Quantity precommitment and Bertrand competition yield Cournot outcomes

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and

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Bertrand's model of oligopoly, which gives perfectly competitive outcomes, assumes that: (1) there is competition over prices and (2) production follows the realization of demand. We show that both of these assumptions are required. More precisely, consider a two-stage oligopoly game where, first, there is simultaneous production, and, second, after production levels are made public, there is price competition. Under mild assumptions about demand, the unique equilibrium outcome is the Cournot outcome. This illustrates that solutions to oligopoly games depend on both the strategic variables employed and the context (game form) in which those variables are employed.

1. Introduction

■ Since Bertrand's (1883) criticism of Cournot's (1838) work, economists have come to realize that solutions to oligopoly games depend critically on the strategic variables that firms are assumed to use. Consider, for example, the simple case of a duopoly where each firm produces at a constant cost b per unit and where the demand curve is linear, $p = a - q$. Cournot (quantity) competition yields equilibrium price $p = (a + 2b)/3$, while Bertrand (price) competition yields $p = b$.

In this article, we show by example that there is more to Bertrand competition than simply "competition over prices." It is easiest to explain what we mean by reviewing the stories associated with Cournot and Bertrand. The Cournot story concerns producers who simultaneously and independently make production quantity decisions, and who *then* bring what they have produced to the market, with the market price being the price that equates the total supply with demand. The Bertrand story, on the other hand, concerns producers who simultaneously and independently name prices. Demand is allocated to the low-price producer(s), who *then* produce (up to) the demand they encounter. Any unsatisfied demand goes to the second lowest price producer(s), and so on.

There are two differences in these stories: how price is determined (by an auctioneer in Cournot and by price "competition" in Bertrand), and when production is supposed to take place. We demonstrate here that the Bertrand outcome requires both price competition and production after demand determination. Specifically, consider the following

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game between expected profit maximizing producers: In a first stage, producers decide independently and simultaneously how much they will produce, and this production takes place. They then bring these quantities to market, each learns how much the other produced, and they engage in Bertrand-like price competition: They simultaneously and independently name prices and demand is allocated in Bertrand fashion, with the proviso that one cannot satisfy more demand than one produced for in the first stage.

In this two-stage game, it is easy to produce one equilibrium. Let each firm choose the Cournot quantity. If each firm does so, each subsequently names the Cournot price. If, on the other hand, either chooses some quantity other than the Cournot quantity, its rival names price zero in the second stage. Since any defection in the first stage will result in one facing the demand residual from the Cournot quantity, and since the Cournot quantity is the best response to this residual demand function, this is clearly an equilibrium. What is somewhat more surprising is that (for the very special parameterization above and for a large class of other symmetric parameterizations) the Cournot outcome is the unique equilibrium outcome. Moreover, there is a perfect equilibrium that yields this outcome. (The strategies above constitute an imperfect equilibrium.) This note is devoted to the establishment of these facts.

One way to interpret this result is to see our two-stage game as a mechanism to generate Cournot-like outcomes that dispenses with the mythical auctioneer. In fact, an equivalent way of thinking about our game is as follows: *Capacities* are set in the first stage by the two producers. Demand is then determined by Bertrand-like price competition, and production takes place at zero cost, subject to capacity constraints generated by the first-stage decisions. It is easy to see that given capacities for the two producers, equilibrium behavior in the second, Bertrand-like, stage will not always lead to a price that exhausts capacity. But when those given capacities correspond to the Cournot output levels, in the second stage each firm names the Cournot price. And for the entire game, fixing capacities at the Cournot output levels is the unique equilibrium outcome. This yields a more satisfactory description of a game that generates Cournot outcomes. It is this language that we shall use subsequently.

This reinterpretation in terms of capacities suggests a variant of the game, in which both capacity creation (before price competition and realization of demand) and production (to demand) are costly. Our analysis easily generalizes to this case, and we state results for it at the end of this article.

Our intention in putting forward this example is not to give a model that accurately portrays any important duopoly. (We are both on record as contending that "reality" has more than one, and quite probably more than two, stages, and that multiperiod effects greatly change the outcomes of duopoly games.) Our intention instead is to emphasize that solutions to oligopoly games depend on both the strategic variables that firms are assumed to employ and on the context (game form) in which those variables are employed. The timing of decisions and information reception are as important as the nature of the decisions. It is witless to argue in the abstract whether Cournot or Bertrand was correct; this is an empirical question or one that is resolved only by looking at the details of the context within which the competitive interaction takes place.

2. Model formulation

■ We consider two identical firms facing a two-stage competitive situation. These firms produce perfectly substitutable commodities for which the market demand function is given by $P(x)$ (price as a function of quantity x) and $D(p) = P^{-1}(p)$ (demand as a function of price p).

The two-stage competition runs as follows. At the first stage, the firms simultaneously and independently *build capacity* for subsequent production. Capacity level x means that up to x units can be produced subsequently at zero cost. The cost to firm i of (initially) installing capacity level x_i is $b(x_i)$.

After this first stage, each firm learns how much capacity its opponent installed. Then the firms simultaneously and independently name prices p_i chosen from the interval $[0, P(0)]$. If $p_1 < p_2$, then firm 1 sells

$$z_1 = \min(x_1, D(p_1)) \tag{1}$$

units of the good at price p_1 (and at zero additional cost), for a net profit of $p_1z_1 - b(x_1)$. And if $p_1 < p_2$, firm 2 sells

$$z_2 = \min(x_2, \max(0, D(p_2) - x_1)) \tag{2}$$

units at price p_2 for a net profit of $p_2z_2 - b(x_2)$. If $p_2 < p_1$, symmetric formulas apply. Finally, if $p_2 = p_1$, then firm i sells

$$z_i = \min\left(x_i, \frac{D(p_i)}{2} + \max\left(0, \frac{D(p_i)}{2} - x_j\right)\right) = \min\left(x_i, \max\left(\frac{D(p_i)}{2}, D(p_i) - x_j\right)\right) \tag{3}$$

at price p_i, for net profits equal to $p_iz_i - b(x_i)$. (In (3), and for the remainder of the article, subscript j means *not* i. Note the use of the *capacity* and *subsequent production* terminology.)

Each firm seeks to maximize the expectation of its profits, and the above structure is common knowledge between the firms. At this point the reader will notice the particular rationing rule we chose. Customers buy first from the cheapest supplier, and income effects are absent. (Alternatively, this is the rationing rule that maximizes consumer surplus. Its use is not innocuous—see Beckmann (1965) and Levitan and Shubik (1972).)

The following assumptions are made:

Assumption 1. The function $P(x)$ is strictly positive on some bounded interval $(0, X)$, on which it is twice-continuously differentiable, strictly decreasing, and concave. For $x \geq X$, $P(x) = 0$.

Assumption 2. The cost function b, with domain $[0, \infty)$ and range $[0, \infty)$, is twice-continuously differentiable, convex, and satisfies $b(0) = 0$ and $b'(0) > 0$. To avoid trivialities, $b'(0) < P(0)$—production at some level is profitable.

3. Preliminaries: Cournot competition

■ Before analyzing the two-stage competition formulated above, it will be helpful to have on hand some implications of the assumptions and some facts about Cournot competition between the two firms. Imagine that the firms engage in Cournot competition with (identical) cost function c. Assume that c is (as b), twice-continuously differentiable, convex, and nondecreasing on $[0, \infty)$. Note that from Assumption 1, for every $y < D(0)$ the function $x \rightarrow xP(x + y) - c(x)$ is strictly concave on $[0, y - x)$. Define

$$r_c(y) = \underset{0 \leq x \leq X-y}{\operatorname{argmax}}\, xP(x + y) - c(x).$$

That is, $r_c(y)$ is the *optimal response function* in Cournot competition if one's rival puts y on the market. It is the solution in x of

$$P(x + y) + xP'(x + y) - c'(x) = 0. \tag{4}$$

Lemma 1. (a) For every c as above, r_c is nonincreasing in y, and r_c is continuously differentiable and strictly decreasing over the range where it is strictly positive.

(b) $r'_c \geq -1$, with strict inequality for y such that $r_c(y) > 0$, so that $x + r_c(x)$ is nondecreasing in x.

(c) If c and d are two cost functions such that $c' > d'$, then $r_c < r_d$.
(d) If $y > r_c(y)$, then $r_c(r_c(y)) < y$.

Proof. (a) For any y, we have

$$P(r_c(y) + y) + r_c(y)P'(r_c(y) + y) - c'(r_c(y)) = 0.$$

Increase y in the above equation while leaving $r_c(y)$ fixed. This decreases the (positive) first term and decreases the second (it becomes more negative). Thus the concavity of $xP(x + y) - c(x)$ in x implies that, to restore equality, we must decrease $r_c(y)$. Where P is strictly positive, the decrease in $r_c(y)$ must also be strict. And the differentiability of r_c follows in the usual fashion from the smoothness of P and c.

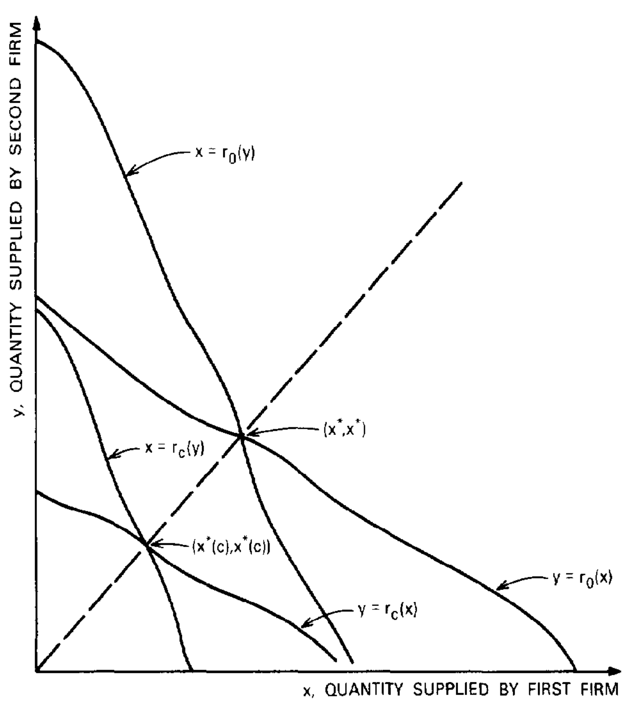
For (b), increase y by h and decrease $r_c(y)$ by h in the equation displayed above. The first (positive) term stays the same, the second increases (becomes less negative), and the third increases. Thus the left-hand side, at $y + h$ and $r_c(y) - h$, is positive. The strict concavity of the profit function ensures, therefore, that $r_c(y + h) > r_c(y) - h$ (with the obvious qualifications about values y for which $r_c(y) = 0$).

For (c) and (d), arguments similar to (b) are easily constructed.

Because of (d), the picture of duopoly Cournot competition is as in Figure 1. For every cost function c, there is a unique Cournot equilibrium, with each firm bringing forward some quantity $x^*(c)$. Moreover, for c and d as in part (c) of the lemma, it is clear that $x^*(c) < x^*(d)$. In the next section, the case where c is identically zero plays an important role. To save on subscripts and arguments, we shall write $r(y)$ for $r_0(y)$ and x^* for $x^*(0)$. Also, we shall write $R(y)$ for $r(y)P(r(y) + y)$, the revenue associated with the best response to y when costs are identically zero.

FIGURE 1

THE PICTURE OF COURNOT COMPETITION UNDER THE ASSUMPTIONS OF THE MODEL



(The astute reader will notice that the analysis to follow does not require the full power of Assumptions 1 and 2. All that is really required is that, for each $y < D(0)$, the functions $x \to xP(x + y) - b(x)$ and $x \to xP(x + y)$ are strictly quasi-concave (on $(0, X - y)$), and that r_b and r appear as in Figure 1. The former does require that $p \to pD(p)$ is strictly concave where it is positive, but this is not quite sufficient. In any event, we shall continue to proceed on the basis of the assumptions given, as they do simplify the arguments that follow.)

4. The capacity-constrained subgames

■ Suppose that in the first stage the firms install capacities x_1 and x_2, respectively. Beginning from the point where (x_1, x_2) becomes common knowledge, we have a *proper subgame* (using the terminology of Selten (1965)). We call this the (x_1, x_2) capacity-constrained subgame—it is simply the Edgeworth (1897) "constrained-capacity" variation on Bertrand competition. It is not *a priori* obvious that each capacity-constrained subgame has an equilibrium, as payoffs are discontinuous in actions. But it can be shown that the discontinuities are of the "right" kind. For subgames where $x_1 = x_2$, the existence of a subgame equilibrium is established by Levitan and Shubik (1972) in cases where demand is linear and marginal costs are constant. Also for the case of linear demand and constant marginal costs, Dasgupta and Maskin (1982) establish the existence of subgame equilibria for all pairs of x_1 and x_2, and their methodology applies to all the cases that we consider. (We shall show how to "compute" the subgame equilibria below.)

The basic fact that we wish to establish is that for each (x_1, x_2), the associated subgame has unique expected revenues in equilibrium. (It is very probably true that each subgame has a unique equilibrium, but we do not need this and shall not attempt to show it.) Moreover, we shall give formulas for these expected revenues.

For the remainder of this section, fix a pair of capacities (x_1, x_2) and an equilibrium for the (x_1, x_2) subgame. Let $\bar{p}_i$ be the supremum of the support of the prices named by firm i; that is, $\bar{p}_i = \inf\{p\text{:firm } i \text{ names less than } p \text{ with probability one}\}$. And let $\underline{p}_i$ be the infimum of the support. Note that if $\min_i x_i \geq D(0)$, then, as in the usual Bertrand game with no capacity constraints, $\bar{p}_i = \underline{p}_i = 0$. And if $\min_i x_i = 0$, we have the monopoly case. Thus we are left with the case where $0 < \min_i x_i < D(0)$.

Lemma 2. For each i, $\underline{p}_i \geq P(x_1 + x_2)$.

Proof. By naming a price p less than $P(x_1 + x_2)$, firm i nets at most px_i. By naming $P(x_1 + x_2)$, firm i nets at worst $P(x_1 + x_2)(x_1 + x_2 - x_j) = P(x_1 + x_2)x_i$.

Lemma 3. If $\bar{p}_1 = \bar{p}_2$ and each is named with positive probability, then

$$\underline{p}_i = \bar{p}_i = P(x_1 + x_2) \quad \text{and} \quad x_i \leq r(x_j), \quad \text{for both} \quad i = 1 \quad \text{and} \quad i = 2.$$

Proof. Suppose that $\bar{p}_1 = \bar{p}_2$ and each is charged with positive probability. Without loss of generality, assume $x_1 \geq x_2$, and suppose that $\bar{p}_1 = \bar{p}_2 > P(x_1 + x_2)$. By naming a price slightly less than $\bar{p}_1$, firm 1 strictly improves its revenues over what it gets by naming $\bar{p}_1$. (With positive probability, it sells strictly more, while the loss due to the lower price is small.) Thus $\bar{p}_1 = \bar{p}_2 \leq P(x_1 + x_2)$. By Lemma 2, we know that $\bar{p}_i = \underline{p}_i = P(x_1 + x_2)$ for $i = 1, 2$.

By naming a higher price p, firm i would obtain revenue $(D(p) - x_j)p$, or, letting $x = D(p) - x_j$, $xP(x + x_j)$. This is maximized at $x = r(x_j)$, so that were $r(x_j) < x_i$, we would not have an equilibrium.

Lemma 4. If $x_i \leq r(x_j)$ for $i = 1, 2$, then a (subgame) equilibrium is for each firm to name $P(x_1 + x_2)$ with probability one.

Proof. The proof of Lemma 3 shows that naming a price greater than $P(x_1 + x_2)$ will not profit either firm in this case. (Recall that $xP(x + x_j)$ is strictly concave.) And there is no incentive to name a lower price, as each firm is selling its full capacity at the equilibrium price.

Lemma 5. Suppose that either $\bar{p}_1 > \bar{p}_2$, or that $\bar{p}_1 = \bar{p}_2$ and $\bar{p}_2$ is not named with positive probability. Then:

(a) $\bar{p}_1 = P(r(x_2) + x_2)$ and the equilibrium revenue of firm 1 is $R(x_2)$;
(b) $x_1 > r(x_2)$;
(c) $\underline{p}_1 = \underline{p}_2$, and neither is named with positive probability;
(d) $x_1 \geq x_2$; and
(e) the equilibrium revenue of firm 2 is uniquely determined by (x_1, x_2) and is at least $(x_2/x_1)R(x_2)$ and at most $R(x_2)$.

Proof. For (a) and (b): Consider the function

$$\Xi(p) = p \cdot [\min (x_1, \max (0, D(p) - x_2))].$$

In words, $\Xi(p)$ is the revenue accrued by firm 1 if it names p and it is undersold by its rival. Under the hypothesis of this lemma, firm 1, by naming $\bar{p}_1$, nets precisely $\Xi(\bar{p}_1)$, as it is certain to be undersold. By naming any price $p > \bar{p}_1$, firm 1 will net precisely $\Xi(p)$. If firm 1 names a price $p < \bar{p}_1$, it will net at least $\Xi(p)$. Thus, if we have an equilibrium, $\Xi(p)$ must be maximized at $\bar{p}_1$.

We must dispose of the case $x_2 \geq D(0)$. Since (by assumption) $D(0) > \min_i x_i$, $x_2 \geq D(0)$ would imply $D(0) > x_1$. Thus, in equilibrium, firm 2 will certainly obtain strictly positive expected revenue. And, therefore, in equilibrium, $\bar{p}_2 > 0$. But then firm 1 must obtain strictly positive expected revenue. And if $x_2 \geq D(0)$, then $\Xi(\bar{p}_1) = 0$. That is, $x_2 \geq D(0)$ is incompatible with the hypothesis of this lemma.

In maximizing $\Xi(p)$, one would never choose p such that $D(p) - x_2 > x_1$ or such that $D(p) < x_2$. Thus, the relevant value of p lies in the interval $[P(x_1 + x_2), P(x_2)]$. For each p in this interval, there is a corresponding level of x, namely $x(p) = D(p) - x_2$, such that $\Xi(p) = x(p)P(x(p) + x_2)$. Note that $x(p)$ runs in the interval $[0, x_1]$. But we know that

$$\operatorname*{argmax}_{x(p) \in [0, x_1]} x(p)P(x(p) + x_2) = r(x_2) \wedge x_1,$$

by the strict concavity of $xP(x + x_2)$. *If* the capacity constraint x_1 is binding (even weakly), then $\bar{p}_1 = P(x_1 + x_2)$, and Lemma 2 implies that we are in the case of Lemma 3, thus contradicting the hypothesis of this lemma. Hence it must be the case that the constraint does not bind, or $r(x_2) < x_1$ (which is (b)), $\bar{p}_1 = P(r(x_2) + x_2)$, and the equilibrium revenue of firm 1 is $R(x_2)$ (which is (a)).

For (c): Suppose that $\underline{p}_i < \underline{p}_j$. By naming $\underline{p}_i$, firm i nets $\underline{p}_i(D(\underline{p}_i) \wedge x_i)$. Increasing this to any level $p \in (\underline{p}_i, \underline{p}_j)$ nets $p(D(p) \wedge x_i)$. Thus, we have an equilibrium only if $D(\underline{p}_i) < x_i$ and $\underline{p}_i$ is the monopoly price. (By the strict concavity of $xP(x)$, moving from $\underline{p}_i$ in the direction of the monopoly price will increase revenue on the margin.) That is, $\underline{p}_i = P(r(0))$. But $\underline{p}_i < \bar{p}_1 = P(r(x_2) + x_2) < P(r(0))$, which would be a contradiction. Thus $\underline{p}_1 = \underline{p}_2$. We denote this common value by $\underline{p}$ in the sequel. This is the first part of (c).

For the second part of (c), note first that $\underline{p} > P(x_1 + x_2)$. For if $\underline{p} = P(x_1 + x_2)$, then by naming (close to) $\underline{p}$, firm 1 would make at most $P(x_1 + x_2)x_1$. Since $x_1 > r(x_2)$ and the equilibrium revenue of firm 1 is $R(x_2)$, this is impossible.

Suppose that the firm with (weakly) less capacity named $\underline{p}$ with positive probability. Then the firm with higher capacity could, by naming a price slightly less than $\underline{p}$, strictly increase its expected revenue. (It sells strictly more with positive probability, at a slightly lower price.) Thus, the firm with weakly less capacity names $\underline{p}$ with zero probability. Since $\underline{p}$ is the infimum of the support of the prices named by the lower capacity firm, this firm must therefore name prices arbitrarily close to *and above* $\underline{p}$. But if its rival named $\underline{p}$ with positive probability, the smaller capacity firm would do better (since $\underline{p} > P(x_1 + x_2)$) to name a price just below $\underline{p}$ than it would to name a price just above $\underline{p}$. Hence, neither firm can name $\underline{p}$ with positive probability.

For (d) and (e): By (c), the equilibrium revenue of firm i must be $\underline{p}(D(\underline{p}) \wedge x_i)$. We know that $\underline{p} < \bar{p}_1 = P(x_2 + r(x_2))$, so that $D(\underline{p}) > D(P(x_2 + r(x_2))) = x_2 + r(x_2)$, and thus $D(\underline{p}) > x_2$. Hence, firm 2 certainly gets $\underline{p}x_2$ in equilibrium. Firm 1 gets no more than $\underline{p}x_1$, so that the bounds in part (e) are established as soon as (d) is shown.

Suppose that $x_2 > x_1$. Then $D(\underline{p}) > x_1$, and firm 1's equilibrium revenue is $\underline{p}x_1$. We already know that it is also $R(x_2)$, so that we would have $\underline{p} = R(x_2)/x_1$, and firm 2 nets $R(x_2)x_2/x_1$. By naming price $P(r(x_1) + x_1)$ ($>\underline{p}_1 = P(r(x_2) + x_2)$), firm 2 will net $R(x_1)$. We shall have a contradiction, therefore, if we show that $x_1 > r(x_2)$ implies $x_1R(x_1) > x_2R(x_2)$.

Let $\Theta(x) = xR(x) = xr(x)P(r(x) + x)$. We have

$$\Theta'(x) = r(x)P(r(x) + x) + xr'(x)P(r(x) + x) + xr(x)P'(r(x) + x)(r'(x) + 1)$$
$$= (r(x) - x)P(r(x) + x) + x(r'(x) + 1)(P(r(x) + x) + r(x)P'(r(x) + x)).$$

The last term is zero by the definition of $r(x)$, so that we have

$$\Theta'(x) = (r(x) - x)P(r(x) + x).$$

Thus $x_2R(x_2) - x_1R(x_1) = \Theta(x_2) - \Theta(x_1) = \int_{x_1}^{x_2} (r(x) - x)P(r(x) + x)dx$. The integrand is positive for $x < x^*$ and strictly negative for $x > x^*$. We would like to show that the integral is negative, so that the worst case (in terms of our objective) is that in which $x_1 < x^*$ and x_2 is as small as possible. Since $x_1 > r(x_2)$, for every $x_1 < x^*$ the worst case is where x_2 is just a bit larger than $r^{-1}(x_1)$. We shall thus have achieved our objective (of contradicting $x_2 > x_1$, by showing that the integral above is strictly negative) if we show that for all $x < x^*$, $\Theta(x) - \Theta(r^{-1}(x)) \geq 0$.

But $\Theta(x) - \Theta(r^{-1}(x)) = xr(x)P(x + r(x)) - r^{-1}(x)xP(r^{-1}(x) + x)$. This is nonnegative if and only if $r(x)P(x + r(x)) - r^{-1}(x)P(r^{-1}(x) + x) \geq 0$, which is certainly true, since $r(x)$ is the best response to x.

Lemma 6. If $x_1 \geq x_2$ and $x_1 > r(x_2)$, there is a (mixed strategy) equilibrium for the subgame in which all the conditions and conclusions of Lemma 5 hold. Moreover, this equilibrium has the following properties. Each firm names prices according to continuous and strictly increasing distribution functions over an (coincident) interval, except that firm 1 names the uppermost price with positive probability whenever $x_1 > x_2$. And if we let $\Psi_i(p)$ be the probability distribution function for the strategy of firm i, then $\Psi_1(p) \leq \Psi_2(p)$: firm 1's strategy stochastically dominates the strategy of firm 2, with strict inequality if $x_1 > x_2$.

Remarks. The astute reader will note that the first sentence is actually a corollary to the previous lemmas and to the (as yet unproven) assertion that every subgame has an equilibrium. The actual construction of an equilibrium is unnecessary for our later analysis, and the casual reader may wish to omit it on first reading. It is, however, of sufficient independent interest to warrant presentation. In the course of this construction, we obtain the second part of the lemma, which is also noteworthy. At first glance, it might be thought that firm 1, having the larger capacity, would profit more by underselling its rival, and therefore it would name the (stochastically) lower prices. But (as is usual with equilibrium logic) this is backwards: Each firm randomizes in a way that keeps the other firm indifferent among its strategies. Because firm 1 has the larger capacity, firm 2 is more "at risk" in terms of being undersold, and thus firm 1 must be "less aggressive."

Proof. Refer to Figure 2. There are five functions depicted there: $pD(p)$, $p(D(p) - x_2)$, $p(D(p) - x_1)$, px_1, and px_2. Note that:

(i) $px_1 = p(D(p) - x_2)$ and $px_2 = p(D(p) - x_1)$ at the same point, namely $P(x_1 + x_2)$.

(ii) $px_1 = pD(p)$ at the point where $p(D(p) - x_1)$ vanishes, and similarly for 2.

(iii) The first three functions are maximized at $P(r(0))$, $P(r(x_2) + x_2)$, and $P(r(x_1) + x_1)$, respectively.

(iv) Because P is concave, the first three functions are strictly concave on the range where they are positive. And every ray from the origin of the form px crosses each of these three functions at most once. (The latter is a simple consequence of the fact that $D(p)$ is decreasing.)

Now find the value $p = P(r(x_2) + x_2)$. This is $\bar{p}_1$. Follow the horizontal dashed line back to the function $p(D(p) \wedge x_1)$. We have drawn this intersection at a point p where $D(p) > x_1$, but we have no guarantee that this will happen. In any event, the level of p at this intersection is $\underline{p}$. Follow the vertical dashed line down to the ray px_2. The height $\underline{p}x_2$ will be the equilibrium revenue of firm 2. Note that even if the first intersection occurred at a point where $x_1 > D(p)$, this second intersection would be at a level $\underline{p}$ where $D(\underline{p}) > x_2$, since $x_2 = D(p)$ at $P(x_2)$, which is to the right of $P(r(x_2) + x_2)$. Also, note that these intersections occur to the right of $P(x_1 + x_2)$, since $R(x_2) > x_1P(x_1 + x_2)$.

Suppose that firm 1 charges a price $p \in [\underline{p}, \bar{p}_1]$. If we assume that firm 2 does not charge this price p with positive probability, then the expected revenue to firm 1 is

$$E_1(p) = \Phi_2(p)p(D(p) - x_2) + (1 - \Phi_2(p))p(D(p) \wedge x_1),$$

where Φ_2 is the distribution function of firm 2's strategy. A similar calculation for firm 2 yields

$$E_2(p) = \Phi_1(p)p[\max(D(p) - x_1, 0)] + (1 - \Phi_1(p))px_2.$$

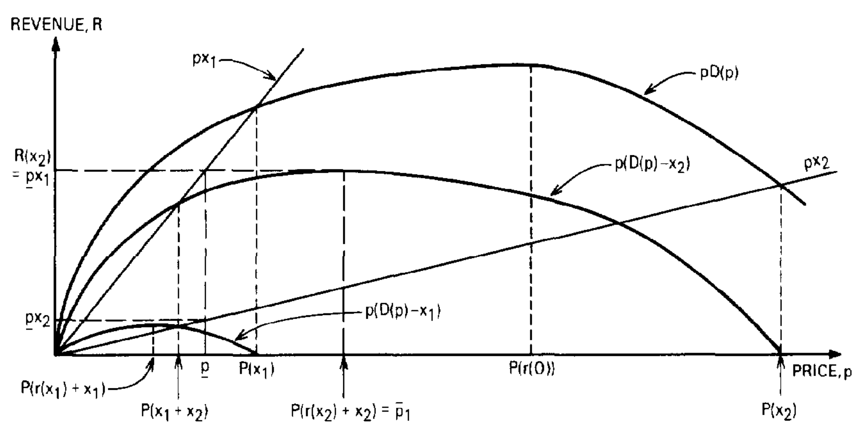
(Note that for $p \in [\underline{p}, \bar{p}_1]$, we know that $D(p) - x_2 > 0$.)

Solve the equations $E_1(p) = R(x_2)$ $(=\underline{p}(D(\underline{p}) \wedge x_1))$ and $E_2(p) = \underline{p}x_2$ in $\Phi_2(p)$ and $\Phi_1(p)$, calling the solutions $\Psi_2(p)$ and $\Psi_1(\bar{p})$, respectively. Note that:

(v) Both functions are continuous and begin at level zero.

FIGURE 2

DETERMINING THE SUBGAME EQUILIBRIUM



(vi) The function $\Psi_2(p)$ is strictly increasing and has value one at $\bar{p}_1$. To see this, note that $p(D(p) - x_2)$ is getting closer to, and $p(D(p) \wedge x_1)$ is getting further from, $R(x_2)$ as p increases. And $R(x_2) = \bar{p}_1(D(\bar{p}_1) - x_2)$.

(vii) The function $\Psi_1(p)$ is strictly increasing, everywhere less than or equal to one, and strictly less than one if $x_1 > x_2$. (If $x_1 = x_2$, then it is identical to $\Psi_2(p)$.) To see this, note first that for $p \geq P(x_1)$, $\Psi_1(p) = 1 - \underline{p}/p$. And for values of p in the range $\underline{p} \leq p < P(x_1)$, we have $R(x_2) = \underline{p}x_1$, and, thus,

$$\Psi_1(p) = \frac{(\underline{p} - p)x_2}{p(D(p) - x_1 - x_2)},$$

and

$$\Psi_2(p) = \frac{(\underline{p} - p)x_1}{p(D(p) - x_1 - x_2)}.$$

That is, for p between $\underline{p}$ and $P(x_1)$, $\Psi_1 = x_2\Psi_2/x_1$. Noting step (vi), the result is obvious.

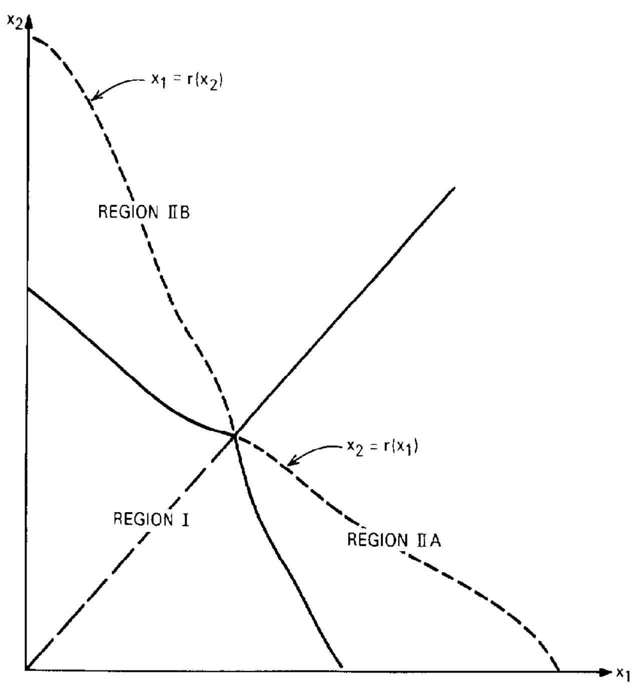
(viii) $\Psi_1(p) \leq \Psi_2(p)$ for all p. This is immediate from the argument above for p in the range $\underline{p} \leq p < P(x_1)$. For $p \geq P(x_1)$, note that $pD(p)$ is receding from $R(x_2)$ more quickly than px_2 is receding from $\underline{p}x_2$ [since $p(D(p) - x_2)$ is still increasing], and $p(D(p) - x_2)$ is increasing, hence approaching $R(x_2)$ more quickly than the constant function zero is approaching $\underline{p}x_2$.

(ix) $\underline{p}x_2 \geq R(x_1)$. To see this, note first that $\underline{p}x_1 \geq R(x_2)$. Thus $\underline{p}x_2 \geq x_2R(x_2)/x_1$. To get the desired result, then, it suffices to show that $R(x_1) \leq x_2R(x_2)/x_1$, or $x_1R(x_1) \leq x_2R(x_2)$ (with strict inequality if $x_1 > x_2$.) Recall that $x_1 > x_2$. If $x_2 \geq x^*$, then the result follows easily from the formula $x_1R(x_1) - x_2R(x_2) = \int_{x_2}^{x_1} (r(x) - x)P(r(x) + x)dx$. If $x_2 < x^*$, then $x_2 > r(x_1)$ (since $(x_1 > r(x_2))$, and the argument from the previous lemma applies.

Putting all these points together, we see that we have an equilibrium of the desired type if firm 1 names prices according to the distribution Ψ_1, and firm 2 names them according to Ψ_2. Each firm is (by construction) indifferent among those strategies that

FIGURE 3

THE DIFFERENT TYPES OF SUBGAME EQUILIBRIA



are in the support of their (respective) distribution functions. The levels of $\bar{p}_1$ and $\underline{p}$ are selected so that firm 1 has no incentive to name a price above the first or below the second. Since firm 2 gets no more than $R(x_1)$, it has no incentive to go above $\bar{p}_1$; neither (by construction) will it gain by naming a price below $\underline{p}$.

Since the construction of the equilibrium took us rather far afield of our main objective, we end this section by compiling the results established above that are important to subsequent analysis:

Proposition 1. (Refer to Figure 3.) In terms of the subgame equilibria, there are three regions of interest.

(a) If $x_i \leq r(x_j)$ for both $i = 1$ and $i = 2$ (which is labelled as region I in Figure 3), the unique equilibrium has both firms naming price $P(x_1 + x_2)$ with certainty. The equilibrium revenues are, therefore, $x_i P(x_1 + x_2)$ for firm i.
(b) If $x_1 \geq x_2$ and $x_1 > r(x_2)$ (labelled region IIA in Figure 3), then, in equilibrium, firm 1 has expected revenue $R(x_2)$, and firm 2 has expected revenue determined by (x_1, x_2) and somewhere between $R(x_2)$ and $x_2 R(x_2)/x_1$. If $x_2 < D(0)$, the equilibrium is the randomized one constructed in Lemma 6; if $x_2 \geqq D(0)$, both firms net zero and name price zero with certainty.
(c) If $x_2 \geq x_1$ and $x_2 > r(x_1)$ (labelled region IIB in Figure 3), then, in equilibrium, firm 2 has expected revenue $R(x_1)$, and firm 1 has expected revenue determined by (x_1, x_2) and somewhere between $R(x_1)$ and $x_1 R(x_1)/x_2$. Similar remarks apply concerning $x_1 \lesseqgtr D(0)$ as appear in (b).
(d) The expected revenue functions are continuous functions of x_1 and x_2.

4. Equilibria in the full game

■ We can now show that in the full game there is a unique equilibrium outcome. We state this formally:

Proposition 2. In the two-stage game, there is a unique equilibrium outcome, namely the Cournot outcome: $x_1 = x_2 = x^*(b)$, and $p_1 = p_2 = P(2x^*(b))$.

Proof. The proposition is established in four steps.

Step 1: preliminaries. Consider any equilibrium. As part of this equilibrium firm i chooses capacity according to some probability measure μ_i with support $S_i \subseteq R$. Let us denote by $\Phi_i(x_1, x_2)$ the (possibly mixed) strategy used by firm i in the (x_1, x_2) subgame. Except for a $\mu_1 \times \mu_2$ null subset of $S_1 \times S_2$, $\Phi_i(x_1, x_2)$ must be an optimal response to $\Phi_j(x_1, x_2)$. That is, $\Omega_i = \{(x_1, x_2)\text{: } \Phi_i(x_1, x_2) \text{ is an optimal response to } \Phi_j(x_1, x_2)\}$ is such that $(\mu_1 \times \mu_2)(\Omega_1 \cap \Omega_2) = 1$. (For subgame perfect equilibria $\Omega_1 \cap \Omega_2 = R^2$, but we do not wish to restrict attention to such equilibria.) In particular, if $E(x_i) = \{x_j : (x_1, x_2) \in \Omega_1 \cap \Omega_2\}$ and $\hat{X}_i = \{x_i \in S_i : \mu_j(E(x_i)) = 1\}$, then $\mu_i(\hat{X}_i) = 1$. Let π_i denote the expected profit of firm i in this equilibrium and $\pi_i(x_i)$ the expected profit when capacity x_i is built. If $X_i = \{x_i \in \hat{X}_i : \pi_i(x_i) = \pi_i\}$, then again $\mu_i(X_i) = 1$. Let $\bar{x}_i$ and $\underline{x}_i$ denote the supremum and infimum of X_i. Because the subgame equilibrium revenue functions are continuous in x_1 and x_2, and because revenues are bounded in any event, $\bar{x}_1$ and $\underline{x}_1$ must yield expected profit π_i if firm j uses its equilibrium quantity strategy μ_j and firms subsequently use subgame equilibrium price strategies.

Assume (without loss of generality) that $\bar{x}_1 \geq \bar{x}_2$.

Step 2: $\bar{x}_1 \geq r_b(\underline{x}_2)$. Suppose contrariwise that $\bar{x}_1 < r_b(\underline{x}_2)$. For every $x_1 < \bar{x}_1$, the subgame equilibrium revenue of firm 2, if it installs capacity $\underline{x}_2$, is $\underline{x}_2 P(x_1 + \underline{x}_2)$. That is,

$$\pi_2 = \int_{\underline{x}_1}^{\bar{x}_1} (\underline{x}_2 P(x_1 + \underline{x}_2) - b(\underline{x}_2))\mu_1(dx_1).$$

If firm 2 increases its capacity slightly, to say, $x_2 + \epsilon$, where it remains true that $\bar{x}_1 < r_b(x_2 + \epsilon)$, then the worst that can happen to firm 2 (for each level of x_1) is that firm 2 will net $(\underline{x}_2 + \epsilon)P(x_1 + \underline{x}_2 + \epsilon) - b(\underline{x}_2 + \epsilon)$. Since for all $x_1 < \bar{x}_1$, $\underline{x}_2 + \epsilon < r_b(x_1)$, it follows that $(\underline{x}_2 + \epsilon)P(x_1 + \underline{x}_2 + \epsilon) - b(\underline{x}_2 + \epsilon) > \underline{x}_2 P(x_1 + \underline{x}_2) - b(\underline{x}_2)$, and this variation will raise firm 2's profits above π_2. This is a contradiction.

Step 3: $\bar{x}_1 \leq r_b(\bar{x}_2)$. Suppose contrariwise that $\bar{x}_1 > r_b(\bar{x}_2)$. By building $\bar{x}_1$, firm 1 nets revenue (as a function of x_2) $R(x_2)$ if $\bar{x}_1 > r(x_2)$ and $\bar{x}_1 P(\bar{x}_1 + x_2)$ if $\bar{x}_1 \leq r(x_2)$, assuming that a subgame equilibrium ensues. That is,

$$\pi_1 = \int_{(r^{-1}(\bar{x}_1),\bar{x}_2]} (R(x_2) - b(\bar{x}_1))\mu_2(dx_2) + \int_{[\underline{x}_2,r^{-1}(\bar{x}_1)]} (\bar{x}_1 P(\bar{x}_1 + x_2) - b(\bar{x}_1))\mu_2(dx_2). \quad (5)$$

Consider what happens to firm 1's expected profits if it lowers its capacity from $\bar{x}_1$ to just a bit less—say, to $\bar{x}_1 - \epsilon$, where $\bar{x}_1 - \epsilon > r_b(\bar{x}_2)$. Then the worst that can happen to firm 1 is that firm 2 (after installing capacity according to μ_2) names price zero. This would leave firm 1 with residual demand $D(p) - x_2$ (where $x_2 \leq \bar{x}_2$). Firm 1 can still accrue revenue $R(x_2)$ if $\bar{x}_1 - \epsilon > r(x_2)$ and $(\bar{x}_1 - \epsilon)P(x_2 + \bar{x}_1 - \epsilon)$ otherwise. Thus, the expected profits of firm 1 in this variation are at least

$$\int_{[r^{-1}(\bar{x}_1-\epsilon),\bar{x}_2]} (R(x_2) - b(\bar{x}_1 - \epsilon))\mu_2(dx_2)$$

$$+ \int_{[\underline{x}_2,r^{-1}(\bar{x}_1-\epsilon))} ((\bar{x}_1 - \epsilon)P(x_2 + \bar{x}_1 - \epsilon) - b(\bar{x}_1 - \epsilon))\mu_2(dx_2). \quad (6)$$

We shall complete this step by showing that for small enough ϵ, (6) exceeds (5), thereby contradicting the assumption.

The difference (6) minus (5) can be analyzed by breaking the integrals into three intervals: $[r^{-1}(\bar{x}_1 - \epsilon), \bar{x}_2]$, $[\underline{x}_2, r^{-1}(\bar{x}_1)]$, and $(r^{-1}(\bar{x}_1), r^{-1}(\bar{x}_1 - \epsilon))$. Over the first interval, the difference in integrands is

$$(R(x_2) - b(\bar{x}_1)) - (R(x_2) - b(\bar{x}_1 - \epsilon)) = \epsilon b'(\bar{x}_1) + o(\epsilon).$$

Note well that $b'(\bar{x}_1)$ is strictly positive. Over the second interval, the difference in integrands is

$$((\bar{x}_1 - \epsilon)P(\bar{x}_1 - \epsilon + x_2) - b(\bar{x}_1 - \epsilon)) - (\bar{x}_1 P(\bar{x}_1 + x_2) - b(\bar{x}_1))$$

$$= \epsilon(b'(\bar{x}_1) - \bar{x}_1 P'(\bar{x}_1) - P(\bar{x}_1 + x_2)) + o(\epsilon).$$

Here the term premultiplied by ϵ is strictly positive except possibly at the lower boundary (where it is nonnegative), since by step 2, $\bar{x}_1 \geq r_b(\underline{x}_2) \geq r_b(x_2)$. Over the third interval, the difference in the integrands is no more than $O(\epsilon)$, because of the continuity of $xP(x + x_2) - b(x)$. Thus as ϵ goes to zero, the integral over the first interval will be strictly positive $O(\epsilon)$ if μ_2 puts any mass on $(r^{-1}(\bar{x}_1), \bar{x}_2]$. The integral over the second interval will be strictly positive $O(\epsilon)$ if μ_2 puts any mass on $(r_b^{-1}(\bar{x}_1), r^{-1}(\bar{x}_1)]$. The integral over the third interval must be $o(\epsilon)$, since it is the integral of a term $O(\epsilon)$ integrated over a vanishing interval. The hypothesis $\bar{x}_1 > r_b(\bar{x}_2)$ implies that μ_2 puts positive mass on either $(r_b^{-1}(\bar{x}_1), r^{-1}(\bar{x}_1)]$ or on $(r^{-1}(\bar{x}_1), \bar{x}_2]$ (or both). Hence for small enough ϵ, the difference between (6) and (5) will be strictly positive. This is the desired contradiction.

Step 4. The rest is easy. Steps 2 and 3 imply that $\bar{x}_1 = r_b(\bar{x}_2) = r_b(\underline{x}_2)$, and hence that firm 2 uses a pure strategy in the first round. But then firm 1's best response in the first round is the pure strategy $r_b(x_2)$. And firm 2's strategy, which must be a best response to this, must satisfy $x_2 = r_b(x_1) = r_b(r_b(x_2))$. This implies that $x_2 = x^*(b)$, and, therefore, $x_1 = r_b(x^*(b)) = x^*(b)$. Finally, the two firms will each name price $P(2x^*(b))$ in the second round (as long as both firms produce $x^*(b)$ in the first round, which they will do with probability one); this follows immediately from Step 1 and Proposition 1.

5. The case $b \equiv 0$

■ When $b \equiv 0$ it is easy to check that the Cournot outcome is an equilibrium. In this case, however, there are other equilibria as well. If imperfect equilibria are counted, then one equilibrium has $x_1 = x_2 = D(0)$ (or anything larger) and $p_1 \equiv p_2 \equiv 0$. Note well that each firm names price zero regardless of what capacities are installed. This is clearly an equilibrium, but it is imperfect, because if, say, firm 1 installed a small capacity and the subgame equilibrium ensued, each would make positive profits.

There are also other perfect equilibria, although it takes a bit more work to establish them. Let $x_1 \geq D(0)$. If firm 2 installs capacity greater than $D(0)$, it will net zero profits (assuming a subgame equilibrium follows). If it installs $x_2 < D(0)$, then its profits (in a perfect equilibrium) are $\underline{p}(x_2)x_2$, where $\underline{p}(x_2) \leq p(0)$ solves the equation $\underline{p}(x_2)D(\underline{p}(x_2)) = R(x_2)$. Hence, in any perfect equilibrium where $x_1 \geq D(0)$, x_2 must be selected to maximize $\underline{p}(x_2)x_2 = R(x_2)x_2/D(\underline{p}(x_2))$. The numerator in the last expression is increasing for $x_2 \leq x^*$ and is decreasing thereafter. (See the proof of Lemma 5.) And as $\underline{p}(x_2)$ decreases in x_2, the denominator increases in x_2. Thus, the maximizing x_2 is less than x^*. But as long as firm 2 chooses capacity less than x^*, the best revenue (in any subgame equilibrium) that firm 1 can hope to achieve is $R(x_2)$, which it achieves with any $x_1 \geq D(0)$. Thus, we have a perfect subgame equilibrium in which firm 1 chooses $x_1 \geq D(0)$ and firm 2 chooses x_2 to maximize $\underline{p}(x_2)x_2$.

6. When both capacity and production are costly

■ In a slightly more complicated version of this game, both capacity (which is installed before prices are named and demand is realized) and production (which takes place after demand is realized) would be costly. Assuming that each of these activities has a convex cost structure and that our assumptions on demand are met, it is easy to modify our analysis to show that the unique equilibrium outcome is the Cournot outcome computed by using the sum of the two cost functions. (This requires that capacity is costly on the margin. Otherwise, imperfect equilibria of all sorts and perfect equilibria of the sort given above will also appear.) It is notable that the cost of capacity need not be very high relative to production cost: the only requirement is that it be nonzero on the margin. Thus, situations where "most" of the cost is incurred subsequent to the realization of demand (situations that will "look" very Bertrand-like) will still give the Cournot outcome. (A reasonable conjecture, suggested to us by many colleagues, is that "noise" in the demand function will change this dramatically. Confirmation or rejection of this conjecture must await another paper.)

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