A Dynamic Model of Leap-Frogging Investments and Bertrand Price Competition

Fyodor Iskhakov, University Technology Sydney
John Rust, University of Maryland
Bertel Schjerning, University of Copenhagen

Conference on Dynamic Aspecits of Economic Decision Making, Copenhagen

June 17, 2010

Road Map for Talk

- Empirical Motivation: damage in a collusion case
- Formulation of the model
- Solving the “End Game”
- Solving the “Full Game”
- Related work and work in progress
- Implications for theoretical and empirical IO: endogenous coordination and a new interpretation for “price wars”
- Computational/empirical implications: the danger of imposing “symmetry” and the role of computational algorithms as “equilibrium selection mechanisms”

Road Map for Talk

- Empirical Motivation: damage in a collusion case
- Formulation of the model
- Solving the “End Game”
- Solving the “Full Game”
- Related work and work in progress
- Implications for theoretical and empirical IO: endogenous coordination and a new interpretation for “price wars”
- Computational/empirical implications: the danger of imposing “symmetry” and the role of computational algorithms as “equilibrium selection mechanisms”

Road Map for Talk

- Empirical Motivation: damage in a collusion case
- Formulation of the model
- Solving the “End Game”
- Solving the “Full Game”
- Related work and work in progress
- Implications for theoretical and empirical IO: endogenous coordination and a new interpretation for “price wars”
- Computational/empirical implications: the danger of imposing “symmetry” and the role of computational algorithms as “equilibrium selection mechanisms”

Road Map for Talk

- Empirical Motivation: damage in a collusion case
- Formulation of the model
- Solving the "End Game"
- Solving the "Full Game"
- Related work and work in progress
- Implications for theoretical and empirical IO: endogenous coordination and a new interpretation for "price wars"
- Computational/empirical implications: the danger of imposing "symmetry" and the role of computational algorithms as "equilibrium selection mechanisms"

Road Map for Talk

- Empirical Motivation: damage in a collusion case
- Formulation of the model
- Solving the "End Game"
- Solving the "Full Game"
- Related work and work in progress
- Implications for theoretical and empirical IO: endogenous coordination and a new interpretation for "price wars"
- Computational/empirical implications: the danger of imposing "symmetry" and the role of computational algorithms as "equilibrium selection mechanisms"

Road Map for Talk

- Empirical Motivation: damage in a collusion case
- Formulation of the model
- Solving the “End Game”
- Solving the “Full Game”
- Related work and work in progress
- Implications for theoretical and empirical IO: endogenous coordination and a new interpretation for “price wars”
- Computational/empirical implications: the danger of imposing “symmetry” and the role of computational algorithms as “equilibrium selection mechanisms”

Road Map for Talk

- Empirical Motivation: damage in a collusion case
- Formulation of the model
- Solving the “End Game”
- Solving the “Full Game”
- Related work and work in progress
- Implications for theoretical and empirical IO: endogenous coordination and a new interpretation for “price wars”
- Computational/empirical implications: the danger of imposing “symmetry” and the role of computational algorithms as “equilibrium selection mechanisms”

Estimating damage in a collusion case

- Economic expert in a civil damage suit
- Case involved a claim for damage by a corporation, C, against two of its key input suppliers, A and B
- The suppliers A and B are near-duopolists in the market for cardboard
- My position: in the absence of collusion, prices would have been those predicted by the Bertrand model
- I argued that a "price war" between A and B that occured just prior to the onset of collusion, was not caused by a breakdown in tacit collusion
- rather, the price war was caused by firm A's attempt to *leap frog* firm B to become the low-cost leader

Estimating damage in a collusion case

- Economic expert in a civil damage suit
- Case involved a claim for damage by a corporation, C, against two of its key input suppliers, A and B
- The suppliers A and B are near-duopolists in the market for cardboard
- My position: in the absence of collusion, prices would have been those predicted by the Bertrand model
- I argued that a "price war" between A and B that occured just prior to the onset of collusion, was not caused by a breakdown in tacit collusion
- rather, the price war was caused by firm A's attempt to *leap frog* firm B to become the low-cost leader

Estimating damage in a collusion case

- Economic expert in a civil damage suit
- Case involved a claim for damage by a corporation, C, against two of its key input suppliers, A and B
- The suppliers A and B are near-duopolists in the market for cardboard
- My position: in the absence of collusion, prices would have been those predicted by the Bertrand model
- I argued that a "price war" between A and B that occured just prior to the onset of collusion, was not caused by a breakdown in tacit collusion
- rather, the price war was caused by firm A's attempt to *leap frog* firm B to become the low-cost leader

Estimating damage in a collusion case

- Economic expert in a civil damage suit
- Case involved a claim for damage by a corporation, C, against two of its key input suppliers, A and B
- The suppliers A and B are near-duopolists in the market for cardboard
- My position: in the absence of collusion, prices would have been those predicted by the Bertrand model
- I argued that a "price war" between A and B that occured just prior to the onset of collusion, was not caused by a breakdown in tacit collusion
- rather, the price war was caused by firm A's attempt to *leap frog* firm B to become the low-cost leader

Estimating damage in a collusion case

- Economic expert in a civil damage suit
- Case involved a claim for damage by a corporation, C, against two of its key input suppliers, A and B
- The suppliers A and B are near-duopolists in the market for cardboard
- My position: in the absence of collusion, prices would have been those predicted by the Bertrand model
- I argued that a "price war" between A and B that occured just prior to the onset of collusion, was not caused by a breakdown in tacit collusion
- rather, the price war was caused by firm A's attempt to *leap frog* firm B to become the low-cost leader

Estimating damage in a collusion case

- Economic expert in a civil damage suit
- Case involved a claim for damage by a corporation, C, against two of its key input suppliers, A and B
- The suppliers A and B are near-duopolists in the market for cardboard
- My position: in the absence of collusion, prices would have been those predicted by the Bertrand model
- I argued that a “price war” between A and B that occured just prior to the onset of collusion, was not caused by a breakdown in tacit collusion
- rather, the price war was caused by firm A’s attempt to *leap frog* firm B to become the low-cost leader

Justification for Bertrand pricing

- cardboard is a highly standardized product
- the consumers of cardboard are firms that are highly rational and interested in buying inputs at least possible cost
- further, firms acquire these inputs via *tenders* that create strong incentives for Bertrand-like price cutting
- In the case, we lacked good data on *aggregate demand* for cardboard facing firms A and B before and after collusion
- but we did have good data on their *costs of production*
- cardboard is made on production lines with machinery that is well-approximated as constant returns to scale with constant marginal costs

Justification for Bertrand pricing

- cardboard is a highly standardized product
- the consumers of cardboard are firms that are highly rational and interested in buying inputs at least possible cost
- further, firms acquire these inputs via *tenders* that create strong incentives for Bertrand-like price cutting
- In the case, we lacked good data on *aggregate demand* for cardboard facing firms A and B before and after collusion
- but we did have good data on their *costs of production*
- cardboard is made on production lines with machinery that is well-approximated as constant returns to scale with constant marginal costs

Justification for Bertrand pricing

- cardboard is a highly standardized product
- the consumers of cardboard are firms that are highly rational and interested in buying inputs at least possible cost
- further, firms acquire these inputs via *tenders* that create strong incentives for Bertrand-like price cutting
- In the case, we lacked good data on *aggregate demand* for cardboard facing firms A and B before and after collusion
- but we did have good data on their *costs of production*
- cardboard is made on production lines with machinery that is well-approximated as constant returns to scale with constant marginal costs

Justification for Bertrand pricing

- cardboard is a highly standardized product
- the consumers of cardboard are firms that are highly rational and interested in buying inputs at least possible cost
- further, firms acquire these inputs via *tenders* that create strong incentives for Bertrand-like price cutting
- In the case, we lacked good data on *aggregate demand* for cardboard facing firms A and B before and after collusion
- but we did have good data on their *costs of production*
- cardboard is made on production lines with machinery that is well-approximated as constant returns to scale with constant marginal costs

Justification for Bertrand pricing

- cardboard is a highly standardized product
- the consumers of cardboard are firms that are highly rational and interested in buying inputs at least possible cost
- further, firms acquire these inputs via *tenders* that create strong incentives for Bertrand-like price cutting
- In the case, we lacked good data on *aggregate demand* for cardboard facing firms A and B before and after collusion
- but we did have good data on their *costs of production*
- cardboard is made on production lines with machinery that is well-approximated as constant returns to scale with constant marginal costs

Justification for Bertrand pricing

- cardboard is a highly standardized product
- the consumers of cardboard are firms that are highly rational and interested in buying inputs at least possible cost
- further, firms acquire these inputs via *tenders* that create strong incentives for Bertrand-like price cutting
- In the case, we lacked good data on *aggregate demand* for cardboard facing firms A and B before and after collusion
- but we did have good data on their *costs of production*
- cardboard is made on production lines with machinery that is well-approximated as constant returns to scale with constant marginal costs

A cardboard corrugator

Technological progress via cost-reducing investments

- in this industry, A and B do minimal amounts of R&D since there is limited scope for new product innovations to replace cardboard
- however the firms do spend considerable amounts on *cost reducing investments*
- these investments consist of building new plants or upgrading existing plants with the latest technology and machinery for producing cardboard
- rather than developing these machines themselves, A and B purchase these machines from other companies that specialize in doing the R&D and product development to develop the machines that produce cardboard at the least possible cost

Technological progress via cost-reducing investments

- in this industry, A and B do minimal amounts of R&D since there is limited scope for new product innovations to replace cardboard
- however the firms do spend considerable amounts on *cost reducing investments*
- these investments consist of building new plants or upgrading existing plants with the latest technology and machinery for producing cardboard
- rather than developing these machines themselves, A and B purchase these machines from other companies that specialize in doing the R&D and product development to develop the machines that produce cardboard at the least possible cost

Technological progress via cost-reducing investments

- in this industry, A and B do minimal amounts of R&D since there is limited scope for new product innovations to replace cardboard
- however the firms do spend considerable amounts on *cost reducing investments*
- these investments consist of building new plants or upgrading existing plants with the latest technology and machinery for producing cardboard
- rather than developing these machines themselves, A and B purchase these machines from other companies that specialize in doing the R&D and product development to develop the machines that produce cardboard at the least possible cost

Technological progress via cost-reducing investments

- in this industry, A and B do minimal amounts of R&D since there is limited scope for new product innovations to replace cardboard
- however the firms do spend considerable amounts on *cost reducing investments*
- these investments consist of building new plants or upgrading existing plants with the latest technology and machinery for producing cardboard
- rather than developing these machines themselves, A and B purchase these machines from other companies that specialize in doing the R&D and product development to develop the machines that produce cardboard at the least possible cost

Leapfrogging by firm A lead to a price war

- the proximate cause of the collusion between A and B was a price war in cardboard
- a key input to cardboard is *paper* and A had a severe cost disadvantage relative to B due to its outdated paper production plant, with machines that had not been replaced/upgraded in decades
- B, on the other hand, has aggressively invested in the latest and most cost-efficient technology and maintained a persistent edge as the low cost leader
- however A planned to invest in a new paper mill, enabling it to produce cardboard at substantially lower costs, thereby leap frogging B to become the low cost leader
- A started cutting prices *before* undertaking the investment to assure sufficient capacity utilization for the new plant

Leapfrogging by firm A lead to a price war

- the proximate cause of the collusion between A and B was a price war in cardboard
- a key input to cardboard is *paper* and A had a severe cost disadvantage relative to B due to its outdated paper production plant, with machines that had not been replaced/upgraded in decades
- B, on the other hand, has aggressively invested in the latest and most cost-efficient technology and maintained a persistent edge as the low cost leader
- however A planned to invest in a new paper mill, enabling it to produce cardboard at substantially lower costs, thereby leap frogging B to become the low cost leader
- A started cutting prices *before* undertaking the investment to assure sufficient capacity utilization for the new plant

Leapfrogging by firm A lead to a price war

- the proximate cause of the collusion between A and B was a price war in cardboard
- a key input to cardboard is *paper* and A had a severe cost disadvantage relative to B due to its outdated paper production plant, with machines that had not been replaced/upgraded in decades
- B, on the other hand, has aggressively invested in the latest and most cost-efficient technology and maintained a persistent edge as the low cost leader
- however A planned to invest in a new paper mill, enabling it to produce cardboard at substantially lower costs, thereby leap frogging B to become the low cost leader
- A started cutting prices *before* undertaking the investment to assure sufficient capacity utilization for the new plant

Leapfrogging by firm A lead to a price war

- the proximate cause of the collusion between A and B was a price war in cardboard
- a key input to cardboard is *paper* and A had a severe cost disadvantage relative to B due to its outdated paper production plant, with machines that had not been replaced/upgraded in decades
- B, on the other hand, has aggressively invested in the latest and most cost-efficient technology and maintained a persistent edge as the low cost leader
- however A planned to invest in a new paper mill, enabling it to produce cardboard at substantially lower costs, thereby leap frogging B to become the low cost leader
- A started cutting prices *before* undertaking the investment to assure sufficient capacity utilization for the new plant

Leapfrogging by firm A lead to a price war

- the proximate cause of the collusion between A and B was a price war in cardboard
- a key input to cardboard is *paper* and A had a severe cost disadvantage relative to B due to its outdated paper production plant, with machines that had not been replaced/upgraded in decades
- B, on the other hand, has aggressively invested in the latest and most cost-efficient technology and maintained a persistent edge as the low cost leader
- however A planned to invest in a new paper mill, enabling it to produce cardboard at substantially lower costs, thereby leap frogging B to become the low cost leader
- A started cutting prices *before* undertaking the investment to assure sufficient capacity utilization for the new plant

Are price wars evidence of tacit collusion?

- The economic experts defending A and B dismissed my Bertrand competition with leap frogging investments hypothesis as completely out of touch with reality
- They claim that there is a huge body of research and empirical work in IO that supports the theory of *tacit collusion* by repeatedly interacting duopolist
- In particular, the theory shows that the duopolists can achieve via tacit collusion the same discounted profits as they could via *explicit collusion.*
- There prices under the counterfactual are *unchanged* and the damage to C is *zero.*
- But if this is the case, and if tacit collusion is *legal,* why would A and B have had an incentive to engage is illegal explicit collusion?

Are price wars evidence of tacit collusion?

- The economic experts defending A and B dismissed my Bertrand competition with leap frogging investments hypothesis as completely out of touch with reality
- They claim that there is a huge body of research and empirical work in IO that supports the theory of *tacit collusion* by repeatedly interacting duopolist
- In particular, the theory shows that the duopolists can achieve via tacit collusion the same discounted profits as they could via *explicit collusion.*
- There prices under the counterfactual are *unchanged* and the damage to C is *zero.*
- But if this is the case, and if tacit collusion is *legal,* why would A and B have had an incentive to engage is illegal explicit collusion?

Are price wars evidence of tacit collusion?

- The economic experts defending A and B dismissed my Bertrand competition with leap frogging investments hypothesis as completely out of touch with reality
- They claim that there is a huge body of research and empirical work in IO that supports the theory of *tacit collusion* by repeatedly interacting duopolist
- In particular, the theory shows that the duopolists can achieve via tacit collusion the same discounted profits as they could via *explicit collusion.*
- There prices under the counterfactual are *unchanged* and the damage to C is *zero.*
- But if this is the case, and if tacit collusion is *legal,* why would A and B have had an incentive to engage is illegal explicit collusion?

Are price wars evidence of tacit collusion?

- The economic experts defending A and B dismissed my Bertrand competition with leap frogging investments hypothesis as completely out of touch with reality
- They claim that there is a huge body of research and empirical work in IO that supports the theory of *tacit collusion* by repeatedly interacting duopolist
- In particular, the theory shows that the duopolists can achieve via tacit collusion the same discounted profits as they could via *explicit collusion.*
- There prices under the counterfactual are *unchanged* and the damage to C is *zero.*
- But if this is the case, and if tacit collusion is *legal,* why would A and B have had an incentive to engage is illegal explicit collusion?

Are price wars evidence of tacit collusion?

- The economic experts defending A and B dismissed my Bertrand competition with leap frogging investments hypothesis as completely out of touch with reality
- They claim that there is a huge body of research and empirical work in IO that supports the theory of *tacit collusion* by repeatedly interacting duopolist
- In particular, the theory shows that the duopolists can achieve via tacit collusion the same discounted profits as they could via *explicit collusion.*
- There prices under the counterfactual are *unchanged* and the damage to C is *zero.*
- But if this is the case, and if tacit collusion is *legal,* why would A and B have had an incentive to engage is illegal explicit collusion?

Paucity of empirical support for tacit collusion

- Tacit collusion is hard to “observe” by the very fact that it is tacit
- We need good data on costs and demands to calculate what the cartel price would be
- Most of the empirical work on tacit collusion comes from laboratory experiments
- Hundreds of experiments done on tacit collusion have found that it is extremely difficult to “grow” tacit collusion in laboratory settings
- There are very few “field studies” that find evidence of tacit collusion outside of Breshnahan’s (1987) JIE paper, “Competition and Collusion in the American Automobile Industry: the 1955 Price War”

Paucity of empirical support for tacit collusion

- Tacit collusion is hard to “observe” by the very fact that it is tacit
- We need good data on costs and demands to calculate what the cartel price would be
- Most of the empirical work on tacit collusion comes from laboratory experiments
- Hundreds of experiments done on tacit collusion have found that it is extremely difficult to “grow” tacit collusion in laboratory settings
- There are very few “field studies” that find evidence of tacit collusion outside of Breshnahan’s (1987) JIE paper, “Competition and Collusion in the American Automobile Industry: the 1955 Price War”

Paucity of empirical support for tacit collusion

- Tacit collusion is hard to "observe" by the very fact that it is tacit
- We need good data on costs and demands to calculate what the cartel price would be
- Most of the empirical work on tacit collusion comes from laboratory experiments
- Hundreds of experiments done on tacit collusion have found that it is extremely difficult to "grow" tacit collusion in laboratory settings
- There are very few "field studies" that find evidence of tacit collusion outside of Breshnahan's (1987) JIE paper, "Competition and Collusion in the American Automobile Industry: the 1955 Price War"

Paucity of empirical support for tacit collusion

- Tacit collusion is hard to “observe” by the very fact that it is tacit
- We need good data on costs and demands to calculate what the cartel price would be
- Most of the empirical work on tacit collusion comes from laboratory experiments
- Hundreds of experiments done on tacit collusion have found that it is extremely difficult to “grow” tacit collusion in laboratory settings
- There are very few “field studies” that find evidence of tacit collusion outside of Breshnahan’s (1987) JIE paper, “Competition and Collusion in the American Automobile Industry: the 1955 Price War”

Paucity of empirical support for tacit collusion

- Tacit collusion is hard to “observe” by the very fact that it is tacit
- We need good data on costs and demands to calculate what the cartel price would be
- Most of the empirical work on tacit collusion comes from laboratory experiments
- Hundreds of experiments done on tacit collusion have found that it is extremely difficult to “grow” tacit collusion in laboratory settings
- There are very few “field studies” that find evidence of tacit collusion outside of Breshnahan’s (1987) JIE paper, “Competition and Collusion in the American Automobile Industry: the 1955 Price War”

Conclusions of meta-study of over 500 experiments

- Christoph Engel (2007) "Tacit Collusion The Neglected Experimental Evidence"
- Econometric meta-analysis of 510 laboratory experiments finds no systematic evidence supporting tacit collusion
- D. Engelmann and W. Müller (2008) "Collusion through price ceilings? A search for a focal point effect"
- "Note that the Folk Theorem (see for example Tirole, 1988) predicts that infinitely many prices can occur as outcomes of collusive equilibria in infinitely repeated games if the discount factor is sufficiently high. This suggests a coordination problem when firms attempt to collude." (p. 2)

Conclusions of meta-study of over 500 experiments

- Christoph Engel (2007) “Tacit Collusion The Neglected Experimental Evidence”
- Econometric meta-analysis of 510 laboratory experiments finds no systematic evidence supporting tacit collusion
- D. Engelmann and W. Müller (2008) “Collusion through price ceilings? A search for a focal point effect”
- “Note that the Folk Theorem (see for example Tirole, 1988) predicts that infinitely many prices can occur as outcomes of collusive equilibria in infinitely repeated games if the discount factor is sufficiently high. This suggests a coordination problem when firms attempt to collude.” (p. 2)

Conclusions of meta-study of over 500 experiments

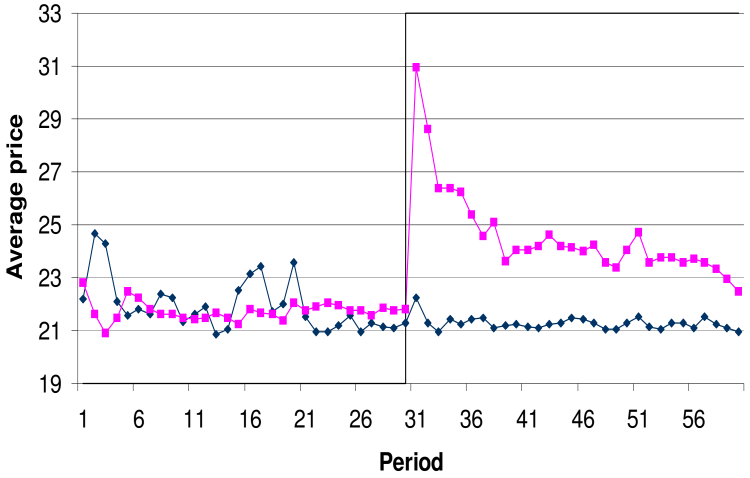
- Christoph Engel (2007) “Tacit Collusion The Neglected Experimental Evidence”
- Econometric meta-analysis of 510 laboratory experiments finds no systematic evidence supporting tacit collusion
- D. Engelmann and W. Müller (2008) “Collusion through price ceilings? A search for a focal point effect”
- “Note that the Folk Theorem (see for example Tirole, 1988) predicts that infinitely many prices can occur as outcomes of collusive equilibria in infinitely repeated games if the discount factor is sufficiently high. This suggests a coordination problem when firms attempt to collude.” (p. 2)

Conclusions of meta-study of over 500 experiments

- Christoph Engel (2007) “Tacit Collusion The Neglected Experimental Evidence”
- Econometric meta-analysis of 510 laboratory experiments finds no systematic evidence supporting tacit collusion
- D. Engelmann and W. Müller (2008) “Collusion through price ceilings? A search for a focal point effect”
- “Note that the Folk Theorem (see for example Tirole, 1988) predicts that infinitely many prices can occur as outcomes of collusive equilibria in infinitely repeated games if the discount factor is sufficiently high. This suggests a coordination problem when firms attempt to collude.” (p. 2)

Results of a laboratory duopoly

(Note: the Bertrand price is 21, the maximum cartel price is 48 and 28 is the price ceiling)



David Rapson's reanalysis of Breshnahan 1987

- David Rapson (2009) "Tacit Collusion in the 1950s Automobile Industry? Revisiting Bresnahan (1987)"
- "This paper reexamines the competitive landscape in the 1950s U.S. automobile industry, and tests the robustness of the famous result from Bresnahan (1987) that firms were engaged in tacit collusion."
- Rapson uses a random coefficients logit model allows for more realistic demand behavior, including a broad set of possible substitution patterns in characteristic space.
- This enables firms to engage in a more realistic set of potential actions, including intrabrand or intrafirm, multiproduct strategic pricing.

David Rapson's reanalysis of Breshnahan 1987

- David Rapson (2009) "Tacit Collusion in the 1950s Automobile Industry? Revisiting Bresnahan (1987)"
- "This paper reexamines the competitive landscape in the 1950s U.S. automobile industry, and tests the robustness of the famous result from Bresnahan (1987) that firms were engaged in tacit collusion."
- Rapson uses a random coefficients logit model allows for more realistic demand behavior, including a broad set of possible substitution patterns in characteristic space.
- This enables firms to engage in a more realistic set of potential actions, including intrabrand or intrafirm, multiproduct strategic pricing.

David Rapson's reanalysis of Breshnahan 1987

- David Rapson (2009) "Tacit Collusion in the 1950s Automobile Industry? Revisiting Bresnahan (1987)"
- "This paper reexamines the competitive landscape in the 1950s U.S. automobile industry, and tests the robustness of the famous result from Bresnahan (1987) that firms were engaged in tacit collusion."
- Rapson uses a random coefficients logit model allows for more realistic demand behavior, including a broad set of possible substitution patterns in characteristic space.
- This enables firms to engage in a more realistic set of potential actions, including intrabrand or intrafirm, multiproduct strategic pricing.

David Rapson's reanalysis of Breshnahan 1987

- David Rapson (2009) "Tacit Collusion in the 1950s Automobile Industry? Revisiting Bresnahan (1987)"
- "This paper reexamines the competitive landscape in the 1950s U.S. automobile industry, and tests the robustness of the famous result from Bresnahan (1987) that firms were engaged in tacit collusion."
- Rapson uses a random coefficients logit model allows for more realistic demand behavior, including a broad set of possible substitution patterns in characteristic space.
- This enables firms to engage in a more realistic set of potential actions, including intrabrand or intrafirm, multiproduct strategic pricing.

David Rapson's reanalysis of Breshnahan 1987

- "Relative to Bresnahan's framework, these improvements increase the likelihood that high price-cost markups will be attributed properly to either strategic oligopoly behavior or collusion." (p. 21).
- "*For no year can either of the forms of Bertrand competition be rejected in favor of tacit collusion.* This stands in contrast to Bresnahan's finding that firms were colluding in 1954 and 1956, with a price war in 1955."
- "These results accentuate the paucity of empirical evidence in favor of tacit collusion.
- Bresnahan's (1987) famous paper is one of the only studies that claim evidence of its occurrence."

David Rapson's reanalysis of Breshnahan 1987

- "Relative to Bresnahan's framework, these improvements increase the likelihood that high price-cost markups will be attributed properly to either strategic oligopoly behavior or collusion." (p. 21).
- "*For no year can either of the forms of Bertrand competition be rejected in favor of tacit collusion.* This stands in contrast to Bresnahan's finding that firms were colluding in 1954 and 1956, with a price war in 1955."
- "These results accentuate the paucity of empirical evidence in favor of tacit collusion.
- Bresnahan's (1987) famous paper is one of the only studies that claim evidence of its occurrence."

David Rapson's reanalysis of Breshnahan 1987

- "Relative to Bresnahan's framework, these improvements increase the likelihood that high price-cost markups will be attributed properly to either strategic oligopoly behavior or collusion." (p. 21).
- "*For no year can either of the forms of Bertrand competition be rejected in favor of tacit collusion.* This stands in contrast to Bresnahan's finding that firms were colluding in 1954 and 1956, with a price war in 1955."
- "These results accentuate the paucity of empirical evidence in favor of tacit collusion.
- Bresnahan's (1987) famous paper is one of the only studies that claim evidence of its occurrence."

David Rapson's reanalysis of Breshnahan 1987

- "Relative to Bresnahan's framework, these improvements increase the likelihood that high price-cost markups will be attributed properly to either strategic oligopoly behavior or collusion." (p. 21).
- "*For no year can either of the forms of Bertrand competition be rejected in favor of tacit collusion.* This stands in contrast to Bresnahan's finding that firms were colluding in 1954 and 1956, with a price war in 1955."
- "These results accentuate the paucity of empirical evidence in favor of tacit collusion.
- Bresnahan's (1987) famous paper is one of the only studies that claim evidence of its occurrence."

The Bertrand Investment Paradox

- why should Bertrand competitors undertake cost-reducing investments?
- suppose a pair of duopolists simultaneously invest in the state of the art low cost production technology with marginal cost c
- Bertrand price competition following these investments will lead to a price of $p = c$ and *zero profits for each firm*
- If each firm earns zero profits *ex post*, why would either have incentive to invest *ex ante*?

The Bertrand Investment Paradox

- why should Bertrand competitors undertake cost-reducing investments?
- suppose a pair of duopolists simultaneously invest in the state of the art low cost production technology with marginal cost c
- Bertrand price competition following these investments will lead to a price of $p = c$ and *zero profits for each firm*
- If each firm earns zero profits *ex post,* why would either have incentive to invest *ex ante*?

The Bertrand Investment Paradox

- why should Bertrand competitors undertake cost-reducing investments?
- suppose a pair of duopolists simultaneously invest in the state of the art low cost production technology with marginal cost c
- Bertrand price competition following these investments will lead to a price of $p = c$ and *zero profits for each firm*
- If each firm earns zero profits *ex post,* why would either have incentive to invest *ex ante*?

The Bertrand Investment Paradox

- why should Bertrand competitors undertake cost-reducing investments?
- suppose a pair of duopolists simultaneously invest in the state of the art low cost production technology with marginal cost c
- Bertrand price competition following these investments will lead to a price of $p = c$ and *zero profits for each firm*
- If each firm earns zero profits *ex post,* why would either have incentive to invest *ex ante*?

Leapfrogging as a solution to the Investment Paradox

- If the duopolist could somehow *coordinate* their investments, they might be able to avoid undertaking simultaneous cost-reducing investments, thereby “solving” the investment paradox
- One form of coordination is *leap frogging:* the firms invest in an alternating fashion, and avoid a “bad” equilibrium outcome of simultaneous investment
- Does leap frogging require explicit communication and collusion, or can it arise “endogenously” as an equilibrium outcome in a dynamic model of competition?

Leapfrogging as a solution to the Investment Paradox

- If the duopolist could somehow *coordinate* their investments, they might be able to avoid undertaking simultaneous cost-reducing investments, thereby "solving" the investment paradox
- One form of coordination is *leap frogging:* the firms invest in an alternating fashion, and avoid a "bad" equilibrium outcome of simultaneous investment
- Does leap frogging require explicit communication and collusion, or can it arise "endogenously" as an equilibrium outcome in a dynamic model of competition?

Leapfrogging as a solution to the Investment Paradox

- If the duopolist could somehow *coordinate* their investments, they might be able to avoid undertaking simultaneous cost-reducing investments, thereby “solving” the investment paradox
- One form of coordination is *leap frogging:* the firms invest in an alternating fashion, and avoid a “bad” equilibrium outcome of simultaneous investment
- Does leap frogging require explicit communication and collusion, or can it arise “endogenously” as an equilibrium outcome in a dynamic model of competition?

A model motivated by the collusion case

- time is discrete, and the horizon is infinite, $t = 1, 2, 3, \ldots$
- there are two firms selling homogenous goods, no entry or exit is allowed
- the firms face two decisions: 1) the price of their product, 2) whether to invest in the state of the art production technology that will allow it to produce at a marginal cost of c at an investment cost of $K(c)$.
- each firm maximizes expected discounted profits and discounts the future at the same discount factor $\beta \in (0, 1)$.
- the state of the art technology evolves as an exogenous Markov process $\{c_t\}$ with transition probability $\pi(c_{t+1}|c_t)$.

A model motivated by the collusion case

- time is discrete, and the horizon is infinite, $t = 1, 2, 3, \ldots$
- there are two firms selling homogenous goods, no entry or exit is allowed
- the firms face two decisions: 1) the price of their product,2) whether to invest in the state of the art production technology that will allow it to produce at a marginal cost of c at an investment cost of $K(c)$.
- each firm maximizes expected discounted profits and discounts the future at the same discount factor $\beta \in (0, 1)$.
- the state of the art technology evolves as an exogenous Markov process $\{c_t\}$ with transition probability $\pi(c_{t+1}|c_t)$.

A model motivated by the collusion case

- time is discrete, and the horizon is infinite, $t = 1, 2, 3, \ldots$
- there are two firms selling homogenous goods, no entry or exit is allowed
- the firms face two decisions: 1) the price of their product, 2) whether to invest in the state of the art production technology that will allow it to produce at a marginal cost of c at an investment cost of $K(c)$.
- each firm maximizes expected discounted profits and discounts the future at the same discount factor $\beta \in (0, 1)$.
- the state of the art technology evolves as an exogenous Markov process $\{c_t\}$ with transition probability $\pi(c_{t+1}|c_t)$.

A model motivated by the collusion case

- time is discrete, and the horizon is infinite, $t = 1, 2, 3, \ldots$
- there are two firms selling homogenous goods, no entry or exit is allowed
- the firms face two decisions: 1) the price of their product,2) whether to invest in the state of the art production technology that will allow it to produce at a marginal cost of c at an investment cost of $K(c)$.
- each firm maximizes expected discounted profits and discounts the future at the same discount factor $\beta \in (0, 1)$.
- the state of the art technology evolves as an exogenous Markov process $\{c_t\}$ with transition probability $\pi(c_{t+1}|c_t)$.

A model motivated by the collusion case

- time is discrete, and the horizon is infinite, $t = 1, 2, 3, \ldots$
- there are two firms selling homogenous goods, no entry or exit is allowed
- the firms face two decisions: 1) the price of their product,2) whether to invest in the state of the art production technology that will allow it to produce at a marginal cost of c at an investment cost of $K(c)$.
- each firm maximizes expected discounted profits and discounts the future at the same discount factor $\beta \in (0, 1)$.
- the state of the art technology evolves as an exogenous Markov process $\{c_t\}$ with transition probability $\pi(c_{t+1}|c_t)$.

A model motivated by the collusion case

- time is discrete, and the horizon is infinite, $t = 1, 2, 3, \ldots$
- there are two firms selling homogenous goods, no entry or exit is allowed
- the firms face two decisions: 1) the price of their product,2) whether to invest in the state of the art production technology that will allow it to produce at a marginal cost of c at an investment cost of $K(c)$.
- each firm maximizes expected discounted profits and discounts the future at the same discount factor $\beta \in (0, 1)$.
- the state of the art technology evolves as an exogenous Markov process $\{c_t\}$ with transition probability $\pi(c_{t+1}|c_t)$.

Timing of decisions

- at the start of each time t the firms observe the current state of the art c_t and make two simultaneous decisions.
- 1. the firms simultaneously set their prices p_1 and p_2
- 2. the firms make simultaneous decisions about whether or not to invest in the current state of the art production technology
- If either of the firms invest, there is a one period lag for *time to build*, before the new investment is operational
- thus, if firm i's marginal cost is $c_{i,t}$ under its legacy technology, and if it invests in the state of the art technology c_t at time t, then $c_{i,t+1} = c_t$, i.e. its marginal cost of production will be c_t *next period.*

Timing of decisions

- at the start of each time t the firms observe the current state of the art c_t and make two simultaneous decisions.
- 1. the firms simultaneously set their prices p_1 and p_2
- 2. the firms make simultaneous decisions about whether or not to invest in the current state of the art production technology
- If either of the firms invest, there is a one period lag for *time to build,* before the new investment is operational
- thus, if firm i's marginal cost is $c_{i,t}$ under its legacy technology, and if it invests in the state of the art technology c_t at time t, then $c_{i,t+1} = c_t$, i.e. its marginal cost of production will be c_t *next period.*

Timing of decisions

- at the start of each time t the firms observe the current state of the art c_t and make two simultaneous decisions.
- 1. the firms simultaneously set their prices p_1 and p_2
- 2. the firms make simultaneous decisions about whether or not to invest in the current state of the art production technology
- If either of the firms invest, there is a one period lag for *time to build*, before the new investment is operational
- thus, if firm i's marginal cost is $c_{i,t}$ under its legacy technology, and if it invests in the state of the art technology c_t at time t, then $c_{i,t+1} = c_t$, i.e. its marginal cost of production will be c_t *next period.*

Timing of decisions

- at the start of each time t the firms observe the current state of the art c_t and make two simultaneous decisions.
- 1. the firms simultaneously set their prices p_1 and p_2
- 2. the firms make simultaneous decisions about whether or not to invest in the current state of the art production technology
- If either of the firms invest, there is a one period lag for *time to build,* before the new investment is operational
- thus, if firm i's marginal cost is $c_{i,t}$ under its legacy technology, and if it invests in the state of the art technology c_t at time t, then $c_{i,t+1} = c_t$, i.e. its marginal cost of production will be c_t *next period.*

Timing of decisions

- at the start of each time t the firms observe the current state of the art c_t and make two simultaneous decisions.
- 1. the firms simultaneously set their prices p_1 and p_2
- 2. the firms make simultaneous decisions about whether or not to invest in the current state of the art production technology
- If either of the firms invest, there is a one period lag for *time to build,* before the new investment is operational
- thus, if firm i's marginal cost is $c_{i,t}$ under its legacy technology, and if it invests in the state of the art technology c_t at time t, then $c_{i,t+1} = c_t$, i.e. its marginal cost of production will be c_t *next period.*

A model of consumer demand

- there are a continuum of consumers who make *static* purchase decisions each period
- there are no consumer switching costs, or reputational or “brand loyalty” frictions
- the consumers choose at most one of the products each period
- we index the *type* of a consumer by a 2×1 vector $\tau = (\tau_1, \tau_2)$ and the utility the consumer gets from purchasing the product of firm i is $u_i = \sigma\tau_i - p_i$
- in some versions of the model we also allow for the possibility of an *outside good* with index 0
- then the consumer type is the 3×1 vector $\tau = (\tau_0, \tau_1, \tau_2)$ and the utility of the outside good is $u_0 = \sigma\tau_0 - p_0$, where p_0 is assumed to be an exogenous parameter.

A model of consumer demand

- there are a continuum of consumers who make *static* purchase decisions each period
- there are no consumer switching costs, or reputational or "brand loyalty" frictions
- the consumers choose at most one of the products each period
- we index the *type* of a consumer by a 2×1 vector $\tau = (\tau_1, \tau_2)$ and the utility the consumer gets from purchasing the product of firm i is $u_i = \sigma\tau_i - p_i$
- in some versions of the model we also allow for the possibility of an *outside good* with index 0
- then the consumer type is the 3×1 vector $\tau = (\tau_0, \tau_1, \tau_2)$ and the utility of the outside good is $u_0 = \sigma\tau_0 - p_0$, where p_0 is assumed to be an exogenous parameter.

A model of consumer demand

- there are a continuum of consumers who make *static* purchase decisions each period
- there are no consumer switching costs, or reputational or "brand loyalty" frictions
- the consumers choose at most one of the products each period
- we index the *type* of a consumer by a 2×1 vector $\tau = (\tau_1, \tau_2)$ and the utility the consumer gets from purchasing the product of firm i is $u_i = \sigma\tau_i - p_i$
- in some versions of the model we also allow for the possibility of an *outside good* with index 0
- then the consumer type is the 3×1 vector $\tau = (\tau_0, \tau_1, \tau_2)$ and the utility of the outside good is $u_0 = \sigma\tau_0 - p_0$, where p_0 is assumed to be an exogenous parameter.

A model of consumer demand

- there are a continuum of consumers who make *static* purchase decisions each period
- there are no consumer switching costs, or reputational or "brand loyalty" frictions
- the consumers choose at most one of the products each period
- we index the *type* of a consumer by a 2×1 vector $\tau = (\tau_1, \tau_2)$ and the utility the consumer gets from purchasing the product of firm i is $u_i = \sigma\tau_i - p_i$
- in some versions of the model we also allow for the possibility of an *outside good* with index 0
- then the consumer type is the 3×1 vector $\tau = (\tau_0, \tau_1, \tau_2)$ and the utility of the outside good is $u_0 = \sigma\tau_0 - p_0$, where p_0 is assumed to be an exogenous parameter.

A model of consumer demand

- there are a continuum of consumers who make *static* purchase decisions each period
- there are no consumer switching costs, or reputational or "brand loyalty" frictions
- the consumers choose at most one of the products each period
- we index the *type* of a consumer by a 2×1 vector $\tau = (\tau_1, \tau_2)$ and the utility the consumer gets from purchasing the product of firm i is $u_i = \sigma\tau_i - p_i$
- in some versions of the model we also allow for the possibility of an *outside good* with index 0
- then the consumer type is the 3×1 vector $\tau = (\tau_0, \tau_1, \tau_2)$ and the utility of the outside good is $u_0 = \sigma\tau_0 - p_0$, where p_0 is assumed to be an exogenous parameter.

A model of consumer demand

- there are a continuum of consumers who make *static* purchase decisions each period
- there are no consumer switching costs, or reputational or “brand loyalty” frictions
- the consumers choose at most one of the products each period
- we index the *type* of a consumer by a 2×1 vector $\tau = (\tau_1, \tau_2)$ and the utility the consumer gets from purchasing the product of firm i is $u_i = \sigma\tau_i - p_i$
- in some versions of the model we also allow for the possibility of an *outside good* with index 0
- then the consumer type is the 3×1 vector $\tau = (\tau_0, \tau_1, \tau_2)$ and the utility of the outside good is $u_0 = \sigma\tau_0 - p_0$, where p_0 is assumed to be an exogenous parameter.

The no outside good case

- if (τ_1, τ_2) are distributed as *IID* Type III extreme value across the population,
- then firm 1's market share, $\Pi_1(p_1, p_2)$ is given by

$$\Pi_1(p_1, p_2) = \frac{\exp\{-p_1/\sigma\}}{\exp\{-p_1/\sigma\} + \exp\{-p_2/\sigma\}}.$$

- The classic Bertrand model is a special case when $\sigma = 0$.

$$\Pi_1(p_1, p_2) = I\{p_1 \leq p_2\}.$$

$$p_1 = p_2 = p = \max[c_1, c_2].$$

The no outside good case

- if (τ_1, τ_2) are distributed as *IID* Type III extreme value across the population,
- then firm 1's market share, $\Pi_1(p_1, p_2)$ is given by

$$\Pi_1(p_1, p_2) = \frac{\exp\{-p_1/\sigma\}}{\exp\{-p_1/\sigma\} + \exp\{-p_2/\sigma\}}.$$

- The classic Bertrand model is a special case when $\sigma = 0$.

$$\Pi_1(p_1, p_2) = I\{p_1 \leq p_2\}.$$

$$p_1 = p_2 = p = \max[c_1, c_2].$$

The no outside good case

- if (τ_1, τ_2) are distributed as *IID* Type III extreme value across the population,
- then firm 1's market share, $\Pi_1(p_1, p_2)$ is given by

$$\Pi_1(p_1, p_2) = \frac{\exp\{-p_1/\sigma\}}{\exp\{-p_1/\sigma\} + \exp\{-p_2/\sigma\}}.$$

- The classic Bertrand model is a special case when $\sigma = 0$.

$$\Pi_1(p_1, p_2) = I\{p_1 \leq p_2\}.$$

$$p_1 = p_2 = p = \max[c_1, c_2].$$

The no outside good case

- if (τ_1, τ_2) are distributed as *IID* Type III extreme value across the population,
- then firm 1's market share, $\Pi_1(p_1, p_2)$ is given by

$$\Pi_1(p_1, p_2) = \frac{\exp\{-p_1/\sigma\}}{\exp\{-p_1/\sigma\} + \exp\{-p_2/\sigma\}}.$$

- The classic Bertrand model is a special case when $\sigma = 0$.

$$\Pi_1(p_1, p_2) = I\{p_1 \leq p_2\}.$$

$$p_1 = p_2 = p = \max[c_1, c_2].$$

Solution concept: Markov-perfect equilibrium

- the *state* of the game at time t is given by $(c_{1,t}, c_{2,t}, c_t)$, where $c_{i,t}$ is the (legacy) marginal cost of production of firm i. We have for each t, $c_{1,t} \geq c_t$ and $c_{2,t} \geq c_t$.
- the *state space* S is the subset of R^3 satisfying the inequalities given above.
- A *stationary Markovian strategy* consists of two pairs of functions $(p_i(c_1, c_2, c), \iota_i(c_1, c_2, c))$, $i = 1, 2$ where $p_i : S \rightarrow R$ is firm i's *pricing decision*, and $\iota_i : S \rightarrow \{0, 1\}$ is firm i's *investment decision*, where $\iota_i = 1$ denotes the decision to invest, and $\iota_i = 0$ is not to invest
- A *Markov-perfect equilibrium* is a pair of strategies (p_i, ι_i) that are mutual best responses for all states in S and at every time period $t = 1, 2, 3, \ldots$

Solution concept: Markov-perfect equilibrium

- the *state* of the game at time t is given by $(c_{1,t}, c_{2,t}, c_t)$, where $c_{i,t}$ is the (legacy) marginal cost of production of firm i. We have for each t, $c_{1,t} \geq c_t$ and $c_{2,t} \geq c_t$.
- the *state space* S is the subset of R^3 satisfying the inequalities given above.
- A *stationary Markovian strategy* consists of two pairs of functions $(p_i(c_1, c_2, c), \iota_i(c_1, c_2, c))$, $i = 1, 2$ where $p_i : S \rightarrow R$ is firm i's *pricing decision*, and $\iota_i : S \rightarrow \{0, 1\}$ is firm i's *investment decision,* where $\iota_i = 1$ denotes the decision to invest, and $\iota_i = 0$ is not to invest
- A *Markov-perfect equilibrium* is a pair of strategies (p_i, ι_i) that are mutual best responses for all states in S and at every time period $t = 1, 2, 3, \ldots$

Solution concept: Markov-perfect equilibrium

- the *state* of the game at time t is given by $(c_{1,t}, c_{2,t}, c_t)$, where $c_{i,t}$ is the (legacy) marginal cost of production of firm i. We have for each t, $c_{1,t} \geq c_t$ and $c_{2,t} \geq c_t$.
- the *state space* S is the subset of R^3 satisfying the inequalities given above.
- A *stationary Markovian strategy* consists of two pairs of functions $(p_i(c_1, c_2, c), \iota_i(c_1, c_2, c))$, $i = 1, 2$ where $p_i : S \rightarrow R$ is firm i's *pricing decision*, and $\iota_i : S \rightarrow \{0, 1\}$ is firm i's *investment decision,* where $\iota_i = 1$ denotes the decision to invest, and $\iota_i = 0$ is not to invest
- A *Markov-perfect equilibrium* is a pair of strategies (p_i, ι_i) that are mutual best responses for all states in S and at every time period $t = 1, 2, 3, \ldots$

Solution concept: Markov-perfect equilibrium

- the *state* of the game at time t is given by $(c_{1,t}, c_{2,t}, c_t)$, where $c_{i,t}$ is the (legacy) marginal cost of production of firm i. We have for each t, $c_{1,t} \geq c_t$ and $c_{2,t} \geq c_t$.
- the *state space* S is the subset of R^3 satisfying the inequalities given above.
- A *stationary Markovian strategy* consists of two pairs of functions $(p_i(c_1, c_2, c), \iota_i(c_1, c_2, c))$, $i = 1, 2$ where $p_i : S \to R$ is firm i's *pricing decision*, and $\iota_i : S \to \{0, 1\}$ is firm i's *investment decision,* where $\iota_i = 1$ denotes the decision to invest, and $\iota_i = 0$ is not to invest
- A *Markov-perfect equilibrium* is a pair of strategies (p_i, ι_i) that are mutual best responses for all states in S and at every time period $t = 1, 2, 3, \ldots$

A specification for exogenous technological process

- We assume the Markov process governing exogenous technological improvement, $\{c_t\}$, has the following form.
- If the current state of the art is c_t at time t, then with probability $p(c_t)$ an improvement in the state of the art occurs, and in this event c_{t+1} is a draw from a Beta distribution on the interval $[0, c_t]$. So we have

$$\pi(c'|c) = \begin{cases} p(c)B(c'|c) & \text{if } c' < c \\ 1 & \text{if } c' \geq c \end{cases}$$

where $B(c'|c)$ is a beta distribution on $[0, c]$.

A specification for exogenous technological process

- We assume the Markov process governing exogenous technological improvement, $\{c_t\}$, has the following form.
- If the current state of the art is c_t at time t, then with probability $p(c_t)$ an improvement in the state of the art occurs, and in this event c_{t+1} is a draw from a Beta distribution on the interval $[0, c_t]$. So we have

$$\pi(c'|c) = \begin{cases} p(c)B(c'|c) & \text{if } c' < c \\ 1 & \text{if } c' \geq c \end{cases}$$

where $B(c'|c)$ is a beta distribution on $[0, c]$.

A specification for exogenous technological process

- We assume the Markov process governing exogenous technological improvement, $\{c_t\}$, has the following form.
- If the current state of the art is c_t at time t, then with probability $p(c_t)$ an improvement in the state of the art occurs, and in this event c_{t+1} is a draw from a Beta distribution on the interval $[0, c_t]$. So we have

$$\pi(c'|c) = \begin{cases} p(c)B(c'|c) & \text{if } c' < c \\ 1 & \text{if } c' \geq c \end{cases}$$

where $B(c'|c)$ is a beta distribution on $[0, c]$.

A specification for exogenous technological process

- We assume the Markov process governing exogenous technological improvement, $\{c_t\}$, has the following form.
- If the current state of the art is c_t at time t, then with probability $p(c_t)$ an improvement in the state of the art occurs, and in this event c_{t+1} is a draw from a Beta distribution on the interval $[0, c_t]$. So we have

$$\pi(c'|c) = \begin{cases} p(c)B(c'|c) & \text{if } c' < c \\ 1 & \text{if } c' \geq c \end{cases}$$

where $B(c'|c)$ is a beta distribution on $[0, c]$.

A specification for exogenous technological process

- We assume the Markov process governing exogenous technological improvement, $\{c_t\}$, has the following form.
- If the current state of the art is c_t at time t, then with probability $p(c_t)$ an improvement in the state of the art occurs, and in this event c_{t+1} is a draw from a Beta distribution on the interval $[0, c_t]$. So we have

$$\pi(c'|c) = \begin{cases} p(c)B(c'|c) & \text{if } c' < c \\ 1 & \text{if } c' \geq c \end{cases}$$

where $B(c'|c)$ is a beta distribution on $[0, c]$.

A specification for exogenous technological process

- We assume the Markov process governing exogenous technological improvement, $\{c_t\}$, has the following form.
- If the current state of the art is c_t at time t, then with probability $p(c_t)$ an improvement in the state of the art occurs, and in this event c_{t+1} is a draw from a Beta distribution on the interval $[0, c_t]$. So we have

$$\pi(c'|c) = \begin{cases} p(c)B(c'|c) & \text{if } c' < c \\ 1 & \text{if } c' \geq c \end{cases}$$

where $B(c'|c)$ is a beta distribution on $[0, c]$.

Private shocks affecting investment decisions

- In each period t each firm incurs additive costs (benefits) from not investing and investing, respectively, given by $\epsilon_{i,t} \equiv \eta(\epsilon_{0,i,t}, \epsilon_{1,i,t})$, where $\{\epsilon_{i,t}\}$ are *IID* bivariate Type III extreme value processes that are contemporaneously independent over the two firms, and $\eta \geq 0$ is a scaling parameter.
- The presence of these privately observed shocks makes this a *dynamic game of incomplete information* when $\eta > 0$. The purpose is to *purify* equilibria in the sense of Harsanyi (1973) and Doraszelski and Escobar (2010).
- That is, all equilibrium strategies in the game of incomplete information are pure, and η serves as a *homotopy parameter* for *path following algorithms* for approximating mixed strategy equilibria in the limit when $\eta = 0$.

Private shocks affecting investment decisions

- In each period *t* each firm incurs additive costs (benefits) from not investing and investing, respectively, given by $\epsilon_{i,t} \equiv \eta(\epsilon_{0,i,t}, \epsilon_{1,i,t})$, where $\{\epsilon_{i,t}\}$ are *IID* bivariate Type III extreme value processes that are contemporaneously independent over the two firms, and $\eta \geq 0$ is a scaling parameter.
- The presence of these privately observed shocks makes this a *dynamic game of incomplete information* when $\eta > 0$. The purpose is to *purify* equilibria in the sense of Harsanyi (1973) and Doraszelski and Escobar (2010).
- That is, all equilibrium strategies in the game of incomplete information are pure, and η serves as a *homotopy parameter* for *path following algorithms* for approximating mixed strategy equilibria in the limit when $\eta = 0$.

Private shocks affecting investment decisions

- In each period *t* each firm incurs additive costs (benefits) from not investing and investing, respectively, given by $\epsilon_{i,t} \equiv \eta(\epsilon_{0,i,t}, \epsilon_{1,i,t})$, where $\{\epsilon_{i,t}\}$ are *IID* bivariate Type III extreme value processes that are contemporaneously independent over the two firms, and $\eta \geq 0$ is a scaling parameter.
- The presence of these privately observed shocks makes this a *dynamic game of incomplete information* when $\eta > 0$. The purpose is to *purify* equilibria in the sense of Harsanyi (1973) and Doraszelski and Escobar (2010).
- That is, all equilibrium strategies in the game of incomplete information are pure, and η serves as a *homotopy parameter* for *path following algorithms* for approximating mixed strategy equilibria in the limit when $\eta = 0$.

Symmetry of equlibria

- Let $V^i(c_1, c_2, c, \epsilon_0, \epsilon_1)$ be firm *i*'s value function in the publicly observed state (c_1, c_2, c) when its private cost shocks are (ϵ_0, ϵ_1).
- A frequently imposed restriction on Markov-perfect equilibria in dynamic games in IO is *symmetry*

$$V^1(c_1, c_2, c, \epsilon_0, \epsilon_1) = V^2(c_2, c_1, c, \epsilon_0, \epsilon_1).$$

  Thus, the *identities* of firms 1 and 2 do not matter, only the values of their production technologies matter for equilibrium strategies and payoffs.
- We show that the symmetry condition unwittingly knocks out most of the interesting equilibria. In particular, pure strategy equilibria, including equilibria involving leap frogging, will *not* satisfy this symmetry restriction.

Symmetry of equlibria

- Let $V^i(c_1, c_2, c, \epsilon_0, \epsilon_1)$ be firm *i*'s value function in the publicly observed state (c_1, c_2, c) when its private cost shocks are (ϵ_0, ϵ_1).
- A frequently imposed restriction on Markov-perfect equilibria in dynamic games in IO is *symmetry*

$$V^1(c_1, c_2, c, \epsilon_0, \epsilon_1) = V^2(c_2, c_1, c, \epsilon_0, \epsilon_1).$$

Thus, the *identities* of firms 1 and 2 do not matter, only the values of their production technologies matter for equilibrium strategies and payoffs.

- We show that the symmetry condition unwittingly knocks out most of the interesting equilibria. In particular, pure strategy equilibria, including equilibria involving leap frogging, will *not* satisfy this symmetry restriction.

Symmetry of equlibria

- Let $V^i(c_1, c_2, c, \epsilon_0, \epsilon_1)$ be firm *i*'s value function in the publicly observed state (c_1, c_2, c) when its private cost shocks are (ϵ_0, ϵ_1).
- A frequently imposed restriction on Markov-perfect equilibria in dynamic games in IO is *symmetry*

$$V^1(c_1, c_2, c, \epsilon_0, \epsilon_1) = V^2(c_2, c_1, c, \epsilon_0, \epsilon_1).$$

  Thus, the *identities* of firms 1 and 2 do not matter, only the values of their production technologies matter for equilibrium strategies and payoffs.
- We show that the symmetry condition unwittingly knocks out most of the interesting equilibria. In particular, pure strategy equilibria, including equilibria involving leap frogging, will *not* satisfy this symmetry restriction.

Symmetry of equlibria

- Let $V^i(c_1, c_2, c, \epsilon_0, \epsilon_1)$ be firm *i*'s value function in the publicly observed state (c_1, c_2, c) when its private cost shocks are (ϵ_0, ϵ_1).
- A frequently imposed restriction on Markov-perfect equilibria in dynamic games in IO is *symmetry*

$$V^1(c_1, c_2, c, \epsilon_0, \epsilon_1) = V^2(c_2, c_1, c, \epsilon_0, \epsilon_1).$$

  Thus, the *identities* of firms 1 and 2 do not matter, only the values of their production technologies matter for equilibrium strategies and payoffs.
- We show that the symmetry condition unwittingly knocks out most of the interesting equilibria. In particular, pure strategy equilibria, including equilibria involving leap frogging, will *not* satisfy this symmetry restriction.

Additive separable representation for V^i

- The independence and additive-separability of the $\{\epsilon_{i,t}\}$ shocks allows us to show that V^i has the following representation

$$V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i) = \max[v_0^i(c_1, c_2, c) + \eta\epsilon_0^i, v_1^i(c_1, c_2, c) + \eta\epsilon_1^i]$$

  where $v_0^i(c_1, c_2, c)$ is the value to firm i if it does not invest, and $v_1^i(c_1, c_2, c)$ is the value to firm i if invests.
- Let $r^1(c_1, c_2)$ be the expected profits that firm 1 earns in period t from the Bertrand-Nash pricing game
- When $\sigma = 0$, the classical Bertrand case, we have

$$r^1(c_1, c_2) = \begin{cases} 0 & \text{if } c_1 \geq c_2 \\ c_2 - c_1 & \text{if } c_1 < c_2 \end{cases}$$

Additive separable representation for V^i

- The independence and additive-separability of the $\{\epsilon_{i,t}\}$ shocks allows us to show that V^i has the following representation

$$V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i) = \max[v_0^i(c_1, c_2, c) + \eta\epsilon_0^i, v_1^i(c_1, c_2, c) + \eta\epsilon_1^i]$$

where $v_0^i(c_1, c_2, c)$ is the value to firm i if it does not invest, and $v_1^i(c_1, c_2, c)$ is the value to firm i if invests.

- Let $r^1(c_1, c_2)$ be the expected profits that firm 1 earns in period t from the Bertrand-Nash pricing game
- When $\sigma = 0$, the classical Bertrand case, we have

$$r^1(c_1, c_2) = \begin{cases} 0 & \text{if } c_1 \geq c_2 \\ c_2 - c_1 & \text{if } c_1 < c_2 \end{cases}$$

Additive separable representation for V^i

- The independence and additive-separability of the $\{\epsilon_{i,t}\}$ shocks allows us to show that V^i has the following representation

$$V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i) = \max[v_0^i(c_1, c_2, c) + \eta\epsilon_0^i, v_1^i(c_1, c_2, c) + \eta\epsilon_1^i]$$

  where $v_0^i(c_1, c_2, c)$ is the value to firm i if it does not invest, and $v_1^i(c_1, c_2, c)$ is the value to firm i if invests.
- Let $r^1(c_1, c_2)$ be the expected profits that firm 1 earns in period t from the Bertrand-Nash pricing game
- When $\sigma = 0$, the classical Bertrand case, we have

$$r^1(c_1, c_2) = \begin{cases} 0 & \text{if } c_1 \geq c_2 \\ c_2 - c_1 & \text{if } c_1 < c_2 \end{cases}$$

Additive separable representation for V^i

- The independence and additive-separability of the $\{\epsilon_{i,t}\}$ shocks allows us to show that V^i has the following representation

$$V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i) = \max[v_0^i(c_1, c_2, c) + \eta\epsilon_0^i, v_1^i(c_1, c_2, c) + \eta\epsilon_1^i]$$

  where $v_0^i(c_1, c_2, c)$ is the value to firm i if it does not invest, and $v_1^i(c_1, c_2, c)$ is the value to firm i if invests.
- Let $r^1(c_1, c_2)$ be the expected profits that firm 1 earns in period t from the Bertrand-Nash pricing game
- When $\sigma = 0$, the classical Bertrand case, we have

$$r^1(c_1, c_2) = \begin{cases} 0 & \text{if } c_1 \geq c_2 \\ c_2 - c_1 & \text{if } c_1 < c_2 \end{cases}$$

Additive separable representation for V^i

- The independence and additive-separability of the $\{\epsilon_{i,t}\}$ shocks allows us to show that V^i has the following representation

$$V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i) = \max[v_0^i(c_1, c_2, c) + \eta\epsilon_0^i, v_1^i(c_1, c_2, c) + \eta\epsilon_1^i]$$

  where $v_0^i(c_1, c_2, c)$ is the value to firm i if it does not invest, and $v_1^i(c_1, c_2, c)$ is the value to firm i if invests.
- Let $r^1(c_1, c_2)$ be the expected profits that firm 1 earns in period t from the Bertrand-Nash pricing game
- When $\sigma = 0$, the classical Bertrand case, we have

$$r^1(c_1, c_2) = \begin{cases} 0 & \text{if } c_1 \geq c_2 \\ c_2 - c_1 & \text{if } c_1 < c_2 \end{cases}$$

The Bellman equations

- The value of not investing is given by

$$v_0^i(c_1, c_2, c) = r^i(c_1, c_2) + \beta EV^i(c_1, c_2, c, 0)$$

where $EV^i(c_1, c_2, c, 0)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it does not invest this period.

- The value of investing is given by

$$v_1^i(c_1, c_2, c) = r^i(c_1, c_2) - K(c) + \beta EV^i(c_1, c_2, c, 1)$$

where $EV^i(c_1, c_2, c, 1)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it *does* invest this period.

The Bellman equations

- The value of not investing is given by

$$v_0^i(c_1, c_2, c) = r^i(c_1, c_2) + \beta EV^i(c_1, c_2, c, 0)$$

where $EV^i(c_1, c_2, c, 0)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it does not invest this period.

- The value of investing is given by

$$v_1^i(c_1, c_2, c) = r^i(c_1, c_2) - K(c) + \beta EV^i(c_1, c_2, c, 1)$$

where $EV^i(c_1, c_2, c, 1)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it *does* invest this period.

The Bellman equations

- The value of not investing is given by

$$v_0^i(c_1, c_2, c) = r^i(c_1, c_2) + \beta EV^i(c_1, c_2, c, 0)$$

where $EV^i(c_1, c_2, c, 0)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it does not invest this period.

- The value of investing is given by

$$v_1^i(c_1, c_2, c) = r^i(c_1, c_2) - K(c) + \beta EV^i(c_1, c_2, c, 1)$$

where $EV^i(c_1, c_2, c, 1)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it *does* invest this period.

The Bellman equations

- The value of not investing is given by

$$v_0^i(c_1, c_2, c) = r^i(c_1, c_2) + \beta EV^i(c_1, c_2, c, 0)$$

where $EV^i(c_1, c_2, c, 0)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it does not invest this period.

- The value of investing is given by

$$v_1^i(c_1, c_2, c) = r^i(c_1, c_2) - K(c) + \beta EV^i(c_1, c_2, c, 1)$$

where $EV^i(c_1, c_2, c, 1)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it *does* invest this period.

The Bellman equations

- The value of not investing is given by

$$v_0^i(c_1, c_2, c) = r^i(c_1, c_2) + \beta EV^i(c_1, c_2, c, 0)$$

where $EV^i(c_1, c_2, c, 0)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it does not invest this period.

- The value of investing is given by

$$v_1^i(c_1, c_2, c) = r^i(c_1, c_2) - K(c) + \beta EV^i(c_1, c_2, c, 1)$$

where $EV^i(c_1, c_2, c, 1)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it *does* invest this period.

The Bellman equations

- The value of not investing is given by

$$v_0^i(c_1, c_2, c) = r^i(c_1, c_2) + \beta EV^i(c_1, c_2, c, 0)$$

  where $EV^i(c_1, c_2, c, 0)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it does not invest this period.

- The value of investing is given by

$$v_1^i(c_1, c_2, c) = r^i(c_1, c_2) - K(c) + \beta EV^i(c_1, c_2, c, 1)$$

  where $EV^i(c_1, c_2, c, 1)$ is firm i's expectation of its next period value function $V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i)$ given that it *does* invest this period.

Representation of the *EV* functions

- Using the "max-stability" property of extreme-value random variables, we get the following representation

$$\int_{\epsilon_0^i}\int_{\epsilon_1^i} V^i(c_1,c_2,c,\epsilon_0^i,\epsilon_1^i)q(\epsilon_0^i)q(\epsilon_1^i)d\epsilon_1^i d\epsilon_0^i =$$
$$\eta \log\left[\exp\{v_0^i(c_1,c_2,c)/\eta\} + \exp\{v_1^i(c_1,c_2,c)/\eta\}\right]$$

Define the function $\phi(v_0^i(c_1,c_2,c), v_1^i(c_1,c_2,c))$ as

$$\phi(v_0^i(c_1,c_2,c), v_1^i(c_1,c_2,c)) =$$
$$\eta \log\left[\exp\{v_0^i(c_1,c_2,c)/\eta\} + \exp\{v_1^i(c_1,c_2,c)/\eta\}\right].$$

We call this the log-sum or the "smoothed max" function.

Representation of the *EV* functions

- Using the "max-stability" property of extreme-value random variables, we get the following representation

$$\int_{\epsilon_0^i}\int_{\epsilon_1^i} V^i(c_1, c_2, c, \epsilon_0^i, \epsilon_1^i) q(\epsilon_0^i) q(\epsilon_1^i) d\epsilon_1^i d\epsilon_0^i =$$

$$\eta \log \left[\exp\{v_0^i(c_1, c_2, c)/\eta\} + \exp\{v_1^i(c_1, c_2, c)/\eta\} \right]$$

Define the function $\phi(v_0^i(c_1, c_2, c), v_1^i(c_1, c_2, c))$ as

$$\phi(v_0^i(c_1, c_2, c), v_1^i(c_1, c_2, c)) \equiv$$

$$\eta \log \left[\exp\{v_0^i(c_1, c_2, c)/\eta\} + \exp\{v_1^i(c_1, c_2, c)/\eta\} \right].$$

We call this the log-sum or the "smoothed max" function.

Representation of the *EV* functions

- Using the “max-stability” property of extreme-value random variables, we get the following representation

$$\int_{\epsilon_0^i}\int_{\epsilon_1^i} V^i(c_1,c_2,c,\epsilon_0^i,\epsilon_1^i)q(\epsilon_0^i)q(\epsilon_1^i)d\epsilon_1^i d\epsilon_0^i =$$
$$\eta \log\left[\exp\{v_0^i(c_1,c_2,c)/\eta\}+\exp\{v_1^i(c_1,c_2,c)/\eta\}\right]$$

Define the function $\phi(v_0^i(c_1,c_2,c),v_1^i(c_1,c_2,c))$ as

$$\phi(v_0^i(c_1,c_2,c),v_1^i(c_1,c_2,c)) \equiv$$
$$\eta \log\left[\exp\{v_0^i(c_1,c_2,c)/\eta\}+\exp\{v_1^i(c_1,c_2,c)/\eta\}\right].$$

We call this the log-sum or the “smoothed max” function.

Representation of the *EV* functions

- Using the "max-stability" property of extreme-value random variables, we get the following representation

$$\int_{\epsilon_0^i}\int_{\epsilon_1^i} V^i(c_1,c_2,c,\epsilon_0^i,\epsilon_1^i)q(\epsilon_0^i)q(\epsilon_1^i)d\epsilon_1^i d\epsilon_0^i =$$
$$\eta \log\left[\exp\{v_0^i(c_1,c_2,c)/\eta\} + \exp\{v_1^i(c_1,c_2,c)/\eta\}\right]$$

Define the function $\phi(v_0^i(c_1,c_2,c), v_1^i(c_1,c_2,c))$ as

$$\phi(v_0^i(c_1,c_2,c), v_1^i(c_1,c_2,c)) \equiv$$
$$\eta \log\left[\exp\{v_0^i(c_1,c_2,c)/\eta\} + \exp\{v_1^i(c_1,c_2,c)/\eta\}\right].$$

We call this the log-sum or the "smoothed max" function.

Representation of the *EV* functions

- Using the "max-stability" property of extreme-value random variables, we get the following representation

$$\int_{\epsilon_0^i}\int_{\epsilon_1^i} V^i(c_1,c_2,c,\epsilon_0^i,\epsilon_1^i)q(\epsilon_0^i)q(\epsilon_1^i)d\epsilon_1^i d\epsilon_0^i =$$
$$\eta \log\left[\exp\{v_0^i(c_1,c_2,c)/\eta\} + \exp\{v_1^i(c_1,c_2,c)/\eta\}\right]$$

Define the function $\phi(v_0^i(c_1,c_2,c), v_1^i(c_1,c_2,c))$ as

$$\phi(v_0^i(c_1,c_2,c), v_1^i(c_1,c_2,c)) \equiv$$
$$\eta \log\left[\exp\{v_0^i(c_1,c_2,c)/\eta\} + \exp\{v_1^i(c_1,c_2,c)/\eta\}\right].$$

We call this the log-sum or the "smoothed max" function.

Log-sum formula is the "smoothed-max" function

- The ϕ function is sometimes referred to as the "log-sum" or "smoothed max" function since we have

$$\lim_{\eta \to 0} \phi(v_0, v_1) = \max\left[v_0, v_1\right].$$

Further, for any $\eta > 0$ we have $\phi(v_0, v_1) > \max[v_0, v_1]$.

- Firm 2's perception of firm 1's probability of investing is given by

$$P_1^1(c_1, c_2, c) = \frac{\exp\{v_1^1(c_1, c_2, c)/\eta\}}{\exp\{v_1^1(c_1, c_2, c)/\eta\} + \exp\{v_0^1(c_1, c_2, c)/\eta\}}$$

As $\eta \to 0$ we have
$P_1^1(c_1, c_2, c) \to I\{v_1^1(c_1, c_2, c) \geq v_0^1(c_1, c_2, c)\}$.

Log-sum formula is the "smoothed-max" function

- The ϕ function is sometimes referred to as the "log-sum" or "smoothed max" function since we have

$$\lim_{\eta \to 0} \phi(v_0, v_1) = \max[v_0, v_1].$$

Further, for any $\eta > 0$ we have $\phi(v_0, v_1) > \max[v_0, v_1]$.

- Firm 2's perception of firm 1's probability of investing is given by

$$P_1^1(c_1, c_2, c) = \frac{\exp\{v_1^1(c_1, c_2, c)/\eta\}}{\exp\{v_1^1(c_1, c_2, c)/\eta\} + \exp\{v_0^1(c_1, c_2, c)/\eta\}}$$

As $\eta \to 0$ we have
$P_1^1(c_1, c_2, c) \to I\{v_1^1(c_1, c_2, c) \geq v_0^1(c_1, c_2, c)\}$.

Log-sum formula is the “smoothed-max” function

- The ϕ function is sometimes referred to as the “log-sum” or “smoothed max” function since we have

$$\lim_{\eta \to 0} \phi(v_0, v_1) = \max\left[v_0, v_1\right].$$

  Further, for any $\eta > 0$ we have $\phi(v_0, v_1) > \max[v_0, v_1]$.

- Firm 2’s perception of firm 1’s probability of investing is given by

$$P_1^1(c_1, c_2, c) = \frac{\exp\{v_1^1(c_1, c_2, c)/\eta\}}{\exp\{v_1^1(c_1, c_2, c)/\eta\} + \exp\{v_0^1(c_1, c_2, c)/\eta\}}$$

  As $\eta \to 0$ we have $P_1^1(c_1, c_2, c) \to I\{v_1^1(c_1, c_2, c) \geq v_0^1(c_1, c_2, c)\}$.

Log-sum formula is the “smoothed-max” function

- The ϕ function is sometimes referred to as the “log-sum” or “smoothed max” function since we have

$$\lim_{\eta \to 0} \phi(v_0, v_1) = \max\left[v_0, v_1\right].$$

Further, for any $\eta > 0$ we have $\phi(v_0, v_1) > \max[v_0, v_1]$.

- Firm 2’s perception of firm 1’s probability of investing is given by

$$P_1^1(c_1, c_2, c) = \frac{\exp\{v_1^1(c_1, c_2, c)/\eta\}}{\exp\{v_1^1(c_1, c_2, c)/\eta\} + \exp\{v_0^1(c_1, c_2, c)/\eta\}}$$

As $\eta \to 0$ we have
$P_1^1(c_1, c_2, c) \to I\{v_1^1(c_1, c_2, c) \geq v_0^1(c_1, c_2, c)\}$.

Log-sum formula is the "smoothed-max" function

- The ϕ function is sometimes referred to as the "log-sum" or "smoothed max" function since we have

$$\lim_{\eta \to 0} \phi(v_0, v_1) = \max\left[v_0, v_1\right].$$

Further, for any $\eta > 0$ we have $\phi(v_0, v_1) > \max[v_0, v_1]$.

- Firm 2's perception of firm 1's probability of investing is given by

$$P_1^1(c_1, c_2, c) = \frac{\exp\{v_1^1(c_1, c_2, c)/\eta\}}{\exp\{v_1^1(c_1, c_2, c)/\eta\} + \exp\{v_0^1(c_1, c_2, c)/\eta\}}$$

As $\eta \to 0$ we have
$P_1^1(c_1, c_2, c) \to I\{v_1^1(c_1, c_2, c) \geq v_0^1(c_1, c_2, c)\}$.

Log-sum formula is the “smoothed-max” function

- The ϕ function is sometimes referred to as the “log-sum” or “smoothed max” function since we have

$$\lim_{\eta \to 0} \phi(v_0, v_1) = \max\left[v_0, v_1\right].$$

  Further, for any $\eta > 0$ we have $\phi(v_0, v_1) > \max[v_0, v_1]$.

- Firm 2’s perception of firm 1’s probability of investing is given by

$$P_1^1(c_1, c_2, c) = \frac{\exp\{v_1^1(c_1, c_2, c)/\eta\}}{\exp\{v_1^1(c_1, c_2, c)/\eta\} + \exp\{v_0^1(c_1, c_2, c)/\eta\}}$$

  As $\eta \to 0$ we have $P_1^1(c_1, c_2, c) \to I\{v_1^1(c_1, c_2, c) \geq v_0^1(c_1, c_2, c)\}$.

Representation of the EV^i functions, continued.

- Using ϕ we can write

$$EV^1(c_1,c_2,c,0) = \int_0^c \Big[P_1^2(c_1,c_2,c)H^1(c_1,c,c')+ (1-P_1^2(c_1,c_2,c))H^1(c_1,c_2,c')\Big]\pi(dc'|c)$$

$$EV^1(c_1,c_2,c,1) = \int_0^c \Big[P_1^2(c_1,c_2,c)H^1(c,c,c')+ (1-P_1^2(c_1,c_2,c))H^1(c,c_2,c')\Big]\pi(dc'|c)$$

where H^1 is given by

$$H^1(c_1,c_2,c) = \phi(v_0^1(c_1,c_2,c), v_1^1(c_1,c_2,c)).$$

Representation of the EV^i functions, continued.

- Using ϕ we can write

$$EV^1(c_1,c_2,c,0) = \int_0^c \Big[P_1^2(c_1,c_2,c)H^1(c_1,c,c')+ (1-P_1^2(c_1,c_2,c))H^1(c_1,c_2,c')\Big]\,\pi(dc'|c)$$

$$EV^1(c_1,c_2,c,1) = \int_0^c \Big[P_1^2(c_1,c_2,c)H^1(c,c,c')+ (1-P_1^2(c_1,c_2,c))H^1(c,c_2,c')\Big]\,\pi(dc'|c)$$

where H^1 is given by

$$H^1(c_1,c_2,c) = \phi(v_0^1(c_1,c_2,c), v_1^1(c_1,c_2,c)).$$

Representation of the EV^i functions, continued.

- Using ϕ we can write

$$EV^1(c_1,c_2,c,0) = \int_0^c \Big[P_1^2(c_1,c_2,c)H^1(c_1,c,c')+ (1-P_1^2(c_1,c_2,c))H^1(c_1,c_2,c')\Big]\pi(dc'|c)$$

$$EV^1(c_1,c_2,c,1) = \int_0^c \Big[P_1^2(c_1,c_2,c)H^1(c,c,c')+ (1-P_1^2(c_1,c_2,c))H^1(c,c_2,c')\Big]\pi(dc'|c)$$

where H^1 is given by

$$H^1(c_1,c_2,c) = \phi(v_0^1(c_1,c_2,c), v_1^1(c_1,c_2,c)).$$

Bellman equation for firm 1

- Using the representation for the EV^i functions above, we can write a system of functional equations for (v_0^1, v_1^1)

$$v_0^1(c_1,c_2,c) = r^1(c_1,c_2) + \beta \int_0^c \Big[P_1^2(c_1,c_2,c)\phi(v_0^1(c_1,c,c'),v_1^1(c_1,c,c')) (1-P_1^2(c_1,c_2,c))\phi(v_0^1(c_1,c_2,c'),v_1^1(c_1,c_2,c'))\Big]\pi(dc'|c).$$

$$v_1^1(c_1,c_2,c) = r^1(c_1,c_2) - K(c) + \beta \int_0^c \Big[P_1^2(c_1,c_2,c)\phi(v_0^1(c,c,c'),v_1^1(c,c,c')) (1-P_1^2(c_1,c_2,c))\phi(v_0^1(c,c_2,c'),v_1^1(c,c_2,c'))\Big]\pi(dc'|c).$$

Bellman equation for firm 1

- Using the representation for the EV^i functions above, we can write a system of functional equations for (v_0^1, v_1^1)

$$v_0^1(c_1, c_2, c) = r^1(c_1, c_2) +$$
$$\beta \int_0^c \Big[P_1^2(c_1, c_2, c)\phi(v_0^1(c_1, c, c'), v_1^1(c_1, c, c'))$$
$$(1 - P_1^2(c_1, c_2, c))\phi(v_0^1(c_1, c_2, c'), v_1^1(c_1, c_2, c'))\Big] \pi(dc'|c).$$
$$v_1^1(c_1, c_2, c) = r^1(c_1, c_2) - K(c) +$$
$$\beta \int_0^c \Big[P_1^2(c_1, c_2, c)\phi(v_0^1(c, c, c'), v_1^1(c, c, c'))$$
$$(1 - P_1^2(c_1, c_2, c))\phi(v_0^1(c, c_2, c'), v_1^1(c, c_2, c'))\Big] \pi(dc'|c).$$

Bellman equation for firm 2

- Using the representation for the EV^i functions above, we can write a system of functional equations for (v_0^2, v_1^2)

$$v_0^2(c_1,c_2,c) = r^1(c_2,c_1) + \beta \int_0^c \Big[P_1^1(c_1,c_2,c)\phi(v_0^2(c,c_2,c'), v_1^2(c,c_2,c')) \\ (1-P_1^1(c_1,c_2,c))\phi(v_0^2(c_1,c_2,c'), v_1^2(c_1,c_2,c')) \Big] \pi(dc'|c).$$

$$v_1^2(c_1,c_2,c) = r^1(c_2,c_1) - K(c) + \beta \int_0^c \Big[P_1^1(c_1,c_2,c)\phi(v_0^2(c,c,c'), v_1^2(c,c,c')) \\ (1-P_1^1(c_1,c_2,c))\phi(v_0^2(c_1,c,c'), v_1^2(c_1,c,c')) \Big] \pi(dc'|c).$$

Bellman equation for firm 2

- Using the representation for the EV^i functions above, we can write a system of functional equations for (v_0^2, v_1^2)

$$v_0^2(c_1, c_2, c) = r^1(c_2, c_1) + \beta \int_0^c \Big[P_1^1(c_1, c_2, c)\phi(v_0^2(c, c_2, c'), v_1^2(c, c_2, c')) (1 - P_1^1(c_1, c_2, c))\phi(v_0^2(c_1, c_2, c'), v_1^2(c_1, c_2, c'))\Big] \pi(dc'|c).$$

$$v_1^2(c_1, c_2, c) = r^1(c_2, c_1) - K(c) + \beta \int_0^c \Big[P_1^1(c_1, c_2, c)\phi(v_0^2(c, c, c'), v_1^2(c, c, c')) (1 - P_1^1(c_1, c_2, c))\phi(v_0^2(c_1, c, c'), v_1^2(c_1, c, c'))\Big] \pi(dc'|c).$$

Solving the "End Game"

- To be a Markov-perfect equilibrium, we must solve for equilibria for *all* (c_1, c_2, c) values, even if some of these will be "off the equilibrium path" (i.e. never reached in equilibrium)
- Similar to chess, there are circumstances where the solution of the game is easier, since there are fewer future options
- Our exogenous Markov specification for technological progress has a natural absorbing state, when $c_t = 0$. So we are interested in solving the "$(c_1, c_2, 0)$ end game" for all possible values of c_1 and c_2
- the Bellman equations simplify considerably when $c = 0$, and it is possible to get some easy insight into the nature of equilibrium in the overall dynamic game

Solving the "End Game"

- To be a Markov-perfect equilibrium, we must solve for equilibria for *all* (c_1, c_2, c) values, even if some of these will be "off the equilibrium path" (i.e. never reached in equilibrium)
- Similar to chess, there are circumstances where the solution of the game is easier, since there are fewer future options
- Our exogenous Markov specification for technological progress has a natural absorbing state, when $c_t = 0$. So we are interested in solving the "$(c_1, c_2, 0)$ end game" for all possible values of c_1 and c_2
- the Bellman equations simplify considerably when $c = 0$, and it is possible to get some easy insight into the nature of equilibrium in the overall dynamic game

Solving the “End Game”

- To be a Markov-perfect equilibrium, we must solve for equilibria for *all* (c_1, c_2, c) values, even if some of these will be “off the equilibrium path” (i.e. never reached in equilibrium)
- Similar to chess, there are circumstances where the solution of the game is easier, since there are fewer future options
- Our exogenous Markov specification for technological progress has a natural absorbing state, when $c_t = 0$. So we are interested in solving the “$(c_1, c_2, 0)$ end game” for all possible values of c_1 and c_2
- the Bellman equations simplify considerably when $c = 0$, and it is possible to get some easy insight into the nature of equilibrium in the overall dynamic game

Solving the "End Game"

- To be a Markov-perfect equilibrium, we must solve for equilibria for *all* (c_1, c_2, c) values, even if some of these will be "off the equilibrium path" (i.e. never reached in equilibrium)
- Similar to chess, there are circumstances where the solution of the game is easier, since there are fewer future options
- Our exogenous Markov specification for technological progress has a natural absorbing state, when $c_t = 0$. So we are interested in solving the "$(c_1, c_2, 0)$ end game" for all possible values of c_1 and c_2
- the Bellman equations simplify considerably when $c = 0$, and it is possible to get some easy insight into the nature of equilibrium in the overall dynamic game

The $(0,0,0)$ End Game

- The easiest end game is when $(c_1, c_2, c) = (0,0,0)$. No further innovation or price reductions will occur in this state, and so the game is fully stationary.

$$\begin{aligned} v_0^i(0,0,0) = & \; r^i(0,0) + \\ & \beta P_1^{\sim i}(0,0,0)\phi(v_0^i(0,0,0), v_1^i(0,0,0)) + \\ & \beta[1 - P_1^{\sim i}(0,0,0)]\phi(v_0^i(0,0,0), v_1^i(0,0,0)) \\ = & \; r^i(0,0) + \beta\phi(v_0^i(0,0,0), v_1^i(0,0,0)) \end{aligned}$$

where $P_1^{\sim i}(0,0,0)$ is a shorthand for firm i's opponent's probability of investing,

$$P_1^{\sim i}(0,0,0) = \frac{\exp\{v_1^{\sim i}(0,0,0)/\eta\}}{\exp\{v_0^{\sim i}(0,0,0)/\eta\} + \exp\{v_1^{\sim i}(0,0,0)/\eta\}}$$

The $(0,0,0)$ End Game

- The easiest end game is when $(c_1, c_2, c) = (0,0,0)$. No further innovation or price reductions will occur in this state, and so the game is fully stationary.

$$
\begin{aligned}
v_0^i(0,0,0) = \quad & r^i(0,0) + \\
& \beta P_1^{\sim i}(0,0,0)\phi(v_0^i(0,0,0), v_1^i(0,0,0)) + \\
& \beta[1 - P_1^{\sim i}(0,0,0)]\phi(v_0^i(0,0,0), v_1^i(0,0,0)) \\
= \quad & r^i(0,0) + \beta\phi(v_0^i(0,0,0), v_1^i(0,0,0))
\end{aligned}
$$

where $P_1^{\sim i}(0,0,0)$ is a shorthand for firm i's opponent's probability of investing,

$$
P_1^{\sim i}(0,0,0) = \frac{\exp\{v_1^{\sim i}(0,0,0)/\eta\}}{\exp\{v_0^{\sim i}(0,0,0)/\eta\} + \exp\{v_1^{\sim i}(0,0,0)/\eta\}}
$$

The $(0, 0, 0)$ End Game

- The easiest end game is when $(c_1, c_2, c) = (0, 0, 0)$. No further innovation or price reductions will occur in this state, and so the game is fully stationary.

$$\begin{aligned} v_0^i(0,0,0) = & \quad r^i(0,0) + \\ & \quad \beta P_1^{\sim i}(0,0,0)\phi(v_0^i(0,0,0), v_1^i(0,0,0)) + \\ & \quad \beta[1 - P_1^{\sim i}(0,0,0)]\phi(v_0^i(0,0,0), v_1^i(0,0,0)) \\ = & \quad r^i(0,0) + \beta\phi(v_0^i(0,0,0), v_1^i(0,0,0)) \end{aligned}$$

where $P_1^{\sim i}(0,0,0)$ is a shorthand for firm i's opponent's probability of investing,

$$P_1^{\sim i}(0,0,0) = \frac{\exp\{v_1^{\sim i}(0,0,0)/\eta\}}{\exp\{v_0^{\sim i}(0,0,0)/\eta\} + \exp\{v_1^{\sim i}(0,0,0)/\eta\}}$$

The $(0,0,0)$ End Game

- The easiest end game is when $(c_1, c_2, c) = (0,0,0)$. No further innovation or price reductions will occur in this state, and so the game is fully stationary.

$$\begin{aligned} v_0^i(0,0,0) = &\quad r^i(0,0) + \\ &\quad \beta P_1^{\sim i}(0,0,0)\phi(v_0^i(0,0,0), v_1^i(0,0,0)) + \\ &\quad \beta[1 - P_1^{\sim i}(0,0,0)]\phi(v_0^i(0,0,0), v_1^i(0,0,0)) \\ = &\quad r^i(0,0) + \beta\phi(v_0^i(0,0,0), v_1^i(0,0,0)) \end{aligned}$$

where $P_1^{\sim i}(0,0,0)$ is a shorthand for firm i's opponent's probability of investing,

$$P_1^{\sim i}(0,0,0) = \frac{\exp\{v_1^{\sim i}(0,0,0)/\eta\}}{\exp\{v_0^{\sim i}(0,0,0)/\eta\} + \exp\{v_1^{\sim i}(0,0,0)/\eta\}}$$

The $(0,0,0)$ End Game, continued

- Due to the fact that $(0,0,0)$ is an absorbing state, it can be easily shown that the value of investing, $v_1^i(0,0,0)$, is given by

$$v_1^i(0,0,0) = v_0^i(0,0,0) - K(0),$$

which implies that

$$P_1^{\sim i}(0,0,0) = \frac{\exp\{-K(0)/\eta\}}{1+\exp\{-K(0)/\eta\}}.$$

- Thus, as $\eta \to 0$, we have $P_1^{\sim i}(0,0,0) \to 0$ and $v_0^i(0,0,0) = r^i(0,0)/(1-\beta)$, and in the limiting case where the two firms are producing perfect substitutes, then $r^i(0,0) = 0$ and $v_0^i(0,0,0) = 0$.

The $(0, 0, 0)$ End Game, continued

- Due to the fact that $(0, 0, 0)$ is an absorbing state, it can be easily shown that the value of investing, $v_1^i(0, 0, 0)$, is given by

$$v_1^i(0, 0, 0) = v_0^i(0, 0, 0) - K(0),$$

which implies that

$$P_1^{\sim i}(0, 0, 0) = \frac{\exp\{-K(0)/\eta\}}{1 + \exp\{-K(0)/\eta\}}.$$

- Thus, as $\eta \to 0$, we have $P_1^{\sim i}(0, 0, 0) \to 0$ and $v_0^i(0, 0, 0) = r^i(0, 0)/(1 - \beta)$, and in the limiting case where the two firms are producing perfect substitutes, then $r^i(0, 0) = 0$ and $v_0^i(0, 0, 0) = 0$.

The $(0,0,0)$ End Game, continued

- Due to the fact that $(0,0,0)$ is an absorbing state, it can be easily shown that the value of investing, $v_1^i(0,0,0)$, is given by

  $$v_1^i(0,0,0) = v_0^i(0,0,0) - K(0),$$

  which implies that

  $$P_1^{\sim i}(0,0,0) = \frac{\exp\{-K(0)/\eta\}}{1+\exp\{-K(0)/\eta\}}.$$

- Thus, as $\eta \to 0$, we have $P_1^{\sim i}(0,0,0) \to 0$ and $v_0^i(0,0,0) = r^i(0,0)/(1-\beta)$, and in the limiting case where the two firms are producing perfect substitutes, then $r^i(0,0) = 0$ and $v_0^i(0,0,0) = 0$.

The $(0, 0, 0)$ End Game, continued

- Due to the fact that $(0, 0, 0)$ is an absorbing state, it can be easily shown that the value of investing, $v_1^i(0, 0, 0)$, is given by

$$v_1^i(0, 0, 0) = v_0^i(0, 0, 0) - K(0),$$

which implies that

$$P_1^{\sim i}(0, 0, 0) = \frac{\exp\{-K(0)/\eta\}}{1 + \exp\{-K(0)/\eta\}}.$$

- Thus, as $\eta \to 0$, we have $P_1^{\sim i}(0, 0, 0) \to 0$ and $v_0^i(0, 0, 0) = r^i(0, 0)/(1 - \beta)$, and in the limiting case where the two firms are producing perfect substitutes, then $r^i(0, 0) = 0$ and $v_0^i(0, 0, 0) = 0$.

The $(0,0,0)$ End Game, continued

- For positive values of η we have

$$v_0^i(0,0,0) = r^i(0,0) + \beta\phi(v_0^i(0,0,0), v_0^i(0,0,0) - K(0)).$$

  This is a single non-linear equation for the single solution $v_0^i(0,0,0)$.

- The derivative of the right hand side of this equation with respect to $v_0^i(0,0,0)$ is 1 whereas the derivative of the right hand side is strictly less than 1, so if $r^i(0,0) > 0$, this equation has a unique solution $v_0^i(0,0,0)$ that can be computed by Newton's method.
- Note that symmetry property *does* hold in the $(0,0,0)$ end game: $v_0^1(0,0,0) = v_0^2(0,0,0)$ and $v_1^1(0,0,0) = v_1^2(0,0,0)$.

The $(0,0,0)$ End Game, continued

- For positive values of η we have

$$v_0^i(0,0,0) = r^i(0,0) + \beta\phi(v_0^i(0,0,0), v_0^i(0,0,0) - K(0)).$$

  This is a single non-linear equation for the single solution $v_0^i(0,0,0)$.

- The derivative of the right hand side of this equation with respect to $v_0^i(0,0,0)$ is 1 whereas the derivative of the right hand side is strictly less than 1, so if $r^i(0,0) > 0$, this equation has a unique solution $v_0^i(0,0,0)$ that can be computed by Newton's method.

- Note that symmetry property *does* hold in the $(0,0,0)$ end game: $v_0^1(0,0,0) = v_0^2(0,0,0)$ and $v_1^1(0,0,0) = v_1^2(0,0,0)$.

The $(0,0,0)$ End Game, continued

- For positive values of η we have

  $$v_0^i(0,0,0) = r^i(0,0) + \beta\phi(v_0^i(0,0,0), v_0^i(0,0,0) - K(0)).$$

  This is a single non-linear equation for the single solution $v_0^i(0,0,0)$.
- The derivative of the right hand side of this equation with respect to $v_0^i(0,0,0)$ is 1 whereas the derivative of the right hand side is strictly less than 1, so if $r^i(0,0) > 0$, this equation has a unique solution $v_0^i(0,0,0)$ that can be computed by Newton's method.
- Note that symmetry property *does* hold in the $(0,0,0)$ end game: $v_0^1(0,0,0) = v_0^2(0,0,0)$ and $v_1^1(0,0,0) = v_1^2(0,0,0)$.

The $(c, 0, 0)$ End Game

- The next simplest end game is $(c, 0, 0)$. In the "pure Bertrand case" (i.e. when $\eta = 0$ and $\sigma = 0$) it is clear that firm 1 would not have any incentive to invest since the investment would not allow it to leap-frog its opponent.
- When $\eta > 0$, there may be transitory shocks that would induce firm 1 to invest and thereby match the 0 marginal cost of production of its opponent.

$$\begin{aligned} v_0^1(c,0,0) &= r^1(c,0) + \beta\phi(v_0^1(c,0,0), v_1^1(c,0,0)) \\ v_1^1(c,0,0) &= r^1(c,0) - K(0) + \beta\phi(v_0^1(0,0,0), v_1^1(0,0,0)). \end{aligned}$$

Substituting the resulting solution for $v_1^1(c,0,0)$ into the first equation above results in another nonlinear equation with a single unique solution $v_0^1(c,0,0)$ that can be computed by Newton's method.

The $(c, 0, 0)$ End Game

- The next simplest end game is $(c, 0, 0)$. In the “pure Bertrand case” (i.e. when $\eta = 0$ and $\sigma = 0$) it is clear that firm 1 would not have any incentive to invest since the investment would not allow it to leap-frog its opponent.
- When $\eta > 0$, there may be transitory shocks that would induce firm 1 to invest and thereby match the 0 marginal cost of production of its opponent.

$$\begin{aligned} v_0^1(c,0,0) &= r^1(c,0) + \beta\phi(v_0^1(c,0,0), v_1^1(c,0,0)) \\ v_1^1(c,0,0) &= r^1(c,0) - K(0) + \beta\phi(v_0^1(0,0,0), v_1^1(0,0,0)). \end{aligned}$$

Substituting the resulting solution for $v_1^1(c,0,0)$ into the first equation above results in another nonlinear equation with a single unique solution $v_0^1(c,0,0)$ that can be computed by Newton’s method.

The $(c, 0, 0)$ End Game

- The next simplest end game is $(c, 0, 0)$. In the "pure Bertrand case" (i.e. when $\eta = 0$ and $\sigma = 0$) it is clear that firm 1 would not have any incentive to invest since the investment would not allow it to leap-frog its opponent.
- When $\eta > 0$, there may be transitory shocks that would induce firm 1 to invest and thereby match the 0 marginal cost of production of its opponent.

$$\begin{aligned} v_0^1(c,0,0) &= r^1(c,0) + \beta\phi(v_0^1(c,0,0), v_1^1(c,0,0)) \\ v_1^1(c,0,0) &= r^1(c,0) - K(0) + \beta\phi(v_0^1(0,0,0), v_1^1(0,0,0)). \end{aligned}$$

Substituting the resulting solution for $v_1^1(c,0,0)$ into the first equation above results in another nonlinear equation with a single unique solution $v_0^1(c,0,0)$ that can be computed by Newton's method.

The $(c, 0, 0)$ End Game

- The next simplest end game is $(c, 0, 0)$. In the "pure Bertrand case" (i.e. when $\eta = 0$ and $\sigma = 0$) it is clear that firm 1 would not have any incentive to invest since the investment would not allow it to leap-frog its opponent.
- When $\eta > 0$, there may be transitory shocks that would induce firm 1 to invest and thereby match the 0 marginal cost of production of its opponent.

$$\begin{aligned} v_0^1(c,0,0) &= r^1(c,0) + \beta\phi(v_0^1(c,0,0), v_1^1(c,0,0)) \\ v_1^1(c,0,0) &= r^1(c,0) - K(0) + \beta\phi(v_0^1(0,0,0), v_1^1(0,0,0)). \end{aligned}$$

Substituting the resulting solution for $v_1^1(c,0,0)$ into the first equation above results in another nonlinear equation with a single unique solution $v_0^1(c,0,0)$ that can be computed by Newton's method.

The $(c, 0, 0)$ End Game, continued

- Note that, as we show below, the probability that firm 2 invests in this case, $P_1^2(c, 0, 0)$ is given by

$$P_1^2(c, 0, 0) = \frac{\exp\{-K(0)/\eta\}}{1 + \exp\{-K(0)/\eta\}} \quad (1)$$

since firm 2 has achieved the lowest possible cost of production and its decisions about investment are governed by the same idiosyncratic temporary shocks, and result in the same formula for the probability of investment as we derived above in the $(0, 0, 0)$ end game.

- Note: It is not hard to show that the symmetry condition holds in the $(c, 0, 0)$ end game as well: $v_0^2(c, 0, 0) = v_0^1(0, c, 0)$, and $v_1^2(c, 0, 0) = v_1^1(0, c, 0)$, where the solutions for the latter functions are presented below.

The $(c, 0, 0)$ End Game, continued

- Note that, as we show below, the probability that firm 2 invests in this case, $P_1^2(c, 0, 0)$ is given by

$$P_1^2(c, 0, 0) = \frac{\exp\{-K(0)/\eta\}}{1 + \exp\{-K(0)/\eta\}} \tag{1}$$

since firm 2 has achieved the lowest possible cost of production and its decisions about investment are governed by the same idiosyncratic temporary shocks, and result in the same formula for the probability of investment as we derived above in the $(0, 0, 0)$ end game.

- Note: It is not hard to show that the symmetry condition holds in the $(c, 0, 0)$ end game as well: $v_0^2(c, 0, 0) = v_0^1(0, c, 0)$, and $v_1^2(c, 0, 0) = v_1^1(0, c, 0)$, where the solutions for the latter functions are presented below.

The $(c, 0, 0)$ End Game, continued

- Note that, as we show below, the probability that firm 2 invests in this case, $P_1^2(c, 0, 0)$ is given by

$$P_1^2(c, 0, 0) = \frac{\exp\{-K(0)/\eta\}}{1 + \exp\{-K(0)/\eta\}} \tag{1}$$

  since firm 2 has achieved the lowest possible cost of production and its decisions about investment are governed by the same idiosyncratic temporary shocks, and result in the same formula for the probability of investment as we derived above in the $(0, 0, 0)$ end game.
- Note: It is not hard to show that the symmetry condition holds in the $(c, 0, 0)$ end game as well: $v_0^2(c, 0, 0) = v_0^1(0, c, 0)$, and $v_1^2(c, 0, 0) = v_1^1(0, c, 0)$, where the solutions for the latter functions are presented below.

The $(c, 0, 0)$ End Game, continued

- Note that, as we show below, the probability that firm 2 invests in this case, $P_1^2(c, 0, 0)$ is given by

$$P_1^2(c, 0, 0) = \frac{\exp\{-K(0)/\eta\}}{1 + \exp\{-K(0)/\eta\}} \tag{1}$$

  since firm 2 has achieved the lowest possible cost of production and its decisions about investment are governed by the same idiosyncratic temporary shocks, and result in the same formula for the probability of investment as we derived above in the $(0, 0, 0)$ end game.
- Note: It is not hard to show that the symmetry condition holds in the $(c, 0, 0)$ end game as well: $v_0^2(c, 0, 0) = v_0^1(0, c, 0)$, and $v_1^2(c, 0, 0) = v_1^1(0, c, 0)$, where the solutions for the latter functions are presented below.

The $(0, c, 0)$ End Game

- In this end game, firm 1 has achieved the lowest cost of production but firm 2 hasn't yet. Clearly firm 1 has no further incentive to invest.
- However in the presence of random cost shocks (i.e. in the case where $\eta > 0$), firm 1 might invest due to idiosyncratic transitory investment shocks. This implies

$$v_1^1(0, c, 0) = v_0^1(0, c, 0) - K(0).$$

$$\begin{aligned} v_0^1(0, c, 0) = {} & r^1(0, c) + \\ & \beta P_1^2(0, c, 0)\phi(v_0^1(0, 0, 0), v_0^1(0, 0, 0) - K(0)) \\ & + \beta[1 - P_1^2(0, c, 0)]\phi(v_0^1(0, c, 0), v_0^1(0, c, 0) - K(0)). \end{aligned}$$

The $(0, c, 0)$ End Game

- In this end game, firm 1 has achieved the lowest cost of production but firm 2 hasn't yet. Clearly firm 1 has no further incentive to invest.
- However in the presence of random cost shocks (i.e. in the case where $\eta > 0$), firm 1 might invest due to idiosyncratic transitory investment shocks. This implies

$$v_1^1(0, c, 0) = v_0^1(0, c, 0) - K(0).$$

$$v_0^1(0, c, 0) = r^1(0, c) + \beta P_1^2(0, c, 0)\phi(v_0^1(0, 0, 0), v_0^1(0, 0, 0) - K(0)) + \beta[1 - P_1^2(0, c, 0)]\phi(v_0^1(0, c, 0), v_0^1(0, c, 0) - K(0)).$$

The $(0, c, 0)$ End Game

- In this end game, firm 1 has achieved the lowest cost of production but firm 2 hasn't yet. Clearly firm 1 has no further incentive to invest.
- However in the presence of random cost shocks (i.e. in the case where $\eta > 0$), firm 1 might invest due to idiosyncratic transitory investment shocks. This implies

$$v_1^1(0, c, 0) = v_0^1(0, c, 0) - K(0).$$

$$v_0^1(0, c, 0) = r^1(0, c) + \\ \beta P_1^2(0, c, 0)\phi(v_0^1(0, 0, 0), v_0^1(0, 0, 0) - K(0)) \\ +\beta[1 - P_1^2(0, c, 0)]\phi(v_0^1(0, c, 0), v_0^1(0, c, 0) - K(0)).$$

The $(0, c, 0)$ End Game, continued

- The probability that firm 2 will invest, $P_1^2(0, c, 0)$ is given by

$$P_1^2(0,c,0) = \frac{\exp\{v_1^2(0,c,0)/\eta\}}{\exp\{v_1^2(0,c,0)/\eta\} + \exp\{v_0^2(0,c,0)/\eta\}} = \frac{\exp\{v_1^1(c,0,0)/\eta\}}{\exp\{v_1^1(c,0,0)/\eta\} + \exp\{v_0^1(c,0,0)/\eta\}},$$

  using the symmetry condition that $v_j^2(0, c, 0) = v_j^1(c, 0, 0)$.

- Substitute $v_0^1(c, 0, 0)$ and $v_1^1(c, 0, 0))$ from the $(c, 0, 0)$ end game into the Bellman equation for $v^1(0, c, 0)$ to obtain a unique solution for $v_0^1(0, c, 0)$.
- Once again, it is not hard to verify that payoff symmetry holds in the $(0, c, 0)$ end game.

The $(0, c, 0)$ End Game, continued

- The probability that firm 2 will invest, $P_1^2(0, c, 0)$ is given by

$$P_1^2(0,c,0) = \frac{\exp\{v_1^2(0,c,0)/\eta\}}{\exp\{v_1^2(0,c,0)/\eta\} + \exp\{v_0^2(0,c,0)/\eta\}} = \frac{\exp\{v_1^1(c,0,0)/\eta\}}{\exp\{v_1^1(c,0,0)/\eta\} + \exp\{v_0^1(c,0,0)/\eta\}},$$

  using the symmetry condition that $v_j^2(0, c, 0) = v_j^1(c, 0, 0)$.
- Substitute $v_0^1(c, 0, 0)$ and $v_1^1(c, 0, 0))$ from the $(c, 0, 0)$ end game into the Bellman equation for $v^1(0, c, 0)$ to obtain a unique solution for $v_0^1(0, c, 0)$.
- Once again, it is not hard to verify that payoff symmetry holds in the $(0, c, 0)$ end game.

The $(0, c, 0)$ End Game, continued

- The probability that firm 2 will invest, $P_1^2(0, c, 0)$ is given by

$$P_1^2(0,c,0) = \frac{\exp\{v_1^2(0,c,0)/\eta\}}{\exp\{v_1^2(0,c,0)/\eta\} + \exp\{v_0^2(0,c,0)/\eta\}} = \frac{\exp\{v_1^1(c,0,0)/\eta\}}{\exp\{v_1^1(c,0,0)/\eta\} + \exp\{v_0^1(c,0,0)/\eta\}},$$

  using the symmetry condition that $v_j^2(0, c, 0) = v_j^1(c, 0, 0)$.

- Substitute $v_0^1(c, 0, 0)$ and $v_1^1(c, 0, 0)$) from the $(c, 0, 0)$ end game into the Bellman equation for $v^1(0, c, 0)$ to obtain a unique solution for $v_0^1(0, c, 0)$.
- Once again, it is not hard to verify that payoff symmetry holds in the $(0, c, 0)$ end game.

The $(0, c, 0)$ End Game, continued

- The probability that firm 2 will invest, $P_1^2(0, c, 0)$ is given by

$$P_1^2(0, c, 0) = \frac{\exp\{v_1^2(0, c, 0)/\eta\}}{\exp\{v_1^2(0, c, 0)/\eta\} + \exp\{v_0^2(0, c, 0)/\eta\}} = \frac{\exp\{v_1^1(c, 0, 0)/\eta\}}{\exp\{v_1^1(c, 0, 0)/\eta\} + \exp\{v_0^1(c, 0, 0)/\eta\}},$$

  using the symmetry condition that $v_j^2(0, c, 0) = v_j^1(c, 0, 0)$.
- Substitute $v_0^1(c, 0, 0)$ and $v_1^1(c, 0, 0))$ from the $(c, 0, 0)$ end game into the Bellman equation for $v^1(0, c, 0)$ to obtain a unique solution for $v_0^1(0, c, 0)$.
- Once again, it is not hard to verify that payoff symmetry holds in the $(0, c, 0)$ end game.

The $(0, c, 0)$ End Game, continued

- The probability that firm 2 will invest, $P_1^2(0, c, 0)$ is given by

$$P_1^2(0,c,0) = \frac{\exp\{v_1^2(0,c,0)/\eta\}}{\exp\{v_1^2(0,c,0)/\eta\} + \exp\{v_0^2(0,c,0)/\eta\}} = \frac{\exp\{v_1^1(c,0,0)/\eta\}}{\exp\{v_1^1(c,0,0)/\eta\} + \exp\{v_0^1(c,0,0)/\eta\}},$$

  using the symmetry condition that $v_j^2(0,c,0) = v_j^1(c,0,0)$.
- Substitute $v_0^1(c,0,0)$ and $v_1^1(c,0,0))$ from the $(c,0,0)$ end game into the Bellman equation for $v^1(0,c,0)$ to obtain a unique solution for $v_0^1(0,c,0)$.
- Once again, it is not hard to verify that payoff symmetry holds in the $(0,c,0)$ end game.

The $(c_1, c_2, 0)$ End Game

- The final case to consider is the end game where both firms have positive marginal costs of production, c_1 and c_2, respectively.
- We will show that in this end game, asymmetric equilibrium solutions are possible. The value to firm 1 of not investing is

$$
\begin{aligned}
v_0^1(c_1, c_2, 0) &= r^1(c_1, c_2) \\
&+ \beta P_1^2(c_1, c_2, 0)\phi(v_0^1(c_1, 0, 0), v_1^1(c_1, 0, 0)) \\
&+ \beta[1 - P_1^2(c_1, c_2, 0)]\phi(v_0^1(c_1, c_2, 0), v_1^1(c_1, c_2, 0)) \\
v_1^1(c_1, c_2, 0) &= r^1(c_1, c_2) - K(0) \\
&+ \beta P_1^2(c_1, c_2, 0)\phi(v_0^1(0, 0, 0), v_1^1(0, 0, 0)) \\
&+ \beta[1 - P_1^2(c_1, c_2, 0)]\phi(v_0^1(0, c_2, 0), v_1^1(0, c_2, 0)).
\end{aligned}
$$

The $(c_1, c_2, 0)$ End Game

- The final case to consider is the end game where both firms have positive marginal costs of production, c_1 and c_2, respectively.
- We will show that in this end game, asymmetric equilibrium solutions are possible. The value to firm 1 of not investing is

$$\begin{aligned} v_0^1(c_1,c_2,0) &= r^1(c_1,c_2) \\ &+ \beta P_1^2(c_1,c_2,0)\phi(v_0^1(c_1,0,0),v_1^1(c_1,0,0)) \\ &+ \beta[1-P_1^2(c_1,c_2,0)]\phi(v_0^1(c_1,c_2,0),v_1^1(c_1,c_2,0)) \\ v_1^1(c_1,c_2,0) &= r^1(c_1,c_2)-K(0) \\ &+ \beta P_1^2(c_1,c_2,0)\phi(v_0^1(0,0,0),v_1^1(0,0,0)) \\ &+ \beta[1-P_1^2(c_1,c_2,0)]\phi(v_0^1(0,c_2,0),v_1^1(0,c_2,0)). \end{aligned}$$

The $(c_1, c_2, 0)$ End Game

- The final case to consider is the end game where both firms have positive marginal costs of production, c_1 and c_2, respectively.
- We will show that in this end game, asymmetric equilibrium solutions are possible. The value to firm 1 of not investing is

$$
\begin{aligned}
v_0^1(c_1, c_2, 0) &= r^1(c_1, c_2) \\
&+ \beta P_1^2(c_1, c_2, 0)\phi(v_0^1(c_1, 0, 0), v_1^1(c_1, 0, 0)) \\
&+ \beta[1 - P_1^2(c_1, c_2, 0)]\phi(v_0^1(c_1, c_2, 0), v_1^1(c_1, c_2, 0)) \\
v_1^1(c_1, c_2, 0) &= r^1(c_1, c_2) - K(0) \\
&+ \beta P_1^2(c_1, c_2, 0)\phi(v_0^1(0, 0, 0), v_1^1(0, 0, 0)) \\
&+ \beta[1 - P_1^2(c_1, c_2, 0)]\phi(v_0^1(0, c_2, 0), v_1^1(0, c_2, 0)).
\end{aligned}
$$

The $(c_1, c_2, 0)$ End Game, continued

- Given the equation for $v_1^1(c_1, c_2, 0)$ depends on known quantities on the right hand side (the values for v_0^1 and v_1^1 inside the ϕ functions can be computed in the $(0, 0, 0)$ and $(0, c, 0)$ end games already covered above), we can treat $v_1^1(c_1, c_2, 0)$ as a linear function of P_1^2 which is not yet "known" because it depends on $(v_0^2(c_1, c_2, 0), v_1^2(c_1, c_2, 0))$ via the identity:

$$P_1^2(c_1, c_2, 0) = \frac{\exp\{v_1^2(c_1, c_2, 0)/\eta\}}{\exp\{v_0^2(c_1, c_2, 0)/\eta\} + \exp\{v_1^2(c_1, c_2, 0)/\eta\}}.$$

- Then we can write $v_1^1(c_1, c_2, 0, P_1^2)$ as an implicit function of P_1^2: the value of v_1^1 that satisfies the Bellman equation for v_1^1 above, for an arbitrary value of $P_1^2 \in [0, 1]$.

The $(c_1, c_2, 0)$ End Game, continued

- Given the equation for $v_1^1(c_1, c_2, 0)$ depends on known quantities on the right hand side (the values for v_0^1 and v_1^1 inside the ϕ functions can be computed in the $(0, 0, 0)$ and $(0, c, 0)$ end games already covered above), we can treat $v_1^1(c_1, c_2, 0)$ as a linear function of P_1^2 which is not yet "known" because it depends on $(v_0^2(c_1, c_2, 0), v_1^2(c_1, c_2, 0))$ via the identity:

$$P_1^2(c_1, c_2, 0) = \frac{\exp\{v_1^2(c_1, c_2, 0)/\eta\}}{\exp\{v_0^2(c_1, c_2, 0)/\eta\} + \exp\{v_1^2(c_1, c_2, 0)/\eta\}}.$$

- Then we can write $v_1^1(c_1, c_2, 0, P_1^2)$ as an implicit function of P_1^2: the value of v_1^1 that satisfies the Bellman equation for v_1^1 above, for an arbitrary value of $P_1^2 \in [0, 1]$.

The $(c_1, c_2, 0)$ End Game, continued

- Given the equation for $v_1^1(c_1, c_2, 0)$ depends on known quantities on the right hand side (the values for v_0^1 and v_1^1 inside the ϕ functions can be computed in the $(0, 0, 0)$ and $(0, c, 0)$ end games already covered above), we can treat $v_1^1(c_1, c_2, 0)$ as a linear function of P_1^2 which is not yet "known" because it depends on $(v_0^2(c_1, c_2, 0), v_1^2(c_1, c_2, 0))$ via the identity:

$$P_1^2(c_1, c_2, 0) = \frac{\exp\{v_1^2(c_1, c_2, 0)/\eta\}}{\exp\{v_0^2(c_1, c_2, 0)/\eta\} + \exp\{v_1^2(c_1, c_2, 0)/\eta\}}.$$

- Then we can write $v_1^1(c_1, c_2, 0, P_1^2)$ as an implicit function of P_1^2: the value of v_1^1 that satisfies the Bellman equation for v_1^1 above, for an arbitrary value of $P_1^2 \in [0, 1]$.

The $(c_1, c_2, 0)$ End Game, continued

- Substituting P_1^2 into the equation for v_0^1, there will be a unique solution $v_0^1(c_1, c_2, 0, P_1^2)$ for any $P_1^2 \in [0, 1]$ since we have already solved for the values $(v_0^1(c_1, 0, 0), v_1^1(c_1, 0, 0))$ in the $(c, 0, 0)$ end game above. Using these values, we can write firm 1's probability of investing $P_1^1(c_1, c_2, 0)$ as

$$P_1^1(c_1, c_2, 0, P_1^2) = \frac{\exp\{v_1^1(c_1, c_2, 0, P_1^2)/\eta\}}{\exp\{v_0^1(c_1, c_2, 0, P_1^2)/\eta\} + \exp\{v_1^1(c_1, c_2, 0, P_1^2)/\eta\}}.$$

- Now, the values for firm 2 $(v_0^2(c_1, c_2, 0), v_1^2(c_1, c_2, 0))$ that determine firm 2's probability of investing can also be written as functions of P_1^1 for any $P_1^1 \in [0, 1]$.

The $(c_1, c_2, 0)$ End Game, continued

- Substituting P_1^2 into the equation for v_0^1, there will be a unique solution $v_0^1(c_1, c_2, 0, P_1^2)$ for any $P_1^2 \in [0, 1]$ since we have already solved for the values $(v_0^1(c_1, 0, 0), v_1^1(c_1, 0, 0))$ in the $(c, 0, 0)$ end game above. Using these values, we can write firm 1's probability of investing $P_1^1(c_1, c_2, 0)$ as

$$P_1^1(c_1, c_2, 0, P_1^2) = \frac{\exp\{v_1^1(c_1, c_2, 0, P_1^2)/\eta\}}{\exp\{v_0^1(c_1, c_2, 0, P_1^2)/\eta\} + \exp\{v_1^1(c_1, c_2, 0, P_1^2)/\eta\}}.$$

- Now, the values for firm 2 $(v_0^2(c_1, c_2, 0), v_1^2(c_1, c_2, 0))$ that determine firm 2's probability of investing can also be written as functions of P_1^1 for any $P_1^1 \in [0, 1]$.

The $(c_1, c_2, 0)$ End Game, continued

- Substituting P_1^2 into the equation for v_0^1, there will be a unique solution $v_0^1(c_1, c_2, 0, P_1^2)$ for any $P_1^2 \in [0, 1]$ since we have already solved for the values $(v_0^1(c_1, 0, 0), v_1^1(c_1, 0, 0))$ in the $(c, 0, 0)$ end game above. Using these values, we can write firm 1's probability of investing $P_1^1(c_1, c_2, 0)$ as

$$P_1^1(c_1, c_2, 0, P_1^2) = \frac{\exp\{v_1^1(c_1, c_2, 0, P_1^2)/\eta\}}{\exp\{v_0^1(c_1, c_2, 0, P_1^2)/\eta\} + \exp\{v_1^1(c_1, c_2, 0, P_1^2)/\eta\}}.$$

- Now, the values for firm 2 $(v_0^2(c_1, c_2, 0), v_1^2(c_1, c_2, 0))$ that determine firm 2's probability of investing can also be written as functions of P_1^1 for any $P_1^1 \in [0, 1]$.

The $(c_1, c_2, 0)$ End Game, continued

- This implies that we can write firm 2's probablity of investing as a function of its perceptions of firm 1's probability of investing, or as $P_1^2(c_1, c_2, 0, P_1^1)$. Substituting this formula for P_1^2 into the equation for P_1^1 we obtain the following fixed point equation for firm 1's probability of investing

$$P_1^1 = \frac{\exp\{v_1^1(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))/\eta\}}{D(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))}$$

where

$$\begin{aligned} D &= \exp\{v_0^1(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))/\eta\} \\ &+ \exp\{v_1^1(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))/\eta\}. \end{aligned}$$

The $(c_1, c_2, 0)$ End Game, continued

- This implies that we can write firm 2's probablity of investing as a function of its perceptions of firm 1's probability of investing, or as $P_1^2(c_1, c_2, 0, P_1^1)$. Substituting this formula for P_1^2 into the equation for P_1^1 we obtain the following fixed point equation for firm 1's probability of investing

$$P_1^1 = \frac{\exp\{v_1^1(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))/\eta\}}{D(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))}$$

where

$$\begin{aligned} D &= \exp\{v_0^1(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))/\eta\} \\ &+ \exp\{v_1^1(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))/\eta\}. \end{aligned}$$

The $(c_1, c_2, 0)$ End Game, continued

- This implies that we can write firm 2's probablity of investing as a function of its perceptions of firm 1's probability of investing, or as $P_1^2(c_1, c_2, 0, P_1^1)$. Substituting this formula for P_1^2 into the equation for P_1^1 we obtain the following fixed point equation for firm 1's probability of investing

$$P_1^1 = \frac{\exp\{v_1^1(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))/\eta\}}{D(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))}$$

where

$$\begin{aligned} D &= \exp\{v_0^1(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))/\eta\} \\ &+ \exp\{v_1^1(c_1, c_2, 0, P_1^2(c_1, c_2, 0, P_1^1))/\eta\}. \end{aligned}$$

End Game Solutions

- By Brouwer's fixed point theorem, at least one equilibrium solution to the fixed point equation exists.
- Further, when $\eta > 0$, the objects entering this equation (i.e. the value functions $v_0^1(c_1, c_2, 0, P_1^2)$, $v_1^1(c_1, c_2, 0, P_1^2)$, $v_0^2(c_1, c_2, 0, P_1^1)$, $v_1^2(c_1, c_2, 0, P_1^1)$ and the logit choice probability function P_1^2 are all C^∞ functions of P_1^2 and P_1^1
- Standard topological index theorems (e.g. Harsanyi, 1973) be applied to show that for almost all values of the underlying parameters, there will be an odd number of separated equilibria.

End Game Solutions

- By Brouwer's fixed point theorem, at least one equilibrium solution to the fixed point equation exists.
- Further, when $\eta > 0$, the objects entering this equation (i.e. the value functions $v_0^1(c_1, c_2, 0, P_1^2)$, $v_1^1(c_1, c_2, 0, P_1^2)$, $v_0^2(c_1, c_2, 0, P_1^1)$, $v_1^2(c_1, c_2, 0, P_1^1)$ and the logit choice probability function P_1^2 are all C^∞ functions of P_1^2 and P_1^1
- Standard topological index theorems (e.g. Harsanyi, 1973) be applied to show that for almost all values of the underlying parameters, there will be an odd number of separated equilibria.

End Game Solutions

- By Brouwer's fixed point theorem, at least one equilibrium solution to the fixed point equation exists.
- Further, when $\eta > 0$, the objects entering this equation (i.e. the value functions $v_0^1(c_1, c_2, 0, P_1^2)$, $v_1^1(c_1, c_2, 0, P_1^2)$, $v_0^2(c_1, c_2, 0, P_1^1)$, $v_1^2(c_1, c_2, 0, P_1^1)$ and the logit choice probability function P_1^2 are all C^∞ functions of P_1^2 and P_1^1
- Standard topological index theorems (e.g. Harsanyi, 1973) be applied to show that for almost all values of the underlying parameters, there will be an odd number of separated equilibria.

End Game Solutions, continued

- Further, the results of Harsanyi (1973) as extended to dynamic Markovian games by Doraszelski and Escobar (2009) show that as $\eta \to 0$ the set of equilibria of the game of incomplete information converge to the set of equilibria of the game of complete information
- Thus, η serves as a "homotopy parameter" for *path following algorithms* that can be used to solve the set of equilibria to the limiting game of complete information, *including mixed strategy equilibria.*
- We use this approach to solve for equilibria of the limiting "pure Bertrand duopoly" case where $\eta = 0$ and $\sigma = 0$.

End Game Solutions, continued

- Further, the results of Harsanyi (1973) as extended to dynamic Markovian games by Doraszelski and Escobar (2009) show that as $\eta \to 0$ the set of equilibria of the game of incomplete information converge to the set of equilibria of the game of complete information
- Thus, η serves as a “homotopy parameter” for *path following algorithms* that can be used to solve the set of equilibria to the limiting game of complete information, *including mixed strategy equilibria.*
- We use this approach to solve for equilibria of the limiting “pure Bertrand duopoly” case where $\eta = 0$ and $\sigma = 0$.

End Game Solutions, continued

- Further, the results of Harsanyi (1973) as extended to dynamic Markovian games by Doraszelski and Escobar (2009) show that as $\eta \to 0$ the set of equilibria of the game of incomplete information converge to the set of equilibria of the game of complete information
- Thus, η serves as a “homotopy parameter” for *path following algorithms* that can be used to solve the set of equilibria to the limiting game of complete information, *including mixed strategy equilibria.*
- We use this approach to solve for equilibria of the limiting “pure Bertrand duopoly” case where $\eta = 0$ and $\sigma = 0$.

End Game Solutions, continued

- We find that there are either 1 or 3 equilibria to the game, depending on the values of the parameters.
- The “trivial equilibrium” is a no-investment equilibrium that occurs when the cost of investment $K(0)$ is too high relative to the expected cost savings, and neither firm invests in this situation.
- However whenever $K(0)$ is below a critical threshold, there will be 3 equilibria to the end game:two pure strategy equilibria and an intermediate mixed strategy equilibrium.

End Game Solutions, continued

- We find that there are either 1 or 3 equilibria to the game, depending on the values of the parameters.
- The "trivial equilibrium" is a no-investment equilibrium that occurs when the cost of investment $K(0)$ is too high relative to the expected cost savings, and neither firm invests in this situation.
- However whenever $K(0)$ is below a critical threshold, there will be 3 equilibria to the end game:two pure strategy equilibria and an intermediate mixed strategy equilibrium.

End Game Solutions, continued

- We find that there are either 1 or 3 equilibria to the game, depending on the values of the parameters.
- The "trivial equilibrium" is a no-investment equilibrium that occurs when the cost of investment $K(0)$ is too high relative to the expected cost savings, and neither firm invests in this situation.
- However whenever $K(0)$ is below a critical threshold, there will be 3 equilibria to the end game: two pure strategy equilibria and an intermediate mixed strategy equilibrium.

End Game Solutions, continued

- We find that there are either 1 or 3 equilibria to the game, depending on the values of the parameters.
- The “trivial equilibrium” is a no-investment equilibrium that occurs when the cost of investment $K(0)$ is too high relative to the expected cost savings, and neither firm invests in this situation.
- However whenever $K(0)$ is below a critical threshold, there will be 3 equilibria to the end game:two pure strategy equilibria and an intermediate mixed strategy equilibrium.

End Game Solutions, continued

- We find that there are either 1 or 3 equilibria to the game, depending on the values of the parameters.
- The "trivial equilibrium" is a no-investment equilibrium that occurs when the cost of investment $K(0)$ is too high relative to the expected cost savings, and neither firm invests in this situation.
- However whenever $K(0)$ is below a critical threshold, there will be 3 equilibria to the end game:two pure strategy equilibria and an intermediate mixed strategy equilibrium.

End Game Solutions, continued

- It turns out that the investment game is isomorphic to a *coordination game.*
- The two pure strategy equilibria correspond to outcomes where firm 1 invests and firm 2 doesn't and firm 2 invests and firm 1 doesn't.
- The mixed strategy equilibrium corresponds to the situation where firm 1 invests with probability π_1 and firm 2 invests with probability π_2.
- It is not hard to see that when $c_1 = c_2$ the game is fully symmetric and we have $\pi_1 = \pi_2$.
- However when $c_1 \neq c_2$, then the game is asymmetric and $\pi_1 \neq \pi_2$.

End Game Solutions, continued

- It turns out that the investment game is isomorphic to a *coordination game.*
- The two pure strategy equilibria correspond to outcomes where firm 1 invests and firm 2 doesn't and firm 2 invests and firm 1 doesn't.
- The mixed strategy equilibrium corresponds to the situation where firm 1 invests with probability π_1 and firm 2 invests with probability π_2.
- It is not hard to see that when $c_1 = c_2$ the game is fully symmetric and we have $\pi_1 = \pi_2$.
- However when $c_1 \neq c_2$, then the game is asymmetric and $\pi_1 \neq \pi_2$.

End Game Solutions, continued

- It turns out that the investment game is isomorphic to a *coordination game.*
- The two pure strategy equilibria correspond to outcomes where firm 1 invests and firm 2 doesn't and firm 2 invests and firm 1 doesn't.
- The mixed strategy equilibrium corresponds to the situation where firm 1 invests with probability π_1 and firm 2 invests with probability π_2.
- It is not hard to see that when $c_1 = c_2$ the game is fully symmetric and we have $\pi_1 = \pi_2$.
- However when $c_1 \neq c_2$, then the game is asymmetric and $\pi_1 \neq \pi_2$.

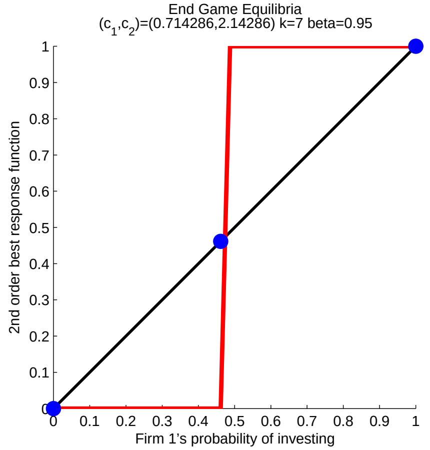
End Game Solutions, continued

- It turns out that the investment game is isomorphic to a *coordination game.*
- The two pure strategy equilibria correspond to outcomes where firm 1 invests and firm 2 doesn't and firm 2 invests and firm 1 doesn't.
- The mixed strategy equilibrium corresponds to the situation where firm 1 invests with probability π_1 and firm 2 invests with probability π_2.
- It is not hard to see that when $c_1 = c_2$ the game is fully symmetric and we have $\pi_1 = \pi_2$.
- However when $c_1 \neq c_2$, then the game is asymmetric and $\pi_1 \neq \pi_2$.

End Game Solutions, continued

- It turns out that the investment game is isomorphic to a *coordination game.*
- The two pure strategy equilibria correspond to outcomes where firm 1 invests and firm 2 doesn't and firm 2 invests and firm 1 doesn't.
- The mixed strategy equilibrium corresponds to the situation where firm 1 invests with probability π_1 and firm 2 invests with probability π_2.
- It is not hard to see that when $c_1 = c_2$ the game is fully symmetric and we have $\pi_1 = \pi_2$.
- However when $c_1 \neq c_2$, then the game is asymmetric and $\pi_1 \neq \pi_2$.

Graph of the Fixed Point Equation for P_1



Foreshadow of leap frogging

- Since the end game is essentially a two-period game, due to the presence of the zero cost absorbing state, it is not rich enough for us to observe *deterministic* leap frogging, excepting the pure strategy equilibrium where firm 1 invests with probability 1 when $c_1 > c_2$
- However, we can show (via numerical solutions) that for the mixed strategy equilibrium, $c_1 > c_2 \implies \pi_1 > \pi_2$.
- We have been unable to prove this yet, but conjecture that the result is true in general.
- Thus, the “high cost follower” has a higher probability of investing and leap frogging the “low cost leader” to attain *permanent* low cost leadership at a marginal cost of $c = 0$.

Foreshadow of leap frogging

- Since the end game is essentially a two-period game, due to the presence of the zero cost absorbing state, it is not rich enough for us to observe *deterministic* leap frogging, excepting the pure strategy equilibrium where firm 1 invests with probability 1 when $c_1 > c_2$
- However, we can show (via numerical solutions) that for the mixed strategy equilibrium, $c_1 > c_2 \Longrightarrow \pi_1 > \pi_2$.
- We have been unable to prove this yet, but conjecture that the result is true in general.
- Thus, the “high cost follower” has a higher probability of investing and leap frogging the “low cost leader” to attain *permanent* low cost leadership at a marginal cost of $c = 0$.

Foreshadow of leap frogging

- Since the end game is essentially a two-period game, due to the presence of the zero cost absorbing state, it is not rich enough for us to observe *deterministic* leap frogging, excepting the pure strategy equilibrium where firm 1 invests with probability 1 when $c_1 > c_2$
- However, we can show (via numerical solutions) that for the mixed strategy equilibrium, $c_1 > c_2 \Longrightarrow \pi_1 > \pi_2$.
- We have been unable to prove this yet, but conjecture that the result is true in general.
- Thus, the "high cost follower" has a higher probability of investing and leap frogging the "low cost leader" to attain *permanent* low cost leadership at a marginal cost of $c = 0$.

Foreshadow of leap frogging

- Since the end game is essentially a two-period game, due to the presence of the zero cost absorbing state, it is not rich enough for us to observe *deterministic* leap frogging, excepting the pure strategy equilibrium where firm 1 invests with probability 1 when $c_1 > c_2$
- However, we can show (via numerical solutions) that for the mixed strategy equilibrium, $c_1 > c_2 \Longrightarrow \pi_1 > \pi_2$.
- We have been unable to prove this yet, but conjecture that the result is true in general.
- Thus, the "high cost follower" has a higher probability of investing and leap frogging the "low cost leader" to attain *permanent* low cost leadership at a marginal cost of $c = 0$.

Normal form representation of the $(c_1, c_2, 0)$ end game

		Firm 2: Invest	Firm 2: Don't Invest
Firm 1	Invest	$-K, c_1 - c_2 - K$	$\beta c_2/(1-\beta) - K, c_1 - c_2$
	Don't Invest	$0, c_1 - c_2 + \beta c_1/(1-\beta) - K$	$\beta V_1, c_1 - c_2 + \beta V_2$

Figure 1: End Game Payoff Matrix in state $(c_1, c_2, 0)$ with $c_1 > c_2$

Explanation of payoff matrix cells

- When both firms invest, they will both achieve the 0 cost absorbing state and make zero profits in every future period. The low-cost leader, firm 2, will earn profits of $c_2 - c_1$ in the period the investment occurs, and both will incur the fixed investment cost K
- When firm 1 invests and firm 2 doesn't, firm 1 attains permanent low cost leadership that allows it to charge of price of c_2. Its discounted payoff net of investment costs is $\beta c_2/(1-\beta) - K$. Firm 2 only earns the single period profit of $c_1 - c_2$.
- When firm 2 invests and firm 1 doesn't, firm 2 attains permanent low cost leadership that allows it to charge a price of c_1. Its discounted payoff is $c_1 - c_2 + \beta c_1/(1-\beta) - K$. Firm 1 earns 0.

Explanation of payoff matrix cells

- When both firms invest, they will both achieve the 0 cost absorbing state and make zero profits in every future period. The low-cost leader, firm 2, will earn profits of $c_2 - c_1$ in the period the investment occurs, and both will incur the fixed investment cost K
- When firm 1 invests and firm 2 doesn't, firm 1 attains permanent low cost leadership that allows it to charge of price of c_2. Its discounted payoff net of investment costs is $\beta c_2/(1-\beta) - K$. Firm 2 only earns the single period profit of $c_1 - c_2$.
- When firm 2 invests and firm 1 doesn't, firm 2 attains permanent low cost leadership that allows it to charge a price of c_1. Its discounted payoff is $c_1 - c_2 + \beta c_1/(1-\beta) - K$. Firm 1 earns 0.

Explanation of payoff matrix cells

- When both firms invest, they will both achieve the 0 cost absorbing state and make zero profits in every future period. The low-cost leader, firm 2, will earn profits of $c_2 - c_1$ in the period the investment occurs, and both will incur the fixed investment cost K
- When firm 1 invests and firm 2 doesn't, firm 1 attains permanent low cost leadership that allows it to charge of price of c_2. Its discounted payoff net of investment costs is $\beta c_2/(1-\beta) - K$. Firm 2 only earns the single period profit of $c_1 - c_2$.
- When firm 2 invests and firm 1 doesn't, firm 2 attains permanent low cost leadership that allows it to charge a price of c_1. Its discounted payoff is $c_1 - c_2 + \beta c_1/(1-\beta) - K$. Firm 1 earns 0.

Deriving the mixing probabilities

- The final case is the case where neither firm invests. Firm 1 earns expected discounted profits of V_1 in this case, and firm 2 earns V_2. For firm 1 we have

$$V_1 = 0 + \beta V_1 \implies V_1 = 0$$

Since firm 1's expected payoffs from not investing are zero regardless of whether firm 2 invests or not, it follows that if firm 2 invests with probability π_2, the expected payoff to firm 1 from investing must also be 0. This implies

$$-K\pi_2 + (1-\pi_2)[\beta c_2/(1-\beta) - K] = 0,$$

or

$$\pi_2 = \frac{\beta c_2/(1-\beta) - K}{\beta c_2/(1-\beta)}. \quad (2)$$

Deriving the mixing probabilities

- The final case is the case where neither firm invests. Firm 1 earns expected discounted profits of V_1 in this case, and firm 2 earns V_2. For firm 1 we have

$$V_1 = 0 + \beta V_1 \implies V_1 = 0$$

Since firm 1's expected payoffs from not investing are zero regardless of whether firm 2 invests or not, it follows that if firm 2 invests with probability π_2, the expected payoff to firm 1 from investing must also be 0. This implies

$$-K\pi_2 + (1-\pi_2)[\beta c_2/(1-\beta) - K] = 0,$$

or

$$\pi_2 = \frac{\beta c_2/(1-\beta) - K}{\beta c_2/(1-\beta)}. \tag{2}$$

Deriving the mixing probabilities

- The final case is the case where neither firm invests. Firm 1 earns expected discounted profits of V_1 in this case, and firm 2 earns V_2. For firm 1 we have

$$V_1 = 0 + \beta V_1 \implies V_1 = 0$$

Since firm 1's expected payoffs from not investing are zero regardless of whether firm 2 invests or not, it follows that if firm 2 invests with probability π_2, the expected payoff to firm 1 from investing must also be 0. This implies

$$-K\pi_2 + (1 - \pi_2)[\beta c_2/(1 - \beta) - K] = 0,$$

or

$$\pi_2 = \frac{\beta c_2/(1 - \beta) - K}{\beta c_2/(1 - \beta)}. \qquad (2)$$

Deriving the mixing probabilities, cont.

- For firm 2 we have the following equation for V_2

$$V_2 = \pi_1(c_1 - c_2) + (1 - \pi_1)(c_1 - c_2 + \beta V_2)$$

which implies that

$$V_2 = \frac{c_1 - c_2}{1 - \beta(1 - \pi_1)}.$$

- In order for firm 2 to be willing to pay a mixed investment strategy, its expected return from investing must also be equal to V_2, so we have

$$V_2 = \pi_1(c_1 - c_2 - K) + (1 - \pi_1)(c_1 - c_2 + \beta c_1/(1 - \beta) - K).$$

Combining these equations, we obtain a quadratic equation, one of whose roots is π_1.

Deriving the mixing probabilities, cont.

- For firm 2 we have the following equation for V_2

$$V_2 = \pi_1(c_1 - c_2) + (1 - \pi_1)(c_1 - c_2 + \beta V_2)$$

which implies that

$$V_2 = \frac{c_1 - c_2}{1 - \beta(1 - \pi_1)}.$$

- In order for firm 2 to be willing to pay a mixed investment strategy, its expected return from investing must also be equal to V_2, so we have

$$V_2 = \pi_1(c_1 - c_2 - K) + (1 - \pi_1)(c_1 - c_2 + \beta c_1/(1 - \beta) - K).$$

Combining these equations, we obtain a quadratic equation, one of whose roots is π_1.

Deriving the mixing probabilities, cont.

- For firm 2 we have the following equation for V_2

$$V_2 = \pi_1(c_1 - c_2) + (1 - \pi_1)(c_1 - c_2 + \beta V_2)$$

which implies that

$$V_2 = \frac{c_1 - c_2}{1 - \beta(1 - \pi_1)}.$$

- In order for firm 2 to be willing to pay a mixed investment strategy, its expected return from investing must also be equal to V_2, so we have

$$V_2 = \pi_1(c_1 - c_2 - K) + (1 - \pi_1)(c_1 - c_2 + \beta c_1/(1 - \beta) - K).$$

Combining these equations, we obtain a quadratic equation, one of whose roots is π_1.

Deriving the mixing probabilities, cont.

- For firm 2 we have the following equation for V_2

$$V_2 = \pi_1(c_1 - c_2) + (1 - \pi_1)(c_1 - c_2 + \beta V_2)$$

which implies that

$$V_2 = \frac{c_1 - c_2}{1 - \beta(1 - \pi_1)}.$$

- In order for firm 2 to be willing to pay a mixed investment strategy, its expected return from investing must also be equal to V_2, so we have

$$V_2 = \pi_1(c_1 - c_2 - K) + (1 - \pi_1)(c_1 - c_2 + \beta c_1/(1 - \beta) - K).$$

Combining these equations, we obtain a quadratic equation, one of whose roots is π_1.

Deriving the mixing probabilities, cont.

- For firm 2 we have the following equation for V_2

$$V_2 = \pi_1(c_1 - c_2) + (1 - \pi_1)(c_1 - c_2 + \beta V_2)$$

which implies that

$$V_2 = \frac{c_1 - c_2}{1 - \beta(1 - \pi_1)}.$$

- In order for firm 2 to be willing to pay a mixed investment strategy, its expected return from investing must also be equal to V_2, so we have

$$V_2 = \pi_1(c_1 - c_2 - K) + (1 - \pi_1)(c_1 - c_2 + \beta c_1/(1 - \beta) - K).$$

Combining these equations, we obtain a quadratic equation, one of whose roots is π_1.

Leapfrogging in the end game

- **Conjecture:** *If $c_1 > c_2$, then in the unique mixed strategy equilibrium of the pure Bertrand dynamic investment and pricing game in state $(c_1, c_2, 0)$ we have $\pi_1 > \pi_2$.*
- We have found this result to hold in all numerical solutions of the game we have examined so far. However the proof of this conjecture turns out to be surprisingly difficult and we have been unable to provide a general proof so far.
- Note that the lack off coordination between the two firms in the mixed strategy equilibrium is very undesirable (from their standpoint), since it implies a positive probability of inefficient simultaneous investment by the two firms.
- The $1000 question is, can more efficient *pure strategy* coordination mechanisms be established as equilibria to the full game?

Leapfrogging in the end game

- **Conjecture:** *If $c_1 > c_2$, then in the unique mixed strategy equilibrium of the pure Bertrand dynamic investment and pricing game in state $(c_1, c_2, 0)$ we have $\pi_1 > \pi_2$.*
- We have found this result to hold in all numerical solutions of the game we have examined so far. However the proof of this conjecture turns out to be surprisingly difficult and we have been unable to provide a general proof so far.
- Note that the lack off coordination between the two firms in the mixed strategy equilibrium is very undesirable (from their standpoint), since it implies a positive probability of inefficient simultaneous investment by the two firms.
- The $1000 question is, can more efficient *pure strategy* coordination mechanisms be established as equilibria to the full game?

Leapfrogging in the end game

- **Conjecture:** *If $c_1 > c_2$, then in the unique mixed strategy equilibrium of the pure Bertrand dynamic investment and pricing game in state $(c_1, c_2, 0)$ we have $\pi_1 > \pi_2$.*
- We have found this result to hold in all numerical solutions of the game we have examined so far. However the proof of this conjecture turns out to be surprisingly difficult and we have been unable to provide a general proof so far.
- Note that the lack off coordination between the two firms in the mixed strategy equilibrium is very undesirable (from their standpoint), since it implies a positive probability of inefficient simultaneous investment by the two firms.
- The $1000 question is, can more efficient *pure strategy* coordination mechanisms be established as equilibria to the full game?

Leapfrogging in the end game

- **Conjecture:** *If $c_1 > c_2$, then in the unique mixed strategy equilibrium of the pure Bertrand dynamic investment and pricing game in state $(c_1, c_2, 0)$ we have $\pi_1 > \pi_2$.*
- We have found this result to hold in all numerical solutions of the game we have examined so far. However the proof of this conjecture turns out to be surprisingly difficult and we have been unable to provide a general proof so far.
- Note that the lack off coordination between the two firms in the mixed strategy equilibrium is very undesirable (from their standpoint), since it implies a positive probability of inefficient simultaneous investment by the two firms.
- The $1000 question is, can more efficient *pure strategy* coordination mechanisms be established as equilibria to the full game?

Solving the full game

- With the end game solutions in hand, we are now ready to proceed to discuss the solution of the full game.
- The end game equilibria give us some insight into what can happen in the full game, but the possibilities in the full game are much richer, since unlike in the end game, if one firm leap frogs its opponent, the game does not end, but rather the firms must anticipate additional leap frogging and cost reducing investments in the future.
- In particular, forms of *dynamic coordination* may be possible that are not present in the end game, which is closer to a "two stage" game than to an infinite horizon game.

Solving the full game

- With the end game solutions in hand, we are now ready to proceed to discuss the solution of the full game.
- The end game equilibria give us some insight into what can happen in the full game, but the possibilities in the full game are much richer, since unlike in the end game, if one firm leap frogs its opponent, the game does not end, but rather the firms must anticipate additional leap frogging and cost reducing investments in the future.
- In particular, forms of *dynamic coordination* may be possible that are not present in the end game, which is closer to a "two stage" game than to an infinite horizon game.

Solving the full game

- With the end game solutions in hand, we are now ready to proceed to discuss the solution of the full game.
- The end game equilibria give us some insight into what can happen in the full game, but the possibilities in the full game are much richer, since unlike in the end game, if one firm leap frogs its opponent, the game does not end, but rather the firms must anticipate additional leap frogging and cost reducing investments in the future.
- In particular, forms of *dynamic coordination* may be possible that are not present in the end game, which is closer to a "two stage" game than to an infinite horizon game.

Equilibrium selection rules

- When there are multiple equilibria to the state-specific investment "stage games", we can construct bigger sets of equilibria in the overall game, which are analogous to *supergame equilibria* in the theory of repeated games
- Thus the state-specific equilibria are the analogs of equilibria in the *stage game* and by adopting various rules for selecting among the various equilibria in the stage games, we can generate different types of equilibrium in the "supergame."
- We will initially *deterministic* equilibrium selection rules, i.e. a function that picks out one of the set of equilibria in each possible state of the game, (c_1, c_2, c).

Equilibrium selection rules

- When there are multiple equilibria to the state-specific investment “stage games”, we can construct bigger sets of equilibria in the overall game, which are analogous to *supergame equilibria* in the theory of repeated games
- Thus the state-specific equilibria are the analogs of equilibria in the *stage game* and by adopting various rules for selecting among the various equilibria in the stage games, we can generate different types of equilibrium in the “supergame.”
- We will initially *deterministic* equilibrium selection rules, i.e. a function that picks out one of the set of equilibria in each possible state of the game, (c_1, c_2, c).

Equilibrium selection rules

- When there are multiple equilibria to the state-specific investment "stage games", we can construct bigger sets of equilibria in the overall game, which are analogous to *supergame equilibria* in the theory of repeated games
- Thus the state-specific equilibria are the analogs of equilibria in the *stage game* and by adopting various rules for selecting among the various equilibria in the stage games, we can generate different types of equilibrium in the "supergame."
- We will initially *deterministic* equilibrium selection rules, i.e. a function that picks out one of the set of equilibria in each possible state of the game, (c_1, c_2, c).

Leapfrogging equilibrium selection rules

- Leapfrogging behavior can be generated via state-specific equilibrium selection rules of the following form:
- If $c_1 > c_2 \geq c$, then only firm 1 invests when c is sufficiently low
- If $c_2 > c_1 \geq c$, then only firm 2 invests when c is sufficiently low
- If $c_1 = c_2$, then firms play the mixed strategy equilibrium.
- The zero expected future profits implied under the mixed strategy equilibrium serves the role of a “punishment threat” if either firm deviates and invests when it is not its “turn”

Leapfrogging equilibrium selection rules

- Leapfrogging behavior can be generated via state-specific equilibrium selection rules of the following form:
- If $c_1 > c_2 \geq c$, then only firm 1 invests when c is sufficiently low
- If $c_2 > c_1 \geq c$, then only firm 2 invests when c is sufficiently low
- If $c_1 = c_2$, then firms play the mixed strategy equilibrium.
- The zero expected future profits implied under the mixed strategy equilibrium serves the role of a "punishment threat" if either firm deviates and invests when it is not its "turn"

Leapfrogging equilibrium selection rules

- Leapfrogging behavior can be generated via state-specific equilibrium selection rules of the following form:
- If $c_1 > c_2 \geq c$, then only firm 1 invests when c is sufficiently low
- If $c_2 > c_1 \geq c$, then only firm 2 invests when c is sufficiently low
- If $c_1 = c_2$, then firms play the mixed strategy equilibrium.
- The zero expected future profits implied under the mixed strategy equilibrium serves the role of a "punishment threat" if either firm deviates and invests when it is not its "turn"

Leapfrogging equilibrium selection rules

- Leapfrogging behavior can be generated via state-specific equilibrium selection rules of the following form:
- If $c_1 > c_2 \geq c$, then only firm 1 invests when c is sufficiently low
- If $c_2 > c_1 \geq c$, then only firm 2 invests when c is sufficiently low
- If $c_1 = c_2$, then firms play the mixed strategy equilibrium.
- The zero expected future profits implied under the mixed strategy equilibrium serves the role of a "punishment threat" if either firm deviates and invests when it is not its "turn"

Leapfrogging equilibrium selection rules

- Leapfrogging behavior can be generated via state-specific equilibrium selection rules of the following form:
- If $c_1 > c_2 \geq c$, then only firm 1 invests when c is sufficiently low
- If $c_2 > c_1 \geq c$, then only firm 2 invests when c is sufficiently low
- If $c_1 = c_2$, then firms play the mixed strategy equilibrium.
- The zero expected future profits implied under the mixed strategy equilibrium serves the role of a "punishment threat" if either firm deviates and invests when it is not its "turn"

Solving the Bellman equations in the full game

- In order to solve the full game, it is helpful to rewrite the firms' Bellman equations in the following way,

$$
\begin{aligned}
v_0^1(c_1,c_2,c) &= r^1(c_1,c_2) \\
&+ \beta\Big[P_1^2(c_1,c_2,c)H^1(c_1,c,c) \\
&+ (1-P_1^2(c_1,c_2,c))H^1(c_1,c_2,c)\Big] \\
v_1^1(c_1,c_2,c) &= r^1(c_1,c_2) - K(c) \\
&+ \beta\Big[P_1^2(c_1,c_2,c)H^1(c,c,c) \\
&+ (1-P_1^2(c_1,c_2,c))H^1(c,c_2,c)\Big]
\end{aligned}
$$

Solving the Bellman equations in the full game

- In order to solve the full game, it is helpful to rewrite the firms' Bellman equations in the following way,

$$\begin{aligned} v_0^1(c_1, c_2, c) &= r^1(c_1, c_2) \\ &+ \beta \Big[P_1^2(c_1, c_2, c) H^1(c_1, c, c) \\ &+ (1 - P_1^2(c_1, c_2, c)) H^1(c_1, c_2, c) \Big] \\ v_1^1(c_1, c_2, c) &= r^1(c_1, c_2) - K(c) \\ &+ \beta \Big[P_1^2(c_1, c_2, c) H^1(c, c, c) \\ &+ (1 - P_1^2(c_1, c_2, c)) H^1(c, c_2, c) \Big] \end{aligned}$$

Solving the full game Bellman equations, cont.

- The function H^1 is given by

$$\begin{aligned} H^1(c_1, c_2, c) &= p(c) \int_0^c \phi(v_0^1(c_1, c_2, c'), v_1^1(c_1, c_2, c')) f(c') dc' \\ &+ (1 - p(c)) \phi(v_0^1(c_1, c_2, c), v_1^1(c_1, c_2, c)), \end{aligned}$$

where $p(c)$ is the probability that a cost-reducing innovation will occur, and $f(c')$ is the Beta/uniformm density of the new (lower) cost of production under the current state of the art conditional on an innovation having occurred.

Solving the full game Bellman equations, cont.

- The function H^1 is given by

$$\begin{aligned} H^1(c_1, c_2, c) &= p(c)\int_0^c \phi(v_0^1(c_1, c_2, c'), v_1^1(c_1, c_2, c'))f(c')dc' \\ &+ (1 - p(c))\phi(v_0^1(c_1, c_2, c), v_1^1(c_1, c_2, c)), \end{aligned}$$

where $p(c)$ is the probability that a cost-reducing innovation will occur, and $f(c')$ is the Beta/uniformm density of the new (lower) cost of production under the current state of the art conditional on an innovation having occurred.

Solving the full game Bellman equations, cont.

- The function H^1 is given by

$$\begin{aligned} H^1(c_1, c_2, c) &= p(c)\int_0^c \phi(v_0^1(c_1, c_2, c'), v_1^1(c_1, c_2, c'))f(c')dc' \\ &+ (1 - p(c))\phi(v_0^1(c_1, c_2, c), v_1^1(c_1, c_2, c)), \end{aligned}$$

where $p(c)$ is the probability that a cost-reducing innovation will occur, and $f(c')$ is the Beta/uniformm density of the new (lower) cost of production under the current state of the art conditional on an innovation having occurred.

Solving the full game Bellman equations, cont.

- If we set the arguments (c_1, c_2, c) to the equation for v_0^1 to (c, c, c), and similarly in equation for v_1^1, we deduce that

$$v_1^1(c, c, c) = v_0^1(c, c, c) - K(c).$$

Clearly, if the firms have all invested and have in place the state of the art production technology, there is no further incentive for either firm to invest.

- For the same reasons we have

$$v_1^1(c, c_2, c) = v_0^1(c, c_2, c) - K(c).$$

Solving the full game Bellman equations, cont.

- If we set the arguments (c_1, c_2, c) to the equation for v_0^1 to (c, c, c), and similarly in equation for v_1^1, we deduce that

$$v_1^1(c, c, c) = v_0^1(c, c, c) - K(c).$$

  Clearly, if the firms have all invested and have in place the state of the art production technology, there is no further incentive for either firm to invest.

- For the same reasons we have

$$v_1^1(c, c_2, c) = v_0^1(c, c_2, c) - K(c).$$

Solving the full game Bellman equations, cont.

- Similar to the strategy we used to solve the value functions (v_0^i, v_1^i) $i = 1, 2$ in the end game, we can Newton's method to compute the unique fixed point $v_0^1(c, c, c)$.
- Similarly, we can solve for $v_0^1(0, c_2, 0)$. Finally, to solve for $v_0^1(c_1, c_2, c)$ we note that we can use the solutions for $v_0^1(c, c, c)$ and $v_0^1(c, c_2, c)$ to obtain $v_1^1(c, c, c)$ and $v_1^1(c, c_2, c)$, we can compute $v_1^1(c_1, c_2, c)$ by substituting these values into the Bellman equation for $v_1^1(c_1, c_2, c)$.
- Then we use this solution and Newton's method to compute $v_0^1(c_1, c_2, c)$.

Solving the full game Bellman equations, cont.

- Similar to the strategy we used to solve the value functions (v_0^i, v_1^i) $i = 1, 2$ in the end game, we can Newton's method to compute the unique fixed point $v_0^1(c, c, c)$.
- Similarly, we can solve for $v_0^1(0, c_2, 0)$. Finally, to solve for $v_0^1(c_1, c_2, c)$ we note that we can use the solutions for $v_0^1(c, c, c)$ and $v_0^1(c, c_2, c)$ to obtain $v_1^1(c, c, c)$ and $v_1^1(c, c_2, c)$, we can compute $v_1^1(c_1, c_2, c)$ by substituting these values into the Bellman equation for $v_1^1(c_1, c_2, c)$.
- Then we use this solution and Newton's method to compute $v_0^1(c_1, c_2, c)$.

Solving the full game Bellman equations, cont.

- Similar to the strategy we used to solve the value functions (v_0^i, v_1^i) $i = 1, 2$ in the end game, we can Newton's method to compute the unique fixed point $v_0^1(c, c, c)$.
- Similarly, we can solve for $v_0^1(0, c_2, 0)$. Finally, to solve for $v_0^1(c_1, c_2, c)$ we note that we can use the solutions for $v_0^1(c, c, c)$ and $v_0^1(c, c_2, c)$ to obtain $v_1^1(c, c, c)$ and $v_1^1(c, c_2, c)$, we can compute $v_1^1(c_1, c_2, c)$ by substituting these values into the Bellman equation for $v_1^1(c_1, c_2, c)$.
- Then we use this solution and Newton's method to compute $v_0^1(c_1, c_2, c)$.

The state-specific fixed point problem

- Following the procedure we used to solve for equilibria in the end game, the set of “pointwise” equilibria for each state (c_1, c_2, c) can be computed from the following fixed point equation

$$P_1^1 = \frac{\exp\{v_1^1(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))/\eta\}}{D(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))}$$

where

$$\begin{aligned} & D(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1)) \\ = \; & \exp\{v_0^1(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))/\eta\} \\ + \; & \exp\{v_1^1(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))/\eta\}. \end{aligned}$$

The state-specific fixed point problem

- Following the procedure we used to solve for equilibria in the end game, the set of "pointwise" equilibria for each state (c_1, c_2, c) can be computed from the following fixed point equation

$$P_1^1 = \frac{\exp\{v_1^1(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))/\eta\}}{D(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))}$$

where

$$\begin{aligned} & D(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1)) \\ = \; & \exp\{v_0^1(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))/\eta\} \\ + \; & \exp\{v_1^1(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))/\eta\}. \end{aligned}$$

The state-specific fixed point problem

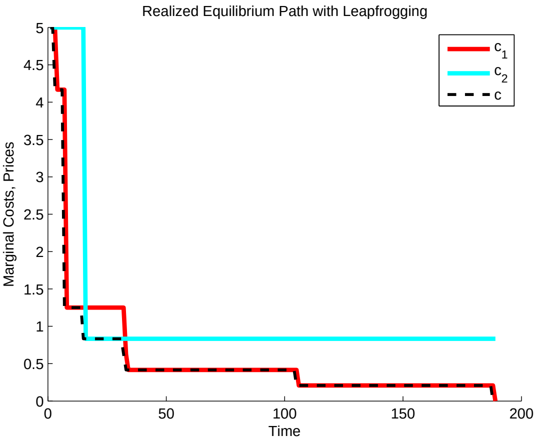
- Following the procedure we used to solve for equilibria in the end game, the set of "pointwise" equilibria for each state (c_1, c_2, c) can be computed from the following fixed point equation

$$P_1^1 = \frac{\exp\{v_1^1(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))/\eta\}}{D(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))}$$

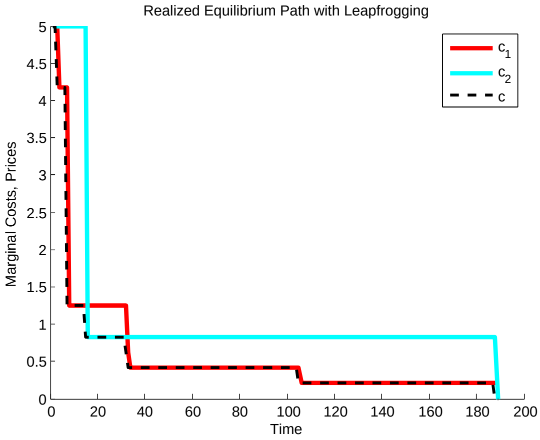
where

$$\begin{aligned} & D(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1)) \\ = \; & \exp\{v_0^1(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))/\eta\} \\ + \; & \exp\{v_1^1(c_1, c_2, c, P_1^2(c_1, c_2, c, P_1^1))/\eta\}. \end{aligned}$$

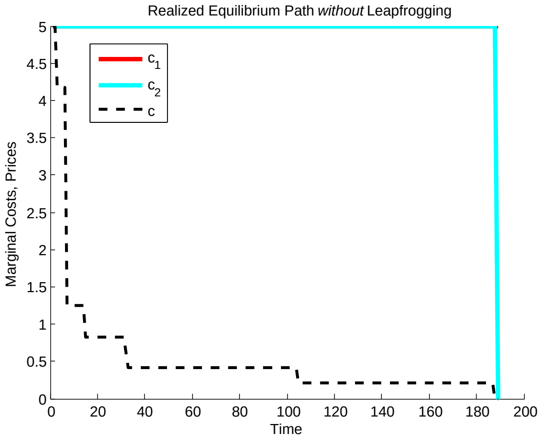
Equilibrium realization with leap frogging



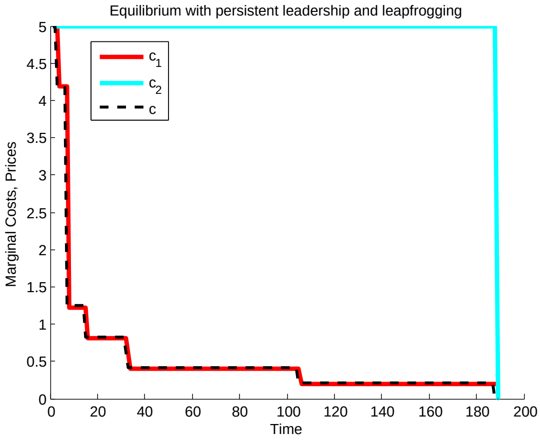
Another equilibrium with leap frogging



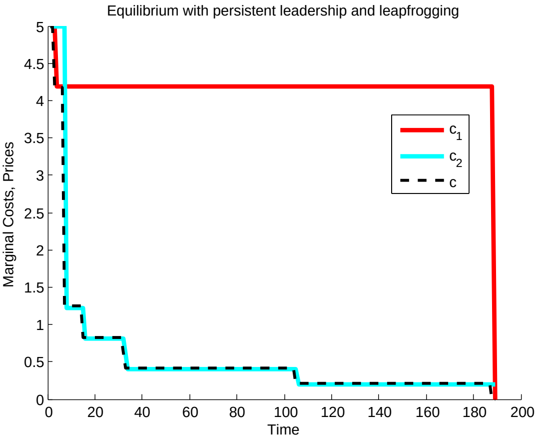
Another equilibrium *without* leap frogging



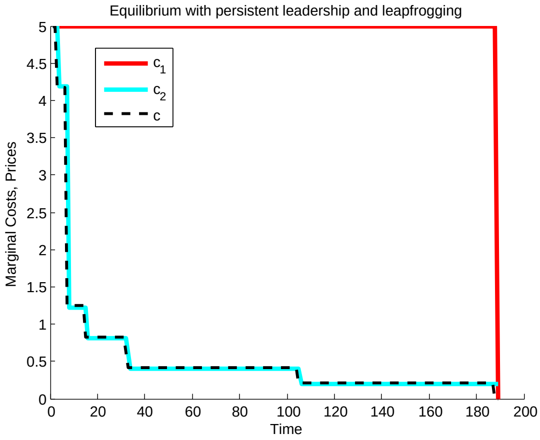
An equilibrium with persistent leadership and “sniping”



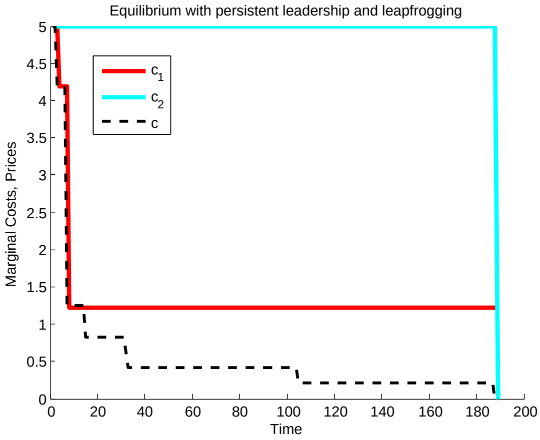
Another equilibrium with leadership and "sniping"



An equilibrium where firm 2 leads and firm 1 snipes



A final equilibrium



Socially optimal investment

- We compare investment and pricing outcomes from various possible equilibria of the Bertrand duopoly game to those that would emerge under the social planning solution
- In the simple static model of Bertrand price competition, the duopoly solution is well known to be efficient and coincide with the social planning solution: both firms earn zero profits and produce at a price equal to marginal cost.
- But a static analysis begs the question of potential redundancy in production costs among the two firms. The static model treats the investment costs necessary to produce the production plant of the two firms as a sunk cost, and it is ignored in the social planning calculation.

Socially optimal investment

- We compare investment and pricing outcomes from various possible equilibria of the Bertrand duopoly game to those that would emerge under the social planning solution
- In the simple static model of Bertrand price competition, the duopoly solution is well known to be efficient and coincide with the social planning solution: both firms earn zero profits and produce at a price equal to marginal cost.
- But a static analysis begs the question of potential redundancy in production costs among the two firms. The static model treats the investment costs necessary to produce the production plant of the two firms as a sunk cost, and it is ignored in the social planning calculation.

Socially optimal investment

- We compare investment and pricing outcomes from various possible equilibria of the Bertrand duopoly game to those that would emerge under the social planning solution
- In the simple static model of Bertrand price competition, the duopoly solution is well known to be efficient and coincide with the social planning solution: both firms earn zero profits and produce at a price equal to marginal cost.
- But a static analysis begs the question of potential redundancy in production costs among the two firms. The static model treats the investment costs necessary to produce the production plant of the two firms as a sunk cost, and it is ignored in the social planning calculation.

Socially optimal investment, cont.

- In a dynamic analysis, the social planner does/should account for these investment costs. Clearly, under our assumptions about production technology (any plant has unlimited production capacity at a constant marginal cost of production) it only makes sense for the social planner to operate just a *single* plant.
- Thus, we expect that duopoly equilibria are typically *inefficient* in the sense that there is redundant investment costs that would not be incurred by a social planner.
- However there are “monopoly” and near-monopoly equilibria where one or the other of the firms does all of the investing.
- How close to full efficiency are these monopoly equilibria?

Socially optimal investment, cont.

- In a dynamic analysis, the social planner does/should account for these investment costs. Clearly, under our assumptions about production technology (any plant has unlimited production capacity at a constant marginal cost of production) it only makes sense for the social planner to operate just a *single* plant.
- Thus, we expect that duopoly equilibria are typically *inefficient* in the sense that there is redundant investment costs that would not be incurred by a social planner.
- However there are “monopoly” and near-monopoly equilibria where one or the other of the firms does all of the investing.
- How close to full efficiency are these monopoly equilibria?

Socially optimal investment, cont.

- In a dynamic analysis, the social planner does/should account for these investment costs. Clearly, under our assumptions about production technology (any plant has unlimited production capacity at a constant marginal cost of production) it only makes sense for the social planner to operate just a *single* plant.
- Thus, we expect that duopoly equilibria are typically *inefficient* in the sense that there is redundant investment costs that would not be incurred by a social planner.
- However there are "monopoly" and near-monopoly equilibria where one or the other of the firms does all of the investing.
- How close to full efficiency are these monopoly equilibria?

Socially optimal investment, cont.

- In a dynamic analysis, the social planner does/should account for these investment costs. Clearly, under our assumptions about production technology (any plant has unlimited production capacity at a constant marginal cost of production) it only makes sense for the social planner to operate just a *single* plant.
- Thus, we expect that duopoly equilibria are typically *inefficient* in the sense that there is redundant investment costs that would not be incurred by a social planner.
- However there are "monopoly" and near-monopoly equilibria where one or the other of the firms does all of the investing.
- How close to full efficiency are these monopoly equilibria?

Socially optimal investment, cont.

- If we assume that consumers have quasi-linear preferences so that the surplus they receive from consuming the good at a price of p is $u - p$, then the social planning solution involves selling the good at marginal cost of production, and adopting an efficient investment strategy that minimizes the expected discounted costs of production.
- Let c_1 be the marginal cost of production of the current (and only) production plant operated by the social planner
- Let c be the marginal cost of production of the current state of the art production process, which we continue to assume evolves as an exogenous first order Markov process with transition probability $\pi(c'|c)$ and its evolution is beyond the purview of the social planner.

Socially optimal investment, cont.

- If we assume that consumers have quasi-linear preferences so that the surplus they receive from consuming the good at a price of p is $u - p$, then the social planning solution involves selling the good at marginal cost of production, and adopting an efficient investment strategy that minimizes the expected discounted costs of production.
- Let c_1 be the marginal cost of production of the current (and only) production plant operated by the social planner
- Let c be the marginal cost of production of the current state of the art production process, which we continue to assume evolves as an exogenous first order Markov process with transition probability $\pi(c'|c)$ and its evolution is beyond the purview of the social planner.

Socially optimal investment, cont.

- If we assume that consumers have quasi-linear preferences so that the surplus they receive from consuming the good at a price of p is $u - p$, then the social planning solution involves selling the good at marginal cost of production, and adopting an efficient investment strategy that minimizes the expected discounted costs of production.
- Let c_1 be the marginal cost of production of the current (and only) production plant operated by the social planner
- Let c be the marginal cost of production of the current state of the art production process, which we continue to assume evolves as an exogenous first order Markov process with transition probability $\pi(c'|c)$ and its evolution is beyond the purview of the social planner.

Socially optimal investment, cont.

- The social planning problem reduces to solving for an *optimal investment strategy* that minimizes the expected discounted costs of producing the good.
- Since consumers are in effect risk-neutral with regard to the price of the good (due to the quasi linearity assumption), there is no benefit to "price stabilization" on the part of the social planner.
- The social planner merely solves and adopts the optimal investment strategy that determines when the current plant should be replaced by a new, cheaper state of the art plant, and it should provide the good to consumers in each period at a price equal to the current marginal cost of production.

Socially optimal investment, cont.

- The social planning problem reduces to solving for an *optimal investment strategy* that minimizes the expected discounted costs of producing the good.
- Since consumers are in effect risk-neutral with regard to the price of the good (due to the quasi linearity assumption), there is no benefit to “price stabilization” on the part of the social planner.
- The social planner merely solves and adopts the optimal investment strategy that determines when the current plant should be replaced by a new, cheaper state of the art plant, and it should provide the good to consumers in each period at a price equal to the current marginal cost of production.

Socially optimal investment, cont.

- The social planning problem reduces to solving for an *optimal investment strategy* that minimizes the expected discounted costs of producing the good.
- Since consumers are in effect risk-neutral with regard to the price of the good (due to the quasi linearity assumption), there is no benefit to "price stabilization" on the part of the social planner.
- The social planner merely solves and adopts the optimal investment strategy that determines when the current plant should be replaced by a new, cheaper state of the art plant, and it should provide the good to consumers in each period at a price equal to the current marginal cost of production.

The Planner's Bellman Equation

- Let $V(c_1, c)$ be the present discounted value of costs of production when the existing plant operated by the social planner has marginal cost c_1 and c is the marginal cost of state of the art production technology
- As in the duopoly problem, the social planner can acquire the state of the art technology with one period delay after incurring an investment cost of $K(c)$.
- The Bellman equation for the social planner is

$$V(c_1, c) = \min \Big[c_1 + \beta \int_0^c V(c_1, c')\pi(dc'|c), \\ c_1 + K(c) + \beta \int_0^c V(c, c')\pi(dc'|c) \Big].$$

The Planner's Bellman Equation

- Let $V(c_1, c)$ be the present discounted value of costs of production when the existing plant operated by the social planner has marginal cost c_1 and c is the marginal cost of state of the art production technology
- As in the duopoly problem, the social planner can acquire the state of the art technology with one period delay after incurring an investment cost of $K(c)$.
- The Bellman equation for the social planner is

$$V(c_1, c) = \min \left[c_1 + \beta \int_0^c V(c_1, c')\pi(dc'|c), \right.$$
$$\left. c_1 + K(c) + \beta \int_0^c V(c, c')\pi(dc'|c) \right].$$

The Planner's Bellman Equation

- Let $V(c_1, c)$ be the present discounted value of costs of production when the existing plant operated by the social planner has marginal cost c_1 and c is the marginal cost of state of the art production technology
- As in the duopoly problem, the social planner can acquire the state of the art technology with one period delay after incurring an investment cost of $K(c)$.
- The Bellman equation for the social planner is

$$
\begin{aligned}
V(c_1, c) &= \min\Big[c_1 + \beta \int_0^c V(c_1, c')\pi(dc'|c), \\
&\qquad c_1 + K(c) + \beta \int_0^c V(c, c')\pi(dc'|c) \Big].
\end{aligned}
$$

Characterizing the optimal investment strategy

- The optimal investment strategy can be easily seen to take the form of a *cutoff rule*
- The social planner invests in the state of the art technology when the current state of the art c falls below a cutoff threshold $\overline{c}(c_1)$, and keeps producing using its existing plant with marginal cost c_1 otherwise.
- The optimal threshold $\overline{c}(c_1)$ is the solution to the following equation

$$K(\overline{c}(c_1)) = \beta \int_0^{\overline{c}(c_1)} \left[V(c_1, c') - V(\overline{c}(c_1), c')\right] \pi(dc'|\overline{c}(c_1)).$$

Characterizing the optimal investment strategy

- The optimal investment strategy can be easily seen to take the form of a *cutoff rule*
- The social planner invests in the state of the art technology when the current state of the art c falls below a cutoff threshold $\overline{c}(c_1)$, and keeps producing using its existing plant with marginal cost c_1 otherwise.
- The optimal threshold $\overline{c}(c_1)$ is the solution to the following equation

$$K(\overline{c}(c_1)) = \beta \int_0^{\overline{c}(c_1)} \left[V(c_1, c') - V(\overline{c}(c_1), c')\right] \pi(dc' | \overline{c}(c_1)).$$

Characterizing the optimal investment strategy

- The optimal investment strategy can be easily seen to take the form of a *cutoff rule*
- The social planner invests in the state of the art technology when the current state of the art c falls below a cutoff threshold $\overline{c}(c_1)$, and keeps producing using its existing plant with marginal cost c_1 otherwise.
- The optimal threshold $\overline{c}(c_1)$ is the solution to the following equation

$$K(\overline{c}(c_1)) = \beta \int_0^{\overline{c}(c_1)} \left[V(c_1, c') - V(\overline{c}(c_1), c') \right] \pi(dc' | \overline{c}(c_1)).$$

Characterizing the optimal investment strategy

- The optimal investment strategy can be easily seen to take the form of a *cutoff rule*
- The social planner invests in the state of the art technology when the current state of the art c falls below a cutoff threshold $\overline{c}(c_1)$, and keeps producing using its existing plant with marginal cost c_1 otherwise.
- The optimal threshold $\overline{c}(c_1)$ is the solution to the following equation

$$K(\overline{c}(c_1)) = \beta \int_0^{\overline{c}(c_1)} \left[V(c_1, c') - V(\overline{c}(c_1), c') \right] \pi(dc' | \overline{c}(c_1)).$$

Marginal cost pricing in the social optimum

- This equation tells us that at the optimal cutoff $\overline{c}(c_1)$, the social planner is indifferent between continuing to produce using its current plant with marginal cost c_1 or investing in the state of the art plant with marginal cost of production $\overline{c}(c_1)$.
- This implies that the decrease in expected discounted production costs is exactly equal to the cost of the investment when c is equal to the cutoff threshold $\overline{c}(c_1)$.
- When c is above the threshold, the drop in operating costs is insufficiently large to justify undertaking the investment, and when c is below the threshold, there is a strictly positive net benefit from investing.
- Note that the pure monopoly solution is socially efficient, although all surplus is captured by the monopolist.

Marginal cost pricing in the social optimum

- This equation tells us that at the optimal cutoff $\overline{c}(c_1)$, the social planner is indifferent between continuing to produce using its current plant with marginal cost c_1 or investing in the state of the art plant with marginal cost of production $\overline{c}(c_1)$.
- This implies that the decrease in expected discounted production costs is exactly equal to the cost of the investment when c is equal to the cutoff threshold $\overline{c}(c_1)$.
- When c is above the threshold, the drop in operating costs is insufficiently large to justify undertaking the investment, and when c is below the threshold, there is a strictly positive net benefit from investing.
- Note that the pure monopoly solution is socially efficient, although all surplus is captured by the monopolist.

Marginal cost pricing in the social optimum

- This equation tells us that at the optimal cutoff $\overline{c}(c_1)$, the social planner is indifferent between continuing to produce using its current plant with marginal cost c_1 or investing in the state of the art plant with marginal cost of production $\overline{c}(c_1)$.
- This implies that the decrease in expected discounted production costs is exactly equal to the cost of the investment when c is equal to the cutoff threshold $\overline{c}(c_1)$.
- When c is above the threshold, the drop in operating costs is insufficiently large to justify undertaking the investment, and when c is below the threshold, there is a strictly positive net benefit from investing.
- Note that the pure monopoly solution is socially efficient, although all surplus is captured by the monopolist.

Marginal cost pricing in the social optimum

- This equation tells us that at the optimal cutoff $\overline{c}(c_1)$, the social planner is indifferent between continuing to produce using its current plant with marginal cost c_1 or investing in the state of the art plant with marginal cost of production $\overline{c}(c_1)$.
- This implies that the decrease in expected discounted production costs is exactly equal to the cost of the investment when c is equal to the cutoff threshold $\overline{c}(c_1)$.
- When c is above the threshold, the drop in operating costs is insufficiently large to justify undertaking the investment, and when c is below the threshold, there is a strictly positive net benefit from investing.
- Note that the pure monopoly solution is socially efficient, although all surplus is captured by the monopolist.

Discussion of previous related work

- Goettler and Gordon (2010) “Does AMD Spur Intel to Innovate More?”.
- The authors solve and estimate a very ambitious dynamic duopoly model with Bertrand pricing and continuous R&D expenditures that increase the chance of technological innovation in processor technology
- The paper is innovative in having forward looking *consumers* as well as forward looking firms.
- Consumers solve optimal stopping problems of when to upgrade their processor, with rational expectations of the probability distribution over innovations

Discussion of previous related work

- Goettler and Gordon (2010) “Does AMD Spur Intel to Innovate More?”.
- The authors solve and estimate a very ambitious dynamic duopoly model with Bertrand pricing and continuous R&D expenditures that increase the chance of technological innovation in processor technology
- The paper is innovative in having forward looking *consumers* as well as forward looking firms.
- Consumers solve optimal stopping problems of when to upgrade their processor, with rational expectations of the probability distribution over innovations

Discussion of previous related work

- Goettler and Gordon (2010) “Does AMD Spur Intel to Innovate More?”.
- The authors solve and estimate a very ambitious dynamic duopoly model with Bertrand pricing and continuous R&D expenditures that increase the chance of technological innovation in processor technology
- The paper is innovative in having forward looking *consumers* as well as forward looking firms.
- Consumers solve optimal stopping problems of when to upgrade their processor, with rational expectations of the probability distribution over innovations

Discussion of previous related work

- Goettler and Gordon (2010) “Does AMD Spur Intel to Innovate More?”.
- The authors solve and estimate a very ambitious dynamic duopoly model with Bertrand pricing and continuous R&D expenditures that increase the chance of technological innovation in processor technology
- The paper is innovative in having forward looking *consumers* as well as forward looking firms.
- Consumers solve optimal stopping problems of when to upgrade their processor, with rational expectations of the probability distribution over innovations

Goettler and Gordon

- Authors claim that equilibrium of their model is *unique* as R&D investment satisfies the UIC condition of Doraszelski and Satterthwaite
- They show that a monopolist chip producer *innovates faster* (and spends more on innovation) compared to the duopoly equilibrium.
- Equilibrium R&D investment is lower under either duopoly or monopoly than the social optimum investment

Goettler and Gordon

- Authors claim that equilibrium of their model is *unique* as R&D investment satisfies the UIC condition of Doraszelski and Satterthwaite
- They show that a monopolist chip producer *innovates faster* (and spends more on innovation) compared to the duopoly equilibrium.
- Equilibrium R&D investment is lower under either duopoly or monopoly than the social optimum investment

Goettler and Gordon

- Authors claim that equilibrium of their model is *unique* as R&D investment satisfies the UIC condition of Doraszelski and Satterthwaite
- They show that a monopolist chip producer *innovates faster* (and spends more on innovation) compared to the duopoly equilibrium.
- Equilibrium R&D investment is lower under either duopoly or monopoly than the social optimum investment

Giovannetti

- The paper by Emanuele Giovannetti (2001 *IER*) “Persistent Leapfrogging in Bertrand Duopoly” is the closest previous work to this paper.
- We were unaware of this work until after we had completed the first draft of this paper
- Giovannetti considers a framework very close to ours, with pure Bertrand price competition and a discrete choice of whether to acquire the state of the art production technology in discrete time, infinite horizon framework
- The major differences are: 1) technological progress is *deterministic* with marginal costs decreasing at a geometric rate in each period, 2) aggregate demand for the good is given by an exogenously specified constant elasticity demand function

Giovannetti

- The paper by Emanuele Giovannetti (2001 *IER*) “Persistent Leapfrogging in Bertrand Duopoly” is the closest previous work to this paper.
- We were unaware of this work until after we had completed the first draft of this paper
- Giovannetti considers a framework very close to ours, with pure Bertrand price competition and a discrete choice of whether to acquire the state of the art production technology in discrete time, infinite horizon framework
- The major differences are: 1) technological progress is *deterministic* with marginal costs decreasing at a geometric rate in each period, 2) aggregate demand for the good is given by an exogenously specified constant elasticity demand function

Giovannetti

- The paper by Emanuele Giovannetti (2001 *IER*) "Persistent Leapfrogging in Bertrand Duopoly" is the closest previous work to this paper.
- We were unaware of this work until after we had completed the first draft of this paper
- Giovannetti considers a framework very close to ours, with pure Bertrand price competition and a discrete choice of whether to acquire the state of the art production technology in discrete time, infinite horizon framework
- The major differences are: 1) technological progress is *deterministic* with marginal costs decreasing at a geometric rate in each period, 2) aggregate demand for the good is given by an exogenously specified constant elasticity demand function

Giovannetti

- The paper by Emanuele Giovannetti (2001 *IER*) "Persistent Leapfrogging in Bertrand Duopoly" is the closest previous work to this paper.
- We were unaware of this work until after we had completed the first draft of this paper
- Giovannetti considers a framework very close to ours, with pure Bertrand price competition and a discrete choice of whether to acquire the state of the art production technology in discrete time, infinite horizon framework
- The major differences are: 1) technological progress is *deterministic* with marginal costs decreasing at a geometric rate in each period, 2) aggregate demand for the good is given by an exogenously specified constant elasticity demand function

Giovannetti

- The paper by Emanuele Giovannetti (2001 *IER*) "Persistent Leapfrogging in Bertrand Duopoly" is the closest previous work to this paper.
- We were unaware of this work until after we had completed the first draft of this paper
- Giovannetti considers a framework very close to ours, with pure Bertrand price competition and a discrete choice of whether to acquire the state of the art production technology in discrete time, infinite horizon framework
- The major differences are: 1) technological progress is *deterministic* with marginal costs decreasing at a geometric rate in each period, 2) aggregate demand for the good is given by an exogenously specified constant elasticity demand function

Giovannetti

- The paper by Emanuele Giovannetti (2001 *IER*) "Persistent Leapfrogging in Bertrand Duopoly" is the closest previous work to this paper.
- We were unaware of this work until after we had completed the first draft of this paper
- Giovannetti considers a framework very close to ours, with pure Bertrand price competition and a discrete choice of whether to acquire the state of the art production technology in discrete time, infinite horizon framework
- The major differences are: 1) technological progress is *deterministic* with marginal costs decreasing at a geometric rate in each period, 2) aggregate demand for the good is given by an exogenously specified constant elasticity demand function

Giovannetti, cont.

- However there are some qualitatively similar equilibrium outcomes in Giovannetti's model and our's
- Giovannetti obtains equilibria with leap frogging. However the leap frogging involves *alternating investments by firms 1 and 2 in every period.*
- Giovannetti also obtains equilibria with persistent investment leadership by one of the firms, which she refers to as *increasing asymmetry*
- Giovannetti does not consider mixed strategy equilibria, or other more complex forms of pure strategy equilibria in this game

Giovannetti, cont.

- However there are some qualitatively similar equilibrium outcomes in Giovannetti's model and our's
- Giovannetti obtains equilibria with leap frogging. However the leap frogging involves *alternating investments by firms 1 and 2 in every period.*
- Giovannetti also obtains equilibria with persistent investment leadership by one of the firms, which she refers to as *increasing asymmetry*
- Giovannetti does not consider mixed strategy equilibria, or other more complex forms of pure strategy equilibria in this game

Giovannetti, cont.

- However there are some qualitatively similar equilibrium outcomes in Giovannetti's model and our's
- Giovannetti obtains equilibria with leap frogging. However the leap frogging involves *alternating investments by firms 1 and 2 in every period.*
- Giovannetti also obtains equilibria with persistent investment leadership by one of the firms, which she refers to as *increasing asymmetry*
- Giovannetti does not consider mixed strategy equilibria, or other more complex forms of pure strategy equilibria in this game

Giovannetti, cont.

- However there are some qualitatively similar equilibrium outcomes in Giovannetti's model and our's
- Giovannetti obtains equilibria with leap frogging. However the leap frogging involves *alternating investments by firms 1 and 2 in every period.*
- Giovannetti also obtains equilibria with persistent investment leadership by one of the firms, which she refers to as *increasing asymmetry*
- Giovannetti does not consider mixed strategy equilibria, or other more complex forms of pure strategy equilibria in this game

Practical Conclusions

- What insights do we learn from this study that could help a judge decide on the appropriate value of damage due to collusion, such as in the collusion case? Does this study yield any new insights of interest to Antitrust authorities?
- The existence of so many equilibria in such a simple extension to the classical Bertrand model is a problem
- We have shown that leap frogging investments are possible in a dynamic duopoly model with Bertand pricing. We have provided a solution to the '*Bertrand investment paradox*'
- However the plethora of equilibria makes it is difficult for an economic expert to use this model to make a definitive prediction of counterfactual outcomes,
- What prices could firm C have expected if its input suppliers, A and B, had behaved "competitively" instead of

Practical Conclusions

- What insights do we learn from this study that could help a judge decide on the appropriate value of damage due to collusion, such as in the collusion case? Does this study yield any new insights of interest to Antitrust authorities?
- The existence of so many equilibria in such a simple extension to the classical Bertrand model is a problem
- We have shown that leap frogging investments are possible in a dynamic duopoly model with Bertand pricing. We have provided a solution to the '*Bertrand investment paradox*'
- However the plethora of equilibria makes it is difficult for an economic expert to use this model to make a definitive prediction of counterfactual outcomes,
- What prices could firm C have expected if its input suppliers, A and B, had behaved "competitively" instead of

Practical Conclusions

- What insights do we learn from this study that could help a judge decide on the appropriate value of damage due to collusion, such as in the collusion case? Does this study yield any new insights of interest to Antitrust authorities?
- The existence of so many equilibria in such a simple extension to the classical Bertrand model is a problem
- We have shown that leap frogging investments are possible in a dynamic duopoly model with Bertand pricing. We have provided a solution to the '*Bertrand investment paradox*'
- However the plethora of equilibria makes it is difficult for an economic expert to use this model to make a definitive prediction of counterfactual outcomes,
- What prices could firm C have expected if its input suppliers, A and B, had behaved "competitively" instead of

Practical Conclusions

- What insights do we learn from this study that could help a judge decide on the appropriate value of damage due to collusion, such as in the collusion case? Does this study yield any new insights of interest to Antitrust authorities?
- The existence of so many equilibria in such a simple extension to the classical Bertrand model is a problem
- We have shown that leap frogging investments are possible in a dynamic duopoly model with Bertand pricing. We have provided a solution to the '*Bertrand investment paradox*'
- However the plethora of equilibria makes it is difficult for an economic expert to use this model to make a definitive prediction of counterfactual outcomes,
- What prices could firm C have expected if its input suppliers, A and B, had behaved "competitively" instead of

Practical Conclusions

- What insights do we learn from this study that could help a judge decide on the appropriate value of damage due to collusion, such as in the collusion case? Does this study yield any new insights of interest to Antitrust authorities?
- The existence of so many equilibria in such a simple extension to the classical Bertrand model is a problem
- We have shown that leap frogging investments are possible in a dynamic duopoly model with Bertand pricing. We have provided a solution to the '*Bertrand investment paradox*'
- However the plethora of equilibria makes it is difficult for an economic expert to use this model to make a definitive prediction of counterfactual outcomes,
- What prices could firm C have expected if its input suppliers, A and B, had behaved "competitively" instead of

Theoretical Conclusions

- In this model equilibrium prices paths are *piecewise flat*
- Thus, there are long periods of *price stability* punctuated by episodes of large price declines
- These episodic price declines could be characterized as *price wars* between the two duopolists
- The standard interpretation of price wars is that it is a *punishment device* for deviations from a tacitly collusive equilibrium
- In this model the interpretation is very different: it is consequence of *leap frogging* and we argue that leap frogging is an efficient dynamic coordination mechanism

Theoretical Conclusions

- In this model equilibrium prices paths are *piecewise flat*
- Thus, there are long periods of *price stability* punctuated by episodes of large price declines
- These episodic price declines could be characterized as *price wars* between the two duopolists
- The standard interpretation of price wars is that it is a *punishment device* for deviations from a tacitly collusive equilibrium
- In this model the interpretation is very different: it is consequence of *leap frogging* and we argue that leap frogging is an efficient dynamic coordination mechanism

Theoretical Conclusions

- In this model equilibrium prices paths are *piecewise flat*
- Thus, there are long periods of *price stability* punctuated by episodes of large price declines
- These episodic price declines could be characterized as *price wars* between the two duopolists
- The standard interpretation of price wars is that it is a *punishment device* for deviations from a tacitly collusive equilibrium
- In this model the interpretation is very different: it is consequence of *leap frogging* and we argue that leap frogging is an efficient dynamic coordination mechanism

Theoretical Conclusions

- In this model equilibrium prices paths are *piecewise flat*
- Thus, there are long periods of *price stability* punctuated by episodes of large price declines
- These episodic price declines could be characterized as *price wars* between the two duopolists
- The standard interpretation of price wars is that it is a *punishment device* for deviations from a tacitly collusive equilibrium
- In this model the interpretation is very different: it is consequence of *leap frogging* and we argue that leap frogging is an efficient dynamic coordination mechanism

Theoretical Conclusions

- In this model equilibrium prices paths are *piecewise flat*
- Thus, there are long periods of *price stability* punctuated by episodes of large price declines
- These episodic price declines could be characterized as *price wars* between the two duopolists
- The standard interpretation of price wars is that it is a *punishment device* for deviations from a tacitly collusive equilibrium
- In this model the interpretation is very different: it is consequence of *leap frogging* and we argue that leap frogging is an efficient dynamic coordination mechanism

Methodological Conclusions

- We find that the *algorithm* used to compute equilibria inadvertently acts as an *equilibrium selection mechanism.*
- We want to find algorithms that can compute or at least help us characterize *all* equilibria, and then use other, more economically motivated equilibrium selection criteria to select particular equilibria of interest
- We have found that imposition of the *symmetry* restriction on equilibria effectively knocks out all of the interesting pure strategy equilibria in this model, leaving only a difficult to compute and "bad" mixed strategy equilibrium.
- Great care should be taken in using dynamic models and Markov-perfect equilibria as a basis for policy recommendations!

Methodological Conclusions

- We find that the *algorithm* used to compute equilibria inadvertently acts as an *equilibrium selection mechanism.*
- We want to find algorithms that can compute or at least help us characterize *all* equilibria, and then use other, more economically motivated equilibrium selection criteria to select particular equilibria of interest
- We have found that imposition of the *symmetry* restriction on equilibria effectively knocks out all of the interesting pure strategy equilibria in this model, leaving only a difficult to compute and "bad" mixed strategy equilibrium.
- Great care should be taken in using dynamic models and Markov-perfect equilibria as a basis for policy recommendations!

Methodological Conclusions

- We find that the *algorithm* used to compute equilibria inadvertently acts as an *equilibrium selection mechanism.*
- We want to find algorithms that can compute or at least help us characterize *all* equilibria, and then use other, more economically motivated equilibrium selection criteria to select particular equilibria of interest
- We have found that imposition of the *symmetry* restriction on equilibria effectively knocks out all of the interesting pure strategy equilibria in this model, leaving only a difficult to compute and "bad" mixed strategy equilibrium.
- Great care should be taken in using dynamic models and Markov-perfect equilibria as a basis for policy recommendations!

Methodological Conclusions

- We find that the *algorithm* used to compute equilibria inadvertently acts as an *equilibrium selection mechanism.*
- We want to find algorithms that can compute or at least help us characterize *all* equilibria, and then use other, more economically motivated equilibrium selection criteria to select particular equilibria of interest
- We have found that imposition of the *symmetry* restriction on equilibria effectively knocks out all of the interesting pure strategy equilibria in this model, leaving only a difficult to compute and "bad" mixed strategy equilibrium.
- Great care should be taken in using dynamic models and Markov-perfect equilibria as a basis for policy recommendations!