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Economics 551b 37 Hillhouse, Rm. 27

PROBLEM SET 1: SOLUTIONS

Bayesian Basics

QUESTION 1

Ans: $\widehat{\theta}_{\text{Optimal}} = m \equiv$median of $p(\theta|x)$.

Proof:

Let $\widehat{\theta}$ be any number such that $\widehat{\theta} > m$.

Then

$$\begin{aligned}
l(\theta,\widehat{\theta}) - l(\theta,m) &= \left|\widehat{\theta}-\theta\right| - |m-\theta| \\
&= \begin{cases} m-\widehat{\theta} & \text{if } \theta \geq \widehat{\theta} \\ \widehat{\theta}-2\theta+m & \text{if } m<\theta<\widehat{\theta} \\ \widehat{\theta}-m & \text{if } \theta \leq m \end{cases}
\end{aligned}$$

Since $(\widehat{\theta}-2\theta+m) > (m-\widehat{\theta})$ when $m<\theta<\widehat{\theta}$, the following result can be obtained:

$$\begin{aligned}
R(\theta,\widehat{\theta}) - R(\theta,m) &= E_{\theta|x}\left[l(\theta,\widehat{\theta}) - l(\theta,m)\right] \\
&= (m-\widehat{\theta})\int_{\widehat{\theta}}^{\infty} p(\theta|x)\,d\theta + \int_m^{\widehat{\theta}} (\widehat{\theta}-2\theta+m)p(\theta|x)\,d\theta \\
&\quad +(\widehat{\theta}-m)\int_{-\infty}^{m} p(\theta|x)\,d\theta \\
&\geq (m-\widehat{\theta})\int_{\widehat{\theta}}^{\infty} p(\theta|x)\,d\theta + (m-\widehat{\theta})\int_m^{\widehat{\theta}} p(\theta|x)\,d\theta \\
&\quad +(\widehat{\theta}-m)\int_{-\infty}^{m} p(\theta|x)\,d\theta \\
&= (\widehat{\theta}-m)\left[\int_{-\infty}^{m} p(\theta|x)\,d\theta - \int_m^{\infty} p(\theta|x)\,d\theta\right] = 0
\end{aligned}$$

Last equality holds since

$$\int_{-\infty}^{m} p(\theta|x)\,d\theta = \int_m^{\infty} p(\theta|x)\,d\theta = \frac{1}{2}$$

by the definition of median.

Inequality will be an equality if, and only if ,

$$\int_m^{\widehat{\theta}} p(\theta|x)\,d\theta = 0$$

This implies that $\widehat{\theta}$ is also a median.

A similar proof can also be given when $\widehat{\theta}$ is any number such that $\widehat{\theta} < m$.

Therefore

$$R(\theta, m) = \min_{-\infty<\widehat{\theta}<\infty} R(\theta, \widehat{\theta}).$$

QUESTION 2

Ans: $\widehat{\theta}_{\text{Optimal}} = c \equiv \frac{b}{a+b}$100th percentile of $p(\theta|x)$.

Proof:

Let $\widehat{\theta}$ be any number such that $\widehat{\theta} > c$.

Then

$$\begin{aligned} & l(\theta,\widehat{\theta}) - l(\theta,c) \\ = & \begin{cases} b\left(\theta-\widehat{\theta}\right) - b(\theta-c) & \text{if } \theta \geq \widehat{\theta} \\ a\left(\widehat{\theta}-\theta\right) - b(\theta-c) & \text{if } c < \theta < \widehat{\theta} \\ a\left(\widehat{\theta}-\theta\right) - a(c-\theta) & \text{if } \theta \leq c \end{cases} \\ = & \begin{cases} b\left(c-\widehat{\theta}\right) & \text{if } \theta \geq \widehat{\theta} \\ a\widehat{\theta} - (a+b)\theta + bc & \text{if } c < \theta < \widehat{\theta} \\ a\left(\widehat{\theta}-c\right) & \text{if } \theta \leq c \end{cases} \end{aligned}$$

Since $a\widehat{\theta} - (a+b)\theta + bc > b\left(c-\widehat{\theta}\right)$ when $c < \theta < \widehat{\theta}$, the following result can be obtained:

$$\begin{aligned} R(\theta,\widehat{\theta}) - R(\theta,c) &= E_{\theta|x}\left[l(\theta,\widehat{\theta}) - l(\theta,c)\right] \\ &= b\left(c-\widehat{\theta}\right)\int_{\widehat{\theta}}^{\infty} p(\theta|x)\,d\theta + \int_c^{\widehat{\theta}} (a\widehat{\theta} - (a+b)\theta + bc) p(\theta|x)\,d\theta \\ &\quad + a\left(\widehat{\theta}-c\right)\int_{-\infty}^{c} p(\theta|x)\,d\theta \\ &\geq b\left(c-\widehat{\theta}\right)\int_{\widehat{\theta}}^{\infty} p(\theta|x)\,d\theta + b\left(c-\widehat{\theta}\right)\int_c^{\widehat{\theta}} p(\theta|x)\,d\theta \\ &\quad + a\left(\widehat{\theta}-c\right)\int_{-\infty}^{c} p(\theta|x)\,d\theta \\ &= \left(\widehat{\theta}-c\right)\left[a\int_{-\infty}^{c} p(\theta|x)\,d\theta - b\int_c^{\infty} p(\theta|x)\,d\theta\right] \\ &= 0 \end{aligned}$$

Last equality holds since

$$\begin{aligned} \int_{-\infty}^{c} p(\theta|x)\,d\theta &= \frac{b}{a+b} \\ \int_c^{\infty} p(\theta|x)\,d\theta &= \frac{a}{a+b} \\ a\int_{-\infty}^{c} p(\theta|x)\,d\theta &= b\int_c^{\infty} p(\theta|x)\,d\theta = \frac{ab}{a+b} \end{aligned}$$

by the definition of $\frac{b}{a+b}100$th percentile.
Inequality will be an equality if, and only if ,

$$\int_c^{\widehat{\theta}} p\left(\theta|x\right) d\theta = 0$$

This implies that $\widehat{\theta}$ is also a $\frac{b}{a+b}100$th percentile.
A similar proof can also be given when $\widehat{\theta}$ is any number such that $\widehat{\theta} < c$.
Therefore

$$R(\theta, c) = \min_{-\infty<\widehat{\theta}<\infty} R(\theta, \widehat{\theta}).$$

QUESTION 3

From definition of posterior distribution,

$$p(\theta|x) = \frac{\Pi_{i=1}^N f(x_i|\theta)p(\theta)}{\int \Pi_{i=1}^N f(x_i|\theta)p(\theta)d\theta}.$$

$$\begin{aligned}
p(\theta|x) &\propto \Pi_{i=1}^N f(x_i|\theta)p(\theta) \\
&= \theta^{\sum x_i} \left(1-\theta\right)^{N-\sum x_i} \frac{1}{B(\alpha,\beta)} \theta^{\alpha-1} \left(1-\theta\right)^{\beta-1} \\
&= \frac{1}{B(\alpha,\beta)} \theta^{\alpha+\sum x_i -1} \left(1-\theta\right)^{\beta+N-\sum x_i -1}
\end{aligned}$$

Since $\int_0^1 \theta^{\alpha+\sum x_i-1} \left(1-\theta\right)^{\beta+N-\sum x_i-1} d\theta = B(\alpha+\sum x_i, \beta+N-\sum x_i)$,
Normalizer is

$$\frac{B(\alpha+\sum x_i, \beta+N-\sum x_i)}{B(\alpha,\beta)}.$$

Therefore,

$$p(\theta|x) = \frac{1}{B(\alpha+\sum x_i, \beta+N-\sum x_i)} \theta^{\alpha+\sum x_i-1} \left(1-\theta\right)^{\beta+N-\sum x_i-1}$$

which is also a Beta distribution $Beta(\alpha+\sum x_i, \beta+N-\sum x_i)$.

QUESTION 4

Sample GAUSS program:

```
/* BETAPLOT.GPR: program to plot posterior distributions from
** a Bernoulli-Beta conjugate family.
** John Rust, Yale University, January, 1997 */
library pgraph;
graphset;
 x=seqa(.001,.001,999); /* points on the x axis to plot prior distribution */
 a=1; /* set hyperparameters of Beta distribution */
 b=1;
 /* now compute beta distribution over the grid of x points */
 betaden=gamma(a+b)*(x^(a-1)).*((1-x)^(b-1))/(gamma(a)*gamma(b));
 truethet=.3;
 /* now compute beta distribution over the grid of x points */
 randat=rndu(500,1);
 n=10;
 d=(randat[1:n].<=truethet);
 a1=a+sumc(d);
 b1=b+n-sumc(d);
 postden1=gamma(a1+b1)*(x^(a1-1)).*((1-x)^(b1-1))/(gamma(a1)*gamma(b1));
 n=100;
 d=(randat[1:n].<=truethet);
 a1=a+sumc(d);
 b1=b+n-sumc(d);
 postden2=gamma(a1+b1)*(x^(a1-1)).*((1-x)^(b1-1))/(gamma(a1)*gamma(b1));
 n=150;
 d=(randat[1:n].<=truethet);
 a1=a+sumc(d);
 b1=b+n-sumc(d);
 postden3=gamma(a1+b1)*(x^(a1-1)).*((1-x)^(b1-1))/(gamma(a1)*gamma(b1));
 /* Now plot out the prior and posterior densities */
#IFUNIX /* this part is only necessary if you are running on
 a Unix system (like I am). It creates the window
 where the plot is printed. You don't need to worry
 about this code for Dos/Windows systems */
 let v = 100 100 640 480 0 0 1 6 15 0 0 2 2;
 wxy = WinOpenPQG(v,"XY Plot","XY");
 call WinSetActive(wxy);
#ENDIF
 _pdate="";
 title("Beta-Bernoulli Prior and Posterior Densities");
 ylabel("Density at x");
 xlabel("X value");
 /* the following commands put legends on the plot so you
 can tell which density curve is which */
 _plegctl=2~5~4.2~4.7;
```

```
_plegstr="Beta prior density (a=1, b=1)\000Beta posterior density (n=10)";
_plegstr=_plegstr$+"\000Beta posterior density (n=100)\000Beta posterior density (n=150)";
/* these commands specify the plotting bounds for the
x and y axes */
y=betaden~postden1~postden2~postden3;
xtics(0,1,.1,3);
ytics(0,maxc(maxc(y)),1,1);
/* finally this command actually does the plot */
xy(x,y);
#IFUNIX

call WinSetActive(1);

#ENDIF
```