Econ 551b Econometrics II
Problem Set 3

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QUESTION 1 Suppose you want to estimate the regression model

$$y = X\beta^* + \varepsilon$$

where X is $n \times K$, using the set of instruments Z that is $n \times J$.

When $J >= K$, the generalized IV estimator is give by:

$$\hat{\beta}_{IV} = (F'Z'X)^{-1}F'Z'y$$

where F is a $J \times K$ weighting matrix.

1. Find the optimal weighting matrix that minimizes variance of the estimator. The estimator using this optimal matrix is the two-stage least squares estimator.
2. Prove the consistency of the two-stage least squares estimator, and find its asymptotic distribution.
3. Show that when $J = K$, the two-stage least squares estimator reduces to the simple IV estimator:
$$\hat{\beta}_{IV} = (Z'X)^{-1}Z'y$$

QUESTION 2 Suppose regression model

$$y = X\beta^* + \varepsilon$$

(where y, ε are $N \times 1$, X is $N \times k$, and β^* is $k \times 1$) with normality assumption on disturbance, $\varepsilon \sim N(0, \Omega)$. Derive the maximum likelihood estimator (MLE) of β^* when Ω is known, and show that MLE is identical to the GLS estimator, $\beta_{GLS} = (X'\Omega^{-1}X)^{-1}X'\Omega^{-1}y$.

QUESTION 3 Show that the Cramer-Rao lower bound, I^{-1}, corresponds to the GLS covariance matrix, $\left(X'\Omega^{-1}X\right)^{-1}$, in Question 2.

QUESTION 4 Let $\{\tilde{X}_1, \ldots, \tilde{X}_N\}$ be *IID* draws from a multinomial distribution with density

$$f(x_i|\theta) = \prod_{k=1}^{K} \theta_k^{I\{x_i=k\}}, \tag{4}$$

where $\theta \in \Delta^{K-1}$, the $K-1$-dimensional simplex (i.e. the set of $\theta \in R^K$ satisfying $\theta_k \geq 0$ and $\sum_{k=1}^{K} \theta_k = 1$).

1. Derive the maximum likelihood estimator for θ.
2. Is the maximum likelihood estimator biased or unbiased?
3. Derive the covariance matrix for θ.
4. Derive the information matrix.
5. Is the maximum likelihood estimator efficient? If not, find a more efficient estimator for θ.

Hint: In parts 2 and 3 you might find the following matrix result useful: let the $(K-1)\times(K-1)$ matrix Σ be given by:

$$\Sigma = \begin{bmatrix} p_1(1-p_1) & -p_1p_2 & \cdots & -p_1p_{K-1} \\ -p_1p_2 & p_2(1-p_2) & \cdots & -p_2p_{K-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -p_1p_{K-1} & p_2p_{K-1} & \cdots & p_{K-1}(1-p_{K-1}) \end{bmatrix},$$

where the $\{p_1, \ldots, p_K\}$ are positive numbers satisfying

$$\sum_{k=1}^{K} p_k = 1.$$

Then verify that Σ is invertible with inverse given by:

$$[\Sigma]^{-1} = \begin{bmatrix} \frac{1}{p_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{p_2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \frac{1}{p_{K-2}} & 0 \\ 0 & \cdots & \cdots & \frac{1}{p_{K-1}} \end{bmatrix} + \frac{1}{(1-p_1-p_2-\cdots-p_{K-1})} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$