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Economics 551b 37 Hillhouse, Rm. 27

PROBLEM SET 1
Regression Basics

QUESTION 1 This question is designed to help you get a better appreciation for what the "Projection Theorem" is and how it relates to doing regression in practical situations.

1. Let $P(y|X)$ denote the projection of the vector y onto the subspace X. Define formally what $P(y|X)$ is.

2. State the Projection Theorem and give a geometrical interpretation of the "orthogonality" between the projection residual $\epsilon = y - P(y|X)$ and the subspace X onto which y is being projected.

3. If $y \in R^N$ and X is the subspace spanned by the $N \times K$ matrix of regressors (which I also denote by X) in a regression, show that
$$P(y|X) = X\hat{\beta} = X(X'X)^{-1}X'y. \tag{1}$$

4. Verify by a direct calculation that $\epsilon = y - P(y|X)$ is orthogonal to X for the closed form expression for $P(y|X)$ given above.

5. Prove that the projection operator satisfies:
$$P(P(y|X)|X) = P(y|X). \tag{2}$$
Interpret this condition geometrically, and relate it to the property of *idempotence* of the projection matrix $X(X'X)^{-1}X'$.

6. Prove the *Law of Iterated Projections* i.e. if X and Z are subspaces of a Hilbert space H and X is a subspace of Z, then for any $y \in H$ we have:
$$P(y|X) = P(P(y|Z)|X). \tag{3}$$

7. Use the Law of Iterated Projections to show that if you first regress y on the variables in the matrices X and Z:
$$y = X\beta_1 + Z\beta_2 + \epsilon \tag{4}$$
and then you use the fitted $\hat{y} = X\hat{\beta}_1 + Z\hat{\beta}_2$ as the dependent variable in the regression
$$\hat{y} = X\gamma + u \tag{5}$$
you will get the same numerical estimate of the estimated regression coefficient $\hat{\gamma}$ as if you regressed y on X
$$y = X\gamma + u \tag{6}$$
However is it generally the case that $\hat{\gamma} = \hat{\beta}_1$? If so, provide a proof, if not, provide a counterexample where $\hat{\gamma} \neq \hat{\beta}_1$?

8. Can you state conditions under which you can guarantee that $\hat{\beta}_1 = \hat{\gamma}$ in the two regression in part 4 above?

9. Let H be $L_2(\Omega, \mathcal{F}, \mu)$ be the classical L_2 space, i.e. the space of all random variables defined on the probability space $(\Omega, \mathcal{F}, \mu)$ that have finite variance, with inner product given by

$$\langle \tilde{X}, \tilde{Y} \rangle = \int X(\omega) Y(\omega) \mu(d\omega). \tag{7}$$

   Let X denote the subspace spanned by the K random variables $(\tilde{X}_1, \ldots, \tilde{X}_K)$. Find a formula for the projection of the random variable $\tilde{y}$ on X, $P(\tilde{y}|X)$, and provide an interpretation of what it means.

10. Let X be the space of all measurable functions of the random variables $(\tilde{X}_1, \ldots, \tilde{X}_K)$ that have finite variance. Show that this is a subspace of $L_2(\Omega, \mathcal{F}, \mu)$. Given any $\tilde{y} \in L_2(\Omega, \mathcal{F}, \mu)$, what is $P(\tilde{y}|X)$?

11. If instead of $L_2(\Omega, \mathcal{F}, \mu)$ we consider the space $H = R^N$, and if X is the space of all measurable functions of K vectors $(X_1, \ldots, X_K)$ in R^N, for any $y \in R^N$ what is $P(y|X)$?

QUESTION 2 Consider the "textbook" regression model:

$$y = X\beta^* + \epsilon$$

where X is regarded as a fixed (non-random) $N \times K$ matrix and the error vector ϵ is a random vector with a $N(0, \sigma^2 I)$ distribution, where 0 is an $N \times 1$ vector of 0's and I is an $N \times N$ identity matrix, and $\sigma^2 > 0$ is a constant.

1. Show that OLS is a *linear* estimator of β^*.

2. Show that an arbitrary linear estimator of β^* must have the form My for some matrix M. What are the dimensions of M?

3. What constraints must be placed on M to result in an unbiased estimator of β^*?

4. What is the matrix M for the OLS estimator? Show that for this choice of M the unbiasedness constraint that you derived above is satisfied.

5. Show that the variance-covariance matrix for a linear estimator of β^* is given by $\sigma^2 M'M$. Does this formula depend on M satisfying the restriction for unbiasedness, or will it hold even for unbiased estimators of β^*?

6. Derive the covariance matrix for $\hat{\beta}$, the OLS estimator.

7. Prove the *Gauss Markov Theorem,* i.e. show that the OLS estimator is the best, linear, unbiased estimator of β^*. **Hint:** for an alternative estimator of the form $\tilde{\beta} = My$ for some matrix M, write M as

$$M = X(x'X)^{-1}X' + C \tag{8}$$

   for some matrix C. Figure out what restrictions C needs to satisfy so that $\tilde{\beta}$ is an unbiased estimator, and then use this to compute the covariance matrix for $\tilde{\beta}$ and show that this exceeds the covariance matrix for $\hat{\beta}$ by a positive semi-definite matrix.