ECON 615/2nd Half
Problem Set 0

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I Consistency and asymptotic normality of two step estimators Suppose we have balanced panel data and the full likelihood function is given by

$$L(\theta) = \prod_{i=1}^{I} \prod_{t=1}^{T} P(d_{i,t}|x_{i,t}, \theta) p(x_{i,t}|x_{i,t-1}, d_{i,t-1}, \theta_2) \tag{1}$$

where I is the number of people in the panel (independently sampled) and T is the number of time periods over which they are observed, and θ_2 is a subset of the full parameter vector θ. Show that maximizing a *partial likelihood function*

$$L_p(\theta_2) = \prod_{i=1}^{I} \prod_{t=1}^{T} p(x_{i,t}|x_{i,t-1}, d_{i,t-1}, \theta_2) \tag{2}$$

results in a consistent and asymptotically normal estimator of θ_2^* under suitable regularity conditions for fixed T as $I \to \infty$. If $\theta = (\theta_1, \theta_2)$ is the full parameter vector to be estimated, show that the second stage estimator of θ_1 that results from the maximization of

$$L_p(\theta_1|\hat{\theta}_2) = \prod_{i=1}^{I} \prod_{t=1}^{T} P(d_{i,t}|x_{i,t}, \theta_1, \hat{\theta}_2) \tag{3}$$

where $\hat{\theta}_2$ is the first stage maximum likelihood estimator of the partial likelihood function (2) above, results in a consistent and asymptotically normal estimator of θ_1^*. Under what conditions will this second stage estimator of θ_1^* be asymptotically efficient? Derive a formula for the covariance matrix of $\hat{\theta}_1$ and show that we need to account for the estimation error in $\hat{\theta}_2$ in order to derive a consistent estimator of the asymptotic covariance matrix for θ_1^* from maximization of the 2nd stage partial likelihood function (3).

II Write down the expression (recursion) for the gradient of the partial loglikelihood function with respect to θ where the log-likelihood for an unbalanced panel is

$$\log(L(\theta)) = \sum_{i=1}^{I} \sum_{t=\underline{t}_i}^{\overline{t}_i} \log(P_t(d_{i,t}|x_{i,t}, \theta)) \tag{4}$$

where

$$P_t(d|x, \theta) = \frac{\exp\{v_t(x, d, \theta)/\sigma\}}{\sum_{d' \in D_t(x)} \exp\{v_t(x, d', \theta)/\sigma\}}, \tag{5}$$

where $v_t(x, d, \theta)$ is given by the recursion,

$$v_t(x, d, \theta) = u_t(x, d, \theta_1) + \beta \int_{x'} \sigma \log \left(\sum_{d' \in D_{t+1}(x')} \exp\{v_{t+1}(x', d', \theta)/\sigma\} \right) p_{t+1}(x'|x, d, \theta_2) dx' \tag{6}$$

where $\theta = (\beta, \sigma, \theta_1, \theta_2)$. Derive a formula for $\frac{\partial}{\partial\theta}\log(L(\theta))$. **Hint:** Show the gradient of the partial likelihood function can be written in terms of $\frac{\partial}{\partial\theta}v_t(x,d,\theta)$ and derive a recursion formula for these partial derivatives similar to the recursion for the functions $v_t(x,d,\theta)$ in equation (6) above.

III Now consider the stationary, infinite horizon case, where $v(x,d,\theta)$ is the solution to the following fixed point equation:

$$v(x,d,\theta) = u(x,d,\theta_1) + \beta\int_{x'}\sigma\log\left(\sum_{d'\in D(x')}\exp\left\{v(x',d',\theta)/\sigma\right\}\right)p(x'|x,d)dx' \tag{7}$$

Derive a formula for $\frac{\partial}{\partial\theta}\log(L(\theta))$ in this case. **Hint:** Consider using the *implicit function theorem* to define $v(x,d,\theta)$ as an implicit function of the model parameters θ.