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Problem Set 3, Due in class Tuesday September 29th

1. Expected Discounted Utility with Uncertain Lifetimes Consider the intertemporal utility maximization problem, but extended to allow for *uncertain lifetimes.* Let $\tilde{T}$ denote the (random) lifetime of a person, in years. Let $f(t)$ denote the *probability density function* for the person's lifetime. Thus, we have

$$f(t) = Pr\{\tilde{T} = t\}, \tag{1}$$

i.e. $f(t)$ is the probability that the person lives for $t-1$ years and dies when they reach t years old.

a. What does the sum $\sum_{t=1}^{\infty} f(t)$ equal?

b. Show that the person's *expected discounted lifetime utility,* allowing for the possibility of dying, is given by

$$E\{U\} = \sum_{t=1}^{\infty}\left[\sum_{s=1}^{t}\beta^s u(c_s)\right] f(t) \tag{2}$$

c. Show that the person's expected discounted lifetime utility can also be written as

$$E\{U\} = \sum_{t=1}^{\infty}[1-F(t-1)]\beta^t u(c_t), \tag{3}$$

where $F(t)$ is the *cumulative probability distribution* corresponding to the probability density $f(t)$, i.e.

$$F(t) = Pr\{\tilde{T} \le t\} = \sum_{s=1}^{t} f(s). \tag{4}$$

d. In words, what is the interpretation of the quantity $[1-F(t-1)]$?

e. Suppose that the person's random age of death $\tilde{T}$ is *geometrically distributed,* i.e.

$$f(t) = p^t(1-p), \quad t = 1,2,\ldots \tag{5}$$

where $p \in (0,1)$ is the probability of surviving in any given year. Show that the expected discounted lifetime utility in this case is

$$E\{U\} = \sum_{t=1}^{\infty} f(t)\sum_{s=1}^{t}\beta^s u(c_s) = \sum_{t=1}^{\infty}[p\beta]^t u(c_t). \tag{6}$$

Hint: Use the rules from calculus on interchanging the orders of integration of an integral over a triangular region,

$$\int_0^{\infty}\left[\int_0^x f(x,y)dy\right]dx = \int_0^{\infty}\left[\int_y^{\infty} f(x,y)dx\right]dy \tag{7}$$

and show that the same reasoning leads to the following analagous formula for interchanging the order of summations in a summation over a triangular region

$$\sum_{t=1}^{\infty}\left[\sum_{s=1}^{t} f(t,s)\right] = \sum_{s=1}^{\infty}\left[\sum_{t=s}^{\infty} f(t,s)\right]. \tag{8}$$

2. Recursive Representation of Lifetime Utilities Consider a discounted sum of utilities for a person with a known lifespan of T years

$$V_0 = \sum_{t=0}^{T} \beta^t u(c_t) = u(c_0) + \beta u(c_1) + \cdots \beta^T u(c_T). \tag{9}$$

Thus, V_0 represents the discounted utility of a person at age $t = 0$, looking ahead over the rest of their life. Now let V_t denote the discounted utility of an *age t person, looking forward from age t onwards.*

a. Write a formula for V_t. What is V_T?

b. Show that the utilities V_t and V_{t+1} are connected *recursively* via the formula

$$V_t = u(c_t) + \beta V_{t+1} \tag{10}$$

c. Show that by age $t = 0$, the recursive representation of V_0, i.e.

$$V_0 = u(c_0) + \beta V_1 \tag{11}$$

gives the same value V_0 as the non-recursive representation of discounted lifetime utility as in the original formula, (**??**).

d. Now let's extend this recursive way of thinking about discounted utilities to expected discounted utilities when there are uncertain lifetimes. Let V_0 be given by

$$V_0 = \sum_{t=0}^{T} [1 - F(t-1)]\beta^t u(c_t), \tag{12}$$

the same formula for expected lifetime utility when there is uncertain mortality as you derived in equation (**??**) above, where $F(t) = Pr\{\tilde{T} \le t\} = \sum_{s=0}^{t} f(t)$ is the cumulative probability of dying *on or before age t* and $f(t) = Pr\{\tilde{T} = t\}$ is the probability of dying *exactly on age t*. Define V_t analagously to V_t in the case where there is no uncertainty about age of death, i.e. V_t is *the expected discounted utility from age t onwards, to whatever random age $\tilde{T}$ the person dies.* Show that the appropriate form for the recursive representation for expected discounted utilities is in this case

$$V_t = u(c_t) + \beta \frac{1 - F(t)}{1 - F(t-1)} V_{t+1} \tag{13}$$

e. Show that

$$\frac{1 - F(t)}{1 - F(t-1)} = 1 - h(t) \tag{14}$$

where $h(t)$ is the *hazard rate,* i.e. the conditional probability of dying at age t given that one has survived to age $t - 1$:

$$h(t) \equiv \frac{f(t)}{1 - F(t-1)} = \frac{f(t)}{f(t) + f(t+1) + \cdots + f(T_m)} \tag{15}$$

where T_m is the maximal possible lifespan, or $T_m = \infty$ if there is no upper bound on the maximal lifespan Thus $1 - h(t)$ is the *survival rate*, i.e. the conditional probability that a person who lives to age $t-1$ will survive another year, to be at least age t or older before they die.

3. Annuities with Uncertain Lifetimes

a. Suppose that a person's lifetime is uncertain, so that the random variable $\tilde{T}$ denotes the random age of death, but that we know that the probability distribution of $\tilde{T}$ is geometric with parameter $p \in (0,1)$. That is, as noted above, $f(t) = p^{t-1}(1-p)$, $t = 1, 2, \ldots$. If a person consumes a flat amount of \$10000 per year until they die, and if the discount factor is $\beta \in (0,1)$, write a formula for the *expected discounted amount that this person will consume over their lifetime.*

b. An *annuity* is a contract such that if a person pays a given amount W up front at the start of their lifetime, the annuity company will in return provide that person with a constant payment of \$$c$ per year over their entire lifetime. Using the result from part 1 above, if a person has endowment of \$1,000,000 when they are born and an annuity is purchased for them, how much will this annuity pay the person if their probabilty of survival is $p = .98$ and the discount factor is $\beta = .95$?

c. Suppose there are no annuity markets and that a person has a lifetime utility function (conditional on living T years) equal to

$$U = \sum_{t=1}^{T} \beta^t \log(c_t) \tag{16}$$

Describe the optimal consumption strategy for this person using dynamic programming, assuming that they are born with an initial endowment of $W = 1,000,000$ and $\beta = .95$ and $p = .98$.

d. Now suppose that there are annuity markets. The person now has the option, at the start of their life, to exchange their entire initial endowment of wealth W for a lifetime annuity. Which option would the person prefer: 1) to exchange W and take the annuity, or 2) not buy the annuity and follow the optimal consumption plan described in part 3 above?

Hint: To solve part 3, use the method of dynamic programming with the following *Bellman equation*

$$V(W) = \max_{0 \le c \le W} \left[\log(c) + p\beta V\left(\frac{W-c}{\beta}\right) \right] \tag{17}$$

Conjecture that $V(W)$ is of the form

$$V(W) = a + b\log(W) \tag{18}$$

and using the Bellman equation above, solve for the coefficients a and b so that the Bellman equation will hold. From this solution you should be able to derive the associated optimal consumption function, $c(W)$, which specifies how much the person will consume each year given that they start that year with total savings of W.

Suggestion: If you can't get this, try working on a 2 or 3 period problem, i.e where $T = 3$ as in part c above. Show that the person would rather take the annuity than to save on their own when they face a random age of death.

4. Consumption and Taxes Suppose a consumer has a utility function $u(x_1, x_2) = \log(x_1) + \log(x_2)$ and an income of $y = 100$ and the prices of the two goods are $p_1 = 2$ and $p_2 = 3$.

a. In a world with no sales or income taxes, tell me how much of goods x_1 and x_2 this consumer will purchase.

b. Now suppose there is a 10% a sales tax on good 1. That is, for every unit of good 1 the person buys, he/she has to pay a price of $p_1(1+.1) = 2.2$, where the 10% of the price, or 20 cents, goes to the government as sales tax. How much of goods 1 and 2 does this person buy now?

c. Suppose instead there is a 5% income tax, so that the consumer must pay 5% of his/her income to the government. If there is no sales tax but a 5% income tax, how much of goods 1 and 2 will the consumer consume?

d. Which would the consumer prefer, a 10% sales tax on good 1, or a 5% income tax? Explain your reasoning for full credit.

e. How big would the sales tax on good 1 have to be for the government to get the same revenue as a 5% income tax? Which of the two taxes would the consumer prefer in this case, or is the consumer indifferent because the consumer has to pay a total tax of \$5 (5% of \$100) in either case?

5. Risk Neutrality and Risk Aversion An person is said to be *risk neutral* if when offered a gamble, their maximum *willingness to pay to undertake the gamble* equals the *expected value of the gamble.* That is, if $\tilde{G}$ denotes a random payoff from a gamble, the maximum "entry fee" F that a risk neutral person would be willing to pay to get the gamble payoff $\tilde{G}$ is

$$F = E\{\tilde{G}\}. \tag{19}$$

A person is *risk neutral* if $F < E\{\tilde{G}\}$ and *risk loving* if $F > E\{\tilde{G}\}$.

a. Suppose a person has a utility function $u(W) = W$, and suppose that initially (before considering taking the gamble) the person has $W = 1000000$ of wealth. Suppose the gamble under consideration is to flip a coin, and if it lands heads the perons wins \$100, and if tails the person gets nothing. What it is the maximum amount F this person would be willing to pay for this gamble? Is this person risk neutral, risk loving, or risk averse?

b. Suppose a person has a utility function $u(W) = \log(W)$ and this person also has $W = 1000000$ in initial wealth before considering taking the gamble. What is the maximum amount F this person would be willing to pay for the gamble?

c. Suppose a third person has a utility function $u(W) = W^2$ and also has initial wealth $W = 1000000$ before considering the gamble. What is the maximum amount this third person would pay for the gamble?

d. Are the persons in cases b and c above risk neutral, risk averse or risk loving?

e. Prove that a person is *risk neutral if their utility function is linear,* $u(W) = a + bW$ *for* $b > 0$, and *risk averse if their utility function is concave,* $u'(W) > 0$ *and* $u''(W) < 0$, and *risk loving if their utility function is convex,* $u'(W) > 0$ *and* $u''(W) > 0$.

Hint: If a person does not take the gamble, they will have utility $u(W)$ from consuming their wealth W. If the person pays an amount F for a gamble $\tilde{G}$, their expected utility would be $E\{u(W-F+\tilde{G})\}$. The *maximum willingness to pay* for the gamble $\tilde{G}$ would be the amount F^* that makes the person indifferent between paying F^* for the gamble and not taking the gamble, i.e. it is the solution to

$$U(W) = E\{u(W-F^*\tilde{G})\}. \tag{20}$$

You can use *Jensen's Inequality* which states that for a concave function u and any random variable $\tilde{X}$ we have

$$E\{u(\tilde{X})\} \leq u(E\{\tilde{X}\}). \tag{21}$$

You should be able to use Jensen's inequality to show that people with concave utility functions are risk averse.

6. St. Petersburg Paradox Consider the following gamble $\tilde{G}$. You flip a fair coin until it lands on tails. Let $\tilde{h}$ denote the number of heads obtained until the first tail occurs and the game stops. Your payoff from playing this game is

$$\tilde{G} = 2^{\tilde{h}} \tag{22}$$

a. Suppose you are risk neutral. What is the maximum amount F^* that you would be willing to pay to play this game?

b. Suppose you have a utility function $u(W) = \log(W)$ and $W = 1000000$ in initial wealth. What is the maximum amount you would be willing to pay to play this gamble in this case?