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Problem set 5
Due in class on Tuesday October 13th

1. **Finite Horizon Dynamic Programming** Consider an individual who is retired and has initial retirement savings W and expects to live for T more years (with no uncertainty over their remaining lifespan). The individual has logarithmic utility function $u(c) = \log(c)$ and has no bequest motive. There is only one savings option for this person: a savings account at the bank that pays interest rate r so that saving one dollar today results in $R = (1+r)$ dollars in the savings account next year. The person maximizes discounted utility over their remaining lifetime and has discount factor $\beta \in (0,1)$. Using dynamic programming, solve for the optimal consumption rule for this retiree, i.e. the function $c_t(W)$ that specifies how much is optimal for a retiree with t more years of life left to live will consume of their wealth W. Also derive a formula for the *value function* $V_t(W)$ that provides the discounted utility for this person over their remaining life assuming they follow an optimal consumption plan.

A. How does your answer change if the person has a CRRA utility function $u(c) = [c^\rho - 1]/\rho$? Show that $\log(c) = \lim_{\rho \to 0}[c^\rho - 1]/\rho$ and derive the consumption function and value function for the full CRRA class of utility functions, not just the special case of log utility when $\rho = 0$.

B. Now consider a general, concave and differentiable utility function $u(c)$. Show that the *Euler equation* must hold, i.e. for each $t \in \{1, \dots, T\}$, the optimal consumption $c_t(W)$ solves the following recursive equation

$$u'(c) = \beta R u'(c_{t-1}(R(W-c)) \qquad (1)$$

and that $c_0(W) = W$.

C. Now consider a *worker* who has T periods to live. The worker earns a deterministic income $y > 0$ each period he/she works, but earns no further wages and has no pension once he/she retires. Instead, the retiree must live off any savings W they have accumulated in the working phase of their life. Assume that retirement is an *absorbing state* — i.e. once a person retires, they cannot "unretire". Write down the Bellman equation for the decision of a worker about whether or not to retire. (**Hint:** you should use your solution for the "retiree's problem" as a sub-component of the "worker's problem", that is, the value of a worker who decided whether or not to retire will depend on the value already solved above for a retiree.)

D. Try to derive the optimal consumption function for a worker and compare it to that of a retiree. Is a retiree more "cautious" and consumes less than a worker of the same age and wealth? Using numerical methods if necessary, plot the optimal consumption function for a worker who earns $y = 20$ when they have $T = 20$ years left to live, with $\beta = .98$ and $R = 1$ for $t = 2, 5, 10, 19$ years prior to their last year of life.

2. **Buffet's problem** Now consider a slight generalization of the problem given above. Consider Warren Buffet who is very rich and retired, so Buffet does not earn any labor income any longer, but Buffet faces a choice of how much to consume and how much of the wealth he invests to invest in risk-free bonds

that earn a deterministic return $R = (1+r)$ and a stock portfolio, which provides *IID* random (normally distributed) returns $\tilde{Z}_t$ where $\tilde{Z}_t$ is the (percentage) return from the stock market in year t which is a lognormal random variable with parameters (μ, σ^2). Suppose Buffet has a constant risk of dying each year, $\rho \in (0,1)$ and he discounts future utility at rate $\beta \in (0,1)$ and has logarithmic utility $u(c) = \log(c)$ and no bequest motive. Write down the Bellman equation equation for Buffet's problem and see if you can solve for $V(W)$, the present discounted value of Buffet's utility over his remaining life if he follows and optimal consumption/savings and investment strategy.

A. Using either numerical or analytical methods, find the optimal consumption rule for Buffet, $c(W)$ and explain why it does not depend on Buffet's age t, whereas the solution in problem 1 above does depend on how many years a person had left to live.

B. Using either numerical or analytical methods, describe Buffet's optimal investment strategy. Let $\mu(W)$ be the fraction of Buffett's saving that he invests in the risk-free bonds. Is Buffett a more cautious investor when he has less wealth, i.e. is μ a decreasing function of W?

C. Suppose an insurance company were to offer Buffett an *actuarially fair annuity.* That is, Buffett can give some or all of his wealth W to the insurance company to purchase in return a constant payment a per year that Buffett reamins alive, and the expected present discounted value of this stream (discounted at the risk free interest rate r and taking into account Buffett's true probability of dying in any given year, ρ) equals the amount he turns over, W. That is, the insurance company makes no net profit from this deal, in the sense that the expected present value of annuity payments exactly equals the amount the company receives from Buffett up front. Calculate a formula for this annuity payment a.

D. If Buffett faced the option to take some or all of his wealth and buy an annuity, or keep his wealth and invest it himself, what would he do? That is, re-solve Buffet's problem but now allowing for his option to buy an annuity and either fully or partially *annuitize* his wealth W.

3. You have a car and the number of miles it has been driven is recorded on the car's *odometer.* Call the odometer reading x. The number of additional miles you drive your car each period is random, but is drawn from an *exponential distribution* with parameter $\lambda > 0$. This means if you have a car with x miles on its odometer today, the number of miles you expect it will have on its odometer after one year is $x + 1/\lambda$. You have a quasi linear utility function $u(x) + y$ where y is your income that you use for other consumption, and $u(x)$ is the happiness you get from using your car. You prefer newer cars to older ones so $u'(x) < 0$. In addition, you must pay maintenance costs to keep your car running each year, $m(x)$, and naturally we have $m'(x) > 0$. Suppose you live forever and will always need one car but have no need for more than one car in any period you live. You discount the future at rate $\beta \in (0,1)$. You can buy a new car for price $\overline{P} > 0$ but there is no secondary market for used cars, so if your car gets too old (i.e. x becomes too large) your only option is to scrap it for amount $\underline{P} > 0$. Naturally we assume that $\overline{P} > \underline{P}$.

A. What is the optimal strategy for how long to keep an existing car and when to buy a new car that maximizes your discounted lifetime utility? Try to characterize the optimal strategy as explicitly as you can and write down Bellman's equation for this problem.

B. If $\overline{P} = 20000$, $\underline{P} = 500$ and $u(x) = 10 - x/10$ and $m(x) = x/10$ where x is the odometer value in thousands of miles, and $\beta = .95$, can you solve either numerically or analytically for the optimal strategy for trading your car?

C. Suppose there is a used car market and you can buy a used car with x miles on it for $P(x)$. A new car has $x = 0$ miles on it and sells for price $\overline{P} = P(0)$ just as above, but $P(x)$ is a continuous downward sloping function that satisfies $P(x) = \underline{P}$ for $x \geq \gamma$ for some positive value $\gamma > 0$. Assume there are no *transactions costs* so that if you decide to sell you existing car with x miles on its odometer and buy another used car with y miles on its odometer the cost for doing this is $P(y) - P(x)$. Can you write down the Bellman equation for the optimal car holding and trading strategy in this case and explain how the existence of a secondary market affects your decision about when to trade your car?

D. Suppose now that there are also positive *transactions costs* such as taxes and transfer costs that you incur when you trade in your old car to buy a new one. Then the cost of selling your existing car with x miles on its odometer and buying another car with y miles on its odometer is $P(y) - P(x) + T(x)$ where $T(x)$ are the transactions costs which are positive and could be a function of x (either an increasing or decreasing function of x, or just a constant, e.g. $T(x) = T$ for all x). How does the presence of transactions costs affect your car holding and trading strategy? Write down Bellman's equation for this version of the problem and explain your answer for full credit.

E. Suppose there are a continuum of individuals in the market and they all have the same preferences $u(x)$ and discount factors β and face the same maintenance costs for cars, $m(x)$ and there are no transactions costs. Can you write down an equation that characterizes the equilibrium price function $P(x)$ in this case, assuming that there is an infintely elastic supply of new cars at an exogenously determined price $\overline{P}$ and an infinitely elastic demand for scrap at an exogenously fixed price $\underline{P} > 0$ and naturally we assume that $\overline{P} > \underline{P}$. If there are no transactions costs, can you write a formula for what $P(x)$ must be to clear the market?

F. If the market is *stationary* and under the conditions given in part E above, can you write an equation for the *stationary distribution of cars in the economy*? That is, let $F(x)$ be the CDF for the fraction of cars in the economy whose odometer is less than or equal to x. If the economy is stationary, can you write an expression for $F(x)$? What does $F(0)$ represent? Can you show whether such an economy must be in *flow equilibrium* — i.e. that the fraction of consumers in this economy who purchase new cars each period equals the fraction of consumers who scrap old ones?

4. Consider a firm that can invest in a capital stock, where k is a measure of the stock of capital it has. Capital is *putty-clay* — that is, the company can buy new capital for \$1 per unit of capital (so installing k units of capital costs the firm \$$k$ dollars to install) but once installed, the capital has no other value and sells for \$0 per unit. Capital also *depreciates* — that is, if the company has k_t units of capital today and invests in i_t additional units of new capital, the total capital it will have tomorrow is

$$k_{t+1} = (1-\delta)k_t + i_t \tag{2}$$

where $\delta \in (0,1)$ is the *depreciation rate* of the capital stock. With k_t units of capital in place at the start of period t, the firm earns $\sqrt{k_t}$ in net profits or *cash flow*. The firm can either use this cash to pay dividends d_t or invest in more capital i_t. Naturally we place the restrictions that $d_t \geq 0$ and $i_t \geq 0$: the firm cannot pay "negative dividends" (e.g. borrow from investors) and it cannot "disinvest" except via letting its capital stock slowly depreciate without making new investment. The firm's objective it to maximize its market value, which equals the present discounted value of its dividend stream, discounted at an interest rate $r > 0$.

A. Characterize the optimal investment and dividend policy of this firm, writing the Bellman equation that determines i_t and d_t.

B. Characterize what we mean by a *steady state capital stock* k_∞ and write an equation for the *optimal steady state capital stock.*

C. Suppose the firm starts with a capital stock $k = 10$ at $t = 0$ and $r = .05$ and $\delta = .02$. Using numerical or analytical methods, can you determine the optimal time path for dividends and investment from this "initial condition" and determine whether the resulting capital stock will converge to the optimal steady state value?

D. Can you determine whether the optimal steady state capital stock is *globally stable.* That is, will the capital stock eventually converge to the optimal steady state capital stock from any initial value k, i.e. will we have

$$\lim_{t\to\infty} k_t = k_\infty \tag{3}$$

given any initial capital stock $k_0 = k > 0$? What about if $k_0 = 0$? Will the firm be able to grow and achieve the optimal steady state capital stock if it starts out with no capital to begin with?

E. Finally consider the case where the firm could make a *one time borrowing decision* at period $t = 0$. That is, the firm can borrown any amount $b > 0$ at an interest rate r and pay back this initial loan as a *consol* i.e. an infinite stream of future interest payments c. What value must c equal so the present value of these interest payments equals the amount the firm borrows up front, b? Assuming the parameters in part C above, would a firm with initial capital stock $k = 10$ want to borrow at time 0? If so, how much would the firm borrow and by how much does the option to borrow increase the value of this firm?