Gaussian quadrature* to numerically calculate the expectations in the expected utility equation (9).

If a person is 62 or older and has not retired yet ($t \geq 62$ and $r = 0$), then the person will have *two* choices (d, rd): where rd is a (0,1) decision whether to retire and d is a (0,1) decision whether to work. Let $(\varepsilon(1), \varepsilon(2), \varepsilon(3), \varepsilon(4))$ be four *IID* Type 1 extreme value random variables corresponding to the *four* possible combinations of the binary d and rd decisions, so the expected utilities for these *four* decision alternatives are

$$\max \left[Eu_t(aw, y, e, r, d, rd) + \varepsilon(\iota(d, rd)) \right], \tag{11}$$

where $\iota(d, rd)$ is simply an indexing function given by

$$\iota(d, rd) = 2 * rd + d + 1 \tag{12}$$

so that $\varepsilon(\iota(1,1)) = \varepsilon(4)$ and so forth (just for notational consistency to max the pair $(d, rd) \to \{1,2,3,4\}$).

If the person has previously retired, so $r > 0$, then the person only has two choices: whether to work or not d (or search for a job if not employed). Then the maximized utility in that period is just

$$\max \left[Eu_t(aw, y, e, r, 0) + \varepsilon(1), Eu_t(aw, y, e, r, 1) + \varepsilon(2) \right] \tag{13}$$

since the person cannot choose to retire again once they have already previously retired.

Let $\beta \in (0,1)$ be the person's discount factor. We assume that individuals choose labor supply and when to retire to maximize their expected discounted lifetime utility.

Conceptual Questions: do *all* of the "pencil and paper" questions below

1. Assuming that individuals are dynamic expected utility maximizers who decide when to retire and when to work (or search for for if unemployed) to maximize their expected discounted lifetime utility with the utility function described above and under the all of the assumptions described above where future utility is discounted at rate $\beta \in (0,1)$, please describe as precisely as you can the dynamic programming problem that agents are solving, writing the *Bellman equations* for the value functions $v_t(y_t, aw_t, e_t, r_t, rd_t, d_t)$ where v_t is the expected discounted utility for a person of age t who has income in the previous year of y_t and average wage at the start of age t of aw_t, whose employment state at the start of age t is e_t and whose retirement state is r_t and whose retirement decision is rd_t (if eligible to retire, so $t \geq 62$ and $r_t = 0$), and whose work decision is d_t.

   answer The *state variables* are the person's age t, and the following variables (aw_t, y_t, e_t, r_t), consisting of the average wage aw_t, income earned in the previous year (or 0 if not employed in the last year) y_t, the employment state in the last year e_t (1 if worked in the last year, 0 if the person did not work), and the retirement state r_t (an integer-valued variable equal to 0 if the person has not yet applied for retirement benefits — or is not yet 62 and thus eligible to apply — and equal to 1 if the person first applied for retirement benefits at age 62, 2 if first applied at age 63, and so on). There are a maximum of two decisions a person makes: an employment decision d_t equal to 1 (to continue to work if $e_t = 1$ or to search for work if $e_t = 0$) or 0 (to stop working if $e_t = 1$ or to stay not working, i.e. not to search for a job, if $e_t = 0$) and a retirement decision rd_t. The retirement decision is only relevant if a) the person is at least 62, $t \geq 62$, and b) has not previously applied for retirement benefits, $r_t = 0$. Let $Eu_t(aw, y, e, r, d, rd)$ be the expected utility a person who is t years old expects if they are in state (aw,y,e,r) and takes decisions (d,rd) (assuming a retirement decision is feasible, otherwise it just depends on the employment decision d which is always a feasible decision). Let $v_t(aw,y,e,r,d,rd)$ be the expected discounted lifetime utility of a person aged t in state (aw,y,e,r) taking a decision (d,rd). Consider an employed person who is younger than 62 (so is not eligible to make a retirement decision). So we have $e = 1$ and $r = 0$ for this person. The the value function can be written recursively using *backward induction* from the last period of life, age $T = 80$ (since we assume that everyone dies with probability 1 after age 80). Since there is no future period of life other than to live out the rest of age 80, let $V_{80}(aw,y,e,r,\varepsilon)$ denote the utility the person gets in their last year of life, $t = 80$ as a function of their state. If $r > 0$ then the person has already retired and has no additional retirement decision left to make (which we indicate by $rd = \emptyset$) so the value function is

$$V_{80}(aw,y,e,r,\varepsilon) = \max\left[Eu_{80}(aw,y,e,r,0)+\varepsilon(1), Eu_{80}(aw,y,e,r,1)+\varepsilon(2)\right]. \tag{14}$$

So the person either chooses $d = 0$ or $d = 1$ depending on which of the two expected utilities is higher, also accounting for the *IID* $(\varepsilon(1),\varepsilon(2))$ shocks (assumed to be observed by the person at the start of age $t = 80$ but not observed by the econometrician). If these shocks have a Type 1 extreme value distribution and we normalize the scale parameter $\sigma = 1$ (since we can multiply all utilities and shocks by a constant postive scalar and this does not affect the choices of such an agent, thuse we must make a "scale normalization" somehwere and it is convenient to normalize the scale parameters of the distribution of the shock terms $(\varepsilon(1),\varepsilon(2))$ entering the utility function), the probability that an 80 year retiree will choose to work is then given by

$$\begin{aligned} &Pr\left\{Eu_{80}(aw,y,e,r,1)+\varepsilon(2) \geq Eu_{80}(aw,y,e,r,0)+\varepsilon(1)\right\} \\ &= \frac{\exp\{Eu_{80}(aw,y,e,r,1)\}}{\exp\{Eu_{80}(aw,y,e,r,1)\}+\exp\{Eu_{80}(aw,y,e,r,0)\}}. \end{aligned}$$

For a non-retiree, $r = 0$, the person has to choose one of the four possible options from the binary choices over d and rd and would choose (d,ed) with the following probability

$$\begin{aligned} &Pr\left\{Eu_{80}(aw,y,e,r,d,rd)+\varepsilon(\iota(d,rd)) \geq \max_{d'\in\{0,1\},rd'\in\{0,1\}} Eu_{80}(aw,y,e,r,d',rd')+\varepsilon(\iota(d',rd'))\right\} \\ &= \frac{\exp\{Eu_{80}(aw,y,e,r,d,rd)\}}{\sum_{d'\in\{0,1\}}\sum_{rd'\in\{0,1\}}\exp\{Eu_{80}(aw,y,e,r,d',rd')\}} \end{aligned}$$

Now consider the person's decision problem at age $t = 79$. For a retiree, $r > 0$ there are only two choices $d \in \{0,1\}$ so the Bellman equation is given by

$$\begin{aligned} &V_{79}(aw,y,e,r,\varepsilon) = \\ &\max\left[Eu_{79}(aw,y,e,r,0)+\varepsilon(1)+\beta EV_{80}(aw,y,e,r,0), Eu_{79}(aw,y,er,1)+\beta EV_{80}(aw,y,e,r,1)\right] \end{aligned}$$

where $EV_{80}(aw,y,e,r,d)$ is the conditional expectation of the person's maximized utility $V_{80}(aw,y,e,r,\varepsilon)$ at age $t = 80$ conditional on their age $t = 79$ state and decision (aw,y,e,r,d). We will define this function in more detail shortly. But first it is important to get the "big picture" and see how we can derive a choice probability for whether the 79 year old works or not. Define the *choice specific value function* $v_{79}(aw,y,e,r,d)$ by

$$v_{79}(aw,y,e,r,d) = Eu_{79}(aw,y,e,r,d)+\beta EV_{80}(aw,y,e,r,d). \tag{15}$$

Then we have the simple relationship between V_{79} (the usual "value function" in the Bellman equation of dynamic programming) and the choice-specific value functions v_{79} given by

$$V_{79}(aw,y,e,r,\varepsilon) = \max\left[v_{79}(aw,y,e,r,0)+\varepsilon(1), v_{79}(aw,y,e,r,1)+\varepsilon(2)\right] \tag{16}$$

and we can write the choice probability for a 79 year old as a binary logit very similar to the way we did for an 80 year old in equation (15) above, but the 79 year old is facing a *two period problem* whereas the 80 year is facing only a *one period problem* since the 80 knows he/she will die at the end of their 80$^{\text{th}}$ year. However if we use v_{79} instead of Eu_{79} as the utility function entering the choice probability, we can still correctly express the probability that a 79 year old will work

$$\begin{aligned} & Pr\{v_{79}(aw,y,e,r,1)+\varepsilon(2) \geq v_{79}(aw,y,e,r,0)+\varepsilon(1)\} \\ & = \frac{\exp\{v_{79}(aw,y,e,r,1)\}}{\exp\{v_{79}(aw,y,e,r,1)\}+\exp\{v_{79}(aw,y,e,r,0)\}}. \end{aligned}$$

For a 79 year old who has not yet retired, $r = 0$ this person has a pair of decisions (d,rd) to make, and it is not hard to see that we can define a corresponding set of choice-specific value functions $v_{79}(aw,y,e,r,d,rd)$ in a similar way, and we will get a corresponding choice probability for the joint choice of (d,rd) as we did for the 80 year old above, but using the choice-specific value functions v_{79} instead of the single period expected utilities Ev_{79}

$$\begin{aligned} & Pr\left\{v_{79}(aw,y,e,r,d,rd)+\varepsilon(\iota(d,rd)) \geq \max_{d'\in\{0,1\},rd'\in\{0,1\}} v_{79}(aw,y,e,r,d',rd')+\varepsilon(\iota(d',rd'))\right\} \\ & = \frac{\exp\{v_{79}(aw,y,e,r,d,rd)\}}{\sum_{d'\in\{0,1\}}\sum_{rd'\in\{0,1\}}\exp\{v_{79}(aw,y,e,r,d',rd')\}}. \end{aligned}$$

Now we show how to derive the expected value functions $EV_{80}(aw,y,e,r,d,rd)$. There are four different equations depending on the values of e and r. First consider a person who is retired and not working, $r > 0$ and $e = 0$ at age $t = 79$. This person could either choose to search for a job ($d = 1$) or stay retired and not working $d = 0$. How much expected utility will this person get in this latter case at age 80? Since the person is already retired, there is no employment decision and since we first consider the decision $d = 0$ (do not search), we know that the person will simply collect their retirement benefits and no labor income, and we have that $e = 0$, $y = 0$ and, since they are retired, their average wage at age 80 will be the same as their average wage at age 79, aw, so the only uncertain quantity to take expectations over is what the unobserved shocks to utility $\varepsilon = (\varepsilon(1),\varepsilon(2)) = (\varepsilon_1,\varepsilon_2)$ will be at age $t = 80$. Since these are independent Type 1 Extreme value random variables, we have a convenient closed for expression for the EV_{80} in this case

$$\begin{aligned} EV_{80}(aw,y,e,r,0) &= (1-p_d(aw,79))\int_{\varepsilon_1}\int_{\varepsilon_2} V_{80}(aw,0,0,r,\varepsilon_1,\varepsilon_2)f(\varepsilon_1)f(\varepsilon_2)d\varepsilon_1 d\varepsilon_2 \\ &= (1-p_d(aw,79))\int_{\varepsilon_1}\int_{\varepsilon_2} \max\left[Eu_{80}(aw,0,0,r,0)+\varepsilon_1, Eu_{80}(aw,0,0,r,1)+\varepsilon_2\right] \\ &= (1-p_d(aw,79))\log\left(\exp\{Eu_{80}(aw,0,0,r,0)\}+\exp\{Eu_{80}(aw,0,0,r,1)\}\right) \end{aligned}$$

where we use the "log-sum formula" (also called the "inclusive value" by McFadden) as a convenient closed form expression for the expected maximum of the age 80 maximum utility over the two extreme value shocks $(\varepsilon_1,\varepsilon_2)$ that will affect the person's decision whether to work or not at that last year of the person's life. Notice we also multiplied by the *survival probability* $p_s(aw,79) = 1 - p_d(aw,79)$ which is one minus the probability that the 79 year old would die and not reach age 80. We assume there is no further utility when someone dies so $p_d(aw,79)$ times the "expected utility of dying" is 0.

Now consider the other possible decision, $d = 1$, which indicates that the person chooses to search for a job at age 79. Now we need to consider the possibiilty that the person may not find a job, or if they do, what their earnings would be. The person's expectations is that their earnings would be a draw y' from the conditional density $f(y'|0,79,aw)$, which is the earnings a 79 year hold who is not currently working (and thus has current income $y = 0$) expects if their average wage is aw. Thus the average wage is a measure of the person's past earnings and thus is natural to include to predict their earnings if they are successful in returning to work in accordance with the wage equation (6). We have

$$\begin{aligned}
&EV_{80}(aw,y,e,r,1) = p_s(aw,79)(1-p_f(aw,79))\int_{\varepsilon_1}\int_{\varepsilon_2} V_{80}(aw,0,0,r,\varepsilon_1,\varepsilon_2)f(\varepsilon_1)f(\varepsilon_2)d\varepsilon_1 d\varepsilon_2 \\
&+p_s(aw,79)p_f(aw,79)\int_{y'}\int_{\varepsilon_1}\int_{\varepsilon_2} V_{80}(aw,y',1,r,\varepsilon_1,\varepsilon_2)f(\varepsilon_1)f(\varepsilon_2)f(y'|0,79,aw)dy'd\varepsilon_1 d\varepsilon_2 \\
&= p_s(aw,79)(1-p_f(aw,79))\log\left(\exp\{Eu_{80}(aw,0,0,r,0)\}+\exp\{Eu_{80}(aw,0,0,r,1)\}\right) \\
&+p_s(aw,79)p_f(aw,79)\int_{y'}\log\left(\exp\{Eu_{80}(aw,y',1,r,0)\}+\exp\{Eu_{80}(aw,y',1,r,1)\}\right)f(y'|0,79,aw)dy'
\end{aligned}$$

Thus, the expected utility now factors in the uncertainty of whether the job search decision $d = 1$ will be successful in getting a job, and if so, the additional uncertainty about amount of wages actually earned y'.

Now consider a retiree who was working at the start of age $t = 79$, $e = 1$. Then the two decisions are $d = 1$ (continue working) and $d = 0$ (quit). If the person chooses to quit, then it is not hard to see that ithe formula for expected utility of the person in this case is the same as the equation for $EV_{80}(aw,y,e,r,0)$ above for a person who is not working at the start of age 79 and who chooses to continue not to work. However for a 79 year old who is working and who chooses to continue to work, we have to factor in the probability that they might be fired, and this is given by the probability $p_u(aw,79)$ below

$$\begin{aligned}
&EV_{80}(aw,y,e,r,1) = p_s(aw,79)p_u(aw,79)\int_{\varepsilon_1}\int_{\varepsilon_2} V_{80}(aw,0,0,r,\varepsilon_1,\varepsilon_2)f(\varepsilon_1)f(\varepsilon_2)d\varepsilon_1 d\varepsilon_2 \\
&+p_s(aw,79)(1-p_u(aw,79))\int_{y'}\int_{\varepsilon_1}\int_{\varepsilon_2} V_{80}(aw,y',1,r,\varepsilon_1,\varepsilon_2)f(\varepsilon_1)f(\varepsilon_2)f(y'|y,79,aw)dy'd\varepsilon_1 d\varepsilon_2 \\
&= p_s(aw,79)p_u(aw,79)\log\left(\exp\{Eu_{80}(aw,0,0,r,0)\}+\exp\{Eu_{80}(aw,0,0,r,1)\}\right) \\
&+p_s(aw,79)(1-p_u(aw,79))\int_{y'}\log\left(\exp\{Eu_{80}(aw,y',1,r,0)\}+\exp\{Eu_{80}(aw,y',1,r,1)\}\right)f(y'|y,79,aw)dy'
\end{aligned}$$

Thus, the expected utility now factors in the uncertainty of whether the worker will be unemployed. If the worker is not fired, this equation captures the uncertainty about the wages the person will earn y'. Notice that since the person is employed at the start of age 79 we know the previous income y and use this to get a better expectation of earnings from remaining in the job via the density $f(y'|y,79,aw)$ in accordance with the wage equation (5) above.

For a non-retiree, $r = 0$, we have to modify the expectations above to allow for the additional retirement decision rd this person has. First, if the person chooses to retire, $rd = 1$, then we know the person will not have a retirement decision to make at age 80, so the equations are basically similar as the four equations given above. The only difference is that the person, having delayed retiring until age 79, will get a higher retirement benefit due to the delayed retirement credit and this expectation is “built in” via the benefit function $b_t(aw,e,d,u,r)$ but other than this, the equations for expected utility in this case are exactly the same as those given in equations above.

The final case is where the age 79 non-retire decides not to retire at age 79, so $r = 0$ and $rd = 0$. Then this person will have a retirement decision to make at age $t = 80$. So there will be four expected utilities to consider at age 80 corresponding to the two binary decisions (d,rd) the person will have in this case. The equations above must be modified to account for the *four* choices available to the person at age $t = 80$. We do this by simply replacing the log-sum formulas in the equations above which are sums only over *two* alternatives (utility of working versus not working) in the case of someone who has already retired to a log-sum over the *four* alternatives (retire or don’t retire, work or don’t work), resulting in formulas such as the one below

$$\log\left(\sum_{d'\in\{0,1\}}\sum_{rd'\in\{0,1\}}\exp\{Eu_{80}(aw,y,e,0,d',rd')\}\right) \tag{17}$$

Having computed $v_{79}(aw,y,e,r,d,rd)$ for all feasible values of the state variables and decisions at age $t = 79$ (since aw and y are continuous state variables, it is necessary to *discretize* these to a *finite grid* and then *interpolate* to compute $v_{79}(aw,y,e,r,d,rd)$ if (aw,y) are not grid points on the pre-determined grid. I will provide computer code to do this.

The backward induction then proceeds to age $t = 78,77,76,\ldots,20$ completing the dynamic optimization. At each age t we will have then computed the choice-specific value functions $v_t(aw,y,e,r,d,rd)$ and each will satisfy the Bellman equation

$$v_t(aw,y,e,r,d,rd) = EU_t(aw,y,e,r,d,rd) + \beta EV_{t+1}(aw,y,e,r,d,rd) \tag{18}$$

where the expected value functions $EV_{t+1}(aw,y,e,r,d,rd)$ will be computed in the same way as outlined for the case $t = 79$ in detail above, except that for $t < 62$ the retirement decision rd is no longer available and there is just a binary decision d over whether to work or not. The conditional choice probabilities will then be either given by

$$P_t(d|aw,y,e,r) = \frac{\exp\{v_t(aw,y,e,r,d)\}}{\exp\{v_t(aw,y,e,r,0)\} + \exp\{v_t(aw,y,e,r,1)\}}, \tag{19}$$

if the retirement decision is not feasible (either because the persion is age $t < 62$ or because $r > 0$), and in the ages and states where the retirement decision is feasible, we have the probabilities for the joint choice of employment and retirement given by

$$P_t(d,rd|aw,y,e,r) = \frac{\exp\{v_t(aw,y,e,r,d,rd)\}}{\sum_{d'\in\{0,1\}}\sum_{rd'\in\{0,1\}}\exp\{v_t(aw,y,e,r,d',rd')\}}. \tag{20}$$

Note that the coinditional choice probabilites P_t will be functions not only of the state variables, but also functions of *all* the parameters in the model, both the utility function parameters $\theta = (\theta_1,\ldots,\theta_4)$ and the α parameters of the wage equation, and the parameters entering the probability of death p_d, of unemployment, p_u, and of finding a job p_f.

2. Using the Bellman equations, write formulas for the *conditional choice probabilities* for the work decision d_t and retirement decision rd_t when the person is eligible to make a retirement choice. That is, write expressions for $P(d|t,y,aw,e,r)$ for individuals who are not eligible to make a retirement decision, and the choice probabilities for the joint choice of retirement and whether to work (or search for work if not employed) $P(d,rd|t,y,aw,e,r)$ as a function of the "state variables" (t,y,aw,e,r).

   answer Already answered as part of the (long) answer to part 1 above.

3. Using the choice probabilities derived in your answer to question 2 and the other probabilities of mortality, unemployment and finding a job described in the problem description above, as well as the equation for the dynamics of wages and average wages, derive a *full information likelihood* for all the possible labor supply and retirement and unemployment and job search outcome events that you observe in the data.

   answer Let β be the agents' common discount factor, θ be the four unknown parameters of peoples' utility functions, α the parameters of the wage equation and let γ be all of the other parameters of the probability of death p_d, unemployment p_u and finding a job p_f. For each person in the sample, $i = 1,\dots,N$ we observe all states and decisions from age $t = 20$ until age $t = 80$ or when they die, whichever is sooner. Let T_i be the age of death of person i or $T_i = 80$ if they survive until the maximum age that we observe 80. Then the data we observe for person i is $\{(aw_{it},y_{it},e_{it},r_{it},d_{it},rd_{it})\}_{t=2-}^{T_i}$. Let $\phi = (\beta,\theta,\alpha,\gamma)$ be the unknown parameters to be estimated. The full likelihood for all N people in the sample is given by

$$L_N(\phi) = \prod_{i=1}^{N} L(\{(aw_{it},y_{it},e_{it},r_{it},d_{it},rd_{it})\}_{t=20}^{T_i},\phi) \tag{21}$$

   so we only need to describe the likelihood for an individual person, $L(\{(aw_{it},y_{it},e_{it},r_{it},d_{it},rd_{it})\}_{t=20}^{T_i},\phi)$. Intuitively we start at age $t-20$ and write the probability that the person decided to work or not work given their "initial condition" at age 20 $(aw_{20},0,0,0)$, i.e. we "endow" each 20 year old with a randomly chosen initial average wage aw_{20} and assume the person is not working $e = 0$ and has never had a job before so $y = 0$. Further they cannot be retired until age 62 so $r = 0$ also. So the only choice we observe at age 20 is d_{20}, whether the person chooses to search for a job or not, and this occurs with probability $P_{20}(d|aw,0,0,0,\phi)$ which can be computed for any given initial guess of ϕ by solving the dynamic programming problem from age $t = 80$ by backward induction as described in the answer to part 1 above. The employment state at age $t = 21$, e_{21} will depend on whether the person chose $d_{20} = 1$ or not. If $d_{20} = 0$ then $e_{21} = 0$ and $y_{21} = 0$ (recall that y_t is the income a person earned in the year *prior* to their current age t). Further since the average wage evolves deterministically, we do not have to write a likelihood equation for that. Thus, if we take the initial state of the person at age 20, $(aw_{20},y_{20},e_{20},r_{20}) = (aw_{20},0,0,0)$ as given, we need to write the likelihood for the observation $(d_{20},e_{21},aw_{21},y_{21})$ that represents the employment, earnings outcome that the person will find himself in at the start of age $t = 21$ resulting from the their job search choices and outcomes at age $t = 20$. We can build the likelihood by writing a *Markov transition probability* $\pi(aw_{t+1},y_{t+1},e_{t+1},r_{t+1}|aw_t,y_t,e_t,r_t,d_t,rd_t,\phi)$, and the probability of the initial choice $P_{20}(d_{20}|aw_{20},y_{20},e_{20},r_{20},\phi)$ which depends on the initial condition $(aw_{20},y_{20},e_{20},r_{20})$. Since a product of the Markov transition probabilities results in the joint probability of the full sequence of observed states and decisions given the initial condition for each person, we can write

$$L(\{(aw_t,y_t,e_t,r_t,d_t,rd_t\}_{t=20}^{T}|(aw_{20},0,0,0),\phi) = \left[\prod_{t=20}^{T-1}(1-p_d(aw_t,t,\gamma)\right] p_d(aw_T,T) \times$$
$$\left[\prod_{t=20}^{T-1} P_{t+1}(d_{t+1},rd_{t+1}|aw_{t+1},y_{t+1},e_{t+1},r_{t+1},\phi)\pi(aw_{t+1},y_{t+1},e_{t+1},r_{t+1}|aw_t,y_t,e_t,r_t,d_t,rd_t,\phi)\right] \times$$
$$P_{20}(d_{20}|(aw_{20},0,0,0,\phi). \tag{22}$$

where $L(\{(aw_t,y_t,e_t,r_t,d_t,rd_t)\}_{t=20}^{T}|(aw_{20},0,0,0),\phi)$ represents the likelihood of observing the sequence $\{(aw_t,y_t,e_t,r_t,d_t,rd_t)\}_{t=20}^{T}$ *conditional* on the initial condition $(aw_{20},0,0,0)$ (i.e. conditional on the 20 year old starting out not previously employed and not retired or working but with an average wage "endowment" equal to aw_{20}). The first factor in equation (22) is the probability of observing an individual who lives from age $t=20$ through age $t=T$ and then dies at age $T+1$, so it is the product of the conditional probability of surviving at ages $t=20,21,\dots,T$ and then the conditional probability $p_d(aw_T,T,\gamma)$ that an individual who has reached age T dies before reaching age $T+1$.

Thus, the likelihood for a given individual in the sample is a product of several factors, including the *conditional probabilities* $P_t(d_t,rd_t|aw_t,y_t,e_t,r_t,\phi)$ for the sequence of employment and retirement choices the person makes over their lifetime, times another product of *transition probabilities* that predict the probability of next period state $(aw_{t+1},y_{t+1},e_{t+1},r_{t+1})$ conditional on the current period state (aw_t,y_t,e_t,r_t) and decision (d_t,rd_t). We can complete the answer by deriving this probability. This transition probability is quite simple in any period/age t where the person chooses to quit an existing job (if $e_t=1$) or remain not working (if $e_t=0$). Then with probability 1 we know that $e_{t+1}=0$ and $y_{t+1}=0$ and average wage will be updated also according to a deterministic formula, so the transition probability π will simply equal 1 in this case. Thus, the more interesting and informative case comes when the person chooses $d_t=1$. When the person is currently employed, $e_t=1$, we need to account for the probability that the person could be laid off, so we have

$$\begin{aligned}\pi(aw_{t+1},y_{t+1},e_{t+1},r_{t+1}|aw_t,y_t,e_t,r_t,d_t,rd_t) &= I\{e_{t+1}=1\}(1-p_u(aw_t,t))f(y_{t+1}|y_t,t,aw_t)+ \\ &\quad (1-I\{e_{t+1}=1\})p_u(aw_t,t).\end{aligned} \tag{23}$$

This equation says that if the person is not laid off, which occrs with probability $1-p_u(aw_t,t)$, the transition probability predicts the realized income earned y_{t+1} via the lognormal conditional probability $f(y_{t+1}|y_t,t,aw_t)$ implied by the wage equation. Otherwise if the person is laid off, which occurs with probability $p_u(aw_t,t)$, we know that $y_{t+1}=0$ with probability 1 and all of the other state variables evolve deterministically so there are no other probabilities to predict in this case.

Similarly in the case where $e_t=0$, then if $d_t=1$ we have a person who is not currently employed searching for a job. The transition probability in this case is

$$\begin{aligned}\pi(aw_{t+1},y_{t+1},e_{t+1},r_{t+1}|aw_t,y_t,e_t,r_t,d_t,rd_t) &= I\{e_{t+1}=1\}p_f(aw_t,t)f(y_{t+1}|y_t,t,aw_t)+ \\ &\quad (1-I\{e_{t+1}=1\})(1-p_f(aw_t,t)).\end{aligned} \tag{24}$$

The explanation for this transition probability is very similar to the one above: if the person is successful in finding a job, which occrs with probability $p_f(aw_t,t)$, the transition probability predicts the realized income earned y_{t+1} via the lognormal conditional probability $f(y_{t+1}|y_t,t,aw_t)$ implied by the wage equation, except that $y_t = 0$ since the person was not employed in the previous period (i.e. at the start of age t). Otherwise if the person is unsuccessful in finding a job, which occurs with probability $(1 - p_f(aw_t,t))$, we know that $y_{t+1} = 0$ and $e_{t+1} = 1$.

Since the laws of motion for retirement state and average wage and benefits are all deterministic and do not depend on any unknown parameters, there is no need to include expressions for these transitions into the likelihood. Instead we simply plug in the values for aw_{t+1}, r_{t+1} that are predicted with probability 1 from the deterministic updating formulas for average wage and the retirement state based on previous states and decisions and the conditional probability of observing these values will just be equal to 1, and thus these variables do not affect the likelihood (or are implicitly included as factors in the likelihood equal to 1). The only probabilistic aspect of retirement is the timing of retirement, and this is captured in the product of the conditional choice probabilities, which will provide a probability of choosing to retire starting at age $t = 62$ until the person actually does retire, after which the person stays retired until they die so the conditional probability of being retired after having previously retired is 1 and thus does not affect the likelihood as explained above.

4. Describe in general terms how you would expect a person's labor supply and retirement behavior would change if they were a *myopic decision maker* with $\beta = 0$ versus if they were a *dynamic decision maker* with $\beta > 0$. If you wanted to distinguish whether people were myopic or dynamic decision makers, would it be possible to tell just by observing a given set of data on their work and retirement behavior, or would it be helpful to be able to conduct an *experiment* that would be able to provide more convincing evidence as to whether people take the future into account in making their decisions? Assuming any experiment you dream up would be feasible to conduct, decribe a *controlled experiment* where a *treatment group* is expoosed to a changed retirement policy or some other change in payment (or retirement bonus, or tax etc) and you can compare the behavior of individuals in the treatment group with those in the *control group* who continue to receive the *status quo* pay and government benefits given above.

Computer/Empirical Questions: do *all* of questions below except you can choose whether to answer 7 *or* 8 *but you do not have to do both.* Then *everyone has to do part 9 which is to try to predict the impact of an increase in the retirement age from 62 to 65 either via a reduced form approach (if you choose to do problem 8 instead of 7) or a structural approach (if you choose to do problem 7 instead of problem 8). Students who get closer to the "true answser" will get more credit in their answer to problem 9.*

5. Using the data in `data.dat` estimate the unknown parameters of the wage equation and the probabilities p_d, p_u and p_f and plot your estimates of these estimated probabilities as functions of their argtuments. Do you need to correct for any endogeneity of sample selection bias in order to consistently estimate these objects?

6. Using the full information likelihood you derived in your answer to question 3 above, describe the *partial likelihoods* that you might use to estimate the parameters of the wage equation and the probabilities p_d p_u and p_f. Can you find a general argument that maximizing a partial likelihood function will result in a consistent estimatof of the underlying parameters? How is your answer affected by the fact that I have given you *prior information* about the functional form of the wage equation but

have not told you explicitly about the functional form of the probabilities p_d p_u and p_f other than to tell you the variables that (potentially) enter these probabilities? Are there issues of *model selection* that you have to confront to estimate these probabilities, and if so, then describe how you went about this model selection problem to settle on your preferred specification for these probabilities? If you estimated these probabilities using non-parametric or semi-parametric methods, describe the method you selected and attempt to provide some justification for you approach.

7. Using the estimated parameters of the wage equation and the probabilities you estimated in your answer to question 6 above, use a *two step approach* to estimate the parameters $(\beta, \theta_1, \theta_2, \theta_3, \theta_4)$ of individuals' utility functions (with everyone assumed to have the same utility function and differ only in their states, e.g. realized incomes, average wages and employment histories, etc). That is, taking the estimated probabilities and wage equations from your answer to part 6 and a *partial likelihood function* for individuals' choices of labor supply and retirement ages, estimate the utility function parameters in the second stage by maximizing this partial likelihood. **Note:** you will need to find a way to solve agent's dynamic programming problems such as numerically solving the Bellman equations you derived in your answer to question 1 above and using the value functions from this DP solution v_t, to compute the choice probabilities and use thse to form your partial likelihood function. Will your estimates be consistent and asymptotically normally distributed? Will your estimated covariance matrix be consistently estimated? Can you conduct a likelihood ratio test of the hypothesis $H_o : \beta = 0$ versus that alternative $H_1 : \beta > 0$ and if so, do you find you can reject H_o?

8. Instead of following a structural approach, can you suggest some simpler *reduced form model* for predicting peoples' labor supply and retirement decisions? Outline your preferred model and approach and using this and the data in `data.dat` estimate your model and evaluate its goodness of fit using the Chi-squared goodness of fit test of Andrews *Econometrica* 1988.

9. The ultimate goal of all of this econometric work is to advise policy makers on the impact of an increase in the retirement age from 62 to 65. As an additional incentive, policy makers propose increasing the delay retirement credit from 5% to 8%, so that for each year of delay in retirement after age 65, retirement benefits will be permanently increased by 8% instead of 5%. Policy makers are interested in knowing the following things: 1) if the retirement age is increased, how many people will choose to retire (start collecting benefits) at the new retirement age 65, and how many will be affected by the increased delayed retirement credit and retire even later than age 65? 2) What is the overall impact of this policy change on the *discounted present value of government taxes and benefits*? To answer the latter question, calculate for this sample of individuals the net present value of taxes and welfare, unemployment and pension benefits over the lifecycle discounted at 2% to age 20 under the *status quo* with the age 62 retirement age and the 5% delayed retirement credit, and then predict how this present value of net benefits would change under an age 65 retirement age with the higher 8% delayed retirement credit. By how much will the net present value of governement outlays be reduced (or increased) by this policy change? 3) Finally, some individuals have charged that the policy change will favor higher income individuals, since these individuals tend to work longer anyway and thus the higher retirement age will not be binding for them and they will benefit additionally from the higher delayed retirement credit. Discuss whether you think these concerns are justified and how you would go about trying to predict how the policy change will affect lifetime inequality and which individuals will be most hurt and which will be most helped by this policy change. **HINT:** you can answer this question either using the structural model (if you answer ques-

tion 7) or your preferred reduced-form or “behavioral model” (if you answered question 8). I do not expect you to provide a numerical answer to question 9-3) above but I do expecte numerical “policy predictions” (ideally with standard errors, not standard errors are not essential) to questions 9-1) and 9-2) above.