Fall 2015 | Professor John Rust
Economics 611 | Georgetown University

Problem set 4
Due in class on Tuesday October 6th

1. Consider the case of a Cournot duopoly with a linear demand function $p = a - bq$ and two firms with equal constant marginal costs of production of c and no fixed costs, is overall social surplus (i.e. the sum of consumer surplus plus total profits) higher if the firms act as ordinary Cournot duopolists or is it higher if firm 1 is a Stackelberg leader and firm 2 acts as a Stackelberg follower? Compute the equilibria in the two cases and show all calculations (including the breakdown of consumer vs. producer surplus in your calculation of overall social surplus) for full credit.

2. Suppose Disneyland is trying to decide the profit-maximizing pricing strategy for its Disneyland theme park. Suppose each ride in the park costs c per ride in terms of electricity and labor and other costs. If Disney is a monopolist and charges a price per ride of p and it believes that all consumers have a utility function for the number of theme park rides r and consumption of all other goods g of $u(r,g) = r^{.1}g^{.9}$ and the average income of a consumer is $y = 20,000$ (20 thousand dollars), compute the total demand for rides and the the profits Disney will earn if $c = 1$ (i.e. a marginal cost of \$1 per ride), and there are 100,000 possible consumers who would be coming to the Disney theme park each year? Using this "demand for rides" function, compute the profit maximizing price per ride, p^*. Now consider which strategy yields higher profits: a) an optimal (linear) monopoly price per ride p^* or b) a two-part tariff consisting of a fixed entrance fee F and a price p per ride (not necessarily equal to the monopoly ride price p^* computed above)? To get full credit, calculate the profits under both pricing schemes, a and b, and show which one is higher.

3. Consider the following two player game. Player 1 (P1) moves first and can make one of three possible moves, which I will call up, middle, or down. Then player 2 (P2) moves and can go either up or down. Suppose that the payoffs (in dollars, with both P1 and P2 risk neutral expected utility maximizers) are

Move by P1	Move by P2	P1's payoff	P2's payoff
U	U	5	-5
U	D	-5	5
M	U	1	-1
M	D	-1	1
D	U	-5	5
D	D	5	-5

A. What is a *subgame perfect equilibrium*? (Hint: see Rust's New Palgrave article on "Dynamic Programming" or do a Google search).

B. Suppose we model this as a game of *complete and perfect information* find at least one *subgame perfect equilibrium of this game.*

C. Suppose we change the game to have *complete, but imperfect information.* The imperfect information is that P2 is not able to observe the move by P1 before taking his/her move. Find *all* Nash equilibria of this game.

D. Suppose we change the game to have *incomplete, but perfect information.* That is, we now assume that P2 does observe the move by P1 before taking his/her move, but P1 and P2 are not certain about each others' payoffs. Each believes that their opponent could be one of two possible "types" with payoffs given in the following table For simplicity denote the types of P2 by $\tau_2 = 1$ (if P2

Move by P1	Move by P2	P1's payoff (Type 1)	P1's payoff (Type 2)	P2's payoff (Type 1)	P2's payoff (Type 2)
U	U	6	-6	5	-5
U	D	-6	6	-5	5
M	U	1	-1	1	-1
M	D	-1	1	-1	1
D	U	-5	5	-5	5
D	D	5	-5	5	-5

is of type 1) and $\tau_2 = 2$ (if P2 is of type 2). Similarly $\tau_1 = 1$ denotes that P1 is a type 1 and $\tau_1 = 2$ denotes that P1 is a type 2. Find a *Bayesian Nash equilibrium* of this game for the case where P1 believes that P2 is a type 1 player with probability $p_1(\tau_2 = 1) = .6$ and P2 is certain that P1 is a type 1 player, i.e. $p_2(\tau_1 = 1) = 1$. Further, each of the beliefs of the two players is *common knowledge* i.e. P1 knows P2's beliefs, P2 knows P1's beliefs, P1 knows that P2 knows P1's beliefs, and so on.

E. Now consider the same game, except that P2 is not completely certain that P1 is a type 1. Let P2's *prior belief* that P1 is a type 1 be 50%, i.e. $p_2(\tau_1 = 1) = .5$. Again assume that the beliefs are common knowledge and thus P1 knows P2's beliefs, and P2 knows P1's beliefs, which are $p_1(\tau_2 = 1) = .6$. Suppose that P2 observes that P1 has chosen to move up. What can P2 infer about P1's type from seeing this move?

F. Now, using the idea that P2 will *update his/her beliefs about P1's type after observing the move P1 makes* find the *Bayesian Nash, subgame perfect equilibrium of this game.*

4. Suppose it is a Friday evening just after work and you and your girlfriend/boyfriend have both lost your cell phones and cannot communicate. You both know there are two things you both like to do on Fridays: go to the bar (B) or to a movie, (M). You are both stuck in traffic leaving work (at opposite sides of town) and want to meet up at one of these two places, and you would prefer to end up at the same place together instead of one of you going to the bar and other to the movie. In fact, suppose you are a rather jealous and also a shy type and do not do well in bars yourself, but your girlfriend/boyfriend is quite social and attractive and really gets lots of attention in the bar, and you would not be happy if he/she ended up at the bar by him/herself. If you are the row player and your girlfriend/boyfriend is the column player, suppose the payoffs (in utils) from this *coordination game* are in table 1 below.

Thus, you would prefer to see the movie with your girlfriend/boyfriend, but your partner would prefer to go to the bar, but they are loyal in the sense that he/she would rather go the movie with you than to the bar alone if he/she knew that you would be at the movie. But if he/she were not sure, your partner would prefer to be at the bar without you than being at the movie without you. On your part,

Payoff Matrix	Your girlfriend/boyfriend's decision	
Your Decision	Bar	Movie
Bar	(10,20)	(5,5)
Movie	(-5,10)	(20,15)

Table 1: Table 1: payoff matrix for coordination game

due to your jealousy, you would be *very unhappy* if you went to the movie and your girlfriend/boyfriend were at the bar, getting lots of attention.

A. Find all Nash equilibrium of this game, including any *mixed strategy* equilibria.

B. Suppose you are the "Stackelberg leader" and your partner knows this. What is the equilibrium to the game if formulated as a Stackelberg game? (Note that even though the "solution concept" is Stackelberg rather than Nash, the game is still a "simultaneous move game", i.e. neither you nor your girlfriend/boyfriend can observe each other's choice. However you do have the ability to "precommit" i.e. if your boyfriend/girlfriend "knows" you will go to the movie, then they will go there too).

C. Now go back to the assumption that this is a Nash equilibrium, but due to traffic conditions, there are random factors affecting payoffs that could cause you and your girlfriend to go to the bar or the movie that only each of you observe. For example, you might be close to the bar and see that there is a huge traffic jam that would make you late if you went to the movie, and similarly for your boyfriend/girlfriend. Suppose that these "other factors" affecting decisions are represented by independent and identically distributed Extreme value "error terms" $\tau_1 = (\varepsilon_{1B}, \varepsilon_{1M})$ and $\tau_2 = (\varepsilon_{2B}, \varepsilon_{2M})$. Thus, you are player 1 and you know your own type τ_1 which consists of the other components to your payoff (due to traffic, hassle of getting from one place to the other) of ε_{1B} if you go to the bar, and ε_{1M} if you go to the movie. While you know these factors for your own situation, you do not know the $\tau_2 = (\varepsilon_{2B}, \varepsilon_{2M})$ for your boyfriend/girlfriend, so there is even an additional layer of uncertainty about what he/she might do. Suppose your beliefs about your boyfriend/girlfriend's uobserved component affect his/her payoff, τ_2, is that τ_2 is a *bivariate Type 3 extreme value distribution.* Suppose your boyfriend/girlfriend has the same belief about your τ_1 vector too, i.e. that it is also a draw from a bivariate Type 3 extreme value distribution, and your types are indepdendently distributed. Thus the payoff matrix for this *simultaneous move game of incomplete information* is Find all of the *Bayesian Nash equilibria* of this game. (Hint: you

Payoff Matrix	Your girlfriend/boyfriend's decision	
Your Decision	Bar	Movie
Bar	$(10+\varepsilon_{1B}, 20+\varepsilon_{2B})$	$(5+\varepsilon_{1B}, 5+\varepsilon_{2M})$
Movie	$(-5+\varepsilon_{1M}, 10+\varepsilon_{2B})$	$(20+\varepsilon_{1M}, 15+\varepsilon_{2M})$

should look at my lecture notes on the prisoner's dilemma game with incomplete information and see how to set up the equilibrium problem. You will probably have to numerically solve for the equilibria by plotting the "best response probability function" (a mapping from the $[0,1]$ interval to the $[0,1]$ interval) and seeing where it crosses the 45 degree line.

5. Consider the following two player game. The game starts with an initial "kitty" of \$100. Player 1 can take any part of this kitty for him/herself. Whatever the player does not take gets passed on to player 2 in the next round, but the amount passed on is *doubled.* Then player 2 decides how much of the kitty, if any, is passed on for player 1 at the next round. Whatever amount is passed on in each round is doubled. The game runs for a total of 4 rounds. Thus, an example of one possible "play" of the game is for player 1 to pass the entire \$100 to player 2 in the first round. This amount is then doubled to \$200 for player 2 in the second round. If player 2 takes \$50 for him/herself at this stage and passes on \$150 to player 1 in round 3, then the \$150 is doubled to \$300 and player 1 decides how much of this to take in round 3. If player 1 takes \$200 in round 3, the remainder, \$100, is doubled, giving player 2 a total of \$200 in the 4th and final round. In this final stage player 2 could take the entire \$200 for him/herself, or give part of it to player 1. If player 2 takes all of the \$200 in this example, then player 2 gets a total of \$250 (\$50 taken in round 2 and the \$200 in round 4) and player 1 gets a total of \$200 from the \$200 he/she took in round 3. Suppose this game is played by two complete strangers who are kept in separate rooms and cannot communicate or collude in any way. If both players are rational and they don't only care about maximizing the amount they personally can earn from this game but they give some weight to how much their opponent will earn (even if the opponent is a complete stranger!), describe the Nash equilibrium outcome of this game. (Hint: the utility function for player i is $u_i(P_i, P_{-i}) = \sqrt{P_i} + \frac{1}{2}\sqrt{P_{-i}}$ where P_i is player i's monetary payoff (in total) and P_{-i} is their opponent's payoff. Use backward induction, starting from player 2's optimal decision in round 4 of the game).

6. Consider a firm selling mufflers. Each day there is a probability p that exactly 1 customer will come to the store to buy a muffler. Suppose the retail price of the muffler (the price the firm can sell to the customer) is p_r and the wholesale price of a muffler (the price the firm can buy mufflers from the manufacturer at) is p_w. Naturally we assume that $p_r > p_w$ so the firm makes profits from selling mufflers. Suppose that each time the firm orders more mufflers to replenish its inventory, it incurs a fixed transport cost K *regardless of how many mufflers it buys from the manufacturer.* Suppose there is also a storage/holding cost of mufflers and if the firm has q mufflers in its inventory, it costs c per muffler to store them. Suppose the firm is an infinite-horizon profit maximizer and its discount factor is $\beta \in (0,1)$.

a. What is the profit maximizing inventory strategy for this firm? Write down the Bellman equation for the firm's optimization problem and characterize the nature of the solution for full credit.

b. Suppose that $\beta = .99$, $c = 0.1$, $p_w = 100$, $p_r = 150$, and $K = 100$. Calculate the optimal policy for the firm numerically, using a computer and report the present discounted value of the firms profits assuming it has in inventory $q = 5$ mufflers.

c. Suppose there are occassional opportunities to buy mufflers at a lower wholesale price than $p_w = 100$. Suppose with probability $q \in (0,1)$ the firm can buy as many mufflers as it wants at a price of $p_{wl} = 75$. Write the Bellman equation in this case. What are the state variables for this problem? What are the decision (control) variables? How does the nature of the solution change compared with the solution you characterized in part a above?

d. Calculate the optimal strategy for the firm numerically using a computer for the modified version of the problem in part c, assuming $q = 0.05$.

7. Consider a seller trying to sell a painting at an auction. Suppose that there are N buyers who will participate in an auction if the seller holds one. Suppose that (normalized to millions) that the seller

knows that the valuations of buyers are random draws from a distribution on the $[0,1]$ interval (where 1 now denotes $1 million dollars, and the cumulative distribution function of the values is

$$F(v) = Pr\{\tilde{v} \leq v\} = v^2 \tag{1}$$

a. What is the expected amount a single buyer would be willing to pay for this painting?

b. Write a formula for the bidding function that the bidders would use in a symmetric Bayesian Nash equilibrium of a *first price auction* for the painting. What is the probability that the painting will be sold, and what is the expected price that the seller will receive for the painting if there are $N = 5$ bidders participating in the auction?

c. Suppose instead that the seller runs a *second price auction.* What is the value that the seller can expect to receive for the painting when there are $N = 5$ bidders participating in this auction?

d. Suppose there are $N = 3$ bidders with valuation for the painting equal to $v_1 = .2$, $v_2 = .8$ and $v_3 = .5$. What price will the seller receive if a) she adopts a first price auction for the painting, b) she adopts a second price auction for the painting, or c) she adopts an *all pay* auction?

e. Suppose that the seller adopts a second price auction, but sets a *reservation price* for the painting of $r = .2$. That is, the seller will not sell the painting unless the highest bid is at least $r = .2$ ($200,000). Describe how the use of the reservation price affects the buyers' bidding strategies in this auction, if at all? What is the probability that the seller will sell the painting when there is a population of $N = 5$ potential bidders whose true values for the painting are given by a probability distribution with cumulative distribution function $F(v) = v^2$? Can you calculate the expected revenue the seller will receive? If so, which is better for the seller, to have no reservation price, or to set a reservation price of $r = 2$?