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Economics 611 Georgetown University

Problem Set 2, Due in class Thursday September 22nd

0. **Characterization of a concave function** Prove this result: *Assume $f : R^n \to R$ is twice continuously differentiable. Prove that f is concave if and only if the the $n \times n$ hessian matrix $\nabla^2 f(x)$ is a symmetric and negative semi-definite matrix for each $x \in R^n$.*

1. **Gradient is orthogonal to the indifference curve** Let $u : R^N \to R$ be a differentiable function (for concreteness you can think of this as a utility function). Let $I(x_0) = \{x \in R^N | u(x) = u(x_0)\}$ be a *level set* of the function (in the utility function case, an *indifference curve*) which provide the same value of the objective as the point x_0. Define the *tangent hyperplane* to the function u at the point $x_0 \in R^n$ to be the set

$$T(x_0) = \{x \in R^n | u(x_0) + \langle x, \nabla u(x_0) \rangle = u(x_0)\}. \tag{1}$$

That is, the tangent hyperplane is the set of $x \in R^n$ lying in a hyperplane that is tangent to the function u at the point x_0. Prove that if $x \in T(x_0)$ then $\langle x, \nabla u(x_0) \rangle = 0$ where ∇u is the *gradient* of u, and also defines the slope of the indifference curve (i.e. tangent hyperplane) at the point x_0.

2. **Lagrangian saddlepoint solution for constrained optimization problems** Consider the following *constrained optimization problem*

$$\max_x u(x) \quad \text{subject to} \quad g(x) \geq 0, \quad \text{and} \quad x \geq 0 \tag{2}$$

where $u : R^n \to R$ is a continuous function and $g : R^n \to R^m$ are *m constraint functions* which are also continuous functions of x. Define the *Lagrangian* $\mathcal{L}(x, \lambda) : R^(n+m) \to R$ by

$$\mathcal{L}(x, \lambda) = u(x) + \lambda' g(x) \tag{3}$$

where

$$\lambda' g(x) = \langle \lambda, g(x) \rangle = \sum_{j=1}^m \lambda_j g_j(x). \tag{4}$$

Definition (x^*, λ^*) is a *saddlepoint* of $\mathcal{L}$ if and only if

$$\begin{array}{rcll} \mathcal{L}(x^*, \lambda^*) & \geq & \mathcal{L}(x, \lambda^*) & \forall x \geq 0 \\ \mathcal{L}(x^*, \lambda^*) & \leq & \mathcal{L}(x^*, \lambda) & \forall \lambda \geq 0 \end{array}$$

Theorem If (x^*, λ^*) is a saddlepoint of $\mathcal{L}$ then x^* solves the constrained optimization problem (2).
Prove this theorem. **Hint:** use the method of *proof by contradiction.*

3. **Prove that if x^* is an *interior solution* that maximizes the consumer's problem below, the indifference curve at x^* is tangent to the budget line.**

$$\max_{x \geq 0} u(x) \quad \text{subject to} \quad \langle p, x \rangle \leq y \tag{5}$$

where $u : R^n \to R$ is a continuously differentiable utility function and $p \in R^n$ are positive prices of the n goods entering the consumer's utility function.

4. **Firm Profit Maximization Problem** Consider a firm whose production function has 2 outputs, y_1 and y_2 and 2 inputs, x_1 and x_2. Suppose that its production function is given by

$$\left[y_1^2 + 4y_2^2\right]^{1/2} = \left[{x_1}^2 + {x_2}^2\right]^{1/2} \tag{6}$$

a. Does this production function have increasing, decreasing, or constant returns to scale? (Hint: if you double both inputs x_1 and x_2 can you double, more than double, or less than double both of the outputs y_1 and y_2?)

b. Suppose for a moment that we fix input levels so that $x_1 = x_2 = 5$. Plot the *output possibility frontier,* i.e. plot (in (y_2, y_1) space) the set of feasible combinations of y_1 and y_2 that can be produced using inputs $x_1 = x_2 = 5$.

c. Continuing the previous question, if the output prices are $p_1 = 6$ for y_1 and $p_2 = 16$ for y_2, and if we assume that the inputs x_1 and x_2 are fixed at 5, what combination of outputs (y_1^*, y_2^*) maximize the firm's revenue? If we were to increase x_1 by a small amount, on the margin, by how much would the revenues of the firm increase (i.e. how much does revenue increase for an increase of amount ε, some small positive number, in input x_1)?

d. Now consider what the optimal level of inputs should be in order to produce the (y_1^*, y_2^*) combination that you computed in part c. If the price of the inputs are $w_1 = 4$ for x_1 and $w_2 = 6$ for x_2, what is the cost-minimizing level of inputs that can produce (y_1^*, y_2^*)? (Hint: recall that when $x_1 = x_2 = 5$ we have $[{x_1}^2 + {x_2}^2]^{1/2} = \sqrt{50}$. So you need to minimize total costs $w_1 x_1 + w_2 x_2$ subject to the constraint that $[{x_1}^2 + {x_2}^2]^{1/2} = \sqrt{50}$). If we needed to increase output y_1 by a small amount, say by .1, approximately how much would it cost the firm to do this?

e. Now step back and look at the firm overall. Is the production plan $(y_1^*, y_2^*, x_1^*, x_2^*)$ that you computed in parts c and d above a profit maximizing production plan for this firm? Why or why not?

f. **Super bonus question:** If you answered in part e that the production plan $(y_1^*, y_2^*, x_1^*, x_2^*)$ computed in parts c and d is not a profit maximizing plan, then find the profit maximizing production plan.

6. **Bertrand Duopoly Problem** Consider those regions in the Washington DC area where households have a choice between two cable tv/internet providers: Comcast and Starpower. Assume that these companies do not engage in price discrimination, but rather provide cable/internet using a simple single per month pricing scheme. Assume also that there are no switching or hookup costs, so that customers can switch from Starpower to Comcast or vice versa (or to not have cable) at zero cost. We now consider the pricing problem faced by these two competing customers, treating their services as imperfect substitutes in the minds of the consumers in the Washington DC area. Thus, a household in this area has the following television "mode" choices:

1. No pay TV (i.e. watch broadcast TV, or don't watch TV or use broadband)

2. cable TV/broadband (via Comcast)

3. cable TV/broadband (via Starpower)

Of course, it is possible for some households to subscribe to both Starpower and Comcast simultaneously, but I assume that this is too expensive relative to the incremental value of having both hooked up, so that virtually no households would subscribe to both at the same time. Thus, I have limited households to the 3 possible choices given above, which I assume are mutually exclusive and exhaustive (having ruleed out the possibility of subcribing to both Comcast and Starpower).

Assume that Starpower and Comcast choose their prices independently and without any collusion as part of a Nash equilibrium in which each tries to maximize its profits, treating the price of its opponent as given. Initially I ignore the presence of explicit or implicit regulatory constraints. I assume that in the DC area where these two companies provide overlapping coverage there are N households. Let $P_c(p_c, p_s)$ denote the fraction of these N households who choose Comcast, and $P_s(p_c, p_s)$ be the fraction who choose Starpower. The remaining fraction, $1 - P_c(p_c, p_s) - P_s(p_c, p_s)$ either watch broadcast TV (which has a price of \$0 per month), or do not watch TV or need broadband internet at all (god forbid!). It is convenient to start with a simple logit representation for the market shares for Comcast and Starpower:

$$\begin{aligned} P_c(p_c, p_s) &= \frac{\exp\{a_c + b_c p_c\}}{1 + \exp\{a_c + b_c p_c\} + \exp\{a_s + b_s p_s\}} \\ P_s(p_s, p_s) &= \frac{\exp\{a_s + b_s p_s\}}{1 + \exp\{a_c + b_c p_c\} + \exp\{a_s + b_s p_s\}} \end{aligned} \tag{7}$$

A more advanced approach would derive these market shares from a household level demand study, using micro data to estimate the consumer choices and accounting for other demographic variables, including household income y, and the characteristics of the "outside alternative", i.e. the characteristics of free to air TV. I assume these market shares are "reduced forms" consistent with the results of a micro level study. This initial "reduced form" approach requires specification of 7 pieces of information in order to predict the prices, profits, and market shares for Comcast and Starpower:

1. the number of households N in the "overlap region" served by both Comcast and Starpower,
2. the 4 market share coefficients (a_c, b_c, a_s, b_s)
3. the 2 marginal cost parameters (k_c, k_s)

Given suggested values for these 7 parameters, your job is to compute the Bertrand Nash equilibruium outcome, i.e. the prices that Comcast and Starpower will charge, their profits, and their equilibrium market shares. Use these parameter values to calculate your answer: $N = 100$ (imagine this as 100,000 households), $(a_o, b_o, a_f, b_f) = (5, -.008, 8, -.01)$ and $(k_c, k_s) = (50, 60)$ (i.e. it costs \$50 per month for Comcast to service each household, and \$60 per month for Starpower, so these constitute the "marginal service costs" of adding additional households, and we ignore fixed costs such as advertising to attract households).

I want you to compare the Bertrand-Nash duopoly outcome with the two possible monopoly outcomes:

1. Comcast has a monopoly in the DC area
2. Starpower has a monopoly in the DC area

7. **Intertemporal utility maximization with certain lifetimes.** Suppose a person has an additively separate, discounted utility function of the form

$$V(c_1, \ldots, c_T) = \sum_{t=1}^{T} \beta_s^t u(c_t) \tag{8}$$

where β_s is a subjective discount factor and $u(c_t)$ is an increasing utility function of consumption c_t in period t. Let the market discount factor is $\beta_m = 1/(1+r)$ where r is the market interest rate.

a. If $\beta_s = \beta_m$ show that the optimal consumption plan in a market where there are no borrowing constraints (i.e. the consumer has unlimited ability to borrow and lend subject to an intertemporal budget constraint) is to have a constant consumption stream over time, i.e. $c_1 = c_2 = \cdots = c_t = c_{t+1} = \cdots = c_T$.

b. If $\beta_s < \beta_m$ will the optimal consumption stream be flat, increasing over time, or decreasing over time, or can't you tell from the information given?

c. How does your answer to part b change if I tell you that the utility function $u(c)$ is convex in c?

8. **Consumption and Taxes** Suppose a consumer has a utility function $u(x_1, x_2) = \log(x_1) + \log(x_2)$ and an income of $y = 100$ and the prices of the two goods are $p_1 = 2$ and $p_2 = 3$.

a. In a world with no sales or income taxes, tell me how much of goods x_1 and x_2 this consumer will purchase.

b. Now suppose there is a 10% a sales tax on good 1. That is, for every unit of good 1 the person buys, he/she has to pay a price of $p_1(1 + .1) = 2.2$, where the 10% of the price, or 20 cents, goes to the government as sales tax. How much of goods 1 and 2 does this person buy now?

c. Suppose instead there is a 5% income tax, so that the consumer must pay 5% of his/her income to the government. If there is no sales tax but a 5% income tax, how much of goods 1 and 2 will the consumer consume?

d. Which would the consumer prefer, a 10% sales tax on good 1, or a 5% income tax? Explain your reasoning for full credit.

e. How big would the sales tax on good 1 have to be for the government to get the same revenue as a 5% income tax? Which of the two taxes would the consumer prefer in this case, or is the consumer indifferent because the consumer has to pay a total tax of $5 (5% of $100) in either case?

9. **Supply and Demand Problem** The supply for corn is given by

$$S = 10 + 5p + .05R \tag{9}$$

where R is the amount of rainfall. The demand for corn is given by

$$D = 5Y^{.2}p^{-.5} \tag{10}$$

where Y is the per capita income.

a. What is the equation for the equilibrium price of corn, assuming this is a competitive market?

b. Solve for the equilibrium price and quantity in this market, using numerical methods (e.g. Newton's method) if necessary, or by any means possible to get numerical answers.

c. Derive a formula for dp/dR, i.e. the effect of an increase in rainfall on the price of corn.

d. Derive a formula for dp/dY, i.e. the effect of an increase in per capita income on the price of corn.