Econ 551b Econometrics II
Suggested Solution to Problem Set 2

Prof. John Rust, Hiu Man Chan

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Question 1

By Jordan decomposition:

$$\Sigma = XDX'$$

where D is a diagonal matrix with eigenvalues on the diagonal:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_K \end{bmatrix}$$

and X is orthogonal. Since Σ is positive definite, all the eigenvalues are positive. Therefore we can define

$$\begin{aligned}
D^{\frac{1}{2}} &= \begin{bmatrix} \sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_K} \end{bmatrix} \\
D^{-\frac{1}{2}} &= \begin{bmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\sqrt{\lambda_K}} \end{bmatrix} \\
P &= XD^{\frac{1}{2}}X' \\
P^{-1} &= XD^{-\frac{1}{2}}X'
\end{aligned}$$

We can then have the following decomposition:

$$\begin{aligned}
\Sigma &= PP' \\
\Sigma^{-1} &= P^{-1\prime}P^{-1}
\end{aligned}$$

Now we know $k \times 1$ vector $Z = P^{-1}Y \sim N(0, I)$, and each element of Z, $z_i \sim N(0,1)$ for all $i = 1, \ldots, k$. Therefore,

$$Y'\Sigma^{-1}Y = Y'P^{-1\prime}P^{-1}Y = Z'Z = \sum_{i=1}^{k} z_i^2 \sim \chi^2(k).$$

Question 2

Given vectors x_1, x_2, x_3,..., we can obtain orthogonal vectors v_1, v_2, v_3,... through the following method:

$$\begin{aligned} v_1 &= x_1 \\ v_j &= x_j - \sum_{i=1}^{j-1} \frac{< v_i, x_j >}{< v_i, v_i >} v_i \text{ for } j > 1; \end{aligned}$$

Given 1, x, x^2, x^3,..., note that

$$\begin{aligned} < 1, 1 > &= \int_{-1}^{1} 1 \times 1 dx = 2 \\ < 1, x > &= \int_{-1}^{1} 1 \times x dx = 0 \\ < 1, x^2 > &= \int_{-1}^{1} 1 \times x^2 dx = \frac{2}{3} \\ < 1, x^3 > &= \int_{-1}^{1} 1 \times x^3 dx = 0 \\ < x, x > &= \int_{-1}^{1} x \times x dx = \frac{2}{3} \\ < x, x^2 > &= \int_{-1}^{1} x \times x^2 dx = 0 \\ < x, x^3 > &= \int_{-1}^{1} x \times x^3 dx = \frac{2}{5} \\ < x^2 - \frac{1}{3}, x^3 > &= \int_{-1}^{1} (x^2 - \frac{1}{3}) \times x^3 dx = 0 \\ < x^2 - \frac{1}{3}, x^2 - \frac{1}{3} > &= \int_{-1}^{1} (x^2 - \frac{1}{3}) \times (x^2 - \frac{1}{3}) dx = \frac{8}{45} \\ < x^3 - \frac{3}{5}x, x^3 - \frac{3}{5}x > &= \int_{-1}^{1} (x^3 - \frac{3}{5}x) \times (x^3 - \frac{3}{5}x) dx = \frac{8}{175} \end{aligned}$$

Plugging into the formula, we can obtain the first four orthogonal vectors (the Legendre Polynomials):

$$\begin{aligned} P_0 &= 1 \\ P_1 &= x \\ P_2 &= x^2 - \frac{1}{3} \\ P_3 &= x^3 - \frac{3}{5}x \end{aligned}$$

Each of these is then normalized to give the final result:

$$\begin{aligned} \hat{P_0} &= \frac{P_0}{||P_0||} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

$$
\begin{aligned}
\hat{P}_1 &= \sqrt{\frac{3}{2}}x \\
\hat{P}_2 &= \sqrt{\frac{45}{8}}(x^2 - \frac{1}{3}) \\
\hat{P}_3 &= \sqrt{\frac{175}{8}}(x^3 - \frac{3}{5}x)
\end{aligned}
$$

Question 3

OLS estimates are obtained from the normal equations:

$$
\begin{bmatrix} x_1'x_1 & x_1'x_2 \\ x_2'x_1 & x_2'x_2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{bmatrix} = \begin{bmatrix} x_1'y \\ x_2'y \end{bmatrix}
$$

Solving the above system gives:

$$
\begin{aligned}
\hat{\beta}_1 &= (x_1'x_1)^{-1}x_1'(y - x_2\hat{\beta}_2) \\
\hat{\beta}_2 &= (x_2'M_{x_1}x_2)^{-1}x_2'M_{x_1}y
\end{aligned}
$$

where

$$
M_{x_1} = I - x_1(x_1'x_1)^{-1}x_1'
$$

Estimates from stepwise regression are given by:

$$
\begin{aligned}
\overline{\beta}_1 &= (x_1'x_1)^{-1}x_1'y \\
\overline{\beta}_2 &= (x_2'M_{x_1}x_2)^{-1}x_2'M_{x_1}y
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\hat{y} &= x_1\hat{\beta}_1 + x_2\hat{\beta}_2 \\
&= x_1(x_1'x_1)^{-1}x_1'(y - x_2\hat{\beta}_2) + x_2\hat{\beta}_2 \\
&= x_1(x_1'x_1)^{-1}x_1'y + (I - x_1(x_1'x_1)^{-1}x_1')x_2\hat{\beta}_2 \\
&= x_1\overline{\beta}_1 + M_{x_1}x_2\overline{\beta}_2 \\
&= x_1\overline{\beta}_1 + (x_2 - P(x_2|x_1))\overline{\beta}_2
\end{aligned}
$$

Stepwise regression can be useful when we care only about the predicted value of the dependent variable, but not the coefficients of the independent variables. Since the predicted value can be obtained by a series of estimations with a smaller number of independent variables, it is computationally less demanding.

Question 4

The solution can be found in Section 1.6 of Brockwell and Davis.