Econ 551: Extra Credit Problem on Markov Processes and Invariant Densities

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Let $\{y_t\}$ be a first order Gaussian process defined by the linear AR(1) system:

$$y_{t+1} = \alpha + \beta y_t + \epsilon_t \tag{1}$$

where $\{\epsilon_t\}$ is and *IID* Gaussian process with mean 0 and variance σ^2 (i.e. ϵ_t and ϵ_s are mutually independent whenever $t \neq s$ and for each t we have $\epsilon_t \sim N(0, \sigma^2)$). Assume that for all t y_t is independent of ϵ_t.

1. Is $\{y_t\}$ a Markov process? If so, write down a formula for its transition density $p(y_{t+1}|y_t)$.

2. Under what conditions on $(\alpha, \beta, \sigma^2)$ is $\{y_t\}$ a stationary, ergodic Markov process?

3. What is the recurrent class for $\{y_t\}$? (The *recurrent class* of a Markov process $\{y_t\}$ is defined as the smallest closed set C such that $Pr\{y_t \in C \text{ i.o.}\} = 1$, where *i.o.* denotes "infinitely often".)

4. Assuming that $\{y_t\}$ is stationary and ergodic, show that its invariant density $p(y)$ is:

$$p(y) = \frac{\sqrt{(1-\beta^2)}}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-(1-\beta^2)(y - \alpha/(1-\beta))^2}{2\sigma^2} \right\}. \tag{2}$$

Hint: Equation (2) states that the invariant density $p(y)$ is normally distributed with mean $\alpha/(1-\beta)$ and variance $\sigma^2/(1-\beta^2)$. Clearly these quantities will make sense only if $|\beta| < 1$, which should give you a pretty good idea about how to answer question 2 above. To see why the invariant density should be a normal distribution, use lag operators and write equation (1) as:

$$(I - \beta L) y_{t+1} = \alpha + \epsilon_t \tag{3}$$

where $Iy_t = y_t$ is the identity operator and $Ly_t = y_{t-1}$ is the lag operator. If $|\beta| < 1$ then the operator $(I - \beta L)$ is invertible and we can write

$$y_{t+1} = [I - \beta L]^{-1}(\alpha + \epsilon_t). \tag{4}$$

By analogy to geometric series the inverse of $(I - \beta L)$ is given by an infinite geometric series:

$$y_{t+1} = \sum_{i=0}^{\infty} (\beta L)^i (\alpha + \epsilon_t) = \frac{\alpha}{(1-\beta)} + \sum_{i=0}^{\infty} \beta^i \epsilon_{t-i} \tag{5}$$

Since $\{\epsilon_t\}$ is an *IID* mean zero Gaussian process, it is clear from equation (5) that $\{y_t\}$ is also Gaussian process, but with mean $\alpha/(1-\beta)$ and variance given by:

$$\text{var}(y_{t+1}) = \text{var}\left[\sum_{i=0}^{\infty} \beta^t \epsilon_{t-i} \right] = \sum_{i=0}^{\infty} \beta^{2i} \sigma^2 = \frac{\sigma^2}{(1-\beta^2)}. \tag{6}$$

Since the time index $t+1$ is arbitrary, it follows from (5) and (6) that y_t normally distributed with mean $\alpha/(1-\beta)$ and variance $\sigma^2/(1-\beta^2)$ for all t. This suggests that y_t is in "stochastic steady state" and that the $N\Big(\alpha/(1-\beta), \sigma^2/(1-\beta^2)\Big)$ distribution for y_t is indeed the invariant distribution of this process. However to be rigorous, we need to verify that $p(y)$ defined in equation (2) satisfies the definition of an invariant density:

$$p(y') = \int_{-\infty}^{+\infty} p(y'|y)p(y)dy. \tag{7}$$

This requires some tedious calculations, but you should be able to verify that $p(y)$ in equation (2) does indeed satisfy equation (7) (in doing the integral in (7) you may need to make use of the fact that a moment generating function for $\tilde{X} \sim N(\mu, \sigma^2)$ is given by $M(t) = E\{\exp(t\tilde{X})\} = \exp(\mu t + t^2\sigma^2/2)$).

Another way to guess that equation (2) gives the invariant density for this case is to assume that stationarity holds, in which case

$$\sigma_y^2 \equiv \mathrm{var}(y_t) = \mathrm{var}(y_s) \tag{8}$$

for all s and t. Similarly

$$\mu_y \equiv E\{y_t\} = E\{y_s\} \tag{9}$$

for all s and t. Using these results and the fact that $\epsilon_t \sim N(0, \sigma^2)$ we have:

$$\mu_y = E\{y_{t+1}\} = \alpha + \beta E\{y_t\} = \alpha + \beta\mu_y = \frac{\alpha}{(1-\beta)}, \tag{10}$$

and

$$\sigma_y^2 = \mathrm{var}(y_{t+1}) = \beta^2 \mathrm{var}(y_t) + \sigma^2 = \beta^2\sigma_y^2 + \sigma^2 = \frac{\sigma^2}{(1-\beta^2)}. \tag{11}$$

So stochastic stationarity implies that y_t has mean $\mu_y = \alpha/(1-\beta)$ and variance $\sigma_y^2 = \sigma^2/(1-\beta^2)$. The fact that $\{y_t\}$ is Gaussian follows from the fact that y_t is equal to a sum of normally distributed random variables as you can see from the moving average representation for y_t given in equation (5).