Econ 611 Solutions to Problem set 5

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1 Consumption/Saving for Retiree

Consider a model where there is a single state variable W (wealth) and two choices: how much to consume c and whether or not to work l. Let $l = 1$ denote the choice to work and $l = 0$ denote the choice to "retire". We initially assume that retirment is absorbing, although it would be easy to extend the model to allow "'unretirement", i.e. a retired person who has run down his/her wealth can decide to return to work. Currently we do not include any state dependence (such as making the decision to work harder for someone who is out of work than for someone who is currently working) and many other elaborations could be added to the model.

We assume that if someone chooses to work they receive a fixed (non-random) wage y but an age-dependent additive disutility or cost of working, d_t at age t. Another extension of the model would include a) allowing for an initial random draw $\tilde{y}_w$ from some initial distribution such as a lognormal to represent a "permanent wage" of the person, and b) in subsequent time periods random wages would be given by $\tilde{y}_t = \tilde{y}_w\tilde{\eta}_t$ where the $\{\tilde{\eta}_t\}$ are *IID* shocks to wages over time, and c) more flexible specifications for decisions such as more than two discrete decisions (e.g. full time, part time or not work, etc).

Let the discount factor at age t be $\beta \in (0,1)$. We believe the closed-form solutions provided below can be extended to the case where β is age-dependent, say $\beta_t \in (0,1)$, which could reflect age-variation due to mortality. We assume that there is a nonstochastic return on savings $\tilde{R} > 1$. We do not consider portfolio decisions, but the model could be be extended to allow returns of the form $R(\mu)$ which depend on a parameter $\mu \in (0,1)$ that can be viewed as a portfolio allocation decision between a riskless security (Treasury bills) and risky securities (stock portfolio). For now we are focusing on two key decisions: a discrete $\{0,1\}$ retirement decision, and an optimal consumption decision during work and retirement.

We consider first the version of the problem where retirement is assumed to be an absorbing state, i.e. once retired we rule out the possibility of subsequent labor market entry. The value function for a person who is age t (with maximum lifespan T) who has not yet retired is $V_t(W,1)$ (where the 1 indicates the state of still working and 0 will indicate the absorbing state of retirement) given by

$$V_t(W,1) = \max\left[\max_{0\le c\le W} u(c) + \beta V_{t+1}(\tilde{R}(W-c),0),\ \max_{0\le c\le W} u(c) - d_t + \beta V_{t+1}(\tilde{R}(W+y-c),1)\right] \quad (1)$$

The value function for a retiree is $V_t(W,0)$, given by

$$V_t(W,0) = \max_{0\le c\le W}\left[u(c) + \beta V_{t+1}(\tilde{R}(W-c),0)\right]. \quad (2)$$

For the class of constant relative risk averse utility functions, $u(c) = (c^{\rho}-1)/\rho$ for $\rho\in[0,1)$ (with $u(c) = \log(c)$ when $\rho = 0$), we have a closed-form solution for $V_{T-t}(W,0)$ where T is the upper bound on lifespan. We have for $\rho\in(0,1)$

$$V_{T-t}(W,0) = \left[\frac{W^{\rho}}{\rho}\right]\left(\sum_{i=0}^{t} K^i\right)^{(1-\rho)} - \frac{1}{\rho}\left(\sum_{i=0}^{t}\beta^i\right) \quad (3)$$

where

$$c_{T-t}(W) = W\left(\sum_{i=0}^{t} K^i\right)^{-1} \quad (4)$$

and

$$K = (\beta[R]^{\rho})^{1/(1-\rho)} \quad (5)$$

so $K\to\beta$ as $\rho\to 0$. Further, as $\rho\to 0$ the value function given in equation (3) above converges (pointwise) to

$$V_{T-t}(W) = \log(W)\left(\sum_{i=0}^{t}\beta^i\right) + A_t \quad (6)$$

where

$$A_t = \left(\sum_{i=0}^{t} i\beta^i\right)[\log(R)+\log(\beta)] - \log\left(\sum_{i=0}^{t}\beta^i\right)\left(\sum_{i=0}^{t}\beta^i\right). \quad (7)$$

Note that equation (6) can be derived by L'Hôpital's rule from (3) in the limit as $\rho\downarrow 0$. Define the optimal retirement threshold at age t, $\overline{w}_t$ by the value of w that makes the person indifferent between retiring and not retiring at that age

$$V_t(\overline{w}_t,0) = V_t(\overline{w}_t,1) \quad (8)$$

Assuming $d_t > 0$ (there is a positive disutility from working), it will be optimal for a person of age t to retire if $w \ge \overline{w}_t$ and work otherwise. We will have a non-convex kink in the value function for working $V_t(W,1)$ at the point $\overline{w}_t$ since we have

$$V_t(W,1) = \max[V_t(W,0),V_t(W,1)] \quad (9)$$

and we can show that in this problem the two functions will only intersect once at $\overline{w}_t$ with $V_t(W,1) > V_t(W,0)$ for $w < \overline{w}_t$ and $V_t(W,1) < V_t(W,0)$ for $w > \overline{w}_t$. Let $c_t(W,0)$ be the optimal consumption of a retiree of age t. This function is given by

$$c_t(W,0) = \underset{0 \le c \le W}{argmax} \left[u(c) + \beta V_{t+1}(\tilde{R}(W-c),0) \right] \tag{10}$$

The optimal consumption of a individual who is still working (not yet retired) is $c_t(W,1)$ given by

$$c_t(W,1) = \underset{0 \le c \le W}{argmax} \left[u(c) - d_t + \beta V_{t+1}(\tilde{R}(W+1-c),1) \right] \tag{11}$$

Let $c_t(W)$ be the optimal consumption function for this individual. It is given by

$$c_t(W) = \begin{cases} c_t(W,1) & \text{if} \quad w_t < \overline{w}_t \\ c_t(W,0) & \text{if} \quad w_t \ge \overline{w}_t \end{cases} \tag{12}$$

We can show that due to the non-convex kink in the value functions that the optimal consumption function $c_t(W)$ will have a discontinuity at $\overline{w}_t$, and

$$c_t(\overline{w}_t,1) > c_t(\overline{w}_t,0). \tag{13}$$

This result follows from the condition that

$$V_t'(\overline{w}_t,1) < V_t'(\overline{w}_t,0) \tag{14}$$

Since there is a kink at $\overline{w}_t$, the derivative $V_t'(\overline{w}_t,1)$ must be interpreted as a left hand derivative (derivative from below $\overline{w}_t$). It will be an important test of various solution methods to see if the solution methods can accurately determine the optimal retirment thresholds $\overline{w}_t$ and capture the discontinuity in the optimal consumption function $c_t(W)$ at these points.

The Bellman equation for value function of retirement $V_t(w,0)$ must be modified when there is a potential for re-entry into the labor force after retirement (something we refer to as "unretirement"). In this case $V_t(w,0)$ is given by

$$V_t(W,0) = \max \left[\max_{0 \le c \le W} u(c) + \beta V_{t+1}(\tilde{R}(W-c),0), \max_{0 \le c \le W} u(c) - d_t - r_t + \beta V_{t+1}(\tilde{R}(W+y-c),1) \right] \tag{15}$$

Notice the value function for a retiree when there is a possibility of labor market re-entry given above differs from the value function given in equation (2) above when retirement is assumed to be an absorbing state by giving the individual an option to return to work, but only by incurring a fixed "labor market re-entry cost" r_t in addition to the usual disutility of working d_t. As $r_t \to \infty$ the problem that allows un-retirement will converge to the problem analyzed above where retirement is assumed to be an absorbing state, since the labor market re-entry option in equation (15) will converge to $-\infty$ and then $V_t(W,0)$ will always equal the first term in the max operator in equation (15), which implies that $V_t(W,0)$ will have the same value as given by equation (2) when the possibility of labor market entry is simply ruled out by assumption.

In the version of the problem with labor market re-entry from retirement, the optimal strategy will consist generally of *two* thresholds, $\overline{w}_{1,t}$ and $\overline{w}_{0,t}$. Further, the consumption decision will depend on the labor market state, which we denote by $c_t(W,0)$ for a retired person, and $c_t(W,1)$ for a worker. The first threshold, $\overline{w}_{1,t}$ is the optimal retirement threshold for someone who is currently working, and is given by the value of w that makes a person who is currently working indifferent between retiring and not retiring. We have

$$u(c_t(\overline{w}_{1,t},1)) + \beta V_{t+1}(\tilde{R}(\overline{w}_{1,t} - c_t(\overline{w}_{1,t},1)),0) = u(c_t(\overline{w}_{1,t},1)) - d_t + \beta V_{t+1}(\tilde{R}(\overline{w}_{1,t} + y - c_t(\overline{w}_{1,t},1)),1). \tag{16}$$

Similarly, the equation for the optimal threshold for "un-retirement" for a person who is currently retired, $\overline{w}_{0,t}$, is given by

$$u(c_t(\overline{w}_{0,t},0)) + \beta V_{t+1}(\tilde{R}(\overline{w}_{0,t} - c_t(\overline{w}_{0,t},0)),0) = u(c_t(\overline{w}_{0,t},0)) - d_t - r_t + \beta V_{t+1}(\tilde{R}(\overline{w}_{0,t} + y - c_t(\overline{w}_{0,t},1)),1). \tag{17}$$

If $r_t > 0$, it is not difficult to show by comparing equations (16) and (17) that $\overline{w}_{0,t} < \overline{w}_{1,t}$. That is, to avoid incurring the labor market re-entry costs, it is optimal for a retiree to return to the labor market at a lower value of wealth $\overline{w}_{0,t}$ than the value of wealth $\overline{w}_{1,t}$ that would induce the opposition transition, i.e. for a worker to retire. However if $r_t = 0$, then it is not hard to see that $\overline{w}_{0,1} = \overline{w}_1, t$, so the two thresholds collapse to each other.

Thus, $V_t(W,1)$ will have a kink at $\overline{w}_{1,t}$ whereas $V_t(W,0)$ will have a kink at $\overline{w}_{0,t}$. In the case where retirement was assumed to be an absorbing state, $V_t(W,0)$ had no kinks and when utility is assumed to be in the CRR class, the formula for this function is given by the close-form solutions above in equation (3) when $\rho > 0$ or by equation (6) when $\rho = 0$. These value functions clearly have no kinks. Thus, the solution to the retirement problem when we allow the possibility of labor market re-entry (and when $r_t > 0$) is more challenging to solve in comparison to the problem where retirement is assumed to be an absorbing state.

Consider the derivation of the optimal retirement thresholds and consumption functions in the special case where $u(c) = \log(c)$. In this case, it is easy to see that in the last period we have $\overline{w}_T = 0$ (i.e. it is optimal for everyone to retire in the last period), and $c_T(w) = w$ (it is optimal to consume all remaining wealth in the last period). Now consider period $T-1$. There is some threshold $\overline{w}_{T-1}$ such that if $w < \overline{w}_{T-1}$ it will be optimal for the person to work, otherwise it will be optimal to retire. We want to derive a formula for $\overline{w}_{T-1}$ and show that this is where there will be a discontinuity in the consumption function. If $w \geq \overline{w}_{T-1}$ it is optimal to retire and so the only decision facing the person is the amount of retirement consumption. This is given by

$$c_{T-1}(w,0) = \underset{0 \leq c \leq w}{argmax} \left[\log(c) + \beta \log(R(w-c))\right] \tag{18}$$

and the solution to this is easily seen from the formulas above to be given by

$$c_{T-1}(w,0) = \frac{w}{(1+\beta)}. \tag{19}$$

Now consider the consumption decision for a worker (i.e. someone with $w < \overline{w}_{T-1}$)

$$c_{T-1}(w,1) = \underset{0 \leq c \leq w}{argmax} \left[\log(c) - d_{T-1} + \beta \log(R(w-c)+y)\right] \tag{20}$$

and the solution to this is given by

$$c_{T-1}(w,1) = \begin{cases} w & \text{if} \quad w < y/R\beta \\ (w+y/R)/(1+\beta) & \text{if} \quad y/\beta \leq w \leq \overline{w}_{T-1} \end{cases} \tag{21}$$

The value function for a worker is

$$V_{T-1}(w,1) = \begin{cases} \log(w) - d_{T-1} + \beta \log(y) & \text{if } w < y/R\beta \\ \log((w+y/R)/(1+\beta)) - d_{T-1} + \beta \log(\beta R(w+y/R)/(1+\beta)) & \text{if } y/R\beta \leq w \leq \overline{w}_{T-1} \end{cases} \tag{22}$$

and the value function for a retiree is

$$V_{T-1}(w,0) = \log(w/(1+\beta)) + \beta \log(\beta R w/(1+\beta)). \tag{23}$$

Equating the values of work and retirement and solving (assuming that $\overline{w}_{T-1} > y/R\beta$) results in the following equation for the optimal retirement threshold $\overline{w}_{T-1}$:

$$V_{T-1}(\overline{w}_{T-1},0) = V_{T-1}(\overline{w}_{T-1},1), \tag{24}$$

and the solution is given by

$$\overline{w}_{T-1} = \frac{(y/R)\exp\{-d_{T-1}/(1+\beta)\}}{1-\exp\{-d_{T-1}/(1+\beta)\}}, \tag{25}$$

provided this is greater than $y/R\beta$ (the threshold below which the consumer is liquidity constrained), otherwise

$$\overline{w}_{T-1} = [y/(R\beta)](1+\beta)^{\frac{(1+\beta)}{\beta}} \exp\{-d_{T-1}/\beta\}. \tag{26}$$

Note that as the disutility of working $d_{T-1} \to \infty$ we have $\overline{w}_{T-1} \to 0$, and as $d_{T-1} \to 0$, then $\overline{w}_{T-1} \to \infty$, i.e. if there is no disutility of working, the person would never choose to retire.

Note also that at $\overline{w}_{T-1}$ there is a kink in the value function: this is a "convex kink" as the max of two concave functions $V_{T-1}(w,0)$ and $V_{T-1}(w,1)$, and this kink in the value function results in a *discontinuity* in the optimal consumption function $c_{T-1}(w)$. There is a drop in consumption equal to $(y/R)/(1+\beta)$ at $\overline{w}_{T-1}$, and with two remaining periods in their life, a retiree has a "marginal propensity to consume" out of wealth equal to $1/(1+\beta)$ the same as a worker. The discontinuous drop in consumption that occurs when the consumer just exceeds the retirement threshold $\overline{w}_{T-1}$ can be seen as their realization that, since they are no longer working in period $T-1$ they will experience a drop in "wealth" equal to the present value of their earnings, y/R, and thus the retiree will rationally cut back on consumption by $(y/R)/(1+\beta)$ which equals the drop in wealth times the marginal propensity to consume out of wealth.

To summarize the solution we found at $T-1$, the optimal retirement threshold is $\overline{w}_{T-1}$ given in equation (25) or (26) depending on the parameter values, and the consumption function is given by

$$c_{T-1}(w) = \begin{cases} w & \text{if} \quad w < y/R\beta \\ (w+y/R)/(1+\beta) & \text{if} \quad y/R\beta \le w \le \overline{w}_{T-1} \\ w/(1+\beta) & \text{if} \quad w > \overline{w}_{T-1} \end{cases} \tag{27}$$

and the value function is given by

$$V_{T-1}(w) = \begin{cases} \log(w) - d_{T-1} + \beta\log(y) & \text{if} \quad w < y/R\beta \\ \log((w+y/R)/(1+\beta)) - d_{T-1} + \beta\log(\beta R(w+y/R)/(1+\beta)) & \text{if} \quad y/R\beta \le w \le \overline{w}_{T-1} \\ \log(w/(1+\beta)) + \beta\log(\beta R w/(1+\beta)) & \text{if} \quad w > \overline{w}_{T-1} \end{cases} \tag{28}$$

Now consider going back one more time period in the backward recursion, to $T-2$. We want to illustrate the possibility of *secondary kinks/discontinuities* in the consumption function for a worker $c_{T-2}(w,1)$ caused by the kinks in $V_{T-1}(w)$. Let $\overline{w}_{T-2}$ denote the *primary kink* due to the retirement threshold at $T-2$

and let $\overline{w}^{r_j}_{T-2}$ denote the secondary kinks, where $j = 1,\ldots,N_{T-2}$ and N_{T-t} is the number of secondary kinks t periods before the end of life at period T.

To see how these secondary kinks arise, consider how the $T-2$ consumption function is determined, as the solution to

$$c_{T-2}(w,1) = \underset{0\leq c\leq w}{argmax}\left[\log(c) - d_{T-2} + \beta V_{T-1}(R(w-c)+y)\right]. \tag{29}$$

As we showed in formula (28), $V_{T-1}(w)$ has two kinks: one at $w = y/R\beta$ where the liquidity constraint stops being binding, and the other at $\overline{w}_{T-1}$ where the worker retires. Assume that the initial wealth of the worker at the start of period $T-1$ is low enough so that the worker will be liquidity constrained in period $T-1$. This implies that $R(w-c)+y < y/R\beta$. Then substituting the liquidity-constrained formula for $V_{T-1}(w)$ from (28) into the period $T-2$ optimization (29), we find that optimal consumption is given by $c_{t-2}(w,1) = (w+y/R)/(1+\beta)$. However imposing the liquidity constraint, we must also have $(w+y/R)/(1+\beta) \leq w$ which implies that $w \leq y/R\beta$, and it is easy to verify that for wealth satisfying this constraint, the worker will be liquidity constrained both in period $T-2$ and in period $T-1$ as well.

However for wealth above $y/R\beta$ the worker is no longer liquidity constrained in period $T-2$ but our derivation of the worker's consumption in period $T-2$ is still contingent on the assumption that the worker is liquidity constrained in period $T-1$. This will be true provided that the savings and earnings the worker brings to the start of period $T-1$, $R\beta(w+y/R)/(1+\beta)$, is less than $y/R\beta$, which is equivalent to the inequality that $w \leq [y/(R\beta)^2](1+\beta-\beta^2)$. It is not hard to show that when $R=1$ we have $y/\beta < (y/\beta^2)(1+\beta-\beta^2)$ so the interval for which the consumer will consume $(w+y)/(1+\beta)$ is non-empty when $R=1$. For $R>1$ if it holds that $y/(R\beta) < [y/(R\beta)^2](1+\beta-\beta^2)$, then this interval will also exist, otherwise the interval is empty and the consumer goes from consuming $c_{T-2}(w,1) = w$ to consuming an amount we derive below.

In this next region, wealth is sufficiently high in period $T-2$ so the consumer is not liquidity constrained at $T-2$ and the saving and earning will keep the consumer out of the liquidity constrained region at $T-1$, but the worker's wealth is not high enough to retire at $T-1$. The relevant expression for $V_{T-1}(w)$ in this case is given by the middle expression in equation (28). This implies an optimal consumption level equal to $c_{T-2}(w,1) = (w+y(1/R+1/R^2))/(1+\beta+\beta^2)$.

As wealth grows even larger, there will come a point where the consumer can save enough in period $T-2$ to retire in period $T-1$, i.e. savings will exceed the $\overline{w}_{T-1}$ threshold. Thus, there is some wealth level $\overline{w}^r_{T-2}$ at which the the relevant expression for the worker's period $T-1$ value function $V_{T-1}(w)$ is given by the last, retirement, formula in (28). The optimal consumption in this region is $c_{T-2}(w,1) = (w+y/R)/(1+\beta+\beta^2)$. It is important to carefully check values of c such that savings, $w+y-c$ is in the "convex region" of $V_{T-1}(w)$ around the $T-1$ retirement threshold, $\overline{w}_{T-1}$. In this region there will be *two local optima* for c, one involving the higher consumption $(w+y(1/R+1/R^2))/(1+\beta+\beta^2)$ and the other involving the lower consumption $(w+y/R)/(1+\beta+\beta^2)$ that enables the worker to retire at $T-1$.

These two solutions are reflected in the two possible solutions to the first order condition for optimal consumption is given by

$$0 = \frac{1}{c} - \begin{cases} (\beta+\beta^2)/(w-c+y(1/R+1/R^2)) & \text{if } R(w-c)+y < \overline{w}_{T-1} \\ (\beta+\beta^2)/(w-c+y/R) & \text{if } R(w-c)+y \geq \overline{w}_{T-1} \end{cases} \tag{30}$$

For $w < \overline{w}^r_{T-2}$ the global optimum will be $c_{T-2}(w,1) = (w+y(1/R+1/R^2))/(1+\beta+\beta^2)$ and the consumer will be working in both periods $T-2$ and $T-1$. However for $w > \overline{w}^r_{T-2}$ the consumer will still work at $T-2$ (provided $w < \overline{w}_{T-2}$, the primary kink point at $T-2$, the wealth threshold at which the consumer retires at $T-2$) but will have enough savings to retire at $T-1$. The optimal consumption in this case will be $c_{T-2}(w,1) = (w+y/R)/(1+\beta+\beta^2)$. It is not hard to show that if $w \leq [y/(R\beta)^2](1+\beta-R\beta^2)$, then the quantity $R(w-c_{T-2}(w,1))+y \leq y/R\beta$, i.e. the consumer will indeed be in the liquidity constrained region $w \leq y/R\beta$ at the start of $T-1$ as we assumed would be the case. We also have that $y/R\beta < [y/(R\beta)^2](1+\beta-R\beta^2)$ provided that $R\beta \leq 1$, which we assume to be the case. Otherwise this region would be empty and the optimal consumption would be given by $c_{T-2}(w,1) = (w+y(1/R+1/R^2))/(1+\beta+\beta^2)$ as derived above. We can check that this consumption function, which is also derived under the assumption that the consumer will not be liquidity constrained at period $T-1$, will result in total savings at $T-1$ that satisfies $R(w-c)+y \geq y/R\beta$ (so the consumer is not liquidity constrained at $T-1$) for wealth at $T-2$ at the lower end of this interval (i.e. at $w = y/R\beta$) provided that $R \leq 1/\beta$.

However, at $w = \overline{w}^r_{T-2}$ the consumer will be indifferent between consuming the larger amount $(w+y(1/R+1/R^2))/(1+\beta+\beta^2)$ knowing they will *not* retire at $T-1$ and consuming the lower amount $(w+y/R)/(1+\beta+\beta^2)$ and knowing they will retire at $T-1$. We find $\overline{w}^r_{T-2}$ as the solution to the following equation

$$\begin{aligned} &\log\left((w+y(1/R+1/R^2))/(1+\beta+\beta^2)\right) + \beta V_{T-1}\left((y+R(w-(w+y(1/R+1/R^2)))/(1+\beta+\beta^2))\right) \\ &= \log\left((w+y/R)/(1+\beta+\beta^2)\right) + \beta V_{T-1}\left(y+R(w-(w+y/R))/(1+\beta+\beta^2)\right). \end{aligned}$$

Thus, at $w = \overline{w}^r_{T-2}$ the consumer is indifferent between consuming the larger amount $(w+y(1/R+$

$1/R^2))/(1+\beta+\beta^2)$ or consuming the smaller amount $(w+y/R)/(1+\beta+\beta^2)$ that provides the additional savings necessary to enable the consumer to retire at $T-1$.

Now we can express the period $T-2$ consumption function as the following piece-wise linear function:

$$c_{T-2}(w,1)=\begin{cases} w & \text{if} \quad w<y/R\beta \\ (w+y/R)/(1+\beta) & \text{if} \quad y/R\beta \leq w \leq [y/(R\beta)^2](1+\beta-R\beta^2) \\ (w+y(1/R+1/R^2))/(1+\beta+\beta^2) & \text{if} \quad [y/(R\beta)^2](1+\beta-R\beta^2) \leq w \leq \overline{w}^r_{T-2} \\ (w+y/R)/(1+\beta+\beta^2) & \text{if} \quad \overline{w}^r_{T-2}<w<\overline{w}_{T-2} \end{cases} \tag{31}$$

To derive the time $T-2$ retirement threshold $\overline{w}_{T-2}$ we solve for the value of w that makes the consumer indifferent between retiring at $T-2$ and working (but with enough wealth so that the person is above the secondary kink $\overline{w}^r_{T-2}$ where their consumption is given by $c_{T-2}(w,1)=(w+y/R)/(1+\beta+\beta^2))$

$$\log(w)(1+\beta+\beta^2)+A_2=\log((w+y/R)/(1+\beta+\beta^2))-d_{T-2}+A_2 \tag{32}$$

where A_2 is defined in equation (7) above. Note that the right hand side of (32) is the value function for a consumer who does not have enough wealth to retire at $T-2$, but since $w>\overline{w}^r_{T-2}$ (the secondary kink point, i.e. the saving threshold that will cause the consumer to retire at $T-1$), it follows that the appropriate formula for $V_{T-1}(w)$ will be the one where $w>\overline{w}_{T-1}$ in equation (28) above. The solution to this equation is $\overline{w}_{T-2}$ given by

$$\overline{w}_{T-2}=\frac{(y/R)e^{-K}}{(1-e^{-K})} \tag{33}$$

where K is given by

$$K=\frac{d_{T-2}}{(1+\beta+\beta^2)}. \tag{34}$$

Notice that if $d_{T-1}\geq d_{T-2}$, then formulas (33) and (25) imply that $\overline{w}_{T-1}<\overline{w}_{T-2}$, i.e. the wealth threshold for retirement decreases as one approaches the end of life, T.

To summarize the solution we found at $T-2$, the optimal retirement threshold $\overline{w}_{T-2}$ is the solution to equation (32), and the optimal consumption function is given by

$$c_{T-2}(w)=\begin{cases} w & \text{if} \quad w<y/R\beta \\ (w+y/R)/(1+\beta) & \text{if} \quad y/R\beta \leq w \leq [y/(R\beta)^2](1+\beta-R\beta^2) \\ (w+y(1/R+1/R^2))/(1+\beta+\beta^2) & \text{if} \quad [y/(R\beta)^2](1+\beta-R\beta^2) \leq w \leq \overline{w}^r_{T-2} \\ (w+y/R)/(1+\beta+\beta^2) & \text{if} \quad \overline{w}^r_{T-2}<w\leq\overline{w}_{T-2} \\ w/(1+\beta+\beta^2) & \text{if} \quad w>\overline{w}_{T-2} \end{cases} \tag{35}$$

The optimal consumption function at $T-2$ has one kink at $w = y/R\beta$ (the level of wealth at which the consumer is no longer liquidity-constrained), and three discontinuities: one at $[y/(R\beta)^2](1+\beta-R\beta^2)$, one at the secondary kink point $\overline{w}^r_{T-2}$ where consumption drops by $(y/R^2)/(1+\beta+\beta^2)$, and the other at the retirement threshold $\overline{w}_{T-2}$ where consumption drops by another $(y/R)/(1+\beta+\beta^2)$. Note that the secondary kink point $\overline{w}^r_{T-2}$ is precisely the amount of wealth where, while the consumer does not yet retire at $T-2$, *they know they will have enough to retire at* $T-1$. Thus, the drop in consumption at this secondary kink point can be regarded as *saving at* $T-2$ *for their anticipated retirement at time* $T-1$.

The value function at $T-2$ can be expressed this way:

$$V_{T-2}(w) = \begin{cases} \log(c_{T-2}(w)) - d_{T-2} + \beta V_{T-1}(R(w - c_{T-2}(w)) + y) & \text{if} \quad w < \overline{w}_{T-2} \\ \log(w)(1+\beta+\beta^2) + A_2 & \text{if} \quad w \geq \overline{w}_{T-2} \end{cases} \tag{36}$$

Thus, depending on whether the person's wealth at $T-2$ is above or below the secondary kink point $\overline{w}^r_{T-2}$, they will know whether they will have enough (with their $T-2$ earnings y) to retire at $T-1$ or not, and will save/consume accordingly.

Now consider solving the problem at $t = T-3$, three periods before the end of life. Now the consumption will have one kink at the level of w where the liquidity constraint no longer binds, and *five* discontinuities, two more discontinuities than $c_{T-2}(w)$, with the two new discontinuities in $c_{T-3}(w)$ corresponding to the two additional kink points in $V_{T-2}(w)$ relative to $V_{T-1}(w)$. One of the new discontinuities in $c_{T-3}(w)$ is added above the end point $[y/(R\beta)^2](1+\beta-R\beta^2)$ of the first linear segment of $c_{T-2}(2)$. We refer to this as a "liquidity constraint related discontinuity." The other new discontinuity corresponds to the secondary kink point $\overline{w}^r_{T-2}$ and we refer to this as a "retirement related discontinuity."

Note the pattern here: $c_{T-1}(w)$ has one kink and one discontinuity, $c_{T-2}(w)$ has one kink and *three* discontinuities, and $c_{T-3}(w)$ will have one kink and *five* discontinuities — two more than $c_{T-2}(w)$. The two new discontinuities in $c_{T-3}(w)$ correspond to the two additional discontinuities present in $c_{T-2}(w)$ which are two more than the single discontinuity in $c_{T-1}(w)$. The important additional point to notice is that c_{T-1}, c_{T-2} and as we show shortly, c_{T-3}, are all *piecewise linear.*

It will be helpful to distinguish the points marking the sequence of linear segments of the consumption function relating to emerging from the liquidity constrained region $[0, y/R\beta]$ from those at higher levels of wealth that related to retirement decisions — both current retirement and anticipated future retirements. Label the first set of liquidity constraint related discontinuities as $\overline{w}^{l_j}_{T-t}$ and the latter set of retirement related discontinuities as $\overline{w}^{r_j}_{T-t}$. Then for $t = 1$ $c_{T-1}(w)$ has no liquidity constraint related kinks but one retirement kink, $\overline{w}_{T-1}$, and at $t = 2$ $c_{T-2}(w)$ has 1 liquidity constraint related kink $\overline{w}^{l_1}_{T-2}$ at $[y/(R\beta)^2](1+\beta - R\beta^2)$ and two retirement related discontinuities $\overline{w}^{r_1}_{T-2}$ and $\overline{w}_{T-2}$, where $\overline{w}_{T-2}$ is the retirement threshold at period $T-2$ and $\overline{w}^{r_1}_{T-2} =$ is the point we referred to above as the "secondary kink" $\overline{w}^{r}_{T-2}$. Thus $\overline{w}^{r_1}_{T-2}$ is the level of wealth that leads the worker to discontinuously reduce consumption at $T-2$ in order to have enough savings to retire at $T-1$.

In period $T-3$ there will be a total of five discontinuities in $c_{T-3}(w)$. The last discontinuity occurs at the retirement threshold $\overline{w}_{T-3}$, but there will be two additional discontinuities at the secondary kink points in the value function V_{T-2}. These are denoted $\overline{w}^{r_1}_{T-3}$ and $\overline{w}^{r_2}_{T-3}$. We have the ordering $\overline{w}_{T-3} > \overline{w}^{r_2}_{T-3} > \overline{w}^{r_1}_{T-3}$. The highest secondary kink point $\overline{w}^{r_2}_{T-3}$ is the level of wealth that leads the consumer to save an amount (including current period wage earnings) of $\overline{w}_{T-2}$, which is the retirement threshold at period $T-2$. Thus at wealth levels that just exceed $\overline{w}^{r_2}_{T-3}$ the consumer works in period $T-3$ but discontinuously reduces consumption in order to have enough resources to retire in period $T-2$. At wealth levels that are just below this $\overline{w}^{r_2}_{T-3}$, the consumer works in both periods $T-3$ and $T-2$, and retires only in period $T-1$.

Similarly, the secondary kink point $\overline{w}^{r_1}_{T-3}$ is th level of wealth in period $T-3$ that leads the consumer to save an amount $\overline{w}^{r}_{T-2}$, which is the secondary kink in the value function at time $T-2$, i.e the level of wealth just beyond which the consumer will have enough saving by period $T-1$ to retire. Thus, in period $T-3$, a consumer who has wealth slightly above $\overline{w}^{r_1}_{T-3}$ will work in periods $T-3$ and $T-2$, but retire in period $T-1$, whereas a consumer with wealth just below $\overline{w}^{r_1}_{T-3}$ will work in all three periods, $T-3$, $T-2$ and $T-1$ and only retires in the last period of life, T.

The consumption function $c_{T-3}(w)$ will also have two *liquidity constraint related discontinuities* $\overline{w}^{l_1}_{T-3}$ and $\overline{w}^{l_2}_{T-3}$, in addition to the liquidity constraint induced kink point at $w = y/R\beta$. The first discontinuity will be $\overline{w}^{l_1}_{T-3} = [y/(R\beta)^2](1+\beta - R\beta^2)$, the level of wealth at which the switches from consuming according the 2nd linear segment of $c_{T-3}(w) = (w + y/R)/(1+\beta)$ to consuming on the third linear segment $c_{T-3}(w) = (w + y(1/R + 1/R^2))/(1+\beta+\beta^2)$. Note at the wealth threshold $\overline{w}^{l_1}_{T-3}$, the implied savings exceeds $y/R\beta$ so that the consumer will be out of the liquidity constrained region in period $T-2$.

At the second liquidity constraint related kink point $\overline{w}^{l_2}_{T-3}$ the worker switches from consuming on the third segement of $c_{T-3}(w) = (w + y(1/R + 1/R^2))/(1+\beta+\beta^2)$ to the fourth segment which is the first of the segments created by the retirement related kink points $\overline{w}^{r_j}_{T-3}$. Thus for wealth that exceeds $\overline{w}^{l_2}_{T-3}$ consumption switches to $c_{T-3}(w) = (w + y(1/R + 1/R^2 + 1/R^3))/(1+\beta+\beta^2+\beta^3)$. Then for still higher levels of wealth the worker consumes according to the various piecewise linear segments demarcated by the successive retirement related kink points $\overline{w}^{r_j}_{T-3}$, $j = 1,2,3$ where $\overline{w}^{r_3}_{T-3} = \overline{w}_{T-3}$ is the retirement threshold at period $T-3$.

Note that the marginal propensity to consume out of wealth is also piecewise linear but also monotonically decreasing in w. In the liquidity constrained region the marginal propensity to consume is 1, and in the first of the liquidity constrained related consumption segements it is $1/(1+\beta)$, and in the second liquidity constrained related segment it is $1/(1+\beta+\beta^2)$. Then in the remaining retirement related consumption segments, the marginal propensity to consume out of wealth is constant and equal to $1/(1+\beta+\beta^2+\beta^3)$.

In summary, the consumption function $c_{T-3}(w)$ is given by

$$c_{T-3}(w) = \begin{cases} w & \text{if} \quad w < y/R\beta \\ (w+y/R)/(1+\beta) & \text{if} \quad y/R\beta \leq w \leq \overline{w}^{l_1}_{T-3} \\ (w+y(1/R+1/R^2))/(1+\beta+\beta^2) & \text{if} \quad \overline{w}^{l_1}_{T-3} \leq w \leq \overline{w}^{l_2}_{T-3} \\ (w+y(1/R+1/R^2+1/R^3))/(1+\beta+\beta^2+\beta^3) & \text{if} \quad \overline{w}^{l_2}_{T-3} \leq w \leq \overline{w}^{r_1}_{T-3} \\ (w+y(1/R+1/R^2))/(1+\beta+\beta^2+\beta^3) & \text{if} \quad \overline{w}^{r_1}_{T-3} \leq w < \overline{w}^{r_2}_{T-3} \\ (w+y/R)/(1+\beta+\beta^2+\beta^3) & \text{if} \quad \overline{w}^{r_2}_{T-3} \leq w < \overline{w}_{T-3} \\ w/(1+\beta+\beta^2+\beta^3) & \text{if} \quad \overline{w}_{T-3} < w \end{cases} \tag{37}$$

The retirement threshold $\overline{w}_{T-3}$ is given by

$$\overline{w}_{T-3} = \frac{(y/R)e^{-K}}{(1-e^{-K})} \tag{38}$$

where

$$K = \frac{d_{T-3}}{(1+\beta+\beta^2+\beta^3)}. \tag{39}$$

We solve for the secondary kinks $\{\overline{w}^{l_i}_{T-3}, \overline{w}^{r_j}_{T-3}\}$, $i = 1,2$ and $j = 1,2$ in the same way as we did for the period $T-2$ secondary kink: we solve for the level of a wealth that makes the consumer indifferent between consuming the higher level of consumption to the “left” of the kink point (more precisely the limit of consumption for wealth approaching the kink point from below) and the lower level of consumption to the “right” of the discontinuity (the limit of consumption for wealth approaching the kink point from above). We do this for each of the discontinuities below the final discontinuity at the optimal retirement threshold $\overline{w}_{T-3}$. So in period $T-3$ there will be *four* such discontinuities, corresponding to the total of four kinks in the value function at $T-2$.

Finally, the value function is given by

$$V_{T-3}(w) = \begin{cases} \log(c_{T-3}(w)) - d_{T-3} + \beta V_{T-2}(R(w - c_{T-3}(w)) + y) & \text{if} \quad w < \overline{w}_{T-3} \\ \log(w)(1 + \beta + \beta^2 + \beta^3) + A_3 & \text{if} \quad w \geq \overline{w}_{T-3} \end{cases} \tag{40}$$

Due to the monotonicity of the saving function, the fact that $\overline{w}_{T-2} > \overline{w}^{r}_{T-2}$ implies that $\overline{w}^{l_1}_{T-3} > w^{l_2}_{T-3} > \overline{w}^{r_1}_{T-3} > \overline{w}^{r_2}_{T-3}$. Similarly, if $d_{T-3} \leq d_{T-2}$, then using formulas (33) and (38), it is not hard to show that $\overline{w}_{T-3} > \overline{w}_{T-2}$.

Having solved for the consumption function explicitly by doing backward induction for 3 periods, it is easy to see the general pattern. At t periods before the end of life at T, i.e. at period $T-t$, the consumption function $c_{T-t}(w)$ will have a total of $2t-1$ discontinuities and $2t+1$ linear segments. Of the $2t-1$ discontinuities, t of them will be retirement related discontinuities and $t-1$ will be liquidity constraint related discontinuities. For every period $T-t$, $t \geq 1$ there will be a kink in the consumption function at $w = y/R\beta$ corresponding to the end of the liquidity constrained region, $[0, y/R\beta]$.

Notationally, we denote the last of the retirement related discontinuities by $\overline{w}_{T-t}$, and the $t-1$ additional retirement related discontinuities as $\overline{w}^{r_j}_{T-t}$, for $j = 1, \ldots, t-1$. The $t-1$ liquidity constraint related discontinuities are denoted by $\overline{w}^{l_j}_{T-t}$, for $j = 1, \ldots, t-1$. The first of the liquidity constrained related kink points is always at the same value of w,

$$\overline{w}^{l_1}_{T-t} = [y/(R\beta)^2](1 + \beta - R\beta^2) \quad \text{for } t \geq 2 \tag{41}$$

The discontinuities in $c_{T-t}(w)$ are ordered as follows

$$y/R\beta < \overline{w}^{l_1}_{T-t} < \overline{w}^{l_2}_{T-t} < \cdots < \overline{w}^{l_{t-1}}_{T-t} < \overline{w}^{r_1}_{T-t} < \overline{w}^{r_2}_{T-t} < \cdots < \overline{w}^{r_{t-1}}_{T-t} < \overline{w}_{T-t} \tag{42}$$

where $\overline{w}_{T-t}$ is given by

$$\overline{w}_{T-t} = \frac{(y/R)e^{-K}}{(1 - e^{-K})}, \tag{43}$$

where K is given by

$$K = \frac{d_{T-t}}{(\sum_{i=0}^{t} \beta^i)}. \tag{44}$$

The values of the last $t-2$ liquidity constraint related secondary kink points $\overline{w}^{l_j}_{T-t}$, $j = 2, \ldots, t-1$ and the first $t-2$ retirement related secondary kink points $\overline{w}^{r_j}_{T-t}$, $j = 1, \ldots, t-2$ are determined by the values of wealth that make the consumer indifferent between consuming according to the linear segments of the consumption function on either side of each of these kink points as described above.

The value function $V_{T-t}(w)$ can be expressed recursively in terms of the already defined value function $V_{T-t+1}(w)$ one period ahead:

$$V_{T-t}(w) = \begin{cases} \log(c_{T-t}(w)) - d_{T-t} + \beta V_{T-t+1}(R(w - c_{T-t}(w)) + y) & \text{if} \quad w < \overline{w}_{T-t} \\ \log(w)\left(\sum_{i=0}^{t} \beta^i\right) + A_t & \text{if} \quad w \geq \overline{w}_{T-t} \end{cases} \tag{45}$$

where A_t was defined in equation (7) above.

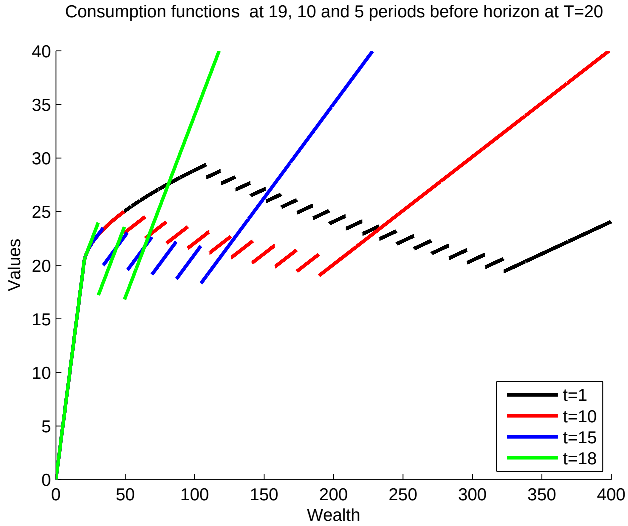
The expression for the piecewise linear consumption function $c_{T-t}(w)$ is given by:

$$c_{T-t}(w) = \begin{cases} w & \text{if} \quad w \leq y/R\beta \\ [w + y/R]/(1+\beta) & \text{if} \quad y/R\beta \leq w \leq \overline{w}_{T-t}^{l_1} \\ [w + y(1/R + 1/R^2)]/(1+\beta+\beta^2) & \text{if} \quad \overline{w}_{T-t}^{l_1} \leq w \leq \overline{w}_{T-t}^{l_2} \\ \cdots & \cdots \quad \cdots \\ \left[w + y\left(\sum_{i=1}^{t-1} R^{-i}\right)\right]\left(\sum_{i=0}^{t-1} \beta^i\right)^{-1} & \text{if} \quad \overline{w}_{T-t}^{l_{t-2}} \leq w \leq \overline{w}_{T-t}^{l_{t-1}} \\ \left(w + \left(\sum_{i=1}^{t} R^{-i}\right)\right)\left(\sum_{i=0}^{t} \beta^i\right)^{-1} & \text{if} \quad \overline{w}_{T-t}^{l_{t-1}} \leq w \leq \overline{w}_{T-t}^{r_1} \\ \left[w + \left(\sum_{i=1}^{t-1} R^{-i}\right)\right]\left(\sum_{i=0}^{t} \beta^i\right)^{-1} & \text{if} \quad \overline{w}_{T-t}^{r_1} \leq w \leq \overline{w}_{T-t}^{r_2} \\ \cdots & \cdots \quad \cdots \\ [w + y(1/R + 1/R^2)]\left(\sum_{i=0}^{t} \beta^i\right)^{-1} & \text{if} \quad \overline{w}_{T-t}^{r_{t-2}} \leq w \leq \overline{w}_{T-t}^{r_{t-1}} \\ [w + y/R]\left(\sum_{i=0}^{t} \beta^i\right)^{-1} & \text{if} \quad \overline{w}_{T-t}^{r_{t-1}} \leq w \leq \overline{w}_{T-t} \\ w\left(\sum_{i=0}^{t} \beta^i\right)^{-1} & \text{if} \quad \overline{w}_{T-t} < w \end{cases} \tag{46}$$

Figure 1 above illustrates optimal consumption functions in a problem where $T = 20$, $\beta = 0.98$, $y = 20$ and the disutility of work is $d_t = 1$ for $t = 1, \ldots, T$. Though the consumption functions are indeed piecewise linear, the jumps in the linear segments for $w \leq \overline{w}_{20}^{l_{36}}$ (the last of the "liquidity constrained" secondary kink points) are sufficiently small that they appear in the graph to join together in the apparently "curved" segments of the consumption functions before they start to "break up" for larger values of wealth in the "retirement region" where the discontinuities at the retirement related kink points $\{\overline{w}_{T-t}^{r_j}\}$ and $\overline{w}_{T-t}$ appear.

This simple example indicates the complexity caused by the combination of the discrete retirement/work decision and the continuous optimal consumption decision, and how multiple discontinuities can arise in the optimal consumption function. Further, these discontinuities can *propogate* was we continue the backward induction solution of the consumer's life cycle problem. Thus, at age $T-3$ there can be *three* discontinuities in the optimal consumption function, corresponding to the primary discontinuity at the retirement threshold $\overline{w}_{T-3}$ plus two additional secondary discontinuities that are inherited from the two discontinuities in $c_{T-2}(w)$ that we derived above, and so on.

Figure 1: Consumption functions for a $T = 20$, $d_t = 1$, $y = 20$, $\beta = 0.98$ and $R = 1$



2 Optimal Replacement Problem

Recall the Bellman equation for the replacement problem is given by

$$V(x) = \min\left[c(0) + K + \beta\int_0^\infty V(y)\lambda e^{-\lambda y}dy,\ c(x) + \beta\int_0^\infty V(x+y)\lambda e^{-\lambda y}dy.\right] \tag{47}$$

Via a change of variables we can rewrite this as

$$V(x) = \min\left[c(0) + K + \beta\int_0^\infty V(y)\lambda e^{-\lambda y}dy,\ c(x) + \beta\int_x^\infty V(y)\lambda e^{-\lambda(y-x)}dy.\right] \tag{48}$$

Notice that the value of not replacing (the first expression in the min in equation **??**) is a constant, independent of x. We will show that if $c'(x) > 0$, then the value of not replacing (the second expression in the min in equation **??**) is also increasing in x. Since we also assume that $K > 0$, it follows that there is a $\gamma > 0$ satisfying

$$c(0) + K + \beta\int_0^\infty V(y)\lambda e^{-\lambda y}dy = c(\gamma) + \beta\int_\gamma^\infty V(y)e^{-\lambda(y-\gamma)}dy \tag{49}$$

So the optimal replacement strategy is to keep the asset (or not replace the bus engine) if $x \in [0,\gamma]$ and trade (or replace the bus engine) otherwise. We refer to the interval $[0,\gamma]$ as the *continuation region* and in the continuation region you can differentiate on both sides of the Bellman equation (**??**) to derive the following *first order differential equation* for $V(x)$ given by

$$V'(x) = -c'(x) + \lambda c(x) + \lambda(1-\beta)V(x) \tag{50}$$

This ODE is referred to as a *free boundary value problem* because the boundary condition

$$V(\gamma) = K + V(0) = -c(\gamma) + \beta V(\gamma) = \frac{-c(\gamma)}{1-\beta} \tag{51}$$

is determined endogenously (i.e. it depends on V, the solution to the ODE). The solution to the free boundary value problem is given by

$$V(x) = \max\left[-c(\gamma)/(1-\beta), -c(\gamma)/(1-\beta) + \int_x^\gamma \left[\frac{c'(y)}{(1-\beta)}\right]\left[1-\beta e^{\lambda(1-\beta)y}\right] dy\right]. \tag{52}$$

and γ is the unique solution to

$$K = \int_0^\gamma \left[\frac{c'(y)}{(1-\beta)}\right]\left[1-\beta e^{-\lambda(1-\beta)y}\right] dy. \tag{53}$$