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Economics 551b 37 Hillhouse, Rm. 27

PROBLEM SET 4
Classical Methods(II)
(Due: April 13, 1997)

QUESTION 1

(Convergence) Show that if X_N converges to X in quadratic mean, then X_N also converges to X in probability.

QUESTION 2

(Spectral density) Show the following equality of spectral density:

$$\begin{aligned} f(\omega) &= \frac{1}{2\pi}\sum_{h=-\infty}^{\infty}\gamma(h)e^{-i\omega h} \\ &= \frac{1}{2\pi}\left[\gamma(0)+2\sum_{h=1}^{\infty}\gamma(h)\cos(\omega h)\right]. \end{aligned}$$

QUESTION 3

(M-estimator) Suppose following linear model:

$$y_i = \beta^* x_i + \varepsilon_i$$

with $i = 1, \ldots, N$, β^* is 1×1 and $E[\varepsilon_i] = 0$.

M-estimator $\widehat{\beta}_N$ is defined by

$$\widehat{\beta}_N = \arg\min_{\beta\in\Theta}\frac{1}{N}\sum_{i=1}^{N}\varphi(y_i, x_i, \beta)$$

where $\varphi(y_i, x_i, \beta)$ is a criterion function.

Let the criterion function be

$$\varphi(y_i, x_i, \beta) = (y_i - \beta x_i)^4.$$

(1) Show the consistency of $\widehat{\beta}_N$.

(2) Derive the asymptotic distibution of $\widehat{\beta}_N$.

(3) Suppose the special case where $x_i = 1$ for all i and $\varepsilon_i \sim N(0, \sigma^2)$. Show that this M-estimator is less efficient (asymptotically) than the OLS estimator.

QUESTION 4

(Generalized Methods of Moment estimator) GMM estimator $\widehat{\theta}_N$ is defined by

$$\widehat{\theta}_N = \arg\min_{\theta\in\Theta}\widehat{g}_N(\theta)' W_N \widehat{g}_N(\theta)$$

where

$$\widehat{g}_N(\theta) = \frac{1}{N}\sum_{i=1}^{N} g(x_i, \theta)$$

with a moment condition

$$E[g(x, \theta^*)] = 0$$

where $\theta^*(k \times 1)$ is the true value of θ, and $g(x, \theta)$ is a $m \times 1$ functions of the data and parameters.

(1) Show the consistency of $\widehat{\theta}_N$.

(2) Derive the asymptotic distibution of $\widehat{\theta}_N$.

(3) What is optimal weight matrix W_N?

Hint for (2): assume the observations $\{x_1, \ldots, x_N\}$ are *IID* and show that

$$\sqrt{N}\left(\widehat{\theta}_N - \theta^*\right) \xrightarrow{d} N\left[(G'WG)^{-1}G'W\Omega WG(G'WG)^{-1}\right]$$

where

$$\begin{aligned} G &= E[\nabla_\theta g(x, \theta^*)] \\ \Omega &= E[g(x, \theta^*)g(x, \theta^*)'] \\ W &= \mathrm{p}\lim_{N\to\infty} W_N. \end{aligned}$$

QUESTION 5 Show that the maximum likelihood estimator $\hat{\theta}$ is *regular* in the sense that if $\{x_1, \ldots, x_N\}$ are *IID* draws from $f(x|\theta_N)$ where $\theta_N = \theta^* + \delta/\sqrt{N}$, then

$$\sqrt{N}[\hat{\theta} - \theta_N] \Longrightarrow N(0, I^{-1}(\theta^*))$$

for all $\delta \in R^k$. What conditions on θ^* and $f(x|\theta)$ did you need to assume to establish this result?