A Simple Theory of Why and When Firms Go Public

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Abstract: This note introduces a simple model of a firm's optimal investment and dividend payout decision where it is possible to obtain a closed form solution for the value function and optimal investment and dividend payout decision functions in certain cases. The analytical solutions are compared to numerical solutions produced by discrete policy iteration and the numerical solutions are shown to be highly accurate. This gives us confidence that conclusions we draw using numerical methods for extensionis of the problem to the investment policy of a "privately held firm" run by an owner who is also solving an intertemporal consumption-smoothing problem on top of the optimal investment decision, are accurate and can be trusted. Using the two solutions, we gain insights on the situations where the owner of a privately held firm can gain by "taking it public" — i.e. by selling off the firm so that it is managed with the goal of maximizing the discounted value of the dividend stream. We show that the optimal investment and borrowing policy of a public firm differs significantly from the investment policy of a private firm, where the owner is concerned about consumption smoothing rather than maximizing the discounted value of dividends.

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1 Introduction

This note introduces a simple "test problem" for which it is possible to obtain an analytical solution for the optimal investment and dividend policy of a "public firm" which invests in a single liquid capital good k. We use the term "public firm" to distinguish it from a "private firm" which we also analyze. The key difference is that a public firm's objective is to adopt an investment and borrowing policy to maximize the discounted stream of dividends, whereas a private firm adopts and investment and borrowing policy to maximize the discounted stream of *utility.*

Debt policy is complex so we initially abstract from debt and assume that the firms are "liquidity constrained" in the sense that any investment they undertake must be financed out of current cash flows. We consider situations where a firm would consider having "retained earnings" but generally, the firms we study will not find it optimal to hold cash balances, but rather either invest all cash, or pay it out as dividends to shareholders. However firms that start with little initial capital may want to borrow. We consider a simplest situation where the firm is allowed to incur debt in the initial period of its existence and pay off any debt it incurs as a *consol* — i.e. a perpetual bond. We show that firms with sufficiently small initial capital stocks (including a new firm with no initial capital stock) will want to borrow as a way of "jump starting" the firm. This significantly shortens the period of time it takes for the firm to achieve "optimal scale" compared to a firm that faces liquidity constraints and is unable to borrow. So the debt option does significantly enhance the value of sufficiently small firms, but has no benefit for firms that are able to enter with a sufficient level of initial capital.

Consider first a firm that has no ability to borrow and which must finance any new investment out of current cash flows. We assume a "putty-clay" production technology where the firm can purchase new capital k using cash flows but once the capital is installed, it cannot be "liquidated" or partially sold to obtain more cash. The firm is constrained to invest using only the new cash flows produced by this capital stock $f(k)$.

At the start of period t, suppose the firm has a capital stock of k_t. It obtains a deterministic cash flow (return) of $f(k_t)$, where $f'(k) > 0$, $f''(k) < 0$ and $\lim_{k\downarrow 0} f'(k) = +\infty$. Using this cash flow, the firm can either pay dividends, $D \geq 0$, or invest an amount $I \geq 0$, subject to the firm budget constraint $D + I \leq f(k_t)$. The amount invested k_t is subject to a deterministic depreciation rate $\delta \in (0,1)$ and the investment is long-term and irreversible, in the sense that the only way to reduce k_t is via depreciation. However the amount invested can be increased by new investment I so that the capital stock follows the law of motion

$$k_{t+1} = k_t(1-\delta) + I. \tag{1}$$

We initially assume that the firm is liquidity constrained and cannot borrow, so it can only use its current cash flow $f(k_t)$ to finance dividend payments and new investment. The firm discounts the future at a constant rate $\beta \in (0,1)$ and its objective is to maximize the present discounted value of future dividend payments.

In the next section we describe the analytical solution to the problem. In section 3 we extend the solution to the case where the firm can make an initial borrowing decision in the first period of its existence, paying off any debit it incurs using a *consol* which is a bond that has no maturity date, but rather involves in infinite stream of interest payments whose discounted value equals the initial amount borrowed. We show how the borrowing option increases the value of sufficiently small firms. Then in section 4 we compare the analytical solution to the numerical solution produced by the the method of “discrete policy iteration” (DPI) and show it is highly accurate. In DPI the value function is approximated by a piece-wise linearly interpolated solution to a linear system of equations over a finite grid of points in the state space, k. In section 4 we introduce the problem of a “private firm” that is subject to borrowing constraints, where an individual invests his/her private wealth in the firm and operates the firm not to maximize the discounted stream of dividends, but rather to maximize the discounted stream of utility from consumption, where consumption includes payment of profits from the firm. We show that the investment policy of a private firm is very different from the investment policy of a public firm due to the consumption smoothing motive of the owner of a privately held firm. In section 5 we consider whether the owner of a private firm would wish to “go public” by selling off their ownership interst in their firm, converting it from a privately owned firm to a publicly owned firm whose objective is to maximize the discounted value of dividends. This decision can be viewed as a simplified model of an “initial public offering” (IPO). We show that similar to borrowing, the IPO decision is generally optimal only for firms that are sufficiently small. When the firm is sufficiently large, the owner would prefer to remain private rather than “take it public.”

2 Publicly held firm: Analytical Solution

Let $V(k)$ denote the value of a publicly held firm when its capital investment is $k \geq 0$. Recall the term “publicly held” signals that the firm’s objective function is to maximize the discounted value of dividend payments to shareholds. The Bellman equation for the firm is given by

$$V(k) = \max_{0 \le I \le f(k)} \left[f(k) - I + \beta V(k(1-\delta) + I) \right]. \tag{2}$$

It is clear that $V(0) = 0$, since when the firm has no capital investment, it generates no cash returns, and thus cannot invest any more funds, and thus will not receive any future cash flows from which it can pay out dividends in the future. Since the marginal return to investment approaches infinity as $k \downarrow 0$, it is reasonable to conjecture that the firm's optimal investment policy has three different regions: 1) an initial region $[0, \underline{k})$ where the firm pays no dividends and devotes all cash flows to investment, 2) an intermediate region $[\underline{k}, \overline{k}]$ where the firm invests and pays dividends, and 3) a final region $(\overline{k}, \infty)$ where the firm has "excess capital" and so it does not invest and pays out all cash flow in the form of dividends. In the intermediate zone where the firm invests and pays dividends, we conjecture that the firm invests just enough to achieve a target or "steady state" level of capital k^* which is the solution to

$$k^* = \underset{k \ge 0}{argmax}\, \beta \frac{f(k) - \delta k}{(1-\beta)} - k \tag{3}$$

The interpretation is that k^* is the level of capital that can be maintained in steady state that maximizes the discounted present value of the firm, net of the cost of the initial investment k. That is, if an investor were to take k in cash and invest it in the firm today, and in all future periods the firm's investment equals the replacement investment (i.e. the depreciation of this invested capital, δk), in order to maintain the capital at level k, then the optimal initial investment that maximizes the difference between the net present value of the firm after the investment (a perpetual dividend stream of $f(k) - \delta k$ that starts with a one period delay) and the initial amount of the investment k is the value k^* given in equation (3). It is not hard to see using calculus that k^* is given by

$$k^* = f'^{-1}(1/\beta - 1 + \delta). \tag{4}$$

where f'^{-1} is the inverse of the marginal return function, $f'(k)$, which is invertible due to our assumption that $f''(k) < 0$.

If we write $\beta = 1/(1+r)$ where $r > 0$ is the one period "market interest rate", then we can rewrite the first order condition determining the optimal steady state capital stock k^* as follows

$$f'(k^*) = r + \delta \tag{5}$$

and observe that this is similar to the equation for the "Golden rule" steady state capital stock in the Solow growth model, except that the population growth rate n is used in place of a "market interest rate" r. The intuition for condition (5): the marginal cash flows produced by the optimal stead state capital stock must be sufficient to cover 1) depreciation of capital, δ, and 2) the opportunity cost of capital, r. Thus, the marginal product of capital equals the sum of these, $r+\delta$, at the optimal steady state level of the capital stock.

Given this, we conjecture that the optimal investment rule $I(k)$ (which describes investment when we are not necessarily at the steady state, k^*) takes the following form

$$I(k)=\begin{cases} f(k) & \text{if } k\in[0,\underline{k}) \\ k^*-(1-\delta)k & \text{if } k\in[\underline{k},\overline{k}] \\ 0 & \text{if } k\in(\overline{k},\infty). \end{cases} \tag{6}$$

It is easy to see that $\overline{k}$ is given by

$$\overline{k}=\frac{k^*}{(1-\delta)} \tag{7}$$

and $\underline{k}$ is given by

$$f(\underline{k})=k^*-(1-\delta)\underline{k} \tag{8}$$

These values of $\underline{k}$ and $\overline{k}$ ensure that the optimal investment function $I(k)$ is a continuous function of k. The optimal dividend function is then determined trivially as follows by assuming that the budget contraint is binding at all k

$$D(k)=f(k)-I(k). \tag{9}$$

Using equation (6) we obtain the following equation for $D(k)$

$$D(k)=\begin{cases} 0 & \text{if } k\in[0,\underline{k}) \\ f(k)+(1-\delta)k-k^* & \text{if } k\in[\underline{k},\overline{k}] \\ f(k) & \text{if } k\in(\overline{k},\infty). \end{cases} \tag{10}$$

Now we verify these conjectures are correct and derive an explicit formula for the value function $V(k)$ by making use of the Bellman equation (2), and showing that these conjectured optimal investment and dividend policies do result from the solution to the firm's Bellman equation.

First, for k in the "unconstrained region" $[\underline{k},\overline{k}]$ there is an interior solution for the optimal level of investment $I(k)$ implied by the Bellman equation (2). That is, assuming that $v(k)$ is differentiable in this region, then $I(k)$ must satisfy the following first order or *Euler equation*

$$1=\beta V'(k(1-\delta)+I(k)). \tag{11}$$

Substituting the optimal investment rule $I(k)$ into the right hand side of the Bellman equation (2) and differentiating with respect to k, making use of the *envelope theorem*, we have

$$\begin{aligned} V'(k) &= f'(k) + (1-\delta)\beta V'(k(1-\delta)+I(k)) \\ &= f'(k) + (1-\delta) \end{aligned}$$

where we used the fact that the Euler equation (11) holds for $k \in [\underline{k}, \overline{k}]$. The envelope equation implies that $V(k)$ is given by

$$V(k) = f(k) + (1-\delta)k + C \tag{12}$$

for some constant C when $k \in [\underline{k}, \overline{k}]$. Notice that at $k = k^*$ the firm generates a perpetual dividend stream of $f(k^*) - \delta k^*$ so this implies that

$$V(k^*) = \frac{f(k^*) - \delta k^*}{(1-\beta)} \tag{13}$$

So using the other formula for $V(k)$ from equation (12), this implies that the unknown constant C is given by

$$C = \frac{\beta\,[f(k^*) + \delta k^*]}{(1-\beta)} - k^* \tag{14}$$

Thus, we can see that C equals the optimized right hand size of the net gain from initial investment in equation (3) which determined the optimal steady state capital stock value k^*. Thus, the value of the firm in the interval $[\underline{k}, \overline{k}]$ is this optimized value, plus $f(k) + (1-\delta)k$. The intuition for this formula is that once the firm is in the interval $[\underline{k}, \overline{k}]$, its investment $I(k) = k^* - (1-\delta)k$ will enable it to achieve the optimal steady state capital level k^* in the following period. So it follows that $V(k)$ equals the net dividends this period, $D(k) = f(k) - I(k) = f(k) - k^* + (1-\delta)k$ plus the present value of all future dividends in all subsequent periods $\beta[f(k^*) - \delta k^*]/(1-\beta)$ where this period's investment has enabled the firm to achieve the optimal steady state capital stock k^*.

Now we need to verify that the optimal investment rule $I(k)$ for $k \in [\underline{k}, \overline{k}]$ really is the formula we conjectured, $I(k) = k^* - (1-\delta)k$. To show that this is correct, we need to show that this satisfies the Euler equation (11). Using the closed form solution for $V(k)$ in equation (12) we can rewrite the Euler equation as

$$1 = \beta\left[f'(k(1-\delta) + I(k)) + (1-\delta)\right] \tag{15}$$

Solving this equation for $I(k)$ we can see that

$$\begin{aligned} I(k) &= f'^{-1}(1/\beta - (1-\delta)) - (1-\delta)k \\ &= k^* - (1-\delta)k \end{aligned}$$

which does indeed match the formula we conjectured in equation (6).

Finally we need to derive formulas for $V(k)$ in the "constrained no dividend region" $[0,\underline{k})$ where $I(k) = f(k)$ and show that it is indeed optimal for the firm to invest all available cash flow and not pay any dividends in this region, and also for the "excess capital, no investment region" we need to derive $V(k)$ and show that it is indeed optimal for the firm to invest zero in this region and pay out all cash flows as dividends, $I(k) = 0$ and $D(k) = f(k)$.

Consider the latter region first. Consider a value of $k > \overline{k} = k^*/(1-\delta)$ that is sufficiently close to $\overline{k}$ so that after depreciation we have $k(1-\delta) \in [\underline{k},\overline{k}]$. In particular, we have $(1-\delta)k > k^*$, so that after depreciation (assuming zero investment) the capital exceeds the optimal steady state capital level k^* in the region where the firm invests and pays dividends.

We claim that for this value of k the value of the firm is given by

$$V(k) = f(k) + \beta\left[f((1-\delta)k) + (1-\delta)^2 k + C\right], \tag{16}$$

and the optimal investment at this value k is $I(k) = 0$. To see this, we consider the value of investing a positive amount $I > 0$

$$V(k,I) = f(K) - I + \beta[f((1-\delta)k + I) + (1-\delta)[(1-\delta)k + I] + C] \tag{17}$$

Notice that for this fixed value of k, the function $V(k,I)$ is strictly concave in I due to our assumption that f is strictly concave. So it is sufficient to show that $\frac{\partial}{\partial I}V(k,I) < 0$ at $I = 0$. The concavity of $V(k,I)$ in I then implies that this partial derivative is negative for all higher values of I which implies that the optimal value of investment is zero at this value of k, $I(k) = 0$. Evaluating the partial derivative of $V(k,I)$ with respect to I at $I = 0$ we have

$$\frac{\partial}{\partial I}V(k,I) = -1 + \beta f'((1-\delta)k) + \beta(1-\delta) \tag{18}$$

However the first order condition for the optimal steady state capital stock level can be written as

$$0 = -1 + \beta f'(k^*) + \beta(1-\delta) \tag{19}$$

Since f is strictly concave and $(1-\delta)k > k^*$, equation (18) implies that $\frac{\partial}{\partial I}V(k,I) < 0$ at $I = 0$, and so we can conclude it is optimal for the firm not to invest at k.

Since we know that no investment is optimal at this point, we conclude that

$$V(k) = f(k) + \beta\left[f((1-\delta)k) + (1-\delta)^2 k + C\right] \tag{20}$$

This will hold for any value of k such that $k^* < (1-\delta)k < k^*/(1-\delta)$. Continuing inductively we can so that if $k(1-\delta) > \overline{k} = k^*/(1-\delta)$ but $k < k^*/(1-\delta)^3$, it will take 2 periods for capital to depreciate to a value $(1-\delta)^2 k \in (k^*, \overline{k})$. We can show that in this interval of k zero investment is optimal as well, using an argument similar to the one above. In fact by a formal induction proof, we can show that $I(k) = 0$ for all $k > \overline{k} = k^*/(1-\delta)$ and $V(k)$ is given by

$$V(k) = \sum_{i=0}^{n-1} \beta^i f((1-\delta)^i k) + \beta^n [f((1-\delta)^n k) + (1-\delta)^n k + C] \quad k \in \left[k^*/(1-\delta)^n, k^*/(1-\delta)^{n+1}\right). \quad (21)$$

Since the equation above satisfies the Bellman equation (2) by construction, it follows that $I(k) = 0$ and $D(k) = f(k)$ is the optimal investment and dividend policy for the firm in the region $k > k^*/(1-\delta)$ and $V(k)$ is given by the formula in (21) once we determine the smallest number of periods n that are required for the capital to depreciate down to a level $k \in (k^*, k^*/(1-\delta))$ where it becomes optimal for the firm to invest again.

Now consider the final interval $k \in [0, \underline{k})$. In this region we claim that it is optimal for the firm to invest all available cash flow and pay no dividends. That is, $D(k) = 0$ and $I(k) = f(k)$. We now verify that this conjecture is correct. Recall that $\underline{k}$ was defined as the solution to the equation

$$f(\underline{k}) + (1-\delta)\underline{k} = k^* \quad (22)$$

Consider a $k < \underline{k}$, but a value not so close to zero so that if the firm invests all cash flow and pays zero dividends, then its capital stock at the start of next period, $f(k) + (1-\delta)k$, satisfies

$$f(k) + (1-\delta)k > \underline{k}. \quad (23)$$

How do we know there is a $k < \underline{k}$ that satisfies inequality (23) above? First notice that $f(k) + (1-\delta)k$ is a strictly concave function of k and notice that at k^*, it is easy to manipulate the first order condition determining the optimal steady state capital stock in equation (3) to show that

$$f'(k^*) + (1-\delta) = 1/\beta > 0. \quad (24)$$

and hence we conclude that $f(k) + (1-\delta)k$ is strictly increasing in k for $k \le k^*$. But since dividends must be positive at k^* we have $f(k^*) - \delta k^* > 0$ which is equivalent to $f(k^*) + (1-\delta)k^* > k^*$. Then since $\underline{k}$ is defined as the value of k that solves $f(k) + (1-\delta)k = k^*$, it follows that $\underline{k} < k*$. If $k < \underline{k}$, then the fact that $f(k) + (1-\delta)k$ is strictly increasing in k implies that $f(k) + (1-\delta)k < f(\underline{k}) + (1-\delta)\underline{k} = k^*$.

Let $k < \underline{k}$ be such that $f(k) + (1-\delta)k > \underline{k}$. We now want to show that it is optimal for the firm to invest all cash flow, $f(k)$, and pay zero dividends. The investment-specific value function is $V(k,I)$ given in equation (17) above. We want to show that $\frac{\partial}{\partial I}V(k,I) > 0$ for all $I \in [0, f(k)]$. This is given by

$$\frac{\partial}{\partial I}V(k,I) = -1 + \beta f'((1-\delta)k + I) + \beta(1-\delta). \tag{25}$$

Noting that $V(k,I)$ is strictly concave in I it is sufficient to show that $\frac{\partial}{\partial I}V(k,I) > 0$ when I takes the maximum possible value, $I = f(k)$. In this case, the partial derivative in equation (25) reduces to

$$\frac{\partial}{\partial I}V(k, f(k)) = -1 + \beta f'((1-\delta)k + f(k)) + \beta(1-\delta). \tag{26}$$

But we know that, from the argument above, that $\frac{\partial}{\partial I}V(k^*, f(k^*)) = 0$ and that $f(k) + (1-\delta)k$ is strictly increasing in k for $k < k^*$. So this implies that $\frac{\partial}{\partial I}V(k, f(k)) > 0$ as claimed.

Let $\underline{k}^1$ be given by the solution to $f(\underline{k}^1) + (1-\delta)\underline{k}^1 = \underline{k}$, or $\underline{k}^1 = 0$ if no solution exists. Then, it is not hard to show using the same argument as above that if $\underline{k}^1 > 0$ we must have $\underline{k}^1 < \underline{k}$. Then for all $k \in [\underline{k}^1, \underline{k})$ we have $I(k) = f(k)$ and $D(k) = 0$ and

$$V(k) = \beta\left[f(k(1-\delta) + f(k)) + (1-\delta)[(1-\delta)K + f(k)] + C\right] \tag{27}$$

If $\underline{k}^1 > 0$ then we can recursively define $\underline{k}^j$, $j = 2, 3, \ldots$ by the formula

$$f(\underline{k}^j) + (1-\delta)\underline{k}^j = \underline{k}^{j-1} \tag{28}$$

until the first value of j is reached where $\underline{k}^j = 0$. Define the function $T(k)$ by

$$T(k) = k(1-\delta) + f(k) \tag{29}$$

and define the composite powers of T, T^2, T^3, etc by

$$T^2(k) = T(T(k)) = T(k)(1-\delta) + f(T(k)) \tag{30}$$

and in general

$$T^j(k) = T^{j-1}(T(k)), \quad j = 1, 2, \ldots \tag{31}$$

where we define $T^0(k) = k$. Then if $k \in [\underline{k}^j, \underline{k}^{j-1})$ (where $\underline{k}^0 = \underline{k}$) we have

$$V(k) = \beta^j\left[f(T^j(k)) + (1-\delta)T^j(k) + C\right]. \tag{32}$$

Using an induction argument, we can show that $I(k) = f(k)$ and $D(k) = 0$ for k in every interval $[\underline{k}^j, \underline{k}^{j-1})$. By construction, $V(k)$ satisfies the Bellman equation (2). We conclude that we have derived a closed form solution for the optimal investment policy in equation (6) and the optimal dividend policy in equation (10) and have an analytic (if recursive) expression for the value function in equations (12), (21) and (32) where the constant C is given by equation (14) and the optimal steady state capital stock k^* is given by equation (4). Further, we can use induction to prove the following result

Theorem *$V(k)$ is strictly concave.*

The proof involves considering V over the three different regions $k \in [0, \underline{k})$, $k \in [\underline{k}, \overline{k}]$ and $k \in (\overline{k}, \infty)$. In the middle region, $V(k) = f(k) + (1-\delta)k + C$ and strict concavity in this region follows from the assumption that $f(k)$ is strictly concave. In the upper region $(\overline{k}, \infty)$ V is given by formula (21) and it is straightforward to see that V is strictly concave in this region as well. Finally in the initial "no dividend" region $[0, \underline{k})$, the concavity follows from an induction argument. We first show by induction that for each $j \geq 1$ that $T^j(k)$ is concave. Then using the properties of compositions of concave functions, it is easy to show from equation (32) that V is strictly concave on $[0, \underline{k})$ as well.

Figure 2 plots the optimal investment and dividend rules for the case $f(k) = \sqrt{k}$. We see that optimal investment intersects the black "replacement investment" line (i.e. the line δk) exactly at k^*, the optimal steady state capital stock level, which equals 23.73 in this example. The level of optimal investment at the steady state is $\delta k^* = 1.1867$, which of course is just enough to offset the corresponding depreciation in capital.

Figure 2 plots the value function for this problem. Notice there are no discontinuities in the value function at the various break points, $\{\underline{k}^j\}$ and $\{k^*/(1-\delta)^j\}$ above and below the cutoffs $\underline{k}$ and $\overline{k}$ defining the region where investment and dividends are positive. The value function is monotonic and strictly concave in k and satisfies $V(0) = 0$.

The dynamics of the capital stock are clear: starting from any k the capital stock converges globally to the unique optimal steady state level k^* in a finite number of periods. For $k \in [\underline{k}, \overline{k}]$ the firm undertakes investment $I(k) = k^* - (1-\delta)k$ enables it to jump to the optimal steady state value k^* in a single period. When initial capital is either below or above this region, the firm has to wait several periods for capital to accumulate above the lower $\underline{k}$ threshold, or depreciate down below the upper $\overline{k}$ threshold.

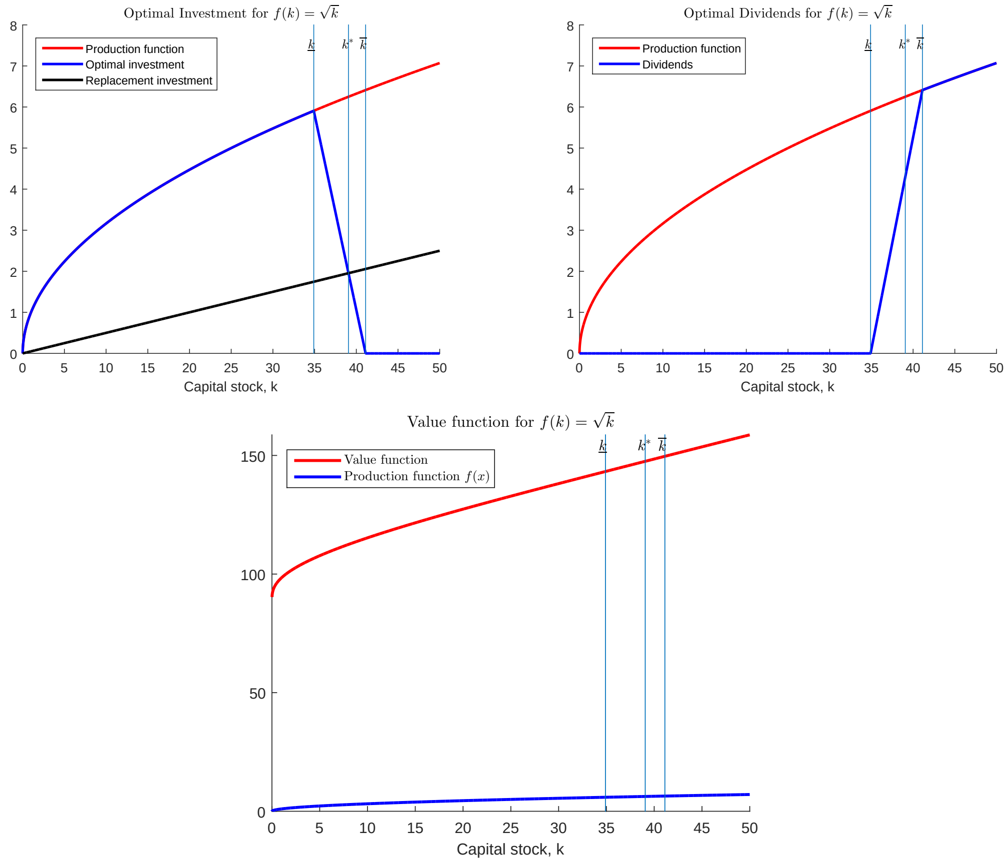

Figure 1: Optimal investment and dividend policy and value function for $f(k) = \sqrt{k}$

3 Extending the model to allow debt

Note that the solution we provided above has the "boundary condition" $V(0) = 0$, i.e. if a firm has no initial capital stock, it will not have any cash flows to invest, and thus it is never able to "get off the ground" even though there may be an attractive production technology $f(k)$ that the firm "owns". We might think of f as the "entrepreneurial idea" but that idea cannot be implemented with an actual cash investment to get the firm going. As long as the firm has some way of getting this initial investment, it is enough to get it going and eventually a sequence of investments will lead it to reach the optimal steady state capital stock k^*, which is the same capital stock it would choose if it had sufficient capital to make a large one time investment at the optimal scale.

We now extend the model to all the firm to make a *one time* borrowing decision at period 0. The firm would borrow enough funds to purchase an initial amount of capital to get the firm going. We assume that the capital cannot be installed instantaneously so that cash flows from the capital stock the firm can achieve with the borrowing are not realized immediately, but start in period 1. Interest payments on the debt also start in period 1 and continue indefinitely because the only means of finance is via a consol. We also assume that there are no "borrowing constraints" so that the firm can choose to borrow enough capital to reach the optimal steady state capital stock level k^*. Will it be optimal for the firm to borrow enough to install a plant of size k^* in period 0?

We will show that it depends on the amount of capital the firm already has in place, and the interest rate at which it can borrow, which we denote by r_b. Recall that $r = 1/\beta - 1$ is the "market rate of interest" at which future dividends payed by the firm at discounted. We will now show that if $r_b = r$ and $k < k^*$, it will be optimal for the firm to borrow an amount $k^* - k$ so that it can attain the optimal steady state capital stock immediately, without any further delay. However if $r_b > r$, then the firm will find it optimal to borrow a lower amount so that its initial capital stock that it reaches after this borrowing, $k^*(r_b)$, is strictly less than k^*. The firm will then invest from its cash flows and in a finite number of periods after this, its sequence of investments will lead it to reach the optimal steady state capital stock k^*.

We work with a simplified "one shot" model of debt because of the complexities of modeling debt in a dynamic programming framework. Any debt with a finite maturity date requires more state variables to describe how many periods are left to pay off the existing debt, at what interest rate, and what level of interest plus principal payments are made over time. Further, if the firm periodically issues new multiperiod debt contracts, we have to keep track separately of all these additional state variables for each separately. It is only convenient to deal with two polar extremes: 1) a one time issuance of perpetual debt (e.g. a consol), or 2) roll-over of single period debt contracts. We will start by discussing the first borrowing option and then at the end of this section consider the case of financing the firm's investments using a sequence of single period debt contracts.

For the case of perpetual debt, if the firm borrows amount b in period 0, it will pay that back over an infinite stream of fixed interest payments e in periods $1, 2, 3, \ldots$. If the rate of interest is r_b, in a single period debt contract the amount due in period 1 would be $(1 + r_b)b$. If the borrowed amount b were financed via a consol, the amount of each payment e has to be calculated so that the present value of a perpetual stream of payments of e per period equals $b(1 + r_b)$, or

$$(1 + r_b)b = e \sum_{t=0}^{\infty} \left(\frac{1}{1 + r_b} \right)^t = \frac{e(1 + r_b)}{r} \tag{33}$$

and hence $e = r_b b$, which is just the interest on the amount borrowed b. Now suppose the firm has initial capital k and wishes to borrow amount b in period 0, paying back the amount borrowed in periods $1,2,3,\ldots$ using a consol with per period payment $e = r_b b$ as discussed above. How much should the firm borrow if it faces no borrowing constraints?

The easiest case to consider is where $r = r_b$, i.e. where the firm can borrow at the same rate of interest as the market discounts its future dividend payments to shareholders. The value of the *equity* of the firm that has borrowed amount b at time period 1 is just the value of the firm at time 1, $V(b+k)$, less the present value of the consol payments, $(1+r)b$, so the optimal borrowing level b^* is the solution to

$$b^* = \underset{b \geq 0}{argmax} \frac{V(k+b) - b(1+r)}{(1+r)}. \tag{34}$$

Note that since cash flows and interest payments from the investment at time 0 do not commence until time period 1, we discount the net value of the firm at time 1, $V(k+b) - b(1+r)$, to obtain the net present value of borrowing an amount b as of period 0.

Assume that k is not too large that there is an interior solution, and conjecture that the solution occurs in the region $[\underline{k}, \overline{k}]$ where $V(k) = f(k) + (1-\delta)k + C$. Then we have that the optimal amount to borrow, b^* satisfies

$$f'(b^* + k) = r + \delta \tag{35}$$

but since $f'(k^*) = r + \delta$ as we have shown above, it follows that

$$b^* = k^* - k, \tag{36}$$

i.e. the firm borrows enough to reach the optimal steady state capital stock k^*. If $k > k^*$, then since V is strictly concave in k, we have $V'(k) < r + \delta$, so we conclude that $b^* = 0$. Thus, we have derived the optimal debt policy for the firm in period 0:

$$b^*(k) = \begin{cases} k^* - k & \text{if } k < k^* \\ 0 & \text{if } k \geq k^* \end{cases} \tag{37}$$

Now consider the case where $r_b > r$. This seems appropriate in many cases where firms can borrow, but at a higher interest rate than the "market interest rate". Though there is no uncertainty in this model, it can reflect market imperfections where it is more costly for firms to borrow, though we will also consider the opposite case where $r_b < r$, which can also arise in real world situations with stochastic returns when a firm's dividend stream is considered to be sufficiently risky that it is discounted at a higher rate than the rate the firm can borrow at.

If the firm has an infinite stream of debt payments equal to $e = r_b b$ to pay out due to borrowing amount b, the present value of this stream *discounted at the market interest rate r back to period 0* is br_b/r. So the firm's problem in this case becomes

$$b^* = \underset{b \geq 0}{argmax} V(k+b)/(1+r) - br_b/r \tag{38}$$

The first order condition for b^* is given by

$$V'(k+b^*) = (1+r)r_b/r, \tag{39}$$

and by the strict concavity of V there is a unique solution to this equation. Define $k^*(r_b)$ as the solution to $V'(k^*(r_b)) = (1+r)r_b/r$, then it is easy to see that $b^* = k^*(r_b) - k$.

Assume that $k^*(r_b) \in [\underline{k}, \overline{k}]$. In this region we have $V(k) = f(k) + (1-\delta)k + C$. In this case the optimal borrowing level b^* satisfies

$$f'(b^*+k) = \delta + r_b + \left(\frac{r_b}{r} - 1\right) > \delta + r \tag{40}$$

which implies that $b^* + k < k^*$. Thus, it is optimal for the firm to borrows an amount that is insufficient to enable it to reach the optimal steady state capital stock k^* right away in period 0. Instead this initial loan helps it to get most of the way there, but it is not optimal to borrow the full amount $k^* - k$ due to the higher cost of borrowing in this case. We conclude that the optimal borrowing by the firm when $r_b > r$ is given by

$$b^*(k) = \begin{cases} k^*(r_b) - k & \text{if } k < k^* \\ 0 & \text{if } k \geq k^* \end{cases} \tag{41}$$

The solution in the case where $r_b < r$ does make a lot of sense in this example. If there are truly no borrowing constraints, the firm should want to borrow an infinite amount. The reason is that for each dollar the firm borrows at period 0, it starts an infinite stream of consol payments in periods $t = 1, 2, 3, \ldots$ that has present value (evaluated at the market interest rate r) of $(1+r)r_b/r$ in period 1. Discounting this back to period 0, the effect of borrowing 1 today on the net present value of the firm (even if the amount borrowed is not used to finance investment) is $1 - r_b/r > 0$, so the firm can increase its value without bound by borrowing an infinite amount if it can. For this reason we do not consider the case $r_b < r$ any further in this simple example.

When the firm has debt its value becomes $V(k^*(r_b)) - (1+r_b)(k^*(r_b) - k)$ in the region $k \in [0, k^*(r_b)]$ and is equal to $V(k)$ for $k > k^*(r_b)$, where V is the solution to the Bellman equation (2) in the case of a firm that does not have a borrowing option. So the borrowing option replaces the strictly convex segment

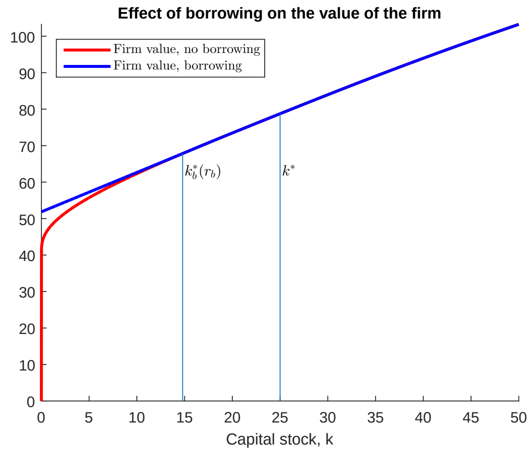

Figure 2: The effect of borrowing on firm value

of $V(k)$ over the interval $[0, k^*(r_b)]$ with the linear segment, and in particular, the firm has a positive value even when $k = 0$ when it can borrow, whereas it has no value when it cannot borrow.

The gains from debt finance are illustrated in figure 2 below which shows an example where $r = 0.05$, $r_b = 0.08$ and $f(k) = \sqrt{k}$. The firm chooses to borrow only enough to reach a capital stock of $k^*(r_b) = 14.7929$, more than 10 less than the optimal steady state capital stock, k^*. The firm uses internal finance (retained earnings) to reach k^*, and since $k^*(r_b) < \underline{k} = 21.446$, it forgoes paying dividends for several period until it can accumulate sufficient capital to enter the zone $[\underline{k}, \overline{k}]$ where it has enough cash flow to finally reach k^* without forgoing all dividend payments to shareholders.

Now consider a firm that can only borrow via a sequence of one period debt contracts. The firm faces a borrowing constraint $\overline{B}$ on the total amount that the market will lend it at interest rate $r_b \geq r$ each period. So if the firm borrows b at period 0, it must pay back principal and interest $(1+r)b$ in period 1. It can borrow some amount $b' \leq \overline{B}$ in period 1 and continue on in this way indefinitely, financing its investments as it needs to via a sequence of one period loans. We now let $V(k,b)$ denote the present value of a public firm that has capital stock k and total debt of b. The Bellman equation for this firm is given by

$$V(k,b) = \max_{\substack{0 \leq b' \leq \overline{B} \\ I \leq f(k)+b'-b}} \left[f(k) - I + b' - b + \beta V(k(1-\delta)+I, b'(1+r)), 0 \right]. \tag{42}$$

In the Bellman equation (42) we have written the value $V(k,b)$ as a maximum of 0 or the value of continuing to operate as an ongoing concern. This effectively serves as a “bankruptcy constraint” that the value of the equity of the firm can never be less than 0 due to limited liability (shareholders cannot be paid "negative dividends" to cover interest and principal on debt, if the cash flows of the firm are not large enough to enable the firm to pay its accumulated debt). In particular, if the debt b is so large that there is no feasible solution to the first optimization problem on the right hand side of (42), then the value of the firm would also be zero. This could correspond to a situation where even if the firm invests zero and pays no dividends, there is not enough cash flow $f(k)$ to repay the current amount due, b, even if the firm's new borrowing b' equals the maximum amount allowed, $\overline{B}$. The firm would effectively be bankrupt at that point, and the bondholders would take control of the firm and operate it in a way to recover as much of the outstanding debt as possible.

Note that with one period debt, and when the borrowing limit $\overline{B}$ is sufficiently large to enable the firm to reach the optimal capital stock $k^*(r_b)$, the firm can replicate the borrowing it could do with a consol by perpetually rolling over one debt. That is, the firm borrows b at period 0 and must pay back $b(1+r+b)$ at $t=1$. But it also borrows b at $t=1$ so its net cash out flow at $t=1$ is just $b-(1+r_b)b=-br_b$. Continuing this way, the firm could maintain a debt load of b indefinitely at the cost of a constant stream of interest payments of br_b per period, the same as it would have to pay if it financed the investment by a consol. However when $r_b>r$, the firm can increase its value by paying off its debt as fast as possible rather than maintaining the debt perpetually as is the case with a consol. Thus, in principle the firm should be able to increase its value via proper debt management when it can finance itself via a sequence of one period debt contracts.

Unfortunately a full treatment of this case appears to be very difficult and is beyond the scope of this note. We can partially solve the problem in certain "easy cases." For example, if $k>\overline{k}$, we know from the optimal solution to the problem without debt that optimal investment is zero in this region. It is natural to conjecture that this is also the case when the firm can borrow. However even this case is complicated. If $r_b=r$, then the time path over which the firm pays off its debts should be irrelevant. It is easy to show that this is indeed the case when $k>\overline{k}$. Then, we conjecture that $V(k,b)=V(k)-b$ where V is the solution to the Bellman equation (2) for the firm problem without debt. Inserting this conjecture into the Bellman equation (42) we obtain

$$V(k,b)=V(k)-b=\max_{\substack{0\le b'\le\overline{B}\\ I\le f(k)+b'-b}}\left[f(k)-I+b'-b+\beta V(k(1-\delta))-b',0\right]. \tag{43}$$

We have already shown that $I(k)=0$ when $k>\overline{k}$ and this continues to be the case in the case with debt above. We also see that b' completely cancels out of the right hand side of equation (43) so the value on the right hand side reduces to $V(k)-b$, verifying that our conjectured solution is correct.

However when $r_b > r$, the debt policy of the firm matters, even when $k > \overline{k}$. In this case $V(k,b)$ cannot be given by the previous conjectured form $V(k,b) = V(k) - b$. To see why, suppose this conjectured form did hold, then we would have a modified version of equation (43) given by

$$V(k,b) = V(k) - b = \max_{\substack{0 \le b' \le \overline{B} \\ I \le f(k)+b'-b}} \left[f(k) - I + b' - b + \beta V(k(1-\delta)) - \beta(1+r_b)b', 0 \right]. \tag{44}$$

Since $\beta(1+r_b) = (1+r_b)/(1+r) > 1$, we now see that the right hand side of equation (44) is strictly decreasing in borrowing b' and so the optimal solution now is $I(k) = 0$ and $b' = \max[b - f(k), 0]$, and this implies that

$$V(k,b) = V(k) + \max[b - f(k), 0] \left(1 - \frac{1+r_b}{1+r} \right) \neq V(k) - b. \tag{45}$$

In general, debt management becomes a much more complex problem when $r_b > r$, and it may in fact be optimal for the firm to invest even in the "no investment region" $k > \overline{k}$ because the investment can result in greater cash flows that can help the firm to retire its debt more quickly.

Due to the complexities identified above, it is not clear how many purely analytical insights we can obtain for the firm's problem with single period debt contracts. It may be that we need to resort to solving the problem numerically, a task we defer to a future paper.

4 Optimal investment for a private firm

Consider an individual who owns the production technology and who has private wealth w that they can invest (partially or fully) in their own firm. The individual has utility function $u(c)$ satisfying $u'(c) > 0$ and $u''(c) < 0$. The suppose the market interest rate is r_m but the individual's personal interest rate is r_p and thus the individual discounts future utility using discount factor $\beta = 1/(1+r_p)$. If the person purchased an annuity with their initial endowment of wealth w they would receive discounted lifetime utility of $u(rw)/(1-\beta)$. Now suppose instead the person invests their wealth to buy an equivalent amount of capital $w = k$ and from each period onward the owner manages the firm to obtain dividends which he/she consumes. What is the optimal investment and dividend policy for this "privately held firm"?

Suppose that $w > k^*$ where k^* is the optimal steady capital stock of the publicly held firm given in equation (4) above. Is it optimal for the owner of the private firm to invest this amount too? Assume that after making an initial capital investment k, the private owner restricts attention to "steady state" investment policies $I(k) = (1-\delta)k$ that will maintain the capital stock of the firm at the initially invested value k forever. What is the optimal value of k that the owner would choose?

This is given by the solution k_p^* to

$$k_p^* = \underset{0 \le k \le w}{argmin} \frac{u(f(k) - \delta k - r(w-k))}{1-\beta}. \tag{46}$$

The first order condition for the optimal steady state policy is

$$f'(k_p^*) = \frac{1}{\beta} - 1 + \delta \tag{47}$$

so we see in fact that $k^* = k_p^*$: the owner of a private firm would invest to the same steady state capital stock value that a publicly held firm would choose if it were to make an initial investment and be able to borrow the funds necessary at the same interest rate as the discount rate the market uses to value the firm (i.e. to discount its future dividend stream).

However assume that $w < k^*$. For the moment, let's conjecture that the owner would choose to invest all of his/her wealth in the firm, so they will receive no annuity income after sinking all of their initial wealth as an investment in their firm. Assuming the owner cannot borrow, the Bellman equation for the privately held firm is given by

$$V(k) = \max_{0 \le I \le f(k)} \left[u(f(k) - I) + \beta V(k(1-\delta) + I) \right]. \tag{48}$$

The first order condition for optimal investment is given by

$$u'(f(k) - I(k)) = \beta V'(k(1-\delta) + I(k)). \tag{49}$$

If we were to assume an "Inada condition" i.e. that $\lim_{c \downarrow 0} u'(c) = +\infty$, then it is easy to see that the optimal investment policy will always entail paying some positive level of dividends, i.e. $I(k) < f(k)$ for all k. However it may still be the case that if the firm had sufficient capital, it may be optimal not to invest, i.e. $I(k) = 0$ for $k \ge \overline{k}$, though the value of $\overline{k}$ may be different than the value $\overline{k} = k^*/(1-\delta)$ at which a public firm stops investing.

Using the Envelope theorem, we have

$$V'(k) = u'(f(k) - I(k)) f'(k) + \beta V'((1-\beta)k + I(k))(1-\delta), \tag{50}$$

but using the first order condition (49) we have

$$V'(k) = u'(f(k) - I(k))[f'(k) + (1-\delta)], \tag{51}$$

and substituting this back into the first order condition (49) we can derive the Euler equation characterizing the private investor's optimal investment policy $I(k)$

$$u'(f(k)-I(k)) = \beta u'\left(f(k(1-\delta)+I(k)) - I(k(1-\delta)+I(k))\right)\left[f'(k(1-\delta)+I(k))+(1-\delta)\right]. \qquad (52)$$

This is a non-linear functional equation for I and it is ordinarily not an easy one to solve via numerical methods. It is not clear there there is a closed form solution in this case, unlike the one we found for the optimal investment policy of a publicly held firm.

However we can show there is a unique steady state solution k_p^* to the Euler equation, and that $k_p^* = k^*$, the same steady state solution for a public firm. Note that any steady state, we have $I(k) = \delta k$ and substituting this for $I(k)$ in the Euler equation above we obtain

$$u'(f(k)-\delta k) = \beta u'\left(f(k)-I(k)\right)\left[f'(k)+(1-\delta)\right], \qquad (53)$$

or $f'(k) = 1/\beta - 1 + \delta$, for which the only solution is $k = k^*$. This suggests that even if the private investor does not have sufficient initial wealth to invest in the firm at the optimal level k^*, the subsequent investment policy will lead the firm to gradually accumulate capital and converge to the optimal steady state asymptotically.

Figure 4 plots the optimal investment and dividend policy functions for a privately held firm and compares them to the ones chosen by a publicly held firm. The solutions for the privately held firm were calculated numerically using the discrete policy iteration algorithm described in section 4 below. We see that both are quite different from each other. The top left panel shows the optimal dividend policies plus the level of replacement investment necessary to keep the capital stock from declining. The intersection of the optimal investment curves and the black replacement investment line defines the optimal steady state capital stock level k^* and as predicted by our analysis above, we see that it is the same for both the public and privately held firm.

Away from the steady state, investment and dividends are quite different from each other. Investment by the privately held firm is less than investment by the public firm for $k \in (0, k^*]$, but investment by the privately held firm is greater than investment by the public firm for $k > k^*$. The pattern for dividends is the opposite: the private firm pays higher dividends than the public firm for $k \in (0, k^*]$, but lower dividends for $k > k^*$, unless capital is sufficiently high that both the public and private firm stop investing, and in this region the dividend payments coincide.

The lower left panel of figure 4 plots the value of the privately held firm $V(k)$ and compares it to the utility the investor would have obtained if they invested all of their wealth in an annuity earning the

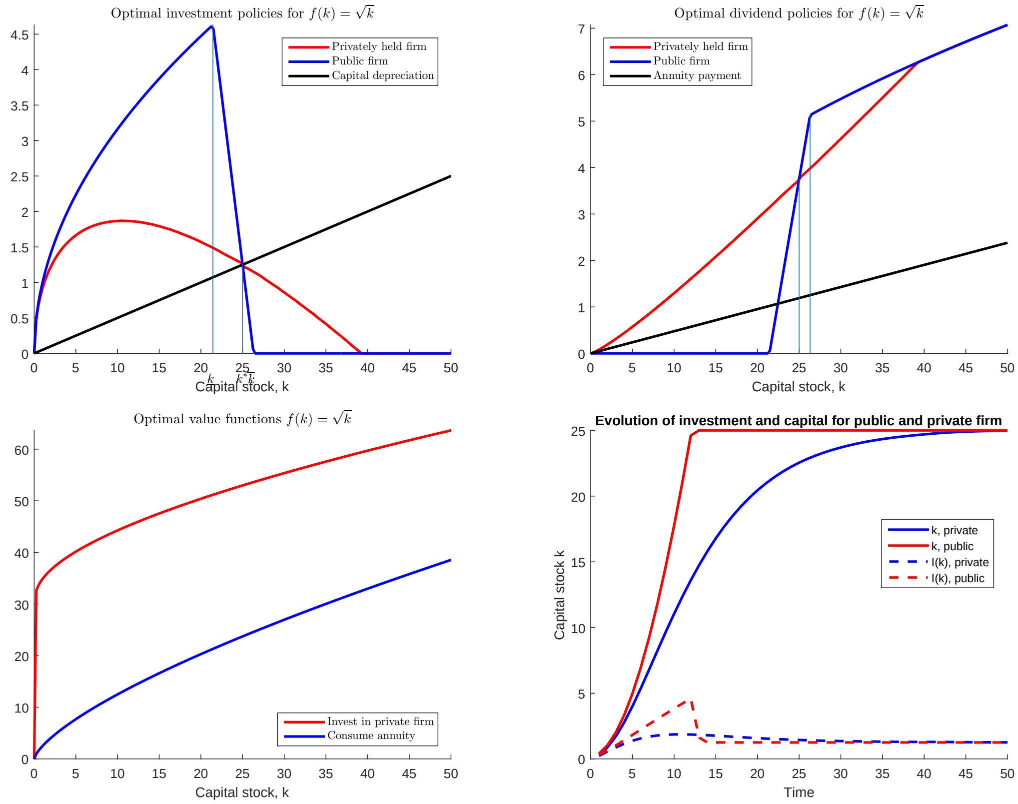

Figure 3: Optimal investment and dividend policy and value function for $f(k) = \sqrt{k}$

market rate of return. We see that at least if investment is framed as an all or nothing choice, it is always preferable for the investor to invest their wealth in the private firm rather than in an annuity. The private firm generates sufficiently greater returns to dominate the return of $r = .05$ that the person could obtain from an annuity. Another way to see this is to look at the black line in the right hand top panel of figure 4. This plots the annuity income the investor would receive each period if they invested all of their wealth into an annuity. We see that the dividend income from investing in a private firm dominates the annuity income they would receive at all levels of initial investment k.

Finally, the lower right hand panel of figure 4 compares the evolution of investment and capital stock for a public and a private firm that each begin life with an initial capital stock of $k = 1$. We see that due to the higher early investment, the public firm reaches the steady state capital stock $k^* = 25$ after only 15 periods, whereas the privately held firm approaches k^* only asymptotically.

Now consider a final question. Suppose the person who “owns” the technology has a third option: instead of investing their own wealth in their firm, the owner could “take their firm public” via an IPO (initial public offering) and after the IPO the firm would be be run by a professional manager whose objective is to maximize the present value of dividends. By selling off a 100% stake in the firm at the IPO, the owner no longer has any operating control over the firm, but the owner can take the proceeds raised by the IPO and buy an annuity and live happily ever after on this annuity income. What will the owner decide to do: sell their firm in an IPO, or keep the firm private?

If there are no transactions costs to doing an IPO, the answer is clear: the owner of the production technology $f(k)$ will do better by investing their initial wealth w to provide the capital to start the firm (so $k = w$), and then immediately hold an IPO. Let $V_m(k)$ be the market value of the firm (which does not have access to borrowing, similar to the privately held firm) given by the solution to the Bellman equation (2) in section 2. This represents the funds the owner would raise if this firm were to be sold in an IPO. The owner can then use these IPO proceeds to purchase an annuity equal to $(1-\beta)V_m(k)$. Thus, the discounted utility to the owner from holding an IPO is given by

$$V_{pub}(k) = \frac{u((1-\beta)V_m(k))}{(1-\beta)}, \tag{54}$$

and the owner compares this value to the value of keeping his firm private and operating it to maximize their lifetime discounted utility. Call this value $V_{pri}(k)$, which is the solution to the private owner's Bellman equation (48).

Figure 4 below plots these two value functions, V_{pub} and V_{pri} as well as the value of simply using their initial wealth to buy an annuity, $u((1-\beta)w)/(1-\beta)$. Though it is slightly hard to see, the value of doing an IPO uniformly dominates the value of running the firm as a private company. The reason is that the owner of a private firm, while undertaking a *privately optimal* dividend and investment policy, are nevertheless adopting a *suboptimal policy* from the standpoint of maximizing the market value of the company. The distorted dividend and investment policies that we illustrated above, plus the slower trajectory of capital accumlation due to a private owner's incentive to pay dividends in every period are costly in terms of lowering the present value of what the owner could consume if he/she sold the market to a professional manager whose objective is to maximize the market value of the firm. In essence, it is better for the owner to use the *annuity market* to smooth their consumption, than to attempt to do this on their own by distorting their investment and dividend policy. By doing an IPO, the owner allows the new management to adopt value maximizing investment and dividend policies and the owner is free to take these proceeds and smooth their consumption stream in the annuity market. This is another example of what is known as a *separation theorem* in the finance literature.

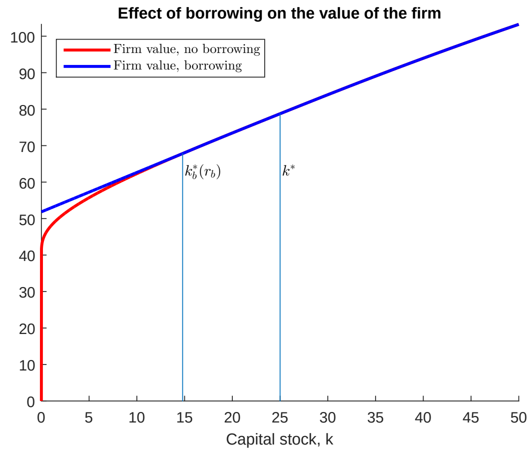

Figure 4: The value of going public with zero transactions costs

There is exactly one point where the owner is exactly indifferent between doing an IPO or keeping his/her firm private. Care to guess what that point is? Yes, you guessed it: k^*. At the optimal steady state capital stock the owner does adopt a value maximizing dividend and investment policy, and stays at that point forever. So at this particular capital stock, the owner would be indifferent between going public or staying private.

Now suppose there are transactions costs associated with doing an IPO. In many countries intermediaries such as investment banks charge hefty proportional and fixed transaction fees. A common proportional fee for doing an IPO is 7% of the proceeds raised, and the fixed costs can often be hundreds of thousands or even millions of dollars depending on the size of the company that is sold. Figure 5 plots the value of staying private and the value of selling in in an IPO in the case where the proportional transactions costs are 7% and the fixed transactions costs are zero.

We see that in the presence of transactions costs, it is no longer better to go public regardless of the initial capital stock of the firm. It is only optimal for *sufficiently small firms to do an IPO*. Once the firm has sufficient capital, it generates enough income from retained earnings to enable investment that can take the private firm close enough to the optimal steady state capital stock k^* that is it not optimal for it to do an IPO: the transactions costs involved in doing the IPO outweigh the benefits from going public.

The model can be extended to allow for debt. If we assume that the firm can take on debt at the same time it goes public, the advantages to doing an IPO are enhanced. This is illustrated by the black line in figure 5. When debt is allowed, the public firm has greater value because it is able to raise more capital to

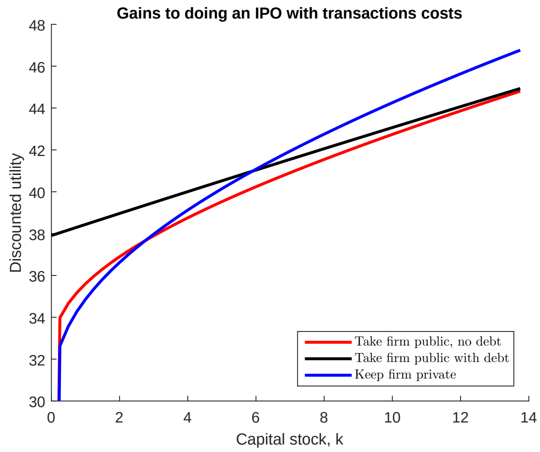

Figure 5: The value of going public with 7% transactions costs

accelerate its investment, enabling it to immediately jump to the steady state capital stock k^* immediately after the IPO. This results in a bigger region of capital stocks over which it is optimal to do the IPO despite the 7% transaction cost.

Of course, we could also assume that the private investor either had sufficient initial wealth or could also borrow and be able to make an initial investment equal to the optimal steady state capital stock k^*. If the owner did this, then as we have shown above, there is no reason to do an IPO since the owner has been able to invest at a scale to achieve a value maximizing investment and dividend policy and so there is no benefit from doing an IPO and only transactions costs.

If the owner of the firm intiially has limited wealth and borrowing potential, then we have shown that these are the conditions where an IPO makes sense, even despite the high transactions costs. The IPO enables the small firm to raise the capital necessary to invest at the efficient scale (or at least approach it more quickly if the firm is not able to borrow as much as it needs immediately after the IPO) and this gain more than offsets the transactions cost of doing the IPO.

5 IPOs with partial cash-outs

Most IPOs do not entail a 100% sell-off of the original owner's stake in the company. Intead, the original owner retains a partial ownership stake in the firm, and only takes part of the IPO proceeds in cash to finance consumption or other investment projects. The other important role of a partial cash-out is that when the original owner continues to own a significant share of the post-IPO company, the share of the

IPO proceeds that the owner does not "cash out" are re-invested in the company, thereby providing a new infusion of capital to the firm after the IPO that is not reflected in our analysis of an IPO with a 100% cash out by the original owner. Thus, an IPO can have two effects that can boost the value of the firm: 1) the IPO can switch the objective function of the firm from utility maximization of the private owner to one of value maximization in the market (an effect we describe as a "moral hazard effect"), and 2) to the extent that the original owner reinvests some of the proceeds of the IPO back into the company, it represents a new source of capital to the firm (an effect we describe as the "financing" or "leverage" effect of the IPO).

Suppose the firm is originally a privately owned firm by a sole owner, and the owner chooses to take the firm public via an IPO and retain only a fraction $\alpha \in (0,1)$ of his/her original 100% ownership stake in the firm. Thus, after the IPO the original owner will own a fraction α of the firm (i.e. α is fraction of shareholdings still owned by the founder of the firm) and the outside investors who bought shares in the new firm will own the remaining fraction $1-\alpha$ of the firm's shares.

The "IPO proceeds" equal the total amount the founder receives from selling shares in the newly public firm to the new "outside investors" and the founder can either reinvest these funds to increase the capital stock (and hence future profit/dividend stream of the firm), or take some or all of the proceeds as a "cash out" for private consumption purposes (e.g. to buy an annuity). Or the founder might want to reinvest some of the IPO proceeds in other nascent investment projects such as to found some other new firm. We will let the symbol $\omega \in [0,1]$ represent the fraction of the IPO proceeds that the owner chooses to take out for consumption or other investment purposes, and thus the fraction $1-\omega$ is reinvested in the firm.

Let $P(k,\alpha,\omega)$ represent the IPO proceeds received by a founder/owner of a private firm who decides to take the firm public when it has initial capital k, and the owner chooses to retain an ownership share α after the IPO, and to "cash out" a fraction ω of the IPO proceeds and reinvest the remaining fraction $1-\omega$. We assume that the fractions α and ω are publicly observable, as a newly public firm must meet various accounting standards that are designed to protect outside investors from fraud such as "take the money and run" schemes that are patent ripoffs of unsuspecting investors. It is one function of intermediaries such as investment banks to do the *due diligence* to investigate a private firm that wishes to go public with an IPO and verify that the company really does exist and the founder will not "take the money and run" after an IPO. Thus, the reputation of the investment bank intermediary, in addition to market regulation (such as is done by government agencies such as the Securities and Exchange Commission) helps to convince outsider investors that an IPO is legitimate and is not a thinly disguised take the money and run scheme.

We assume that an investment bank intermediary incurs costs of doing the due diligence and insuring that a private firm that wants to go public via an IPO is legitimate. The investment bank recovers the costs of providing these services by charging a proportional fee $\rho \in (0,1)$ plus, possibly, a fixed fee F. Thus if the gross proceeds of the IPO are $P(k,\alpha,\omega)$, the net proceeds received by the founder from the investment bank (after it deducts its fees) are $(1-\rho)P(k,\alpha,\omega)-F$. Initially we will study the IPO in a "frictionless market" setting where the costs of doing due diligence are zero, and hence we initially assume that $\rho = F = 0$. In this case, the gross and net proceeds of the IPO coincide.

In a market where k, α and ω are public information, and where the operations of a public firm are sufficiently regulated by both government regulators and the discipline of market competition, the public will also have a rational expectation that the newly public firm operates to maximize the discounted stream of dividend payments to its shareholders. In this case we can write an equation for the new proceeds of the IPO as

$$P(k,\alpha,\omega) = (1-\alpha)V(k+(1-\omega)[P(k,\alpha,\omega)(1-\rho)-F]). \tag{55}$$

where $V(k)$ is the value of a public company with capital stock k as defined in the Bellman equation (2) in section 2 above, which is the value of the company after an IPO with $\alpha = 1$. In equation (2) we assume that the net proceeds $P(k,\alpha,\omega)(1-\rho)-F \geq 0$, otherwise it is not clear that the founder would see any benefit to doing the IPO. Further we assume that $k > (1-\omega)F$. This implies that the function $V(P) = (1-\alpha)V(k,(1-\omega)[P-F])$ satisfies $V(0) > 0$, and together with the strict concavity of V implies that there is a unique solution $P(k,\alpha,\omega)$ to equation (55). Furthermore, it is easy to see from the strict concavity, that at this solution we have $1 > (1-\alpha)(1-\omega)(1-\rho)V'(k+(1-\omega)[P(k,\alpha,\omega)(1-\rho)-F])$. This implies that, in equilibrium, if an additional dollar were raised in the IPO, the amount of this extra dollar, net of IPO costs and the fraction of proceeds taken out by the founder, will raise the market value of the fraction of the shares held by the outside investors, $1-\alpha$, by less than 1 dollar.

Equation (55) tells us that the IPO proceeds will equal the value of the outside shareholders' share of the firm *after* the original founder has reinvested the fraction $1-\omega$ of the net proceeds $P(k,\alpha,\omega)(1-\rho)-F$ received from the investment bank as new capital for the newly public firm. The IPO proceeds is implicitly defined as the solution to equation (55) above. Due to the strict concavity of $V(k)$, there is a unique solution to (55) for each $k \geq 0$ and each $\alpha \in (0,1)$, and $\omega \in [0,1]$. The Implicit Function Theorem guarantees that $P(k,\alpha,\omega)$ is continuously differentiable in its arguments k, α and ω for almost all values of k, α and ω

with derivatives

$$
\begin{aligned}
\frac{\partial}{\partial k}P(k,\alpha,\omega) &= \frac{(1-\alpha)V'(k+(1-\omega)[P(k,\alpha,\omega)(1-\rho)-F])}{1-(1-\alpha)(1-\omega)(1-\rho)V'(k+(1-\omega)[P(k,\alpha,\omega)(1-\rho)-F])} \\
\frac{\partial}{\partial \alpha}P(k,\alpha,\omega) &= \frac{-V'(k+(1-\omega)[P(k,\alpha,\omega)(1-\rho)-F])}{1-(1-\alpha)(1-\omega)(1-\rho)V'(k+(1-\omega)[P(k,\alpha,\omega)(1-\rho)-F])} \\
\frac{\partial}{\partial \omega}P(k,\alpha,\omega) &= \frac{-V'(k+(1-\omega)[P(k,\alpha,\omega)(1-\rho)-F])}{1-(1-\alpha)(1-\omega)(1-\rho)V'(k+(1-\omega)[P(k,\alpha,\omega)(1-\rho)-F])}
\end{aligned}
\tag{56}
$$

It follows that $P(k,\alpha,\omega)$ is increasing in k and decreasing in α and ω, as we would naturally expect.

We can view $P(k,\alpha,\omega)/(1-\alpha)$ as the market's rational expectation of the total value of the firm following an IPO where it has full knowledge of the fraction of the firm owned by the founder after the IPO, and the fraction of the IPO proceeds that the founder cashed out for consumption or other purposes, leaving only the fraction $1-\omega$ of the net proceeds as the amount of new investment the firm actually undertakes as a result of the IPO. It also is contingent on the assumption that after the IPO the firm will be run in a discounted profit maximizing manner, even if the owner retains a majority stake in the company after the IPO. Thus, our theory of rational market valuation following an IPO and partial cash out encompasses both the moral hazard and financing/leverage effects that an IPO can have on the valuation of a company that we discussed above.

Figure 6 illustrates how an IPO can be used as leverage, substantially increasing a firm's value by reinvesting a fraction of the IPO proceeds to acquire more capital, which further increases the value of the firm. We focus on a small private firm that has an initial capital stock of $k=3$ when it decides to go public via an IPO. We plot the value of the firm as a fraction of α, the fraction of the firm that the founder chooses to own after the IPO. We assume that $\omega=0$, so that the owner does not divert any of the IPO proceeds to any other purposes except reinvestment in the firm. The left panel of figure 6 plots the total value of the firm, the amount reinvested, and the value of the share of the firm owned by the founder as a function of α.

Notice that the value of the founder's interest in the firm is zero when $\alpha=0$. Clearly it would make no sense for the founder to sell off his/her entire ownership interest and then reinvest all proceeds back into a firm he/she no longer owns: this would be a nice gift to the new shareholders but not something that the founder would want to do absent a peculiar sense of altruism to outside investors. The case where $\alpha=1$ corresponds to a situation where the founder decides to take the company public but without raising any new capital from outside investors. There is no new investment resulting from the IPO in this case, and the value of the firm is equal to the value we already calculated in section 4 under 100% sell off option,

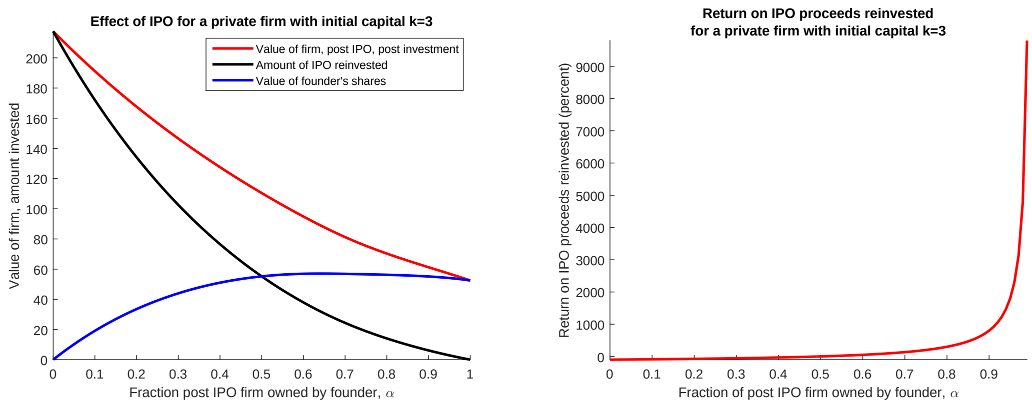

Figure 6: Value of a firm after IPO, and return to investment from IPO proceeds

namely \$52.47.

The black line in the left hand panel of figure 6 shows that amount of the IPO that is reinvested in the firm as a function of α. If the owner was to be so nice to sell off his/her entire ownership interest to outside invstors and reinvest the entire IPO proceeds in the firm, the firm would attain its maximum value of \$217.58, which equals the amount of new capital the original owner reinvests in the firm. However if the owner were to retain 50% ownership, the value of the firm is \$110.41, which is double the amount the original owner reinvests in the firm when $\alpha = .5$. Thus, the owner obtains a 100% return from doing an IPO and reinvesting half of their ownership stake in the firm, even though the outside investors will also benefit from this investment made by the founder. The founder's net worth after this deal is \$55.21, which exceeds the founder's net worth from the option of selling off his entire ownership stake and not reinvesting any of the IPO proceeds back in the company, \$52.47. Thus, some degree of apparent "altruism" towards the outside shareholders by the founder is actually in the founder's self-interest.

If we consider which value of α maximizes the founder's net worth after the IPO (assuming ω is fixed at 0), we find that $\alpha = .64$ and the founder's net worth (i.e. the value of his/her ownership stake in the post IPO firm) is \$57.47. The founder invests \$32.08, and the total value of the firm is \$89.13 after this investment. Thus, the return on this investment is equal to $(57.47 - 32.08)/32.08 = .7915$. This represents a very high return even though the founder is not able to capture all of the benefit from this investment: the outside investors reap 36% of the increase in the firm value resulting from the founder's reinvestment of the \$32.08 in IPO proceeds back into the capital stock of the firm.

The right hand panel of figure 6 plots the rate of return on the marginal dollar the owner reinvests in the firm, as a function of α. The first dollar reinvested has an exceptionally high rate of return in this example. Naturally there are diminishing returns to investment and so the return falls as α (the fraction the owner cashes out) decreases quick to zero as α tends to zero. However even when $\alpha = 0.64$, the founder obtains a 79% return on their investment as we noted above.

To complete the model, we now discuss the founder's choice of α and ω. It is simplest to consider the case where the only motive for a cash out is to buy an annuity to smooth consumption. If the owner retains ownership of a fraction α of the company following the IPO, the owner could initially invest this fraction of the IPO proceeds in the company (to benefit from the effective leverage or financing effect of the IPO) and then immediately sell off this residual stake after the IPO and the investment in new capital is completed. Then the owner could purchase an annuity with the total proceeds. Under this formulation of the owner's problem we have that the optimal values of α^* and ω^* solves

$$(\alpha^*(k), \omega^*(k)) = \underset{\alpha\in[0,1]}{argmax}\, \underset{\omega\in[0,1]}{argmax}\, u((1-\beta)(\omega + \alpha/(1-\alpha)P(k,\alpha,\omega))/(1-\beta). \tag{57}$$

Notice that the optimal fraction to cash out depends on the size of owner's initial capital stock k when the firm is privately held, just prior to doing the IPO. Since u is monotonically increasing, the founder's problem reduces to simply maximizing the value of his/her net worth following the IPO, where the net worth is a combination of the cash taken out of the IPO proceeds, $\omega P(k,\alpha,\omega)$, plus the value of the founder's sharedholdings in the post-IPO company, $\alpha V(k + (1-\omega)[P(k,\alpha,\omega)(1-\rho) - F]) = \alpha P(k,\alpha,\omega)/(1-\alpha)$. Thus, the founder's problem reduces to

$$(\alpha^*(k), \omega^*(k)) = \underset{\alpha\in[0,1]}{argmax}\, \underset{\omega\in[0,1]}{argmax} (\omega + \alpha/(1-\alpha))P(k,\alpha,\omega) \tag{58}$$

Figure 7 plots the net worth of the founder, $(\omega + \alpha/(1-\alpha))P(k,\alpha,\omega)$ as a function of (α,ω). It turns out that this function is symmetric as a function of (α,ω) about the diagonal line $\alpha = \omega$. As a result we find two symmetrically located optimal solutions, $(\alpha^*(k), \omega^*(k)) = (.42, .38)$ and $(\alpha^*(k), \omega^*(k)) = (.38, .48)$, and both yield the optimal level of net worth for the founder equal to \$57.04. We see that when we fix α, if α is sufficiently small, the founder's net worth is initially increasing in ω and then decreasing, so there is an optimal value of the cash out fraction $\omega^*(k,\alpha)$ for any fixed α. By symmetry, there is also an optimal value of the fraction of ownership $\alpha^*(k,\omega)$ that the founder should retain for any fixed cash out fraction ω provided ω is not too close to 1.

However if we fix a value for α that is sufficiently large, say $\alpha > .7$, then the net worth of the founder is monotonically decreasing in ω and thus the optimal value $\omega^*(k,\alpha) = 0$ when α is sufficiently large.

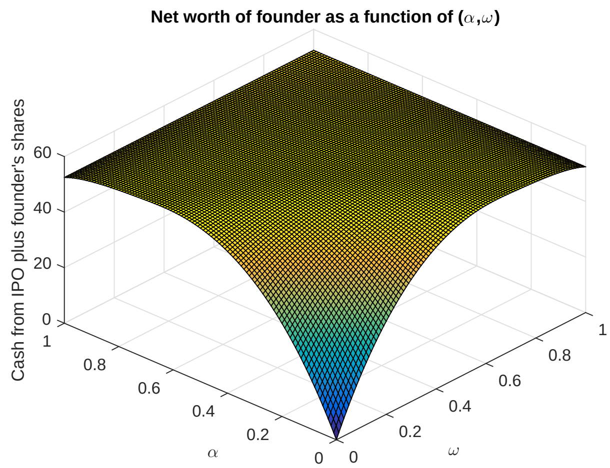

Figure 7: Net worth of founder, as a function of (α, ω)

That is, the founder does not want to cash out if he/she decides to retain a sufficiently large stake in the firm: the return to reinvesting in the firm is higher. By symmetry this is also true for α when ω is fixed as a value that is sufficiently large: the optimal ownership stake is zero $\alpha^*(k, \omega) = 0$. Thus, if the founder precommits to cashing out a sufficiently large share of the IPO proceeds, the founder will also will not find it optimal to retain any ownership interest in the firm.

The optimal combination $(\alpha^*(k), \omega^*(k))$ represents the tradeoff between the founder's desire to reinest in the firm, but tempered by the disincentive effect of the fact that the larger the amount the founder sells to outside investors, the less the founder benefits from reinvesting the IPO proceeds back into the firm.

6 Solving the model using Discrete Policy Iteration

This section describes the numerical solution of the model using the Howard (1960) *policy iteration* algorithm. This algorithm was originally developed to solve infinite horizon stationary Markovian dynamic programming problems (often abbreviated as MDPs for Markovian Decision Problems) on a *finite* state spaces. The optimal investment and dividend problem is superficially not a finite state MDP in the following senses: 1) the state space is continuous (the entire positive real line, $k \geq 0$), and 2) the problem is deterministic, rather than stochastic. Despite these differences, we show that policy iteration can still be applied, but to solve the problem on a finite subset or *grid* of points in the state space and then to apply *linear interpolation* to construct an approximate value function and decision rule essentially by "connecting the dots" where the "dots" are the calculated value function and optimal investment/divident policy at values of k on a pre-defined grid of points $\{k_1,\ldots,k_n\}$ where $k_1 = 0$ and $k_j < k_{j+1}$, $1 = 1,\ldots,n-1$. There is quite a bit of flexibility in how one chooses a grid, but we will show that even for relatively small n and a "naive" choice of equally spaced grid points, it is possible to obtain a very accurate approximation of $V(k)$, $I(k)$ and $D(k)$. The most important choice is the value k_n which constitutes an effective "upper bound" on the capital stock. It is important to "guess" a value for k_n that is large enough so that $I(k_n) = 0$. Otherwise if the guess of the upper bound is too small and $I(k_n) > 0$, this poor initial choice of upper bound can lead to substantial errors in the calculated V, I and D functions.

The basic ideal of how policy iteration works is explained well for the case of finite state spaces in Howard (1960) or Bertsekas (1987), but for the case where the state space has uncountably many states, policy iteration can also be defined but it takes somewhat more advanced functional analysis, see e.g. Puterman (1978). We will describe policy iteration first in the case where the state space is continuous, but it is important to consider a "truncated" version of the problem on a finite interval $[0,K]$ for some $K > 0$ sufficiently large. The reason for truncating the problem is that much of the standard functional analysis machinery is based on use of the *sup norm* $\|V\| = \sup_k |V(k)|$ but this will equal ∞ if the state space is the entire positive real line $[0,\infty)$ if the function V is not bounded.

However once we consider a bounded interval, we can define the Banach space $C(K)$ of all bounded, continuous functions on the interval $[0,K]$, and for this function space, the sup-norm is well defined. In particular if we define the *Bellman operator* $\Gamma : C(K) \to C(K)$ by

$$\Gamma(V)(k) = \max_{0 \le I \le k} [f(k) - I + \beta V(k(1-\delta)+I)] \tag{59}$$

we can show that Γ is a *contraction mapping*, i.e. it satisfies

$$\|\Gamma(V) - \Gamma(W)\| \le \beta \|V - W\| \tag{60}$$

and via the well-know Banach Fixed Point Theorem (also known as the Contraction Mapping Theorem), Γ has a unique fixed point $V = \Gamma(V)$. This unique fixed point is the value function for the truncated problem given by the Bellman equation (2).

Policy iteration is an iterative method for finding the solution to the Bellman equation which is equivalent to finding the fixed point to the Bellman operator Γ. The standard method for finding a fixed point is the method of *successive approximation* and it is based on any initial guess V_0 and an updated estimate V_1 is produced by evaluating Γ on the initial guess V_0, or $V_1 = \Gamma(V_0)$. Then we use V_1 to produce another estimate $V_2 = \Gamma(V_1)$ and we continue this iteration in general as

$$V_j = \Gamma(V_{j-1}) \quad j = 1, 2, \dots \tag{61}$$

until we find that the changes in the successive iterates are less than a specified convergence tolerance ε, i.e. until some iteration j such that $\|V_j - V_{j-1}\| < \varepsilon$. The Contraction Mapping Theorem guarantees that for any $V_0 \in C(K)$ we have

$$\lim_{j\to\infty} V_j = \Gamma(V_{j-1}) = V = \Gamma(V) \tag{62}$$

so the method of successive approximations is guaranteed to converge from any initial guess V_0. A drawback of successive approximations is that is converges only *geometrically,* that is, we have

$$\|V_j - V\| \le \beta^j \|V_0 - V\| \tag{63}$$

so that for β close to 1, the rate of convergence of the estimated value function V_j to the true value function V is very very slow.

However policy iteration is a much faster algorithm that usually converges to the *exact solution V in a finite number of iterations, regardless of how close β is 1.* This is technically true only in finite state MDPs, but for continuous state MDPs there is a close analog of this result, namely that *policy iteration is equivalent to Newton's method and will converge at a quadratic rate.* This implies that the error in approximating the fixed point, $\|V_j - \Gamma(V_j)\|$, where V_j is the j^{th} iterate produced by the policy iteration algorithm will be very small after only a "small" number of iterations j even for β very close to 1.

Policy iteration is a combination of two "sub-iterations": 1) policy improvement and 2) policy valuation. We explain policy valuation first. Policy valuation is a method to find the value function V_I corresponding to any given investment policy I. Given that we are considering only truncated investment problems, we will initially consider only a subclass of decision rules that satisfy the constraints 0) $I(k)$ is a continuous function of $k \in [0, K]$, i.e. $I \in C(K)$, 1) $0 \le I(k) \le f(k)$ (feasibility), and 2) $k(1-\delta) + I(k) \le K$ for all $k \in [0, K]$. The latter constraint ensures that the mapping Γ_I defined by

$$\Gamma_I(V)(k) = f(k) - I(k) + \beta V(k(1-\delta) + I(k)) \tag{64}$$

makes Γ_I a well defined operator on $C(K)$, i.e. for any $W \in C(K)$ we have $\Gamma_I(W) \in C(K)$. Further Γ_I can be shown to be a constraction mapping, and thus it has a unique fixed point $V_I = \Gamma_I(V_I)$. We now show that Γ_I is an *affine operator,* that is it is a "shifted linear operator" given by

$$\Gamma_I(W)(k) = D_I(k) + \beta E_I(W)(k) \tag{65}$$

where $W \in C(K)$, $D_I(k) = f(k) - I(k)$, and E_I is a linear operator on $C(K)$ defined by

$$E_I(W)(k) = W(k(1-\delta) + I(k)) \tag{66}$$

The constraint on the set of allowable investment rules I implies that if $W \in C(K)$ then $E_I(W) \in C(K)$ so that E_I is an operator on $C(K)$ and further it is a *linear operator* since we have

$$E_I(V+W)(k) = V(k(1-\delta)k + I(k)) + W(k(1-\delta) + I(k)) = E_I(V)(k) + E_I(W)(k) \tag{67}$$

Now define the *norm* of the linear operator E_I by $\|E_I\|$ as follows

$$\|E_I\| = \sup_{V \neq 0} \frac{\|E_I(V)\|}{\|V\|} \tag{68}$$

It is not hard to show that for any $V \in C(K)$ we have $\|E_I(V)\| \leq \|V\|$ which implies that $\|E_I\| \leq 1$, and further, using the example of a function $W(k) = 1$ for $k \in [0,K]$, it is trivially true that $\|E_I(W)\| = 1$, which implies that $\|E_I\| = 1$.

Since we have established that E_I is a linear operator, we can write the equatino for V_I, the fixed point of the operator Γ_I as

$$V_I(k) = D_I(k) + \beta E_I(V)(k) \tag{69}$$

or

$$[\mathscr{I} - \beta E_I](V)(k) = D_I(k) \tag{70}$$

where $\mathscr{I}$ is the *identity operator* on $C(K)$, i.e. $\mathscr{I}(W) = W$ for all $W \in C(K)$ (we use the funny scripted version of capital letter "I" here, $\mathscr{I}$, to distinguish the identity operator from the investment function I). It is easy to show that $\mathscr{I}$ is a linear operator, and thus $[\mathscr{I} - \beta E_I]$ is also a linear operator. Suppose that this linear operator is *invertible.* Then we have the solution

$$V_I = [\mathscr{I} - \beta E_I]^{-1} D_I \tag{71}$$

where $[\mathscr{I} - \beta E_I]^{-1}$ is the inverse operator of $[\mathscr{I} - \beta E_I]$, which is itself also a linear operator. We can show that the inverse operator exists by a geometric series argument. We conjecture that

$$[\mathscr{I} - \beta E_I]^{-1} = \sum_{j=0}^{\infty} \beta^j E_I^j \tag{72}$$

where E_I^j is the linear operator formed as the $j-$fold composition of the operator E_I, i.e. $E_I^2 = E_I(E_I)$ and recursively, $E_I^j = E_I(E_I^{j-1})$. Since $\beta \in (0,1)$ and $\|E_I\| = 1$, it follows that the norm of the right hand side of the *Neumann series expansion* of the inverse operator $[\mathscr{I} - \beta E_I]^{-1}$ in equation (72) is finite (and equals $1/(1-\beta)$) and this establishes that the inverse operator exists.

So in summary policy valuation enables us to obtain a "closed form" expression for $V_I = [\mathscr{I} - \beta E_I]^{-1} D$, which is the value of the firm implied by a given feasible investment policy I. This is why we call this *policy valuation* because V_I represents the *value of the investment policy I*.

Now consider the next sub-iteration of the policy iteration algorihm: *policy improvement.* Using the value function V_I we now seek an *improved policy I'* given by

$$I'(k) = \underset{\substack{0 \le \iota \le f(k) \\ k(1-\delta)+\iota \le K}}{argmax} \; [f(k) - \iota + \beta V_I(k(1-\delta)+\iota)] \tag{73}$$

where we use the notation ι to denote a candidate investment value that we are optimizing over in order to find a new better policy $I'(k)$. Given I' we can now return to the policy valuation step to find the value $V_{I'}$ of this new, improved policy I'

$$V_{I'} = [\mathscr{I} - \beta E_{I'}]^{-1} D_{I'} \tag{74}$$

We can show that $V_{I'} \ge V_I$, i.e. I' really is an *improved* policy that results in a higher value for the firm. However if $V_{I'} = V_I$, then the new policy is not a *strict improvement* over the previous policy I at any $k \in [0,K]$, and at that point *policy iteration has converged.* It is not hard to show that at convergence, $V_I = \Gamma(V_I)$, i.e. V_I is a solution to the Bellman equation, and since this is unique by the Contraction Mapping Theorem, the policy iteration algorithm has succeeded to find the fixed point to Bellman's equation.

The formulas above seem "theoretical" since they involve inversion of linear operators on $C(K)$ which are infinite-dimensional objects. However we can approximate these "infinite dimensional" operators with finite-dimensional operators on a large but finite dimensional Euclidean space R^n. We achieve this via the device of *discretization* and solving the problem on a finite grid of n points $\{k_1,\dots,k_n\} \subset [0,K]$.

When we have a finite grid, we can produce a continuous piecewise-linear approximation by *linear interpolation.* For example suppose we have a given function $W(k)$ but suppose that we only have access to values of this function at n grid points $\{k_1,\dots,k_n\} \subset [0,K]$. That is we know the n values $\{w_1,\dots,w_n\}$ where $w_j = W(k_j)$, $j = 1,\dots,n$. How can we approximate the true value $W(k)$ at some $k \in [0,K]$ that is not one of these grid points? This is quite easy: k must lie between two successive grid points, i.e. $k \in (k_{j-1}, k_j)$ and so we can represent k as a convex combination of these grid points using a weight (or it could be interpreted as a "probability") $p(k)$ given by

$$p(k) = \frac{k - k_{j-1}}{k_j - k_{j-1}} \tag{75}$$

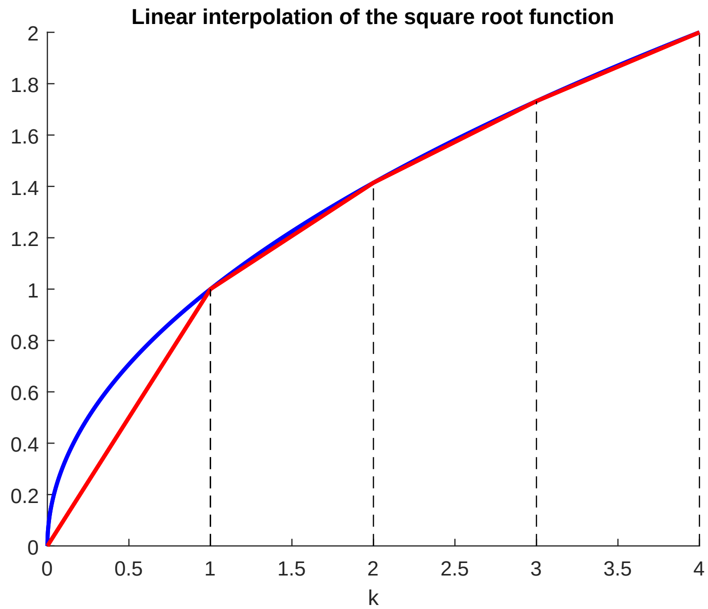

Figure 8: Interpolated square root function $f(k) = \sqrt{k}$

so we can write $k = p(k)k_j + (1 - p(k))k_{j-1}$. Then using the weight $p(k)$ we can produce the following approximate value $\hat{W}(k)$

$$\hat{W}(k) = p(k)w_j + (1 - p(k))w_{j-1} \tag{76}$$

Figure 8 below shows the square root function on the interval $[0,4]$ and its linear interpolation using a grid of five equally spaced points $\{k_1, k_2, k_3, k_4, k_5\} = \{0, 1, 2, 3, 4\}$.

Using interpolation, we can carry out policy iteration over a grid of n points on the interval $[0,K]$ and nearly all of the operations become *finite* because the set of piecewise linear functions with nodes (or "knot points") at a grid of n points $\{k_1, \ldots, k_n\}$ is an *n-dimensional subspace of* $C(K)$. Our goal is to try to approximate the true value function $V \in C(K)$ with an approximate value function V_n that lives in the n-dimensional subspace of $C(K)$ that consists of all continuous functions whose values over the entire interval $[0,K]$ are linearly interpolated from their values at the n points $\{k_1, \ldots, k_n\}$.

So suppose we are given an investment policy I whose values are known at each of the grid points $\{k_1, \ldots, k_n\}$ and are determined by linearly interpolation of the known values $\{I(k_1), \ldots, I(k_n)\}$ for other values of $k \in [0,K]$. Recall our general equation for the policy valuation step

$$V_I = D_I + \beta E_I V_I \tag{77}$$

where E_I is the (infinite-dimensional) linear operator that "implements" the evaluation of V at a given point $k(1-\delta)+I(k) \in [0,K]$, i.e.

$$E_I(V)(k) \equiv V(k(1-\delta)k+I(k)) \tag{78}$$

Now consider restricting the domain of allowable values of k to just the n grid points $\{k_1,\dots,k_n\}$. Then the infinite-dimensional version of the policy-evaluation equation (77) above become a *system of n linear equations in n unknowns in* R^n

$$V_I = D_I + \beta E_I V_I \tag{79}$$

where now we have V_I and D_I are *vectors* in R^n given by

$$V_I = \begin{bmatrix} V_I(k_1) \\ V_I(k_2) \\ \\ \cdots \\ \\ V_I(k_{n-1}) \\ V_I(k_n) \end{bmatrix}, \tag{80}$$

and

$$D_I = \begin{bmatrix} D_I(k_1) \\ D_I(k_2) \\ \\ \cdots \\ \\ D_I(k_{n-1}) \\ D_I(k_n) \end{bmatrix}, \tag{81}$$

and E_I is an $n \times n$ *transition probability matrix* which implements the interpolation operation. That is, consider the first row of E_I. It will be all zeros except for at most two adjacent non-zero elements with values $1-p_I(k_1)$ and $p_I(k_1)$, repectively. Recall that we can interpolate $V_I(k_1(1-\delta)+I(k_1))$ using its known values $(V_I(k_1),\dots,V_I(k_n))$ on the grid $(k_1,\dots,k_n)$ as follows

$$V_I(k_1(1-\delta)+I(k_1)) = p_I(k_1)V_I(k_j)+(1-p_I(k_1))V_I(k_{j-1}) \tag{82}$$

where j indexes the grid point k_j such that $k_1(1-\delta)+I(k_1) \in [k_{j-1},k_j]$ and $p_I(k_1)$ is given by

$$p_I(k_1) = \frac{k_1(1-\delta)+I(k_1)-k_{j-1}}{k_j-k_{j-1}}. \tag{83}$$

Thus the first row of E_I will have $p_I(k_1)$ in its j^{th} column and $1-p_I(k_1)$ in its $(j-1)^{\text{st}}$ column and all other columns equal zero. It follows that the first row will sum to 1 by construction. This same idea applies to all other rows of E_I so we conclude it as the form of a Markov *transition probability matrix* i.e. all elements are between 0 and 1 and each row sums to 1.

Using E_I and D_I it is now a matter of linear algebra to solve for $V_I \in R^n$

$$V_I = [\mathscr{I} - \beta E_I]^{-1} D_I \tag{84}$$

except that now $\mathscr{I}$ is the $n \times n$ *identity matrix* (which is also the "identity operator" on R^n). Using $V_I \in R^n$ we can extend it to a continuous function of k over all of the interval $[0,K]$ via linear interpolation, so we can also interpret V_I as an element of the n-dimensional subspace of $C(K)$ of functions which are linearly interpolated from their values at the n grid points $\{k_1,\ldots,k_n\}$.

Given V_I we can now do the *policy improvement step* to see if we can find a better investment policy $I'(k)$ by optimizing over investment at each of the n grid points k_j, $j = 1,\ldots,n$.

$$\begin{aligned} I'(k_j) &= \underset{\substack{0 \le \iota \le f(k_j) \\ k_j(1-\delta)+\iota \le K}}{argmax} \left[f(k_j) - \iota + \beta \hat{V}_I(k_j(1-\delta)+\iota) \right] \\ &= \underset{\substack{0 \le \iota \le f(k_j) \\ k_j(1-\delta)+\iota \le K}}{argmax} \left[f(k_j) - \iota + \right. \\ &\qquad \left. \beta \left[p(k_j(1-\delta)+\iota) V_I(k_l) + (1 - p(k_j(1-\delta)+\iota) V_I(k_{l-1})) \right] \right] \end{aligned}$$

where $k_j(1-\delta)+\iota \in [k_{l-1},k_l]$ for some index $l \in \{1,\ldots,n\}$. If $I'(k) = I(k)$ for all grid points $k \in \{k_1,\ldots,k_n\}$ (or equivalently if $V_{I'} = V_I$), then **stop:** policy iteration as converged to a V that solves the Bellman equation (though restricted to the finite dimensional subspace of $C(K)$ of functions defined by linear interpolation at the n grid points $\{k_1,\ldots,k_n\}$). If not, then using the improved policy I' we return to the policy valuation step (79) to calculate $V_{I'}$ and continue until the policy iteration process converges.

Figure 9 presents the approximate decision rules for investment and dividends computed by poicy iteration with $n = 301$ grid points, equally spaced from $k_1 = 0$ to $k_{301} = 30$, a spacing of 0.1 apart. Policy iteration converged after 20 iterations, resulting in a (sup norm) change in value functions of 5.97×10^{-13}. We see that the computed solutions look virtually identical to the true solutions plotted in figure 2 in section 2. Figure 10 also plots the interpolated value function from policy iteration and it also looks virtually identical to the true value function in figure 2.

There are approximation errors but they are small. Figure 11 plots the approximation errors at the grid points $\{k_1,\ldots,k_n\}$ for two different solutions, one using policy iteration with $n = 150$ grid points, and the other using $n = 300$ grid points, in both cases equally spaced over the interval $[0,30]$. Generally we would expect that using a "finer grid" i.e. a larger number of grid points n, should result in a better

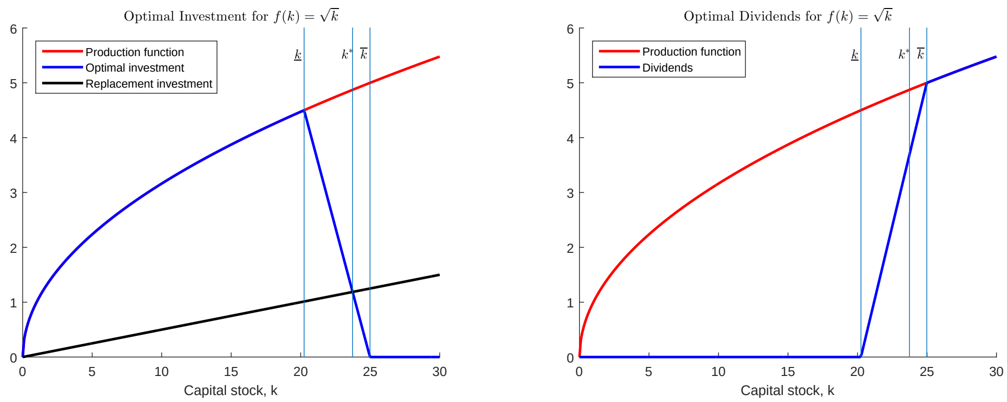

Figure 9: Approximate optimal investment and dividend policy for $f(k) = \sqrt{k}$

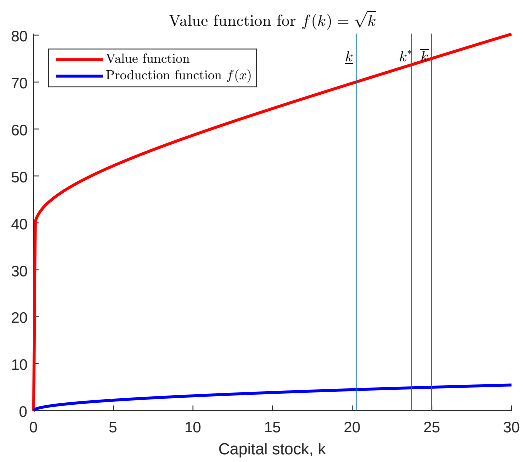

Figure 10: Value function for $f(k) = \sqrt{k}$

approximation. This is the case here, though it required a more sophisticated version of interpolation than simple linear interpolation of the ordinates $\{V(x_1), \ldots, V(x_n)\}$ in the policy improvement step. We used *piecewise cubic hermit polynomial interpolation* as implemented in the `pchip` function of Matlab. The `pchip` function interpolates the ordinates in a way that gaurantees continuous differentiability of the interpolated function at the grid points (unlike what happens with simple linear interpolation, where the derivatives are generally discontinuous at the grid points) and the interpolated function is *shape preserving* which is particularly important in this case to preserve the concavity of the value function over the entire domain.

In each policy improvement step we used the Matlab `fminbnd` function to numerically search for the

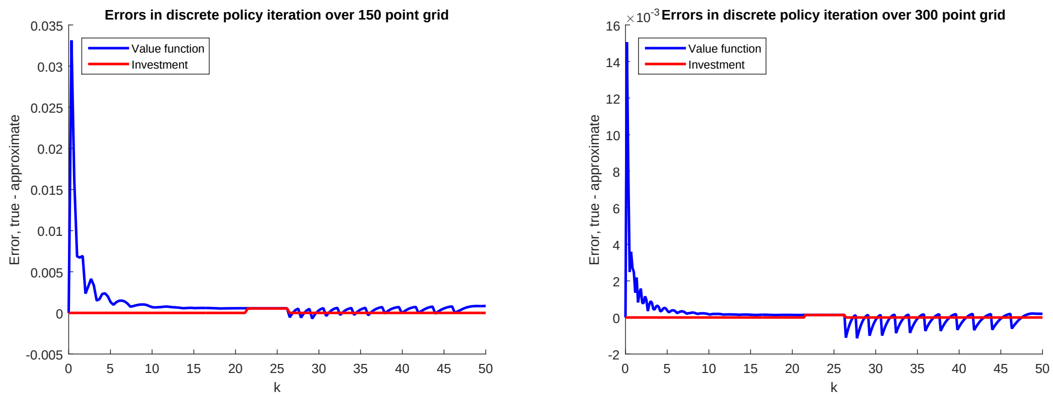

Figure 11: Approximation errors in value function and investment rule with $n = 150$ and $n = 300$ grid points

optimal value of investment $\iota \in [0, f(k_j)]$ at each grid point k_j, $j = 1, \ldots, n$. When we used simple linear interpolation, the interpolated value function has more and more discontinuities in its derivative as n gets large. This appears to create problems for the Matlab optimizer, and when n gets sufficiently large, the approximation actually starts to degrade. This is not the case when the `pchip` interpolater was used. The approximation error reduces as the number of grid points increases, though there is diminshing returns to increasing n. Further accuracy can be achieved by using the strict concavity of the value functions and using a Newton or bisection algorithm to find optimal investment as a solution to the first order condition $1 = \beta V'(k(1-\delta) + I)$, using the fact that the `pchip` interpolated results in a piecewise quadratic expression for V' that makes it easy to employ Newton's method to solve for the value of I that satisfies the first order condition.

Overall, we have demonstrated that the DPI algorithm seems to be capable of finding a good approximation to the true value function and decision rules.