Fall 2015 John Rust
Economics 611 Georgetown University

Problem Set 1, Due in class Tuesday September 15th

1. Monopoly profit maximization with linear demand: Bertrand vs. Cournot approaches Consider a monopolist that has a constant returns to scale production function and can produce any (continuous) amount of a good $q \geq 0$ at a constant marginal $c \geq 0$. Suppose the monopolist faces a linear demand function for its product, $q_d = a - bp$, where q_d is the quantity of the monopolist's good that customers demand when the price is p and $a > 0$ and $b > 0$ are positive constants. However we can also compute the inverse demand function as $p = \frac{1}{b}(a - q)$ and interpret p as the per unit price the monopolist could receive if the monopolist produced an amount q and put the entire amount q up for auction in the market under the requirement that all consumers pay the same per unit price p (i.e. no price discrimination).

a. Compute the monopolist's optimal price and quantity under the assumption that the monopolist is a *price setter,* i.e. the monopolist chooses the price that maximizes its profits. We refer to this as the *Bertrand model* of the monopolist's behavior.

answer: Write the monopolist's profit as a function of p as $\Pi(p)$ given by

$$\Pi(p) = (a - bp)(p - c) - F \tag{1}$$

This equation for profit states that total profits are the product of the quantity sold at the price p, $(a - bp)$ times the profit per unit (or *markup*) $(p - c)$ less any fixed costs that the firm incurs, F (fixed costs are costs that are independent of the number of units the firm produces, such as "front office" costs, insurance, and so forth). Notice that $\Pi(p)$ is a *quadratic* function of p and you can check that its second derivative is negative, equal to $-2b$. This implies that the profit function is *strictly concave* and therefore has a *unique profit maximizing price* p^*. We can find p^* by taking the derivative of $\Pi(p)$ with respect to p an setting it to zero:

$$\Pi'(p^*) = (a - bp^*) - b(p^* - c) = 0. \tag{2}$$

Solving this equation for p^* we find

$$p^* = \left(\frac{a + bc}{2b}\right). \tag{3}$$

Plugging this formula for p^* into the demand equation, the implied quantity sold is q^* given by

$$q^* = a - bp^* = \left(\frac{a - bc}{2}\right). \tag{4}$$

To check that these formulas are sensible, note that in a *competitive equilibrium* the price must be equal to marginal cost of production, or $p_c = c$, where p_c is the competitive price. The implied competitive quantity is $q_c = a - bc$. So we see a monopolist produces exactly one half of the quantity that would be produced in a competitive market. Note that the highest price a monopolist could charge consumers is a/b since at this price, quantity demanded is zero. So the price p^* charged by the monopolist turns out to be halfway between the competitive price, c, and the highest price that monopolist can charge consumers, a/b. We assume that $a/b > c$ (otherwise $a - bc < 0$ and then the monopoly quantity would be negative, which does not make sense) and so this implies that the monopolist sets a price that is *higher* than the competitive price $p_c = c$, which also makes sense.

b. Compute the monopolist's optimal price and quantity under the assumption that the monopolist is a *quantity setter,* i.e. the monopolist chooses the quantity that maximizes its profits. We refer to this as the *Cournot model* of the monopolist's behavior.

answer Now we use the inverse demand curve $p = \frac{1}{b}(a-q)$ to express the monopolist's profits only as a function of q rather than as a function of p. That is, we write the monopolists as $\Pi(q)$ where

$$\Pi(q) = \frac{1}{b}(a-q)q - cq - F = \left(\frac{1}{b}(a-q) - c\right) q - F \tag{5}$$

So once again we have expressed the monopolist's profits as the product of quantity sold times the markup less fixed costs, but we have done it in a way that is only a function of q since we have "substituted out" p using the inverse demand curve. Now we observe that this function is also a quadratic function and its second derivative equals $\frac{-2}{b}$ so this way of expressing profits also results in a strictly concave profit function. We therefore conclude that there is a unique value of q^* that maximizes the firm's profits (you should try to prove the implicit Theorem here, namely that a strictly concave function has a unique maximizer, since proving this Theorem will be assigned as a problem on the next problem set), and we can solve for q^* by taking the derivative of $\Pi(q)$ with respect to q and setting it to zero

$$\Pi'(q^*) = -\frac{q^*}{b} + \frac{(a-q^*)}{b} - c = 0. \tag{6}$$

Solving this equation for q^* we obtain

$$q^* = a - bp^* = \left(\frac{a-bc}{2}\right). \tag{7}$$

c. Show that the Bertrand and Cournot solutions are the same in this case.

answer Notice that the equation for the monopolist's optimal quantity q^* in part b. above is the same quantity that we calculated in part a. above. Plugging q^* into the inverse demand curve and simplifying, we also obtain that the implied monopoly price p^* is the same as above. Plugging q^* into $\Pi(q)$ and simplifying and plugging p^* into $\Pi(p)$ and simpliifying, we find that

$$\Pi(q^*) = \Pi(p^*) = \frac{1}{4}\left(\frac{a}{b} - c\right)(a-bc) - F. \tag{8}$$

We conclude that the Bertrand and Cournot solutions result in the same optimal price, quantity and profits, so for a monopolist it does not matter whether we model the monopolist as a price setter or as a quantity setter.

d. Now suppose that instead of a *linear* demand function the monopolist faces a general demand function $q = D(p)$ where $D(p)$ is a differentiable function of p satisfying $D'(p) < 0$. Will the Cournot and Bertrand solutions be the same in the case of a general demand function? (If you say yes, then provide a proof that they result in a same profits, price and quantity produced, otherwise if you say no, then provide an example of a demand function $q = D(p)$ where the Bertrand and Cournot solutions are different.)

answer Since the demand function is monotonic (by the assumption $D'(p) < 0$), the inverse demand function $p = D^{-1}(q)$ exists. We can use the demand and inverse demand functions to express the monopolist's profits either as $\Pi(p)$ or as $\Pi(q)$ where

$$\begin{aligned}\Pi(p) &= (p-c)D(p) - F \\ \Pi(q) &= (D^{-1}(q)-c)q - F\end{aligned}$$

Now we can use the approach of *proof by contradiction* to establish the following proposition.

Proposition *If p^* maximizes $\Pi(p)$, then $q^* = D(p^*)$ maximizes $\Pi(q)$ for the two expressions of profit given above.*

The proof works as follows. Suppose that $q^* = D(p^*)$ *didn't* maximize $\Pi(q)$. Then there exists some $\hat{q} \neq q^*$ such that $\Pi(\hat{q}) > \Pi(q^*)$. Let $\hat{p} = D^{-1}(\hat{q})$. Then we have

$$\Pi(\hat{p}) = (\hat{p}-c)D(\hat{p}) - F = (D^{-1}(\hat{q})-c)D(D^{-1}(\hat{q})) - F = (D^{-1}(\hat{q})-c)\hat{q} - F = \Pi(\hat{q}). \quad (9)$$

Similarly, we can compute $\Pi(p^*)$ as

$$\Pi(p^*) = (p^*-c)D(p^*) - F = (D^{-1}(q^*)-c)D(D^{-1}(q^*)) - F = (D^{-1}(q^*)-c)q^* - F = \Pi(q^*). \quad (10)$$

Since $\Pi(\hat{q}) > \Pi(q^*)$, it follows from the two equations above that $\Pi(\hat{p}) > \Pi(p^*)$, but this contradicts the hypothesis of the Proposition that p^* maximizes $\Pi(p)$.

Note that in the proof we used the property of inverse functions:

$$\begin{aligned}D(D^{-1}(q)) &= q \\ D^{-1}(D(p)) &= p.\end{aligned}$$

e. How do your answers to parts a to d above change if the monopolist faces *a fixed capacity constraint* K? That is, the monopolist can produce at constant returns to scale with marginal cost of c for any $q \leq K$, but at least in the short run (i.e. assuming that we are considering a period of time too short for the monopolist to have time to invest and increase its production capacity K) the monopolist cannot produce any more than K.

answer Now we must formulate the monopolist's problem as a *constrained optimization problem* where the constraint is $q \leq K$, where K is the maximum that the monopolist can produce. It is not hard to show, following the arguments above, that the solution to this problem will be the same regardless of whether the monopolist is choosing q or p. If the monpolist is choosing p then the constraint is $D(p) \leq K$. So the two constrained optimization problems can be written as

$$\begin{aligned}p^* &= \underset{\{p\geq 0|D(p)\leq K\}}{argmax}\ (p-c)D(p) - F \\ q^* &= \underset{\{q\geq 0|q\leq K\}}{argmax}\ (D^{-1}(q)-c)q - F\end{aligned}$$

here the term *argmax* means "argument that maximizes" and below the *argmax* are the sets over which the optimizations are restricted to occur. For example the set $\{p \geq |D(p) \leq K\}$ denotes the set of prices p that are not negative but also satisfy the constraint $D(p) \leq K$. Thus, the monopolist is constrained to choose a positive price but one that is not too low that would imply a demand for the product that exceeds its production capacity K. In the other case, where the monopolist chooses q, the monopolist must choose a non-negative value of q that is less than or equal to K.

Now to describe the solutions to these constrained optimization problems, consider first the constrained optimization problem where the monopolist chooses prices. Let q_u^* denote the *unconstrained* profit maximizing value of q, i.e. when the monopolist does not face the capacity constraint K. It should not be hard for you to show that if $q < q_u^*$, then $\Pi'(q) > 0$. That is, if the monopolist is producing at less than the unconstrained profit maximizing quantity, the monopolist would like to increase output to move up to that profit maximizing quantity. This fact implies the following

Proposition *Suppose that the monopolist faces a demand function $q = D(p)$ and constant marginal costs of production c and assume that if there were no capacity constraints, the firm's profit function Π is a strictly concave function of q and $\Pi'(0) > 0$ and $\Pi'(q) < 0$ when q is sufficiently large. Then if q_u^* represents the unconstrained profit maximizing quantity over we have: $0 < q_u^*$ and if $q^*(K)$ denotes the profit maximizing quantity when the firm can produce at most K units of the good, we have*

$$q^* = \begin{cases} q_u^* & \text{if } q_u^* \leq K \\ K & \text{if } q_u^* > K \text{ and } (D^{-1}(K) - c)K - F \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

What this equation is telling us is that the optimal quantity a monopolist should produce when the monopolist is subject to a production capacity constraint is the unconstrained optimal quantity q_u^* if the monopolist's capacity is big enough to produce the unconstrained optimal quantity. Otherwise the monopolist should produce at the maximum capacity K *provided* that the monopolist can make a positive profit, otherwise the monopolist should shut down and not produce anything. It is not hard to show that if the monopolist is a price setter, this (Bertrand) solution is just the same as the quantity solution above but translated into prices. Thus, the monopolist should set the unconstrained optimal price p_u^* provided that the quantity demanded at this price, $q_u^* = D^{-1}(p_u^*) \leq K$, otherwise the monopolist must choose a higher price $p^* = D^{-1}(K)$ that implies a demand equal to the monopolist's capacity K, again provided that the monopolist can make a profit at this value, otherwise the monopolist shuts down and in effect, is charging a price of $D^{-1}(0)$, i.e. the price at which demand for the good is zero (which could be infinity).

The main point is that the existence of a capacity constraint does not alter the basic conclusion that whether the monopolist is a price setter or a quantity setter, the monopolist will choose the same price and quantities in either scenario. So *capacity constraints do not affect the conclusion that the Bertrand solution and the Cournot solution are the same in the case of a monopoly.*

2. Duopoly solution with linear demand: Bertrand vs. Cournot approaches Assume that instead of a monopoly, the market is served by 2 firms that both have constant returns to scale production functions with identical marginal costs of production of c for each firm. Assue that the market demand function is linear and the firms are producing *perfect substitutes* and consumers are perfectly informed and face no *switching or transportation costs* of choosing one firm's product versus the other's. This implies that under either the Cournot or Bertrand models, the price of the good must be the same for both firms, since no consumer would buy from the firm with the higher price. Thus, if the market price of both firms' products is p we assume the *total demand* is still linear, $q_d = a - bp$, where q_d is the total amount demanded. Since supply must equal demand, and if q_1 is the amount produced by firm 1 and q_2 is the amount produced by firm 2, then we have $q_1 + q_2 = q_d$.

a. Compute the *Cournot equilibrium quantities* for each firm sets quantity of production as its decision variable. Note that in a Cournot equilibrium each firm's optimal quantity q_i, $i \in \{1,2\}$ depends on its *expectation* of the profit maximizing quantity that its opponent will choose. In a Cournot equilibrium both firms must have *correct expectations,* that it, the amounts each firm expects that the other firm will produce must actually equal the amount that its opponent *actually* will produce and sell.

answer: When there are two firms, since total quantity produced is the sum of the outputs produced by the two firms, $q_d = q_1 + q_2$, each firm's beliefs about the prices it will receive (and hence its profits) will depend not only on its own production quantity decision, but also on its *expectation of its opponent's quantity decision.* Thus, let $\Pi_1(q_1,q_2)$ be the profits firm 1 expects to get if it produces q_1 and it expects firm 2 to produce q_2. We have

$$\Pi_1(q_1,q_2) = (p(q_1,q_2) - c)q_1 = (a/b - (q_1+q_2) - c)q_1, \tag{12}$$

where $p(q_1,q_2)$ is the price firm 1 expects to get when it produces q_1 and firm 2 produces q_2. Taking derivatives with respect to q_1 we can derive the optimal output for firm, *which will depend on its expectations of what firm 1 expects firm 2 will produce.* For this reason we refer to firm 1's optimal output $q_1^*(q_2)$ as a *reaction function* or a *best response function* since it specifies the profit maximizing output for firm 1 as a function of its expectation of what firm 2 will produce.

$$\frac{\partial}{\partial q_1}\Pi(q_1,q_2) = 0 = -q_1/b + a/b - (q_1+q_2) - c. \tag{13}$$

Solving this equation for the optimal $q_1^*(q_2)$ we see explicitly how firm 1's optimal output depends on its expectation of what firm 2 will produce, q_2

$$q_1^*(q_2) = \frac{a - q_2 - cb}{2} \quad \text{(firm 1's reaction function)} \tag{14}$$

Similarly, maximizing firm 2's profit $\Pi_2(q_2,q_1)$ over its choice of output q_2 yields its reaction function which depends on firm 2's expectations of the amount q_1 that firm 1 will produce

$$q_2^*(q_1) = \frac{a - q_1 - cb}{2} \quad \text{(firm 2's reaction function)} \tag{15}$$

The *Cournot Nash equilibrium* is found by solving these two reaction functions as a pair of simultaneous linear equations for the unique solution (q_1^*, q_2^*). Why is this solution a Nash equilibrium? Because at this solution each firm is maximizing its own profits (i.e. each firm's strategy is a best-response to its opponent's strategy) and thus at this equilibrium point, neither firm would want to change its output decision. So at this equilibrium, both firms have correct or *rational expectations* of what the opponent will produce. So now to actually solve these two equations, there are several ways to go about this, but one way is to assume a *symmetric equilibrium* ($q_1^* = q_2^*$) and add equations (14) and equation (15) to get

$$(q_1^* + q_2^*) = a - cb + \frac{1}{2}(q_1^* + q_2^*) \tag{16}$$

Solving this equation for $2q^*$, the total quantity produced in the symmetric Cournot equilibrium, we get

$$2q^* = \frac{2}{3}(a - bc) \tag{17}$$

or

$$q^* = \frac{a - bc}{3}. \qquad (18)$$

This implies that total amount supplied to the market is $\frac{2}{3}$ of the competitive amount $q_c = a - bc$ (recall that in a perfectly competitive market equilibrium price equals $p_c = c$ so the amount supplied/demanded is the value of q that solves $p = (a-q)/b$ or $q = a - bc$). The implied price under Cournot duopoly is

$$p^* = (a - 2q^*)/b = \frac{1}{3}(a/b) + \frac{1}{3}c \qquad (19)$$

so the duopoly price is $\frac{2}{3}$ of the way between the perfectly competitive price, $p_c = c$ and the highest possible market price a/b. Note that under monopoly the price is $p_m^* = (a/b + c)/2$ so the monopolist's price is halfway between a/b and c and since we assume that $a/b > c$, it follows that the Cournot duopoly price is lower than the monopoly price but higher than the perfectly competitive price.

b. Compute the *Bertrand equilibrium prices* for each firm now under the assumption that instead of choosing quantities of production, the firms choose their respective prices for selling their goods. Just at in the Cournot equilibrium case, in equilibrium the two firms must have correct expectations about the price their opponent will charge its customers.

answer We proved this in class, but for the record, I used a *proof by contradiction* to prove the following proposition

Proposition *Suppose two duopolists produce perfectly homogeneous goods at constant marginal costs of production c and have no fixed costs of production and no capacity constraints. Suppose also that consumers are rational, fully informed about the prices, and face no switching or transactions costs (and hence all consumers wish to buy from the firm selling the good at a lower price). Then the Bertrand equilibrium prices are $p_1^* = p_2^* = c$ and both firms make zero profits in equilibrium and the competitive quantity of goods is supplied to the market.*

Similar to our proof by contradiction in the monopoly problem (see solution to problem 1 part d above) assume that at the Bertrand equilibrium we have at least one of the prices p_1^* or p_2^* note equal to marginal cost, c. For example suppose that $p_1^* > p_2^*$ and $p_1^* > c$. Then I claim this cannot be a Nash equilibrium since firm 2 could set a price just below firm 1's price p_1^* but higher than c and make higher prices. This is because we assumed the good is homogeneous and we assume all consumers are informed and rational and have no switching costs, all consumers will abandon firm 1 and switch to firm 2. So firm 2's profits must be higher than at the claimed Bertrand Nash eqeuilibrium prices (p_1^*, p_2^*) since firm 2 charges a higher price *and* gets more customers. But this contradicts the assumption that $(p_1^*, p^* - 2)$ is a Bertrand Nash equilibrium, since by definition, at an equilibrium neither firm would want to change its prices. By this type of reasoning we can dismiss any set of prices (p_1^*, p_2^*) where either p_1^* or p_2^* is higher than c as being a Bertrand Nash equilibrium. But what about the claimed pair $(p_1^*, p_2^*) = (c, c)$? Why is this a Bertrand Nash equilibrium? In this case both firms make zero profits and if either firm lowered their price the would make a loss (thus neither wants to lower their price) and neither firm has an incentive to raise its price *unilaterally* since no consumers would buy from the firm charging a higher price. So we conclude that $p_1^* = p_2^* = c$ is indeed a Bertrand Nash equilibrium of the duopoly pricing game.

The key thing to notice that in duopoly, unlike in the monopoly case, the Bertrand and Cournot solutions are different. In the Bertrand equilibrium neither firm makes any profits and the competitive price is charged and the competitive level of output is produced. In the Cournot equilibrium, the firms both make positive profits, and the price is higher than the competitive price c and the total ouput of the two duopolists is only $\frac{2}{3}$ of the competitive output. Thus it makes a huge difference in moving from monopoly to duopoly where Bertrand and Cournot models predict very different outcomes.

c. Repeat parts a and b above in the case where the two duopolists have different marginal costs of production, c_1 and c_2. Compute the prices, production/sales, and profits for the two firms in both the Bertrand and Cournot cases.

answer It is not hard to show that the Bertrand Nash equilibrium prices are $p_1^* = p_2^* = \max[c_1, c_2]$. Essentially, the low cost firm is able to serve the entire market but it charges a price just slightly below the marginal cost of the high cost firm, so the high cost firm makes zero profits but the low cost firm makes positive profits. For example if $c_1 < c_2$ then firm 1 is the low cost firm and it will be able to charge a markup of $c_2 - c_1$ and thus set a price of $p_1 = c_2$ and make positive profits equal to $\Pi_1(p_1^*, p_2^*) = (c_2 - c_1)D(c_2) > 0$. Firm 1 sells nothing, and firm 2 sells an amount equal to $D(c_2)$.

In the Cournot Nash equilibrium both firms generally produce and sell positive amounts of the good, unless the fixed costs of production are high enough to cause one of the firms to exit and leave the other firm with a monopoly (and if the fixed costs of the other firm are extremely high, it might want to exit too even when it has a monopoly). Let's assume that fixed costs are not too high and both firms want to produce. What are the equations for the Cournot-Nash equilibrium? Now we can guess that the equilibrium will no longer be symmetric, i.e. the two firms will produce different quantities. Further we can guess that the firm with the lower maringal cost of production should produce more and earn a higher profit. The equations we need to solve now are as before, to solve the two reaction functions decribing the optimal outputs of each firm as a function of its expectations of what its opponent will produce:

$$q_1^*(q_2) = \frac{a - q_2 - bc_1}{2} \quad \text{(firm 1's reaction function)} \tag{20}$$

$$q_2^*(q_1) = \frac{a - q_1 - bc_2}{2} \quad \text{(firm 2's reaction function)} \tag{21}$$

Solving these two equations for q_1^* and q_2^* we get

$$q_1^* = \frac{1}{3}(a + bc_2 - 2bc_1), \tag{22}$$

$$q_2^* = \frac{1}{3}(a + bc_1 - 2bc_2), \tag{23}$$

As a check, notice that these two equations reduce to the previously given solution for $q_1^* = q_2^*$ in the symmetric case when $c_1 = c_2 = c$, see equation (18) above. Adding the two quantities produced to get total market quantity produced we obtain

$$q = q_1^* + q_2^* = \frac{2}{3}\left(a - b(c_1 + c_2)/2\right). \tag{24}$$

Putting this total quantity into the demand function $p = a - bq$ to determine the implied Cournot-Nash duopoly market price we get

$$p^* = (a - q)/b = \frac{1}{3}(a/b) + \frac{2}{3}\left(\frac{c_1 + c_2}{2}\right). \tag{25}$$

Thus, the Cournot-Nash price is a weighted average of the highest possible price, a/b with a weight of $\frac{1}{3}$, and the average marginal cost, $(c_1 + c_2)/2$, with a weight of $\frac{2}{3}$. Thus, we can say that the Cournot Nash price is $\frac{2}{3}$ of the way between a/b and the average marginal cost $(c_1 + c_2)/2$. It is easy to use this to write the formulas for the profits earned by the two duopolists and show that the firm with the lower marginal cost makes more profits. I don't bother to do this here, but you should carry out the algebra just to make sure you understand this.

d. Now consider parts a and b in the case of a general demand function $q_d = D(p)$, where $D'(p) < 0$. Can you say anything about the Cournot and Bertrand solutions in the general case when the firms have the same marginal costs of production c? In particular, are the Bertrand and Cournot prices and outputs the same or different from each other? If they are the same, provide a proof of this, if they are different provide an example or an argument as to why they are different.

answer In the case of general demand function (as opposed to a linear demand function considered above) the Bertrand equilibrium does not change. The price is still given by $p_1^* = p_2^* = \max[c_1, c_2]$ independent of the demand curve $D(p)$. The demand curve just changes the quantity produced but does not affect the price. In the Cournot-Nash equilibrium, however, the shape of the demand curve does affect the prices that result. We can no longer find a general "closed for" solution for q_1^* and q_2^* as we did in the case where the demand is a linear function, but we can write the equations that need to be solved (by computer if necessary) to find the Cournot-Nash equilibrium quantities, and from these quantities, the market price, and firm profits, etc. Using the inverse demand curve the profit function for firm 1 is

$$\Pi_1(q_1, q_2) = (D^{-1}(q_1 + q_2) - c_1)q_1 - F_1, \tag{26}$$

and the profit function for firm 2 is

$$\Pi_2(q_2, q_1) = (D^{-1}(q_1 + q_2) - c_2)q_2 - F_2. \tag{27}$$

We now derive the reaction functions for the two firms. In general these will be *nonlinear functions* instead of linear functions as was the case when the demand function was linear (and the implied profit functions quadratic). Let $q_1 = \eta_1(q_2)$ be the best response function for firm 1. It is given by

$$\eta_1(q_2) = \underset{q_1}{argmax}(D^{-1}(q_1 + q_2) - c_1)q_1 - F_1 \tag{28}$$

Similarly, let $q_2 = \eta_2(q_1)$ be the best response function for firm 2. It is given by

$$\eta_2(q_1) = \underset{q_2}{argmax}(D^{-1}(q_1 + q_2) - c_2)q_2 - F_2 \tag{29}$$

The Cournot-Nash equilibrium can be found as the set of all intersections of these two best response functions. One way to find these is to substitute the η_2 best response function into the η_1 best response function to get

$$q_1 = \eta_1(\eta_2(q_1)). \tag{30}$$

If we define the function $f(q) = \eta_1(\eta_2(q))$, then the equation (30) above defines q_1 as a *fixed point* of the mapping f. You should be able to prove the following

Proposition *If q_1^* is a fixed point of the composition mapping $f(q) = \eta_1(\eta_2(q))$, where η_1 and η_2 are the best response functions defined in equations (28) and (29) above, and if $q_2^* = \eta_2(q_1^*)$, then (q_1^*, q_2^*) is a Cournot-Nash equilibrium of the duopoly problem.*

e. Which of the two equilibrium concepts, Bertrand equilibrium or Cournot equilibrium, do you think is more realistic? That is, when you think about what happens in the real world, which of these models provides a better approximation to the way two firms actually make decisions and compete with each other in actual situations?

answer The Bertrand assumption that firms set prices rather than quantities is typically more realistic of what most firms actually do. There are some firms, however, that may be better described as quantity setters, however. For example NuCor is a highly efficient steel producer that runs its plants often at full capacity, producing as much as possible but accepting whatever price the steel it produces will fetch in a highly internationally competitive wholesale market for steel. However the assumption that the products are perfectly homogeneous goods, and that customers are always fully informed of all prices, and face no switching or transaction costs underlying both models is not a good one either. We often observe *price dispersion* (i.e. different prices for nearly identical goods, such as gasoline), and it is rare where one firm can capture *all* of its competitor's customers by just slightly undercutting its competitor's price.

3. Cournot and Bertrand oligopoly Now suppose that there are N firms competing in the market. Initially assume that there is linear aggregate demand, $q_d = a - bp$, the firms produce a homogenous good, and that suppy equals demand so that if firms $i = 1, \ldots, N$ produce amounts $q_1, q_2, \ldots, q_N$, respectively, we have $q_d = q_1 + q_2 + \cdots + q_{N-1} + q_N$. Assume initially that all firms have constant returns to scale production functions with identical marginal costs of production c.

a. Compute the Bertrand and Cournot oligopoly solutions. Show that the Bertrand equilibrium is *competitive* for any finite value of $N > 1$, i.e. $p_B^* = c$ where p_B^* is the Bertrand equilibrium price. However show that under the Cournot equilibrium, show that the Cournot equilibrium price p_C^* satisfies $p_C^* > c$, but that as $N \to \infty$, we have $p_C^* \to c$.

answer The Bertrand-Nash equilibrium with N firms is the same for any $N > 1$ when the marginal costs of production are the same for all firms and equal to a common value c. The equilibrium is simply for $p_1^* = p_2^* = \cdots = p_N^* = c$. The division of production among the N is indeterminant though. Since each firm makes zero profits for any level of production it does not matter to any firm how much it produces. So in this sense the Bertrand-Nash has a *continuum of equilibria* because there are infinitely many ways to divide up the total quantity demanded, $D(c)$ among the N firms.

In the Cournot case, since all firms have the same marginal cost of production, we can conjecture that there will be a symmetric equilibrium solution where all firms produce the same amount, $\overline{q}$. In fact it is not hard to show that the Cournot-Nash equilibrium *must* be symmetric in this case. Let q_{-i} denote the list of amounts produced by all firms except firm i

$$q_{-i} = (q_1, q_2, \ldots, q_{i-1}, q_{i+1}, \ldots, q_N). \tag{31}$$

Using this notation we can write firm i's profit function as $\Pi_i(q_i, q_{-i})$ where

$$\Pi_i(q_i, q_{-i}) = (a/b - \sum_{i=1}^{N} q_i/b - c)q_i \tag{32}$$

Taking the first order condition with respect to q_i and setting it to zero we have

$$\frac{\partial}{\partial q_i}\Pi_i(q_i,q_{-i}) = (a/b - \sum_{i=1}^{N} q_i/b - c) - q_i/b = 0. \tag{33}$$

Solving this equation for q_i we get

$$q_i = \frac{1}{b}\left(a - \sum_{i=1}^{N} q_i - bc\right). \tag{34}$$

Since the right hand side of this equation is the same for all firms (all $i = 1,\ldots,N$), let $\overline{q}$ be the common value of output for all of the firms. Then we can rewrite the equation above as

$$\overline{q} = \frac{1}{b}\left(a - N\overline{q} - bc\right). \tag{35}$$

Solving this equation for $\overline{q}$ we get

$$\overline{q} = \frac{1}{(N+1)}\left(a - bc\right). \tag{36}$$

Total output is $N\overline{q}$, or

$$N\overline{q} = \frac{N}{(N+1)}\left(a - bc\right). \tag{37}$$

Note that as $N \to \infty$ we have

$$\lim_{N\to\infty} N\overline{q} = (a - bc). \tag{38}$$

Recall that $(a - bc)$ is the quantity of the good sold in competitive equilibrium, so we see that the Cournot-Nash equilibrium quantity (total output for whole market) converges to the competitive equilibrium vale, but the output for each firm tends to zero. The Cournot-Nash equilibrium price is given by $p_N = (a - N\overline{q})/b$. Using the formula for $N\overline{q}$ above we have

$$p_N = \frac{1}{(N+1)}(a/b) + \frac{N}{(N+1)}c \tag{39}$$

This tells us that the Cournot-Nash equilibrium price is a weighted average of the maximum price a/b (with weight $\frac{1}{(N+1)}$) and the marginal cost of production c (with weight $\frac{N}{(N+1)}$). As $N \to \infty$ the weight on the marginal cost c tends to 1 and the weight on a/b tends to zero, so we have

$$\lim_{N\to\infty} p_N = c. \tag{40}$$

b. Repeat part a except explain what happens if one half of the firms (say firms i where i is an even number) have a constant marginal cost of production c_1 and the other half (say firms i where i is an odd number) have a marginal cost $c_2 > c_1$. Characterize what the Bertrand and Cournot equilibrium prices, and profits and output would be in this case, both for finite numbers of firms N and as $N \to \infty$.

answer The Bertrand case is easy, the price will be equal to c_1 and the high cost firms produce and sell nothing and the low cost firms produce generally positive amounts but make zero profits. The reason is due to the continuum of possible Bertrand equilibrium when there are multiple firms with the lowest cost of production: there are infinitely many ways to allocate the production among these firms and the firms are indifferent since each allocation gives each firm zero profits. Note that unlike the 2 firm case, the Bertrand equilibrium price cannot equal the higher marginal cost of production c_2. This is because if one of the low cost firms tried to charge a price higher than its marginal cost c_1, one of the other firms wold undercut it and steal all of its business.

The Cournot case is more complicated. It is tempting to think that we should have a solution where the Cournot-Nash price is a weighted average of the highest possible price a and the average of the marginal costs of production of the two types of firms, $(c_1+c_2)/2$, similar to the Cournot-Nash equilibrium solution we obtained in equation (25) in the answer to problem 2-c above. However this would not be correct for large values of N. The intuition is that when there are many competing low cost firms, they should eventually (i.e. when N is sufficiently large) derive the high cost firms out of business, and thus the price in the market should eventually be driven down to the marginal cost of production of the low cost firm, c_1.

We will first solve the general Cournot Nash equilibrium for N firms with *any* configuration of constant marginal costs of production, $(c_1,\dots,c_N)$. We will use matrix algebra to drive this solution and I do not expect students in the class to know matrix/linear algebra. So I will first present the solution without using matrix algebra and let you verify that this is a Cournot-Nash equilibrium. Note that the profit function for firm i is $\Pi_i(q_i,q_{-i})$ where, as I previously defined q_{-i} in equation (31) above,

$$q_i^*(q_{-i}) = \underset{q_i}{argmax}\,\Pi_i(q_i,q_{-i}) = \underset{q_i}{argmax}(a/b-\sum_{i=1}^N q_i/b-c_i)q_i \tag{41}$$

Taking the derivative of $\Pi_i(q_i,q_{-i})$ with respect to q_i and solving, we get this equation for the best response function

$$q_i = q_i^*(q_{-i}) = a - bc_i - \sum_{i=1}^N q_i, \quad i=1,\dots,N \tag{42}$$

This is a system of N linear equations in N unknowns and shortly I will use *matrix algebra* to present a solution to this system of equations. But another way is to take the sum over all of the best response function equations (42) to get

$$q_i^* = a - N\overline{q} - bc_i \quad i=1,\dots,N \tag{43}$$

where

$$\overline{q} = \frac{1}{N+1}(a-b\overline{c}) \tag{44}$$

and

$$\overline{c} = \frac{1}{N}\sum_{i=1}^N c_i. \tag{45}$$

You should be able to insert the solution given in equation (43) above into the system of linear equations defining the best response functions for the N firms in equation (42) and verify that this is in fact a *potential* solution to the problem. Why do I qualify this by saying *potential solution*? The reason is that we cannot have a solution to the Cournot-Nash equilibrium that would ever imply that any firm produces a *negative quantity of output* $q_i^* < 0$. However we can see from equation (43 that when N is sufficiently large, if a firm has a marginal cost that is above the mean cost $\overline{c}$, then the predicted Cournot-Nash output q_i^* is *negative* which is impossible. Thus, if we impose a non-negativity constraint on firm i's choice of q_i, we will find for large enough values of N that the implied q_i^* would be negative, and this cannot be a solution. So what we do is remove those firms that have negative predicted outputs and resolve the Cournot Nash equilibrium with the smaller number of surviving firms. We keep resolving the problem this way until we obtain a solution where all of the resulting values of q_i^* are non-negative.

To summarize: when we take the non-negativity constraint into account, the Cournot-Nash equilibrium quantities are given by

$$q_i^* = \begin{cases} (a - bc_i - N_+\overline{q}_+), & \text{if } q_i^* > 0 \\ 0 & \text{otherwise} \end{cases} \qquad i = 1, \dots, N \tag{46}$$

where

$$\overline{q}_+ = \frac{1}{N_+} \sum_{\{i|q_i^*>0\}} q_i^*, \tag{47}$$

where $\overline{c}$ is now redefined as the average of the marginal costs for those firms that are producing a non-negative amount of the good

$$\overline{c} = \frac{1}{N_+} \sum_{\{i|q_i^*>0\}} c_i, \tag{48}$$

and

$$N_+ = \sum_{i=1}^{N} I\{q_i^* > 0\}. \tag{49}$$

So with this insight, when there are a large number of only two types of firms, a high cost type firm with marginal cost of production c_2 and a low cost type of firm with marginal cost or production $c_1 < c_2$, then when the total number of firms get sufficiently large, there will come a point (as N increases) where the high cost firms can no longer compete with the large number of low cost firms and the high cost firms will have to go out of business leaving only the $N/2$ low cost firms competing with each other. But when this happens, all of the $N/2$ remaining firms will have the same lower marginal cost of production c_1, so the Cournot Nash equilibrium will be the same as the one we previously computed in problem 3-a above, but instead of N firms with the same marginal cost c in this case we have $N/2$ firms with the same marginal cost c_1. But otherwise the solution will be the same. Adapting equation (39) above, we obtain

$$p_N = \frac{a/b}{N/2+1} + \left(\frac{N/2}{N/2+1}\right) c_1 \tag{50}$$

when N is sufficiently large. However for smaller values of N when all N firms are able to produce positive quantities and make positive profits, the Cournot-Nash equilibrium price is given by

$$p_N = \frac{a/b}{N+1} + \left(\frac{N}{N+1}\right) \overline{c}. \tag{51}$$

The remainder of this answer is optional, for the benefit of students who know linear algebra. I am going to use bold math notation for matrices and vectors to make things as clear as possible. So $\mathbf{q}^*$ denotes the $N \times 1$ vector of Cournot equilibrium quantities of the N firms

$$\mathbf{q}^* = \begin{bmatrix} q_1^* \\ q_2^* \\ \cdots \\ q_{N-1}^* \\ q_N^* \end{bmatrix}. \tag{52}$$

Further, define the $N \times 1$ vector $\mathbf{v}$ by

$$\mathbf{v} = \begin{bmatrix} (a - bc_1) \\ (a - bc_2) \\ \cdots \\ (a - bc_{N-1}) \\ (a - bc_N) \end{bmatrix}. \tag{53}$$

Then we can re-write the reaction function equations for the N firms in equation (42) as follows

$$q_i^* + \sum_{j=1}^{N} q_j^* = (a - bc_i), \quad i = 1, \ldots, N \tag{54}$$

We can "stack" these N linear equations and re-write the in matrix notation as follows

$$(\mathbf{I} + \mathbf{E}) * \mathbf{q}^* = \mathbf{v}, \tag{55}$$

where $\mathbf{I}$ is the $N \times N$ *identity matrix*

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{56}$$

Or in words, $\mathbf{I}$ is a square matrix of numbers with 1's on its diagonal elements and 0's everywhere else. The matrix $\mathbf{E}$ is just an $N \times N$ matrix with every element equal to 1,

$$\mathbf{E} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ & & \cdots & & \\ 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{bmatrix}. \tag{57}$$

Notice that we have

$$\mathbf{I} * \mathbf{q} = \mathbf{q} \tag{58}$$

since the identity matrix mutiplied by any vector just returns the same vector. Notice also that

$$\mathbf{E} * \mathbf{q} = \begin{bmatrix} \sum_{i=1}^{N} q_i \\ \sum_{i=1}^{N} q_i \\ \sum_{i=1}^{N} q_i \\ \cdots \\ \sum_{i=1}^{N} q_i \\ \sum_{i=1}^{N} q_i \end{bmatrix}, \tag{59}$$

which is an $N \times 1$ vector whose elements are just the sum of the elements of the vector $\mathbf{q}$. Thus $(\mathbf{I}+\mathbf{E}) * \mathbf{q}^*$ is just the stacked vector form of the left hand sides of the re-written reaction (first order conditions) in equation (54). Suppose the $N \times N$ matrix $(\mathbf{I}+\mathbf{E})$ is *invertible*, then we can write the Cournot-Nash equilibrium solution in one matrix equation as follows

$$\mathbf{q}^* = (\mathbf{I}+\mathbf{E})^{-1} * \mathbf{v} \tag{60}$$

Further, it is not hard to show that the inverse of the matrix $(\mathbf{I}+\mathbf{E})$ is given by

$$(\mathbf{I}+\mathbf{E})^{-1} = (\mathbf{I}-\mathbf{E}/(N+1)). \tag{61}$$

(**Extra credit assignment:** for those of you who know matrix algebra, prove that $(\mathbf{I}-\mathbf{E}/(N+1))$ is indeed the inverse of the matrix $(\mathbf{I}+\mathbf{E})$, where $\mathbf{E}/(N+1)$ is an $N \times N$ matrix each of whose elements equals $1/(N+1)$).

Using the formula for $\mathbf{q}^*$ in equation (60) combined with the formula for the inverse of the matrix $(\mathbf{I}+\mathbf{E})$ in equation (61) you should be able to show that this results in the formula I provided above for the Cournot-Nash equilibrium quantities in equation (43) above. From this, you should also be able to derive the formula for total market output and the general formula for the Cournot-Nash price p_N in equation (51).

4. Bertrand duopoly with capacity constraints Suppose there are two firms with identical marginal costs of production c and both can produce at constant returns to scale up to a capacity constraint K_1 for firm 1 and a capacity constraint K_2 for firm 2. What is the Bertrand equilibrium in this case? For simplicity, you can assume that the market demand function is linear.

answer The answer is very simply, Bertrand price is the price at which "supply=demand" where the "supply curve" is a flat curve at the marginal cost c until quantity q reaches the capacity constraint $K_1 + K_2$ at which point the supply curve goes vertical as in figure 2 below. Why is this the case? Suppose demand is not very strong, such as the red line in figure 2, so that the demand curve intersects the black supply curve in the horizontal segment where supply equals c, the constant marginal cost of both firms. Then in this case there is *excess capacity* and I will now argue that when there is excess capacity, Bertrand price competition works just like in the case where there are no capacity constraints, and drives the price down to $p = c$ and both firms have zero profits.

Now we claim in this case, $p = c$ is the Bertrand equilibrium, just like the case where neither firm had any capacity constraints. Why? Clearly there is no incentive for either firm to lower its price, because they are both getting zero profits when $p = c$, and thus any further lowering of price would cause the firm that did this to lose money. Would either firm want to raise its price above c?

We can use a proof by contradiction argument to show that the Bertrand equilibrium price p cannot exceed marginal cost, c. To do a proof by contradiction we suppose that the equilibrium price did exceed marginal cost c and then we show that at this price, at least one of the firms has an incentive to lower its price below p — i.e. it can earn more profits by charging a price below p than it could earn by charging p.

Remember that in a Bertrand equilibrium the firms must simultaneously choose a price p without being able to see what price the other firm would choose. But if the firms anticipated an equilibrium where $p > c$, then they would know that at least one of them would have excess capacity at that price. Though the model does not really specify how demand would be allocated between the two firms at the price p, at least one of them would have to expect to have excess capacity at the posited Bertrand equilibrium price $p > c$, because otherwise there could not be excess capacity in the market as a whole at this price. So whichever (or possibly both) firms (call this firm i, $i = 1$ or $i = 2$) who expect to have excess capacity will realize that if they cut their price by an infinitesimally small amount ε below this p, they will be able to "sell out" and produce an amount $q = K_i$, i.e. they would be producing at their capacity since they had price undercut their competitor. If the firm had expected to sell an amount $q < K_i$ at the posited equilibrium price $p > c$, they will earn revenue $p * q$ and profit of $(p - c)q$, but if they lower their price by a small amount ε they will earn revenue of $(p - \varepsilon)K_i$ and profit of $(p - \varepsilon - c)K_i$ and for small enough value of ε you can show that $(p - c)q < (p - \varepsilon - c)K_i$ when $q < K_i$. So this is a contradiction of the hypothesis that $p > c$ is a Bertrand equilibrium price: i.e. we have shown that contrary to our hypothesis, at least one of the firms *can* profitably deviate by lowering their price.

Thus, it follows that in the "low demand" scenario, that $p = c$ is a Bertrand equilibrium because at this price neither firm can gain from raising their price (the firm that did this would lose most of its business to the other firm), and they cannot gain by lowering their price. There is also no Bertrand equilibrium at a price $p > c$ as we showed by our argument above.

Now consider the "high demand" scenario illustrated by the blue line in figure 2 which crosses the vertical segement of the supply curve at a price of $p = 260$ which is above marginal cost of $c = 200$. In this case the market is capacity constrained, since the amount sold, $q = 80$ equals the sum of the two firm's capacities, $K_1 + K_2$. I claim this price of $p = 260$, which is the "competitive price" where supply equals demand with this capacity-constrained supply curve, is the Bertrand equilibrium price in this case. Similar to the low demand case we can argue that no price $p > D(K_1 + K_2)$ can be a Bertrand equilibrium outcome, because a higher price will correspond to a quantity sold $q < K_1 + K_2$ and thus a situation where there is excess capacity in the market. We have already showed above that at least one of the firms would have an incentive to lower its price in that case.

However the difficult part is to show that $p = D(K_1 + K_2) > c$ is a Bertrand equilibrium. Again we can ask whether either firm can expect to increase its profits by lowering its price. Unlike in the low demand scenario, neither firm expects to increase its profits by lowering its price because *both firms are producing at their maximum capacity K_i.* Would either firm have an incentive to raise its price? Here we have a difficulty, because one of the firms might reason that if its opponent set a price of $p = D(K_1 + K_2)$, it could raise its price and increase its profits. How could this happen? Suppose that firm 1 keeps its price at $p = D(K_1 + K_2) = 260$ and firm 2 raises its price to $p_2 = 270$. Then all consumers will rush to buy from firm 1, but firm 1 can only supply a quantity of $q = K_1 = 40$, whereas the quantity demanded at a price of $p = 270$ is $q = D^{-1}(270) = 76\frac{2}{3}$. So if firm 1 expected to capture the "residual demand" of $q - 40 = 36.666$, its total profits would be $(270 - 200) * 36.666 = 2566.666$ which exceed the profits it would get by keeping its price at $p = 260$: $(260 - 200) * 40 = 2400$. So it appears that both firms would have an incentive to deviate by *raising* their price from the posited Bertrand equilibrium price of $p = 260 = D(K_1 + K_2)$.

However notice that firm 2 could expect to profit only if it assumes that it can capture the "residual demand" of customers who are not served by firm 1, which is producing at capacity, $K_1 = 40$. If we assume that all customers would go to firm 1 and take their chances in a sort of "lottery" about whether firm 1 will sell them the goods they demand, then firm 2 might expect its demand to fall do zero in this case, and it would then lose by raising its price. The Bertrand model is not clear about how demand is allocated between the two firms if their prices are different, and what the "rationing rule" is when one firm charges a lower price and the consumers flock to it. The Kreps and Scheinkman article makes its own more complicated assumption about the rationing rule that results in a very complicated analysis: here I make another, perhaps less "realistic" assumption about consumer behavior that results in a simpler solution for the Bertrand equilibrium. The classical model of consumers who are willing to instantly shift from one firm to another if its price is just slightly less is not really "realistic" in the first place. Later in Econ 425 we will consider more "realistic" models of consumer choice that posit that there are "transactions costs" and customers have idiosyncratic reasons for preferring one firm or product over another one even if the price is higher. One easy way to see how this can happen is if the products of the two firms are *imperfect substitutes* such as Coke and Pepsi. Though they may be close substitutes for each other, some customers may have idiosyncratic preferences for Coke that cause them to be willing to pay more to drink Coke over Pepsi, and vice versa. We will see later in Econ 425 how the Bertrand model of price competition can be extended to allow for products that are imperfect substitutes for each other.

But for the rest of this problem let's just go with the simpler version of the Bertrand equilibrium when there are capacity constraints which posits that the firm that raises its price over the price charged by its rival that it expects to lose enough business (even if its rival is capacity constrained, since consumers may just all flock to the rival to take the chance of the "lottery" of being able to buy from the rival at the lower price) that it would have lower profits because its residual demand would be sufficiently low.

Now this implies that the Bertrand equilibrium takes the following form:

$$p = \begin{cases} D(K_1+K_2) & \text{if } D(K_1+K_2) > c \\ c & \text{otherwise} \end{cases} \tag{62}$$

Now consider a two stage game in which in stage 1 the firms simultaneously choose their capacities K_1 and K_2 and in stage 2 they play the capacity-constrained Bertrand price equilibrium given above. Then what capacities would they choose in stage 1?

Here it is easy to see that the firms would not want to choose excessive capacity in stage 1 of the game because doing so could result in a situation where $D(K_1+K_2) < c$ and thus the zero profit Bertrand equilibrium when there is excess capacity would result. So we conjecture that the firms would rationally both choose to have a capacity low enough that the market is capacity-constrained and results in a Bertrand equilibrium price $p = D(K_1+K_2) > c$, and thus positive profits for both firms.

Now suppose at first that there are zero costs of choosing different capacities in stage 1. Then the stage 1 Nash equilibrium would be the solution to

$$\begin{aligned} K_1^* &= \underset{K_1}{argmax}(D(K_1+K_2)-c)K_1 \\ K_2^* &= \underset{K_2}{argmax}(D(K_1+K_2)-c)K_2 \end{aligned}$$

We can see that if a solution to the above exists, call the solution (K_1^*, K_2^*), that this is the same as the *Cournot equilibrium quantities* (q_1^*, q_2^*) in the Cournot quantity setting game:

$$\begin{aligned} q_1^* &= \underset{q_1}{argmax}(D(q_1+q_2)-c)q_1 \\ q_2^* &= \underset{q_2}{argmax}(D(q_1+q_2)-c)q_2 \end{aligned}$$

In the case where there are up-front costs of investment, $I_1(K_1)$ and $I_2(K_2)$, then the two stage Bertrand equilibrium needs to have the first stage choice of (K_1^*, K_2^*) modified to

$$\begin{aligned} K_1^* &= \underset{K_1}{argmax}(D(K_1+K_2)-c)K_1 - I(K_1) \\ K_2^* &= \underset{K_2}{argmax}(D(K_1+K_2)-c)K_2 - I(K_2) \end{aligned}$$

and in general, the higher the investment costs, the lower the capacities that the firms will choose in the equilibrium of this game. We can extend the second stage to an infinitely repeated Bertrand pricing game. Then after investing in stage 1, in subsequent periods the firms sell to customers and discount their profits from period 2 to infinity at discount factor $\beta \in (0,1)$. Then in this case it is easy to see that the equilibrium is the solution to

$$\begin{aligned} K_1^* &= \underset{K_1}{argmax}\frac{\beta}{1-\beta}(D(K_1+K_2)-c)K_1 - I(K_1) \\ K_2^* &= \underset{K_2}{argmax}\frac{\beta}{1-\beta}(D(K_1+K_2)-c)K_2 - I(K_2) \end{aligned}$$

The final subtlety is to consider the case when the investment costs $I_1(K_1)$ and $I_2(K_2)$ are so high that there "is only room for one firm". That is, if one of the firm was a monopoly, it would be profitable for them to invest and install capacity K_i, but if both firms were to invest, then the firms would both lose money, so there could not be any equilibrium where both firms invested and entered the market. This results in an *anti-coordination game* similar to the leapfrogging problem discussed in class where the first stage investment equilibrium has 3 possible equilibria: 1) firm 1 invests and firm 2 doesn't, 2) firm 2 invests and firm 1 doesn't, and 3) both firms play a mixed strategy, investing with probabilities π_1 and π_2, respectively.

You should also consider what the *Stackelberg equilibrium* to the first stage investment game would be, i.e. if firm 1 were able to move first and make an investment in capacity level K_1 and then firm two would consider whether to invest, seeing the choice of capacity K_1 made by firm 1. Would you expect to get *pre-emption* by firm 1 in this case? That is, would firm 1 have an incentive to invest in sufficient production capacity K_1 to deter firm 2 from entering the market at all, resulting in firm 1 being a monopolist?