Delegated Investment in a Dynamic Agency Model[*]

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Abstract

We analyze a continuous-time principal-agent model where, in addition to managing the day-to-day business, the agent has to invest in the firm's future profitability. The interaction of both tasks endogenously creates distortions in delegated investment, which depend on past performance as well as the profitability of the investment technology. Delegated investment is distorted upwards relative to first-best for highly profitable technologies and distorted downwards for less profitable ones. These distortions decrease in past performance giving rise to either positive or negative investment-cash-flow sensitivities. Implications for the dynamics of compensation as well as the impact of better corporate governance are discussed.

Keywords: Continuous time contracting, multiple tasks, delegated investment, managerial compensation.

JEL Classification: D86 (Economics of Contract: Theory), D92 (Intertemporal Firm Choice, Investment, Capacity, and Financing), J33 (Compensation Packages, Payment Methods).

1 Introduction

A manager responsible for a firm's operations usually has some form of discretion in running the day-to-day business which relies on his specific skills or private information. Due to the separation of ownership and control a standard agency problem arises which has been the focus of much of the recent dynamic financial contracting literature.[1] Yet, in a dynamic world, managers also have to take strategic decisions and invest in order to maintain the firm's long-term profitability. Typically, this investment process also relies on information of the same manager or is even delegated to him with discretion.[2] Thus, the manager's (hidden) actions will affect both the firm's current period payoffs as well as its long-term profitability. To capture this idea the present paper describes a continuous time cash flow diversion model with delegated investment in the firm's future profitability. We analyze how the need to incentivize the manager to meet both short-term as well as long-term targets affects the optimal compensation scheme as well as the efficiency of the delegated investment process.

In our model, investment, which stochastically determines the firm's long-term profitability, does not cause the manager any direct costs or benefits. The need to incentivize him to invest in the owners' interest arises only indirectly through the interaction of delegated investment with a standard cash-flow diversion problem. That is, in order to prevent diversion, the agent's income has to be tied to short-term earnings, i.e., his future discounted expected payoff (his "continuation payoff") must increase after high cash flow realizations and decrease after low cash flow realizations. Since investment expenditures are not observable to the principal, this *endogenously* creates an incentive for the agent to misuse his investment budget in order to increase current period cash flows. As a consequence, the principal has to rely on an imperfect signal about the agent's investment expenditures, indicating whether or not investment was successful in maintaining the firm's long-term profitability, in order to provide incentives for investment. The agent

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[1] E.g. DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007) and Biais et al. (2007). See also Biais et al. (2011) for an excellent survey.

[2] In fact, many types of investment, like measures to improve product quality or manufacturing processes, are not directly contractible. Other examples include managerial effort and time, entrepreneurial talent, internal personnel resources, or the development of new styles and products that must be kept secret from competitors (cf. Bebchuk and Stole 1993). These "soft" investment expenditures are – unlike "hard" investment in plants, property or other equipment – notoriously hard to measure based on a firms accounting figures (cf. e.g. Dutta and Reichelstein 2003).

is rewarded for an investment success and punished following investment failure.[3]

Now, as is well known (cf. DeMarzo and Sannikov 2006), the need to provide incentives for truthful reporting of realized cash flows makes the risk neutral principal effectively risk averse with respect to variation in the agent's "continuation payoff".[4] Thus, incentivizing investment – thereby inducing additional volatility in the agent's continuation payoff – causes additional agency costs. As a consequence, investment in our dynamic multi-task model will in general be distorted away from its first best level. We show how the sign of the distortion depends both on the firm's performance record, as captured by the entire history of cash flows and investment outcomes, as well as on the "returns to investment".[5] When the incumbent manager's track record is sufficiently positive, optimal investment is distorted upwards and decreasing in past performance if returns to investment are relatively high and distorted downwards and increasing if investment returns are rather low.

To see this in more detail note that, when setting the recommended investment schedule, firm owners optimally take into account the resulting agency costs of delegated investment. In particular, starting from any investment level, a marginal increase in investment first requires to provide stronger incentives, which ceteris paribus induces more variation in the agent's continuation payoff and is, thus, costly. Second, it increases the probability of an investment success and, accordingly, the probability that the agent has to be rewarded. This is costly as well. Third, raising investment reduces the probability of a failure and, thus, of having to punish the agent, which is beneficial for the principal. Whether marginal agency costs are positive (giving rise to underinvestment) or negative (implying overinvestment) at the optimal investment level, then depends on a quantitative comparison of these three effects. More concretely, if "returns to investment" – and accordingly the first-best investment level – are rather high, incentives are optimally provided by harsh punishment in the event of a then unlikely failure. This makes investment failure particularly costly such that the principal optimally distorts investment upwards in order to further increase the probability of success. If, by contrast, investment returns are moderate so that at

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[3]The fact that investment outcome is observable and, thus, there is symmetric information with respect to the success of past investment projects allows us to focus on the moral hazard problem with respect to the manager's instantaneous investment choice.

[4]This is a consequence of the agent's relative impatience together with limited liability, which requires to terminate the contract if the agent's continuation payoff hits a lower bound in which case the principal incurs a cost in order to replace the agent.

[5]The term "returns to investment" here refers to the difference in firm profitability after an investment success relative to that following an investment failure.

the first-best investment level a success is rather unlikely relative to a failure, delegated investment will be too low, as optimally rewarding successful investment is particularly expensive.

While the sign of the investment distortion, thus, depends crucially on the underlying investment technology, the absolute value of the distortion decreases in firm performance as the manager's track record improves and with it his stake in the firm increases mitigating the agency problem.[6] Thus, our model predicts that investment should increase in realized cash flows only for firms that invest inefficiently little, while it should decrease in realized cash flows for firms with inefficiently high investment. We confirm this basic intuition using several numerical examples. When studying empirically the relation of investment and realized cash flows, the analysis should, in light of our model, account for the sign of the investment distortion. Therefore, as put forward also in DeMarzo et al. (2012), the history dependence of an optimal dynamic contract suggests that it is the entire history (stock) of cash flows and investment outcomes that determines the optimal investment level rather than merely period cash flows.

Moreover, within our multi-task setting, we also analyze how the severity of the underlying cash flow diversion problem affects investment distortions under the optimal contract. To this end we vary the rate at which the agent benefits from diversion, which can be interpreted as a measure for corporate governance. We find that better corporate governance leads to higher investment levels (i.e. less underinvestment) if investment was initially below the first-best level, and to lower investment levels (i.e. less overinvestment) if investment was initially too high relative to the first-best level. Again the intuition is that the costs of providing incentives for investment decrease with better corporate governance and with it the need to distort investment.

Our model further provides interesting implications for the dynamics of managerial compensation. In order to incentivize a given level of investment, the optimal contract specifies a combination of reward and punishment that minimizes the agency costs of delegated investment subject to incentive compatibility. To this end, for every possible performance history as reflected in the agent's continuation payoff, it has to be taken into account whether it is relatively cheaper to provide incentives for investment through reward

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[6]In fact for sufficiently high levels of the agent's continuation payoff, delegated investment in our model approaches first-best.

or punishment. We show that the resulting punishment-to-reward ratio under the optimal contract is inverse U-shaped in the agent's continuation payoff. On the one hand, if the agent has accumulated sufficiently much wealth inside the firm it becomes increasingly cheaper to provide investment incentives via rewards as this helps to further relax the agency problem. On the other hand, for low realizations of the agent's continuation payoff, punishment is effectively restricted by limited liability. In this case the agent is immediately fired following an investment failure.

Related Literature. Optimal investment with dynamic agency is also analyzed in the two-period model of Gertler (1992) as well as the multiperiod discrete time models in Quadrini (2004), Clementi and Hopenhayn (2006), and DeMarzo and Fishman (2007). Further, DeMarzo et al. (2012) analyze dynamic agency in a neoclassical investment setting using continuous time methods. All of these contributions consider capital investment and further assume that investment expenditures are verifiable and the optimal long term contract can condition on it.[7] In contrast we consider the case where the agent controls investment and incentives have to be provided for the agent to undertake investment in the principal's interest. In this respect, our analysis is also related to Zhu (2012) who considers a discrete-time model of persistent moral hazard where the agent can choose between a "short-term" and a "long-term" action in every period.[8] He characterizes the optimal contract that always induces the "long-term" action. Our focus is instead on determining the optimal interior investment level at every possible history while fully abstracting from problems of (persistent) private information.

While we frame the agency conflict as one between outside investors and a (single)

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[7]Philippon and Sannikov (2007) consider a different investment technology within a continuous time diversion model where the principal can exercise a growth option and He (2009) presents a model where the manager's hidden action determines the expected growth rate of the firm.

[8]There is a sizable literature on managerial short-sightedness where it is argued that many common incentive schemes are too short-term, thereby inducing managers to focus excessively on current period outcomes. This short-term bias has been attributed to career concerns (Narayanan 1985), takeover threats (Stein 1988), concerns about the firm's stock prices over a near-term horizon (Stein 1989), mispricing of long-term assets (Shleifer and Vishny 1990), reputational herding (Zwiebel 1995) and short-term financing (von Thadden 1995). In particular, there is also ample empirical and anecdotal evidence that managers cut R&D spending (Baber et al. 1991, Bushee 1998, Cheng 2004, Brown and Krull 2008) or marketing expenditures (Mizik and Jacobson 2007, Mizik 2010) – at the expense of investors' long-term profits – in order to meet short term earnings targets. Furthermore Graham et al. (2005) find that the majority of managers would decide not to implement a positive NPV project in order to avoid missing the current quarter's consensus earnings forecast.

manager, one could also apply our model of delegated investment, where the manager has discretion over the firm's funds, to agency models within the firm as is standard in the capital budgeting literature (cf. most prominently Harris and Raviv 1996). In this literature it usually is firm headquarters that allocates capital to division managers who have private information about the investment technology. An agency problem arises because managers have some incentive to misrepresent information about capital productivity in order to attract more capital, usually due to assumed empire building preferences, giving rise to distortions in capital allocation. In our model, the manager's incentives to over- or underinvest arise endogenously as a consequence of a cash flow diversion problem, without assuming private costs of investment or the desire to build an empire.

That the interaction of different managerial tasks may endogenously cause distortions in investment is also analyzed by Inderst and Klein (2007) and Bernardo et al. (2009) in a static setting.[9] There the optimal compensation scheme needs to balance the incentives to exert effort in order to create new investment opportunities as well as to report their quality truthfully. In contrast we consider a fully dynamic multi-task model incorporating both a short-term (cash flow diversion) as well as a long-term action (delegated investment), thus allowing us to derive interesting implications about the dynamics of delegated investment and compensation.

Finally, our modelling of delegated investment also shares some ideas with the literature on how to motivate innovative activities. Bergemann and Hege (2005) study the financing of innovation in a model with learning about the quality of the project. However, they focus on a single task and further only consider implementation with a sequence of short-term contracts. Related is also the work in Manso (2011) who examines incentives for experimentation and finds that the optimal contract exhibits high-powered incentives in the long run together with tolerance for early failure. These features are, however, derived within a single-task setting, while in our model the principal has to balance incentives for investment in future profitability with those for the short-term task of reporting period cash flows truthfully. Hellmann and Thiele (2011) analyze a related (static) multi-task model, where the agent has to perform a standard task as well as an innovative activity which cannot be contracted upon and requires ex-post bargaining. In contrast to this setup

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[9]The insight that providing incentives along one task can distort an agent's incentives on another goes back to Homström and Milgrom (1991).

we take a complete contracting approach and consider a fully dynamic problem where the agent can divert cash flow for private benefits as well as move funds within the firm in order to meet either short- or long-term targets.

The remainder of the paper is organized as follows. We introduce the model in Section 2. In Section 3 we derive the optimal dynamic contract and Section 4 discusses optimal compensation and investment. Section 5 provides a numerical assessment of the dynamics of investment, compensation and the impact of corporate governance. Section 6 concludes. All proofs are delegated to Appendix A and Appendix B contains some additional material.

2 The Model

2.1 Cash Flow Process and Investment Technology

At the heart of our analysis is an infinite horizon continuous time principal-agent model of a firm whose owners (the principal) hire a manager (the agent) to operate the business. The manager is protected by limited liability and has limited wealth. Hence, owners have to bear the costs of setting up the firm and cover operating losses.[10] Firm owners as well as the manager are risk neutral and discount future consumption at rates r and γ, respectively, with $r < \gamma$. The firm's cash flows net of investment, I_t, evolve according to

$$dY_t = (\mu_t - I_t)\, dt + \sigma dZ_t, \tag{1}$$

where $\mu_t \in \left\{\mu^l, \mu^h\right\}$, with $0 < \mu^l < \mu^h$, denotes the drift rate of cash flows, which we will refer to as the firm's "profitability", σ the instantaneous volatility, and Z is a standard Brownian motion on the complete probability space $(\Phi, \mathcal{F}, Q)$.

Investment persistently affects the firm's future profitability by determining its ability to adopt and commercialize new technologies as they become available. The arrival of new technologies is governed by a Poisson process N with arrival rate ν, representing unpredictable, exogenous "technology shocks". In case of a technology shock $(dN_t = 1)$, the firm is able to incorporate the new, more efficient technology with probability $p(I_t)$, implying that its profitability after t will be high $(\mu_{t+} := \lim_{s\downarrow t} \mu_s = \mu^h)$. With probability $1-p(I_t)$ the firm is, however, unable to adopt the new technology and its future profitability

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[10]In order to save on notation, we will assume set up costs of zero.

will be low ($\mu_{t^+} = \mu^l$). In between two shocks, where $dN_t = 0$, the firm's profitability remains unchanged. Firm profitability hence follows a two-state Markov switching process where the transition rate for switching from μ^h to μ^l is given by $\nu[1 - p(I_t)]$ and that for switching from μ^l to μ^h is given by $\nu p(I_t)$.

Current profitability thus is a function of past investment expenditures, which can be interpreted as a firm's choice of "absorptive capacity" in the sense of Cohen and Levinthal (1990), describing its capability of "assimilating new, external information and apply it to commercial ends".[11] To complete the description of the investment technology, we assume that the success probability $p(\cdot) \in [0, 1]$ is an increasing and strictly concave function of the investment amount $I_t \in [0, \overline{I}]$, satisfying $p(0) = 0$ as well as $p(\overline{I}) = 1$. [12]

Now, before describing the informational frictions leading to an agency problem of delegated investment, let us, first, derive the optimal investment policy in the absence of any agency problem, which will serve as a useful benchmark in the remainder of the paper.

First-Best Benchmark. Absent any agency problem, that is, if the principal can observe cash flows Y directly and also controls investment I, his optimization problem is given by

$$
\begin{aligned}
rf^h &= \max_{I^h} \left\{ \mu^h - I^h + \nu\left(1 - p(I^h)\right)\left[f^l - f^h\right] \right\}, \\
rf^l &= \max_{I^l} \left\{ \mu^l - I^l + \nu p(I^l)\left[f^h - f^l\right] \right\},
\end{aligned}
$$

where $i \in \{l, h\}$ indexes firm profitability, as captured by the prevailing drift rate of cash flows μ^i, and f^i denotes the respective firm value. Optimal first-best investment, I^{FB}, if interior, must then satisfy in both states $i \in \{l, h\}$ the first order condition

$$
1 = \nu p'(I^{FB})\left(f^h - f^l\right) = \nu\Delta p'(I^{FB}), \tag{2}
$$

where Δ denotes the difference in first-best values

$$
\Delta := f^h - f^l = \frac{1}{r + \nu}\left(\mu^h - \mu^l\right). \tag{3}
$$

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[11] Such investment decisions are likely to require a considerable understanding of the firm's internal processes as well as the market it is operating in, which we assume is specific to the manager running the firm. The same investment technology is employed in Board and Meyer-ter-Vehn (2013). In their model of firm reputation, investment determines future product quality.

[12] Note that we allow for $\overline{I} \to \infty$.

As usual, the first-best investment level equates the marginal costs of investment and the expected discounted gain, properly accounting for the Poisson structure. In the following we will assume that

$$p'(\overline{I}) < \frac{1}{\nu\Delta} < p'(0), \tag{4}$$

which guarantees an interior level of first-best investment, implying $p(I^{FB}) \in (0,1)$.

2.2 Contracting Problem

We now proceed with the analysis of the main model, where, in case firm owners decide to start the firm, they have to hire a manager to run the business. Due to the separation of ownership and control an agency problem arises as firm owners do not observe Y, but only the manager's reports $\hat{Y}$.[13] Hence, there is scope for diversion of cash flows for private consumption. The manager can consume fraction $\lambda \in (0,1]$ of each unit diverted, which captures the costs of concealing and taking funds out of the firm. Furthermore, firm owners also do not directly observe investment expenditures,[14] allowing the manager to deviate from the recommended investment level such as to manipulate net cash flows Y.[15] However, because technology shocks, dN_t, are directly observable to all players (irrespective of whether μ_t changes or not), at any point in time and, in particular, independent of the current level of profitability $\mu_t \in \{\mu^l, \mu^h\}$, firm owners can observe whether there has been an investment success (switching to μ^h or staying there) or a failure (switching to μ^l or staying there), which can be used to provide incentives for investment. In what follows it is thus convenient to define the good news process N^g, governing investment success, with intensity $\nu p(I_t)$, as well as the analogous bad news process N^b, governing investment failure, with intensity $\nu(1 - p(I_t))$.

The manager's employment starts at time $t = 0$ and ends at an endogenously determined stopping time τ. The principal can commit to a long-term contract $\{U, \tau\}$ specifying – based on reported cash flows and investment outcomes – transfers dU to the agent, as well as the (random) time τ, when the agent is fired and replaced, which incurs the principal

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[13]Recall that the principal covers (reported) operating losses ($d\hat{Y} < 0$).

[14]Cf. the Introduction, in particular footnote 2 for further motivation of this assumption.

[15]Alternatively one could assume that the agent reports cash flows gross of investment and is given an investment budget which he can in turn either invest or divert for private consumption. However, because all incentive compatibility constraints will be binding under the optimal contract, the two settings are equivalent as long as the manager benefits from diversion at the same rate.

fixed costs k. Due to limited liability, the cumulative wage process $U = \{U_t : 0 \leq t \leq \tau\}$ has to be non-decreasing. We further assume that the agent can not save privately, implying that $dY_t - d\hat{Y}_t \geq 0$ i.e. he can only underreport cash flows.[16] Hence, the agent's consumption flow at time t is given by

$$dC_t = dU_t + \lambda \left[dY_t - d\hat{Y}_t dt \right].$$

To complete the description of the agency problem, let us give a short and heuristic outline of the timing of events taking place in any infinitesimal time interval $[t, t+dt]$ prior to liquidation: (i) The agent decides on investment (I_t), current cash flows are realized (Y_t) and reported to the principal $(\hat{Y}_t)$, (ii) with probability νdt there is a technology shock shock (dN_t), (iii) the agent gets compensated (dU_t) and the continuation decision is taken $(\mathbf{I}_{\{\tau > t\}})$. Note in particular that the agent chooses investment prior to observing whether a technology shock occurred or not, which is the main important "sequentiality" in our continuous time model, while compensation and possible liquidation occur thereafter.[17]

Upon liquidation, the manager receives his outside option, which is normalized to zero. This leads to the agent's total expected wealth at $t = 0$ of[18]

$$w_0 = E^S \left[\int_0^\tau e^{-\gamma t} dC_t \right], \tag{5}$$

where E^S denotes the expectation under the probability measure Q^S induced by the agent's strategy $S = \left\{ \hat{Y}_t, I_t : 0 \leq t \leq \tau \right\}$.[19] If the agent is replaced, the principal receives L_τ, which represents his expected profit from the relationship with a new agent net of replacement costs k. Hence, at $t = 0$, the principal's total expected profit, delivering the agent

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[16]Note that this assumption is without loss of generality given that the agent is risk-neutral and more impatient than the principal.

[17]Formally, I is predictable with respect to $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$, the filtration generated by (Z, N^g, N^b), while U is adapted to the filtration generated by $\left(\hat{Y}, N^g, N^b\right)$, and τ is a $(\hat{Y}, N^g, N^b)$-measurable stopping time.

[18]The concrete value of $w_0 > 0$ in (5) is determined by the two parties' relative bargaining power. In what follows, we will assume that the principal enjoys all bargaining power and the agent accepts any contract with $w_0 \geq 0$.

[19]If obvious, we will not explicitly state the measure associated with the expectation operator in the following.

an expected wealth of w_0, given $\mu_0 \in \{\mu^l, \mu^h\}$, is:

$$f_0 = E\left[\int_0^\tau e^{-rt}\left(d\hat{Y}_t - dU_t\right) + e^{-r\tau}L_\tau\right]. \tag{6}$$

Although replacing the agent is costly, as $k > 0$, it is needed and indeed part of the ex-ante optimal contract in order to incentivize the agent who is protected by limited liability.

Given a contract $\{U, \tau\}$, the agent then chooses a feasible strategy S to maximize his initial expected payoff w_0. A strategy S is then called *incentive compatible* if it maximizes w_0 given the contract. Hence, an incentive compatible contract can be described by the triple $\{S^*, U, \tau\}$, where S^* is an incentive compatible strategy that the principal wants to induce. We can thus write the global incentive constraint as:

$$E^{S^*}\left[\int_0^\tau e^{-\gamma t}dC_t\right] \geq E^{\widetilde{S}}\left[\int_0^\tau e^{-\gamma t}dC_t\right], \quad \text{given } \widetilde{S} \neq S^*. \tag{7}$$

Note that we can simplify the analysis considerably by relying on a version of the revelation principle, which allows us to restrict attention to truth-telling contracts, implementing $\hat{Y}_t = Y_t$, $t \geq 0$.[20] The contracting problem then is to find an incentive compatible truth-telling contract maximizing the principal's expected profit f_0, while delivering w_0 to the agent and satisfying the limited liability constraint.[21] We will refer to the solution of this constrained maximization problem as the *optimal contract*.

3 Model Solution

3.1 Continuation Payoff and Local Incentive Compatibility

For any truth-telling contract, define the agent's continuation payoff, w_t, as his future expected discounted payoff at time t, given he will follow strategy S^* from t onwards, i.e.,

$$w_t = E_t^{S^*}\left[\int_t^\tau e^{-\gamma(s-t)}dC_s\right].$$

As the agent's investment decision has to be predictable with respect to technology shocks (dN_t), it is convenient to define the left-limit of his continuation payoff w_t by $w_{t^-} :=$

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[20]For a formal argument compare Lemma 1 and Proposition 2 in DeMarzo and Sannikov (2006).

[21]Note that there is no relevant participation constraint, as the agent enjoys a positive rent, once he is hired.

$\lim_{s \uparrow t} w_s$. Intuitively, while w_t is the agent's continuation payoff after observing whether a technology shock occurs in t or not, w_{t-} denotes the respective value before this uncertainty is resolved. This variable will serve, together with μ_t, as the state variable in the recursive formulation of the contracting problem. Applying standard martingale techniques, the evolution of w_{t-} can be characterized as follows:[22]

Lemma 1. *If the manager follows the recommended strategy* $S^* = \{Y_t, I_t^* : 0 \leq t \leq \tau\}$, *then there exist some predictable processes* $\alpha = \{\alpha_t : 0 \leq t \leq \tau\}$ *and* $\beta^j = \{\beta_t^j : 0 \leq t \leq \tau\}$, $j \in \{g, b\}$, *such that the agent's continuation payoff at any moment of time evolves according to:*

$$
\begin{aligned}
dw_{t-} = {} & \gamma w_{t-} dt - dU_t + \alpha_t \left(d\hat{Y}_t - (\mu_t - I_t^*) \, dt \right) \\
& + \beta_t^g \left(dN_t^g - \nu p(I_t^*) dt \right) + \beta_t^b \left(dN_t^b - \nu \left(1 - p(I_t^*) \right) dt \right),
\end{aligned}
\tag{8}
$$

where $d\hat{Y}_t - (\mu_t - I_t^*) \, dt$ *is the increment of a Brownian motion if the agent follows* S^* *and* dN_t^g (dN_t^b) *is the increment of a standard Poisson processes with arrival rate* $\nu p(I_t)$ $(\nu (1 - p(I_t)))$.

To build some intuition let us discuss in more detail the evolution of w_{t-} in (8). Due to promise keeping, the agent's promised wealth, w_{t-}, has to grow at his discount rate, γ, while it must decrease with direct payments, dU_t. It also depends on his reporting strategy via the sensitivity α_t and on his investment strategy via the sensitivities β_t^g and β_t^b. So, if the agent underreports cash flows, he can immediately consume $\lambda \left(dY_t - d\hat{Y}_t \right)$, while his continuation payoff is reduced by $\alpha_t \left(dY_t - d\hat{Y}_t \right)$. As for deviations from the recommended investment level, if the agent raises I_t marginally above I_t^*, his continuation payoff changes in expectation by $\left[\nu p'(I_t^*) \left(\beta_t^g - \beta_t^b \right) - \alpha_t \right] dt$. That is, the probability of an investment success – triggering "reward" β_t^g – increases by $\nu p'(I_t^*) dt$, while the probability of a failure – triggering "punishment" β_t^b – decreases by $\nu p'(I_t^*) dt$. At the same time, w_{t-} is reduced by $\alpha_t dt$ reflecting a reduction in net current cash flows dY_t due to the additional investment expenses. In a similar manner, if the agent reduces I_t marginally to inflate net current cash flows dY_t, his continuation payoff changes in expectation by $- \left[\nu p'(I_t^*) \left(\beta_t^g - \beta_t^b \right) - \alpha_t \right] dt$. These observations lead to the local incentive compatibility conditions in Lemma 2 below.

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[22] Cf. Theorem III 4.34 in Jacod and Shiryaev (2003) for a statement of the relevant martingale representation theorem and Piskorski and Tchistyi (2010, Proposition 1) for its application to dynamic contracts.

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Lemma 2. *The truth-telling contract $\{S^*, U, \tau\}$ with $I_t^* \in (0, \overline{I})$ is incentive compatible if and only if*

$$\alpha_t \geq \lambda \tag{9}$$

and

$$\beta_t^g - \beta_t^b = \frac{\alpha_t}{\nu p'(I_t^*)} \tag{10}$$

holds for all $t \in [0, \tau)$, almost surely.[23] Further, the limited liability constraint implies for all $t \in [0, \tau)$ that[24]

$$w_{t^-} + \beta_t^j \geq 0, \; j \in \{g, b\}. \tag{11}$$

Note that (10) establishes a link between the delegated investment problem and the cash flow diversion problem: To provide incentives for investment, the agent's continuation payoff has to increase following an investment success (β^g) and decrease following a failure (β^b). More precisely, according to (10), the sensitivity of the agent's continuation payoff with respect to the investment outcome, $\beta^g - \beta^b$, has to increase with the sensitivity to reported cash flows, α, for any given level of investment. Thus, a more severe cash flow diversion problem, reflected by a greater λ, would require to impose more risk on the agent's income in two respects: First, from (9), the sensitivity α has to increase, because with higher diversion benefits, the agent's income has to be linked more closely to reported cash flows in order to induce truthful reporting. Second, with his income being more sensitive to realized cash flows, the agent has an incentive to inflate cash flows by deviating from the recommended investment level. Hence, according to (10), his income has to be more responsive to the investment outcome as well. Clearly, if the principal wants to induce a marginally higher investment level I^*, according to (10), he has to increase $\beta_t^g - \beta_t^b$ by $\frac{-p''(I^*)\alpha_t}{(p'(I^*))^2}$.

3.2 Optimal Contract

The optimal contract can now be derived using the dynamic programming approach. Denote by $f^i(w)$, $i \in \{l, h\}$ the principal's value function, that is, the highest profit the

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[23]If $I_t^* = 0$ (10) is replaced by $\nu p'(I_t)\left(\beta_t^g - \beta_t^b\right) \leq \alpha_t$, in case $I_t^* = \overline{I}$ we instead require $\nu p'(I_t)\left(\beta_t^g - \beta_t^b\right) \geq \alpha_t$.

[24]Note, that (11) is only necessary for U to be non-decreasing. Still, in a slight abuse of language, we will sometimes refer to condition (11) as the "limited liability" constraint.

principal can attain under any incentive compatible contract delivering w to the agent and given the prevailing drift rate of cash flows, μ^i. Further, for derivatives with respect to w, we adopt the notation $f_w^i(w) := \frac{\partial}{\partial w} f^i(w)$.[25]

Let us first consider the optimal compensation policy. Clearly, the principal can always compensate the agent directly by paying him a lump-sum of $dU > 0$ (at marginal costs of -1) and then move to the optimal contract with reduced continuation payoff $(w - dU)$. However, deferring compensation may be valuable: The higher w, the smaller is the probability that it falls to zero, which causes an inefficiency as the agent has to be replaced at costs $k > 0$. This benefit declines, however, as w increases, while the costs of deferring compensation, which are due to the wedge in discount rates $(\gamma > r)$, stay constant. This trade off is reflected in the concavity of $f^i(w)$, which will be shown formally in the proof of Proposition 1 below. As a consequence compensation is optimally deferred until the threshold $\overline{w}^i$ is reached, where $f_w^i(\overline{w}^i) = -1$ and the agent is paid cash.

Now, since the principal discounts at rate r, his expected flow of value at time t must be $rf^i(w)dt$. This has to be equal to the expected instantaneous net cash flow $(\mu^i - I^i)\, dt$ plus the expected change in the principal's value function. Hence, using Itô's lemma and the change of variables formula for jump processes, we find that, over the interval $w \in [0, \overline{w}^i]$, the principal's value function must satisfy the following coupled HJB equations

$$rf^i(w) = \max_{\alpha,\beta,I} \left\{ \begin{array}{c} \mu^i - I^i + \left[\gamma w - \nu \left(\beta^{g,i} p(I^i) + \beta^{b,i}\left(1 - p(I^i)\right)\right)\right] f_w^i(w) + \frac{1}{2}\sigma^2(\alpha^i)^2 f_{ww}^i(w) \\ +\nu p(I^i)\left[f^h(w + \beta^{g,i}) - f^i(w)\right] + \nu\left(1 - p(I^i)\right)\left[f^l(w + \beta^{b,i}) - f^i(w)\right] \end{array} \right\} \tag{12}$$

for $i \in \{h, l\}$. Formally, the principal's problem is to maximize (12) subject to the two incentive compatibility constraints in (9) and (10) as well as the limited liability constraint in (11). Proposition 1 characterizes the solution to this maximization problem, the *optimal contract*.

Proposition 1. *Under the optimal truth-telling contract with delegated investment, the principal's value function satisfies*

$$f(w) := f^l(w) = f^h(w) - \Delta, \tag{13}$$

where Δ is given by (3); $f(w)$ is concave, strictly so for $0 \le w < \overline{w}$, and solves, for

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[25]To simplify notation we will further drop time subscripts in the following.

$w \in [0, \overline{w}]$:

$$(r + \nu) f(w) = \mu^l - I(w) + \frac{1}{2}\sigma^2\lambda^2 f_{ww}(w) \tag{14}$$
$$+ \left[\gamma w - \nu \left[\beta^g(w)p(I(w)) + \beta^b(w)(1 - p(I(w))) \right] \right] f_w(w)$$
$$+ \nu p(I(w)) \left[f(w + \beta^g(w)) + \Delta \right] + \nu \left[1 - p(I(w)) \right] f(w + \beta^b(w)),$$

with boundary conditions $f(0) = f(w^*) - k$, where $w^* \in \arg\max_w \{f(w)\}$, $f_w(\overline{w}) = -1$ and $f_{ww}(\overline{w}) = 0$.[26] Optimal investment $I(w)$ as well as the sensitivities $\alpha(w) = \lambda$ and $\beta^j(w)$, $j \in \{g, b\}$, are state-independent and chosen as maximizers in (12). The incumbent agent's continuation payoff evolves according to (8) with $\alpha_t = \lambda$, $\beta_t^g = \beta^g(w_{t-})$ and $\beta_t^b = \beta^b(w_{t-})$ $\forall t$ and given some starting values (w_{0-}, μ_0). When $w_{t-} \in [0, \overline{w})$, $dU_t = 0$; when $w_{t-} \geq \overline{w}$, payments dU_t cause w_{t-} to reflect at $\overline{w}$.

Note, first, that the principal's value functions in the high and in the low cash flow state differ only by an additive constant. As in the first-best, this constant (Δ) captures the difference in value due purely to the difference in profitability ($\mu^h - \mu^l$), properly discounted accounting for the Markov switching structure. This intuitive result follows from the fact that the basic cash flow diversion problem is independent of the currently prevailing state $i \in \{h, l\}$ and, furthermore, also the agency problem with respect to investment is symmetric across states; in any instant, independent of the current level of μ^i, there is an investment success ($dN^g = 1$) with probability $\nu p(I)dt$, while there is an investment failure ($dN^b = 1$) with probability $\nu(1 - p(I))dt$. Together with the symmetric structure of the investment technology, this implies that also the optimal investment policy is independent of the prevailing profitability level μ^i, $i \in \{h, l\}$.

Next, the agent's incentive constraint with respect to cash flow diversion (9) binds under the optimal contract, which is intuitive, because it is costly to provide incentives that are too strong. Formally, this follows from the concavity of $f(w)$ and the observation that increasing α would increase the instantaneous volatility of w. Finally, the first boundary condition ("value matching") reflects the fact that upon firing the incumbent manager at $w = 0$, the principal receives $f(w^*)$ from the relation with a new manager and bears replacement costs k. The "smooth pasting" condition is motivated by the optimal deferral of compensation until $\overline{w}$, where $f_w(\overline{w}) = 1$. The optimal choice of $\overline{w}$ is guaranteed by the

[26]To the right of $\overline{w}$, $f(w)$ extends linearly according to $f(w) = f(\overline{w}) - (w - \overline{w})$.

third boundary condition ("super contact"). The next section offers an in-depth discussion of the optimal $\beta^g(w)$ and $\beta^b(w)$ and the resulting investment profile $I(w)$.

4 Optimal Compensation and Investment

In order to build intuition, it is instructive to first consider the benchmark setting where the principal controls the investment process I.

Contractible Investment Benchmark. With contractible investment, the evolution of the agent's continuation payoff w is still given by (8), and the principal's value function has to satisfy HJB (12). However, there is no need to incentivize the agent to invest according to I^* and, thus, the only relevant constraints are the incentive constraint for truth-telling (9) and the limited liability constraint (11). The optimal contract in this simplified setting is then characterized in the following Proposition.

Proposition 2. *If investment is contractible, then, under the optimal truth-telling contract, the agent's continuation payoff evolves according to (8) with $\alpha_t = \lambda$ as well as $\beta_t^g = \beta_t^b = 0$, $\forall t$ and given some starting values (w_{0-}, μ_0). When $w_{t-} \in [0, \overline{w})$, $dU_t = 0$; when $w_{t-} \geq \overline{w}$, payments dU_t cause w_{t-} to reflect at $\overline{w}$. The principal's value function $f(w)$ satisfies (13), is concave, strictly so for $0 \leq w < \overline{w}$, and solves, for $w \in [0, \overline{w}]$:*

$$rf(w) = \mu^l + \nu p(I^{FB})\Delta - I^{FB} + \frac{1}{2}\sigma^2\lambda^2 f_{ww}(w) + \gamma w f_w(w),$$

with boundary conditions $f(0) = f(w^*) - k$, where $w^* \in \arg\max_w \{f(w)\}$; $f_w(\overline{w}) = -1$ and $f_{ww}(\overline{w}) = 0$.[27] Optimal investment is for all w equal to the first-best level, I^{FB}, as characterized, if interior, in (2).

With contractible investment, there is clearly no need to reward or punish the agent following a technology shock, and the optimal contract can perfectly "smooth" the marginal costs of compensating the agent (as reflected in the slope of $f(w)$) across states:[28]

$$
\begin{aligned}
f_w(w) &= f_w\left(w + \beta^j(w)\right), \quad j \in \{h, l\} \quad &(15)\\
&\Longleftrightarrow \quad \beta^j(w) = 0, \quad j \in \{h, l\}.
\end{aligned}
$$

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[27] To the right of $\overline{w}$, $f(w)$ extends linearly according to $f(w) = f(\overline{w}) - (w - \overline{w})$.

[28] Cf. Hoffmann and Pfeil (2010) for further discussion.

Then, as agency costs in our model are independent of the prevailing level of μ^i, $i \in \{h, l\}$, and, hence, absent the need to provide incentives for investment ($\beta^g = \beta^b = 0$), irrelevant for the optimal investment policy, contractible investment is equal to the first-best level.[29] However, if investment is delegated to the manager who has to be incentivized appropriately, a perfect smoothing of marginal compensation costs according to (15) is no longer possible, due to the restrictions imposed by incentive compatibility, thus causing additional agency costs. This has interesting implications for the optimal investment and compensation policy, which we will discuss in what follows.

4.1   Optimal Incentives for Investment

From now on we focus on the multi-task case where investment is not contractible. The need to provide incentives for investment requires some form of reward following an investment success or punishment after a failure. However, this causes additional variation in the agent's continuation payoff relative to the contractible investment benchmark in Proposition 2 and, thus, additional agency costs as the principal is effectively risk-averse with respect to variations in w. Precisely, according to (10), incentive compatibility imposes a restriction on the agent's continuation payoff after a success *relative* to his continuation payoff following a failure, $\beta^g - \beta^b$. Optimizing subject to this restriction, interior values of β^b and $\beta^g = \beta^b + \frac{\lambda}{\nu p'(I(w))}$ then satisfy the first-order condition

$$p\left(I(w)\right)\left[f_w(w) - f_w(w + \beta^g(w))\right] = \left(1 - p\left(I(w)\right)\right)\left[f_w(w + \beta^b(w)) - f_w(w)\right], \qquad (16)$$

equalizing the expected costs of providing incentives for investment through reward (left hand side) with the expected costs of providing incentives through punishment (right hand side). Providing incentives for investment is costly as it is no longer possible to perfectly smooth marginal compensation costs as in the case with contractible investment (cf.(15)).[30]

Due to the concavity of $f(w)$, (16) then implies that both rewards and punishments are used for all w except for $w = 0$, where compensation is restricted due to limited liability and only rewards are used as the agent is "too poor to be punished". For $w = \overline{w}$, on the

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[29]This is in contrast to agency models with neoclassical investment as in DeMarzo et al. (2012), where investment is a scaling of cash flows. There agency costs reduce the return to investment such that the principal optimally underinvests as a consequence of the basic underlying agency problem.

[30]In contrast to the case with contractible investment, now, with delegated investment, marginal compensation costs can only be kept constant in expectation.

other hand, it is optimal to rely only on rewards to incentivize investment, as these can be paid out directly in form of a cash payment, causing no further variation in w and, thus, no additional agency costs. These observations are summarized in the next proposition.

Proposition 3. *Under the optimal contract of Proposition 1, the state independent sensitivities $\beta^j(w)$, $j \in \{g, b\}$ for positive levels of investment are determined from the incentive constraint (10) together with first order condition (16), if interior. The agent is always rewarded for a success, $\beta^g(w) > 0$ for $w \in [0, \overline{w}]$. The agent is punished for a failure, $\beta^b(w) < 0$, for $w \in (0, \overline{w})$, while $\beta^b(w) = 0$ for $w \in \{0, \overline{w}\}$. Further, there exists $\widehat{w} \in (0, \overline{w})$ such that the limited liability constraint for $\beta^b(w)$ binds on $[0, \widehat{w}]$. Hence, on this interval the agent is instantly fired following a failure.*

To gain more intuition about the optimal compensation policy for a given level of $I(w)$, let us rewrite (16) as

$$p(I(w)) \int_w^{w+\beta^g(w)} f_{ww}(x)dx - (1 - p(I(w))) \int_{w+\beta^b(w)}^w f_{ww}(x)dx = 0, \qquad (17)$$

reflecting the importance of the principal's aversion to fluctuations in w. If the costs of providing incentives were constant in w, i.e., $f_{www} = 0$, then (17) would simplify to $p(I)\beta^g + (1 - p(I))\beta^b = 0$, which is equivalent to minimizing the instantaneous variance of w associated with the technology shock subject to (10). In general, however, $f_{www}(w) \neq 0$ and the relative use of punishments to rewards then also reflects whether the costs of providing incentives are increasing or decreasing in the agent's continuation payoff.

For the analysis of equilibrium investment below, it is important to understand how $\beta^g(w)$ and $\beta^b(w)$ comove with investment $I(w)$.[31] As is immediate from inspection of (16), when the probability of success $p(I(w))$ goes up, optimal compensation $\beta^j(w)$, $j \in \{g, b\}$, has to adjust such that the costs of providing incentives through reward, $f_w(w) - f_w(w + \beta^g(w))$, decrease relative to the costs of providing incentives through punishment, $f_w(w + \beta^b(w)) - f_w(w)$. That is, to implement a given investment level $I(w)$, the sensitivities $\beta^g(w)$ and $\beta^b(w)$ are chosen such that (i) incentive compatibility (10) holds and (ii) the investment outcome that is more likely to occur given $I(w)$ is less costly in terms of incentive provision.

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[31] Although $I(w)$ and $\beta^j(w)$ will be determined simultaneously in equilibrium, we here still ask which combination of $\beta^g(w)$ and $\beta^b(w)$ is optimally used to implement *any* given investment schedule $I(w)$.

Corollary 1. *Consider the optimal compensation policy from Proposition 3 and $w \in (0, \overline{w})$. When the limited liability constraint on $\beta^b(w)$ is slack, then rewards for investment success and punishments for investment failure are chosen such that the ratio of the costs of providing incentives through punishment relative to the costs of providing incentives through reward,*

$$\frac{f_w(w + \beta^b(w)) - f_w(w)}{f_w(w) - f_w(w + \beta^g(w))},$$

is strictly increasing in $I(w)$. Further, $\beta^g(w) \to 0$ as $p(I(w)) \to 1$, while for $p(I(w)) \to 0$, when the agency problem vanishes, $\left[-\beta^b(w)/\beta^g(w)\right] \to 0$, i.e., the first unit of investment is incentivized through rewards only.

Intuitively, Corollary 1 states that, the higher the implemented investment level – and therefore the more likely a success is relatives to a failure – the higher are the costs of providing incentives through punishment relative to the costs of providing incentives through reward under the optimal compensation scheme.[32] This intuition will now prove useful for the following discussion of the optimal investment profile.

4.2   Optimal Investment Profile

In determining the optimal level of investment, the principal trades off the potential for higher profitability with the costs that accrue from having to provide incentives for delegated investment. The resulting optimal investment profile is characterized in the following Proposition.

Proposition 4. *Under the optimal contract of proposition 1 and given $\beta^j(w), j \in \{h, l\}$ from Proposition 3, the optimal level of investment $I(w)$ is independent of the current state $i \in \{h, l\}$ and satisfies, if interior,*

$$\nu p'(I(w))\Delta - 1 = \left(\frac{\lambda}{\nu}\frac{-p''(I(w))}{[p'(I(w))]^2}\right)\delta^{gb}(w) + \nu p'(I(w))\delta^g(w) - \nu p'(I(w))\delta^b(w), \quad (18)$$

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[32]Let us stress here that further analytical implications about the relative use of punishments to rewards could be drawn if $f_{www}(w)$ were equal to zero. Then, indeed, $\left|\beta^b(w)/\beta^g(w)\right|$ is increasing in $I(w)$. However, as $f_{ww}(w)$ in our model changes with w, this amounts to a quantitative comparison for which we have to employ numerical methods, as will be done in Section 5 below.

where

$$\delta^{gb}(w) \quad : \quad = p(I(w))\,(1 - p(I(w)))\,\nu\left[f_w(w + \beta^b(w)) - f_w(w + \beta^g(w))\right] \geq 0, \quad (19)$$

$$\delta^g(w) \quad : \quad = f(w) + \beta^g(w)\,f_w(w) - f(w + \beta^g(w)) \geq 0, \quad (20)$$

$$\delta^b(w) \quad : \quad = f(w) + \beta^b(w)\,f_w(w) - f(w + \beta^b(w)) \geq 0. \quad (21)$$

The left hand side of first order condition (18) represents the potential efficiency gains from investment (cf. the first-best benchmark in (2)). The right hand side captures the *marginal agency costs of delegated investment*. As discussed previously, the additional agency costs from delegated investment arise as incentive compatibility requires to tie the agent's continuation payoff to the investment outcome, which is costly as the principal is effectively risk-averse with respect to variations in w. A marginal increase in I now affects the variability of w in three ways: The first term on the right hand side of (18) captures the fact that from incentive compatibility $\beta^g - \beta^b$ has to increase. The second term reflects the fact that the probability of success increases in I and the third term reflects the fact that, at the same time, the probability of a failure decreases. We will now provide a detailed discussion of each of these three terms.

To understand the first term on the right hand side in (18), recall the incentive compatibility constraint (10). A marginal increase in $I(w)$ requires to increase $(\beta^g(w) - \beta^b(w))$ by $-\frac{\lambda}{\nu}p''(I(w))/\left[p'(I(w))\right]^2$, which is costly due to the concavity of the principal's value function as captured by δ^{gb}. Here the term in square brackets in (19) is equal to the sum of the costs of providing incentives through punishment, $f_w(w + \beta^b(w)) - f_w(w)$, and reward, $f_w(w) - f_w(w + \beta^g(w))$. The factor $p(I(w))\,(1 - p(I(w)))$ captures the variance of the investment outcome in the event of a technology shock with arrival rate ν.[33] Therefore, $\delta^{gb}(w)$ tends to be bigger if the investment outcome is more volatile and it vanishes for sufficiently high investment levels, where investment almost surely leads to a success $(p(I(w)) \to 1)$, and for sufficiently low levels, where investment almost surely leads to a failure $(p(I(w)) \to 0)$.[34]

The second term of the marginal agency costs of delegated investment on the right hand side in (18) stems from the fact that by raising investment marginally the chance of

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[33] Recall that the investment outcome in the event of a technology shock is determined by a Bernoulli random variable with success probability $p(I)$.

[34] In the limit incentives are given "off equilibrium".

an investment success increases by $\nu p'(I(w))$. The associated costs are captured by $\delta^g(w)$, which denotes the efficiency loss that accrues when rewarding the agent for a success. From Corollary 1, $\delta^g(w)$ vanishes for $p(I(w)) \to 1$, as, then, all incentives are optimally given off-equilibrium through punishments as long as the limited liability constraint does not bind.[35]

Analogously to $\delta^g(w)$, the term $\delta^b(w)$ denotes the reduction in value when punishing the agent for a failure, again due to the concavity of the principal's value function. Note that increasing investment reduces the probability of a failure by $\nu p'(I(w))$, in which case $\delta^b(w)$ would have to be borne, thus *reducing* this component of the agency costs. As $\delta^b(w)$ is large for high investment levels, where incentives are given mainly through punishments, this "marginal benefit" eventually dominates the other two components of the marginal agency costs as will be shown below.

From the discussion so far, we conjecture that the agency costs of delegated investment may both increase as well as decrease in the implemented investment level, with a decrease more likely for high levels of investment. The following Corollary confirms this intuition.

Corollary 2. *Under the optimal contract of Proposition 1, the marginal agency costs of delegated investment are positive for $p(I(w)) \to 0$ and negative for $p(I(w)) \to 1$, as long as the limited liability constraint on $\beta^b(w)$ does not bind.*

As the agency costs of delegated investment are, hence, non-monotonic in investment expenditures, equilibrium investment may be distorted both upwards as well as downwards relative to the first-best level. Still, it is far from trivial to make any general analytical statement about the direction of investment distortions, as this hinges, for interior $I(w)$, upon a quantitative comparison of δ^{gb} and δ^j, $j \in \{g, b\}$. Now, while the sign of the distortions δ^{gb} and δ^j follows directly from concavity of $f(w)$, their relative magnitudes can only be computed numerically, as this comparison depends on how the curvature of $f(w)$ changes over the interval $\left[w + \beta^b, w + \beta^g\right]$.[36] Nevertheless, based on the previous discussion, we should be more likely to have overinvestment relative to first-best if I^{FB} is high, while underinvestment should be more likely to occur when I^{FB} is rather low. The numerical analysis in Section 5 indeed confirms this intuition. By varying the "return to investment", $\mu^h - \mu^l$, and thus also I^{FB}, it can be shown that both equilibria with

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[35]This requires w to be large enough, precisely, $w \geq \lambda / \left(\nu p'(\bar{I})\right)$.

[36]Note that there does not exist an analytical solution of $f(w)$.

overinvestment and equilibria with underinvestment relative to first-best are possible, with the former occurring for relatively high values of the "return to investment". Before proceeding, let us, however, provide some more analytical results on investment distortions for extreme values of the state variable w.

Proposition 5. *Under the optimal contract of Proposition 1, there exists $\check{w} \in (0, \overline{w})$ such that there is underinvestment for all $w < \check{w}$, while investment is efficient at $w = \overline{w}$.*

Recall from Proposition 3 that in $\overline{w}$, the agent will never be punished, as $\beta^b(\overline{w}) = 0$, and rewards are directly paid cash, leaving the agent's continuation payoff unchanged at $\overline{w}$. Thus, the agent can be given incentives to invest without distorting the marginal costs of compensation after a shock. This, in turn, implies that investment will be first-best as can be seen formally by substituting $\overline{w}$ in (18) and noting that $f(w)$ extends linearly to the right of $\overline{w}$. Further, when the agent's continuation payoff is low, he can barely be punished as he is protected by limited liability. In this case, when the limited liability constraint is binding, $\delta^g(w) - \delta^b(w)$ will be positive and we will, thus, always have underinvestment for sufficiently low values of w.

5 Numerical Analysis

In order to shed some more light on the properties of the optimal contract, we now provide a quantitative exploration of our model based on numerical examples, for which we assume that $p(I) = \alpha\sqrt{I}$, with $\alpha > 0$ and hence $I \in [0, 1/\alpha^2]$.[37] It turns out that already a relatively simple dynamic model with cash flow diversion and delegated investment, as the one presented above, gives rise to rich investment dynamics. While we do not attempt to take our model to the data directly, we still believe that some of its implications could prove useful in understanding the empirically documented relation between investment expenditures and past cash flows, the impact of corporate governance on investment as well as the relation of investment to executive compensation and managerial turnover.

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[37] A sketch of the algorithm can be found in Appendix B. Note that the examples provided in the following are just an illustration. We have experimented with a wide range of parameterizations and alternative functional specifications and the main (qualitative) results seem to be robust. We assume throughout that $\overline{w} > \lambda / (\nu p'(\overline{I}))$, which ensures that, for high enough values of w, the limited liability constraint on $\beta^b(w)$ does not bind, and is more likely to hold the better the principal's signal about the agent's investment behavior.

In the following we compare two examples illustrating a case with low "return to investment" (upper panel of Figure 1), and another one with high "return to investment" (lower panel of Figure 1). It is important to note that the only difference between the two cases is the "return to investment" as measured by the term $\mu^h - \mu^l$, which is twice as high in the lower than in the upper panel. In particular, all parameters affecting the agency problem directly are left unchanged. Thus, while $\mu^h - \mu^l$ clearly affects investment, it does not alter the optimal contract used to implement a given investment schedule.[38]
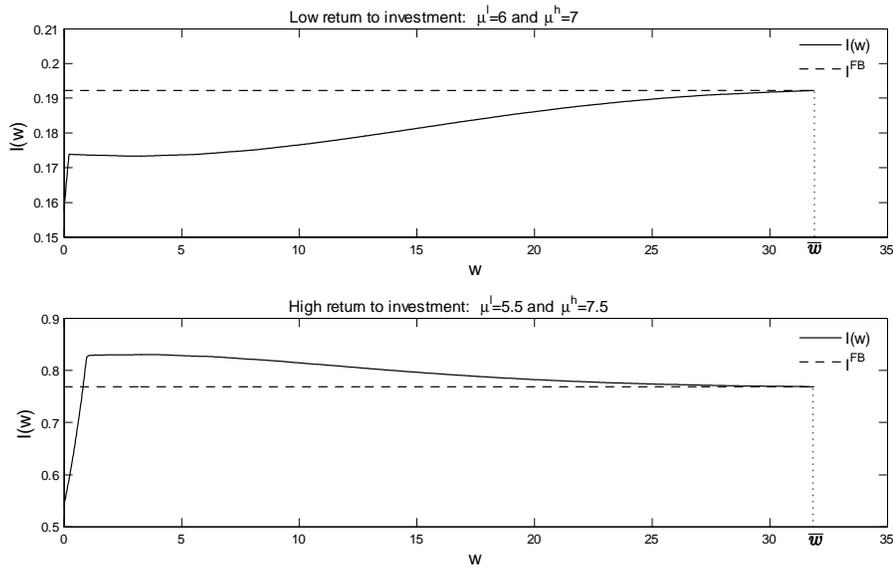

Figure 1: Numerical examples for the optimal investment $I(w)$ (and the respective first-best investment levels I^{FB} as dashed lines) for low and high return to investment, captured by $\mu^h - \mu^l$. The parameter values used for calibration are: $\nu = 1.2$, $\sigma = 10$, $\lambda = 0.7$, $r = 0.1$, $\gamma = 0.15$, $k = 15$, $p(I) = 0.95\sqrt{I}$.

Investment Distortions. While Section 4 qualitatively discussed the agency costs of delegated investment and their implications for the optimal investment profile, it also made clear, that it is hard to make any general analytical statements about the sign of possible distortions, except at particular values of the state variable w (cf. Proposition 5). However, the numeric analysis confirms our previous intuition that, for the example with low return to investment, equilibrium investment is always below its first-best level

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[38]Intuitively, if one had to implement in both cases the same (exogenously given) investment schedule, then also the optimal contract would be identical.

(upper panel of Figure 1), while the example with high return to investment turns out to give rise to overinvestment for some values of w. That is, the optimally chosen $I(w)$ increases in $\mu^h - \mu^l$ and, as the marginal agency costs of delegated investment eventually turn negative (cf. Corollary 2), there will be overinvestment for sufficiently profitable investment technologies.

Further, the graphs confirm the theoretical predictions for extreme values of w derived in Proposition 5, as in both panels and, hence, independent of the "return to investment", there is underinvestment for sufficiently low values of w, while investment approaches the first-best level as w approaches $\overline{w}$. With the agent's continuation payoff tracking both the firm's realized cash flows as well as its investment history, these features imply that firms looking back on a history of sufficiently low cash flow and bad investment outcomes under the incumbent manager invest inefficiently little, while for firms with a successful past record the investment distortion eventually vanishes. For firms with neither an extremely negative nor particularly positive recent history, investment may both be inefficiently high as well as inefficiently low, depending on the "return to investment".

Observation 1. *The optimal investment level will be distorted*
a) downwards relative to first-best if the return to investment is low, and
b) upwards relative to first-best if the return to investment is high, given that the manager's track record is sufficiently positive.

Investment Dynamics. In both the example with under- as well as the one with over-investment in Figure 1, investment increases in w after sufficiently poor results, i.e., as long as the limited liability constraint binds. If the agent's track record is sufficiently positive, that is if his continuation payoff is sufficiently above the firing threshold, investment increases in w in the example with underinvestment, while it decreases in the example with overinvestment. Thus, our model shows that the investment-cash flow sensitivity may well be negative for firms operating in industries where being in a position of technological leadership matters a lot and, hence, the return to investment is high (e.g. the pharmaceutical or high tech industry).[39]

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[39] In line with these predictions Hovakimian (2009) finds that firms with negative investment-cash flow sensitivities have higher growth opportunities, as well as higher investment levels and R&D expenditures. Similarly, Chen and Chen (2012) document a negative investment-cash flow sensitivity for R&D investment in general.

Observation 2. *Given that the manager's track record is sufficiently positive, the investment-cash flow sensitivity is*

a) positive if the return to investment is low, and

b) negative if the return to investment is high.

There is a substantial empirical literature analyzing the relation between cash flows and investment, mainly focussing on the question of whether positive investment-cash flow sensitivities indicate that firms are financially constrained. Fazzari et al. (1988) were the first to use firms' investment-cash flow sensitivity as a measure for financing constraints, which rests on the implicit assumption that the investment-cash flow sensitivity increases with a firm's financing constraints (i.e. with the wedge in the costs of internal and external funds). This monotonicity assumption was challenged repeatedly in the past, starting with Kaplan and Zingales (1997). Newer research that follows this debate has not only found that investment-cash flow sensitivities may also be negative,[40] but also that they seem to have disappeared in the last decade.[41] We are not aware of any empirical study that systematically analyzes the relation between a firm's investment-cash flow sensitivity and the profitability of its investment technology which, in light of our results, might add to the understanding of these findings.[42]

Furthermore, the focus of most of the empirical literature on financing constraints is to understand how current cash flow influences investment, while our model suggests that it is w and, thus, the entire history (stock) of cash flow and investment outcomes that determines the optimal investment level.[43] This history dependence of the optimal investment level is at the heart of the dynamic contracting approach, however, it is often ignored in empirical work.

Because the agent's continuation payoff increases after an investment success and decreases after an investment failure, there is a similar relation between investment outcome and future investment.

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[40]Cf. e.g. Almeida and Campello (2007), Bhagat et al. (2005), Cleary et al. (2007), Guariglia (2008) or Hovakimian (2009).

[41]Cf. e.g. Brown and Petersen (2009) or Chen and Chen (2012).

[42]The result that comes closest to this is in fact Hovakimian (2009) who finds that firms with negative investment-cash flow sensitivities appear to have rather high R&D expenses.

[43]This point is also put forward in DeMarzo et al (2012).

Observation 3. *Given that the manager's track record is sufficiently positive, the investment level will*

a) increase (decrease) following a success (failure) if the return to investment is low, and

b) decrease (increase) following a success (failure) if the return to investment is high.

Taken together, our model shows how the need to provide incentives for investment can generate positive as well as negative investment-cash flow sensitivities depending on both the profitability of the investment technology as well as the entire history of firm performance under the incumbent manager. These results are derived within a framework where financing constraints are derived endogenously from a fully dynamic multi-task model with delegated investment. This should help to better understand the effects of financial constraints on investment behavior, and to explain seemingly conflicting findings in the empirical literature.

Dynamic Incentives for Investment. We will now analyze the dynamics of incentive provision under the optimal contract. As stipulated in Corollary 1, the costs of providing incentives through reward relative to the costs of providing incentives through punishment decrease in the implemented investment level. As a result, the punishment-to-reward ratio, $|\beta^b/\beta^g|$, varies with realized cash flows and the history of investment outcomes. Figure 2 plots the ratio $|\beta^b/\beta^g|$ as a function of w for both the underinvestment case (upper panel) as well as the example with overinvestment (lower panel).[44] In line with Proposition 3, the ratio approaches zero as the manager's continuation payoff gets either close to the firing threshold or approaches the compensation threshold. For low values of w this reflects the binding limited liability constraint, that is, the manager is fired following a failure. For values close to the compensation threshold on the other hand, i.e., when the firm is looking back at a sequence of high cash flow realizations and/or successful investment, it is optimal not to punish the manager for the failure of an investment project. Figure 2 further shows that for intermediate values of w, the punishment-to-reward ratio $|\beta^b/\beta^g|$ is strictly positive and may well be greater than one for high investment levels. Indeed, in our example with overinvestment, the reduction in continuation payoff following a failure may be up to six times higher than the increase after a success.

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[44]Note once more that the optimal contract in both cases differs only because it is optimal to incentivize a different level of investment.
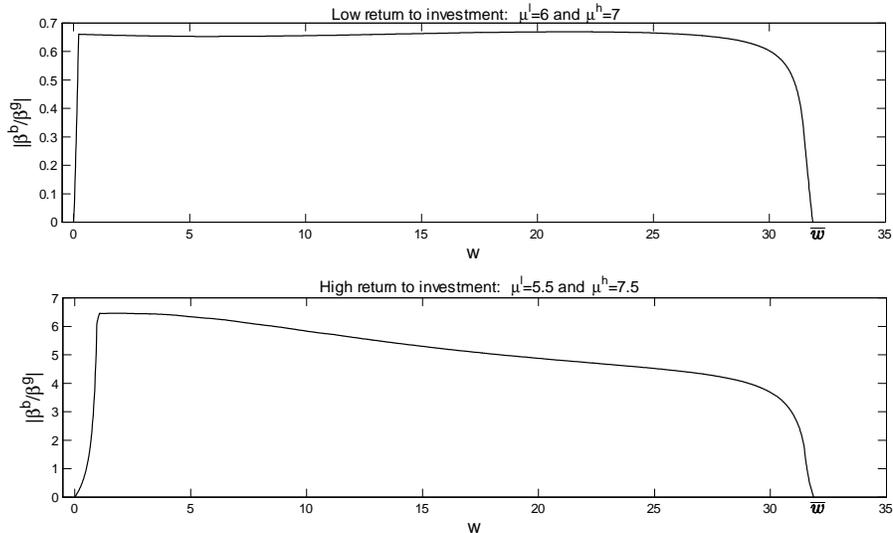
Figure 2: Relative importance of punishment and reward for the example with low, and high return to investment, captured by $\mu^h - \mu^l$. The parameter values used for calibration are: $\nu = 1.2$, $\sigma = 10$, $\lambda = 0.7$, $r = 0.1$, $\gamma = 0.15$, $k = 15$, $p(I) = 0.95\sqrt{I}$.

Observation 4. *The relative strength of punishments to rewards, $|\beta^b/\beta^g|$, is inverse U-shaped in the agent's continuation payoff.*

Corporate Governance and Activist Shareholders. The key measure for the severity of agency conflicts in our model is the agent's benefit from diversion since it determines how much participation in the firm's earnings is needed to induce truth-telling which in turn determines the reward and punishment necessary to incentivize investment. In the following, we conduct a comparative analysis in the agent's benefits from diversion, λ, which can be interpreted as a measure of the quality of corporate governance.

Note, first, that an increase in λ requires to make the agent's continuation payoff more volatile as it has to be tied stronger to reported cash flows (cf. the incentive constraint in (9)). This increases the risk that the agent has to be replaced and, thus, reduces the principal's expected profits.[45] As can be seen in Figure 3, this is partly offset by raising the compensation threshold $\overline{w}$, where the agent is paid cash.

Now, if the underlying cash flow diversion problem becomes less severe (λ decreases), less variation in the agent's continuation payoff in response to technology shocks is needed

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[45]This result can be shown analytically based on the comparative statics approach developed in De-Marzo and Sannikov (2006). The derivations are available from the authors upon request.
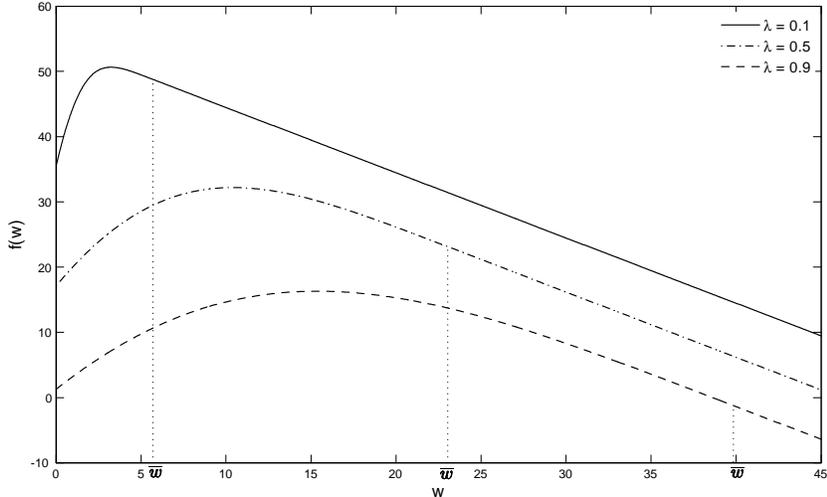
Figure 3: Numerical examples for the principal's value function $f(w)$ for different levels of λ. The other parameter values used for calibration are: $\mu^l = 6$, $\mu^h = 7$, $\nu = 1.2$, $\sigma = 10$, $r = 0.1$, $\gamma = 0.15$, $k = 15$, $p(I) = 0.95\sqrt{I}$.

to implement a given investment level (cf. the incentive constraint in (10)). This reduces the agency costs of delegated investment, implying that the implemented investment level should be closer to the first-best benchmark. This holds for both low "return to investment" (the underinvestment case in Figure 4), as well as for a high one (the overinvestment case in Figure 5). In general, a more severe cash flow diversion problem, hence, should result in more severe investment distortions.[46] One would thus observe empirically, that better corporate governance tends to reduce investment for firms with high investment expenditures and that it should increase investment for firms with low investment.

Observation 5. *Given that the manager's track record is sufficiently positive, an improvement in corporate governance (lower λ) implies that for a given w,*

a) investment increases (less underinvestment) if the return to investment is low, and

b) investment decreases (less overinvestment) if the return to investment is high.

Both the negative and the positive relation are documented in different empirical studies. First, Wahal and McConnell (2000) as well as Lev and Nissim (2003) find ownership

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[46]Note, however, that the *direction* of these distortions does not depend on the underlying cash flow diversion problem. That is, whether we are in the overinvestment or in the underinvestment case depends only on characteristics of the investment technology.
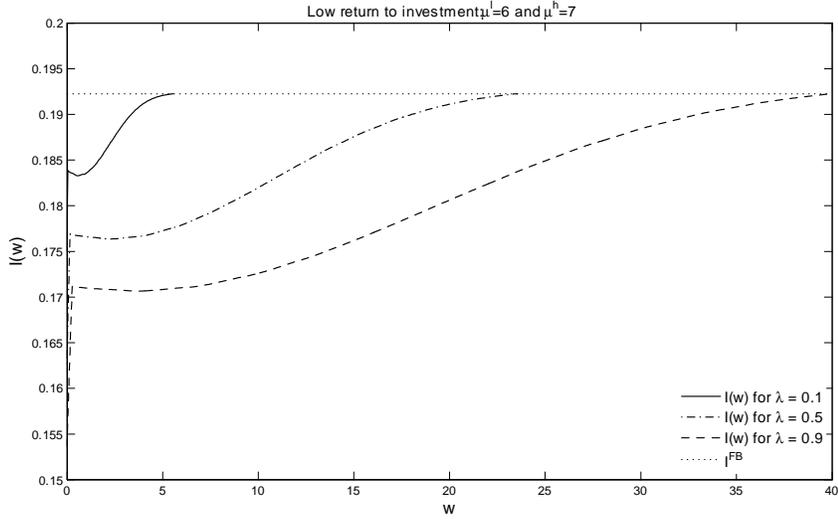
Figure 4: Numerical examples for the optimal investment level $I(w)$ for the case with low return to investment, $\mu^l = 6$ and $\mu^h = 7$. The importance of the cash flow diversion problem is captured by the parameter λ. The parameter values used for calibration are: $\nu = 1.2$, $\sigma = 10$, $\lambda = 0.7$, $r = 0.1$, $\gamma = 0.15$, $k = 15$, $p(I) = 0.95\sqrt{I}$.
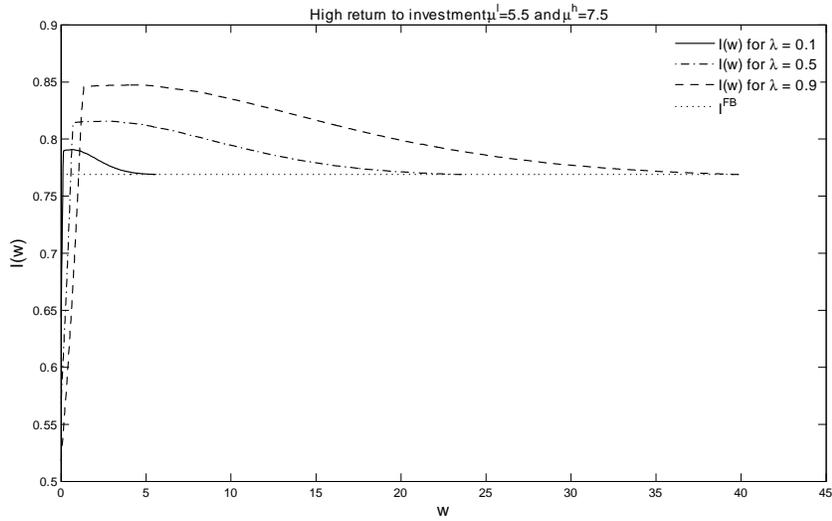

Figure 5: Numerical examples for the optimal investment level $I(w)$ for the case with high return to investment, $\mu^l = 5.5$ and $\mu^h = 7.5$. The importance of the cash flow diversion problem is captured by the parameter λ. The parameter values used for calibration are: $\nu = 1.2$, $\sigma = 10$, $\lambda = 0.7$, $r = 0.1$, $\gamma = 0.15$, $k = 15$, $p(I) = 0.95\sqrt{I}$.

of institutional investors to be associated with higher investment levels. Aghion et al. (2009) in turn find a strong positive relation between institutional ownership and firms' returns to R&D (measured by the number of future patents). Bushee (1998) finds that managers in firms with a long-term oriented institutional investor are less inclined to cut R&D expenditure in response to earnings declines. Using the corporate governance index of Gompers et al. (2003), Bohren et al. (2007) find that better governed firms' investment is higher and more closely aligned with investment opportunities.

Other empirical studies find the opposite result, namely that better governed firms tend to invest less (see e.g. Gompers et al. 2003, Philippon 2006 Albuquerque and Wang 2008), suggesting that better governance counteracts managers' empire building preferences (cf. Jensen 1986) and thus reduces overinvestment. Moreover, Cella (2010) shows that ownership by institutional investors reduces investment and particularly so for firms that have too high investment levels compared to their investment opportunities or average industry investment.

Our model describes a novel channel on how corporate governance might impact investment expenditures based on dynamic agency costs, which can endogenously create both a positive as well as a negative relationship without assuming private costs of investment or the desire to build an empire.

6   Conclusion

In this paper we analyze a continuous time cash flow diversion model with delegated investment. Investment in our model requires the manager's expertise and specific skills and can be thought of as the choice of "absorptive capacity". For example, in the event of a technological innovation in the industry the firm is operating in, future profitability, as measured by expected cash flows, is determined by the firm's ability to adopt the new technology and, thus, ultimately by its investment expenditures. A successful investment outcome implies that the firm is "at the research frontier" and, hence, its future cash flows are high, whereas, in case of an investment failure, the firm falls behind the technology leaders in its industry and expects to have low cash flows in the future. As technological breakthroughs are both rare and unpredictable, the investment technology in our model is stochastic and creates persistent effects of investment on future profitability.

The discretion the manager has over the firm's funds gives rise to a fully dynamic "accounting problem" where he can divert cash flow for private consumption as well as move funds within the firm in order to meet either short- or long-term targets. While the optimal contract in the (single-task) cash flow diversion model is by now well understood, the focus of the present paper is on how the underlying cash flow diversion problem impacts optimal *delegated* investment. The need to provide incentives for truthful reporting of realized cash flows makes the principal effectively risk averse with respect to variation in the agent's continuation payoff. Now, in order to incentivize the manager to invest in the principal's interest some form of reward for an investment success and punishment in case of a failure is required, thus inducing additional volatility in the agent's continuation payoff.

The resulting additional agency costs of delegated investment cause optimal investment in our model to depend on the entire history of cash flows and investment outcomes as summarized by the agent's continuation payoff. While investment will in general be distorted away from its first-best level, the sign of the distortion depends both on the firm's performance as well as the profitability of the investment technology. When the incumbent manager does not have to fear being fired in the foreseeable future, optimal investment is distorted upwards and decreasing in past performance if the investment technology is relatively profitable and distorted downwards and increasing for unprofitable technologies.

We further characterize the optimal incentive scheme and its dynamics. The model implies that the relative importance of punishment and reward in the provision of incentives for investment also changes with the agent's continuation payoff. Rewards are particularly important for sufficiently low as well as sufficiently high levels of the agent's promised wealth, while punishment is used more heavily in an intermediate range of the continuation payoff. Finally, we show that better corporate governance tends to reduce the distortion in investment and may thus both reduce investment (overinvestment case) as well as increase investment (underinvestment case). Overall these implications should prove useful in understanding better the empirically documented relation between investment expenditures and past cash flows, the impact of corporate governance on investment as well as the relation of investment to executive compensation and managerial turnover.

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Appendix A Omitted Proofs

Proof of Lemma 1. For $S = S^*$, we define the manager's expected lifetime utility evaluated conditional on time t information by:

$$v_t = \int_0^t e^{-\gamma s} dU_s + e^{-\gamma t} w_t, \tag{A.1}$$

which, by construction, is a martingale with respect to the filtration generated by (Z, N^j), $j \in \{g, b\}$ under the probability measure Q^{S^*}. By the martingale representation theorem, we can express v_t for some predictable processes α and β^j, $j \in \{g, b\}$ as

$$
\begin{aligned}
v_t &= v_0 + \int_0^t e^{-\gamma s} \alpha_s \left(d\hat{Y}_s - (\mu_s - I_s^*) ds \right) + \int_0^t e^{-\gamma s} \beta_s^g \left(dN_s^g - \nu p(I_s^*) ds \right) \\
&\quad + \int_0^t e^{-\gamma s} \beta_s^b \left(dN_s^b - \nu (1 - p(I_s^*)) ds \right),
\end{aligned}
$$

where, under Q^{S^*}, $\hat{Y}_t - \int_0^t (\mu_s - I_s^*) ds$ is a standard Brownian motion and $N_t^g - \int_0^t \nu p(I_s^*) ds$ as well as $N_t^b - \int_0^t \nu (1 - p(I_s^*)) ds$ are compensated Poisson processes. Differentiating both this representation and the definition of v_t in (A.1) yields (8). **Q.E.D.**

Proof of Lemma 2. Consider any feasible policy of the agent, $S = \left\{ \hat{Y}_t, I_t : 0 \leq t \leq \tau \right\}$, with $dY_t - d\hat{Y}_t \geq 0$ and $I_t \geq 0$. The associated expected lifetime utility is given by

$$
\begin{aligned}
v_t &= v_0 + \int_0^t e^{-\gamma s} \alpha_s \left(d\hat{Y}_s - (\mu_s - I_s^*) ds \right) + \int_0^t e^{-\gamma s} \lambda \left(dY_s - d\hat{Y}_s \right) \\
&\quad + \int_0^t e^{-\gamma s} \beta_s^g \left(dN_s^g - \nu p(I_s^*) ds \right) + \int_0^t e^{-\gamma s} \beta_s^b \left(dN_s^b - \nu (1 - p(I_s^*)) ds \right),
\end{aligned} \tag{A.2}
$$

where $d\hat{Y}_t - (\mu_t - I_t^*) dt = \sigma dZ_t$ for $S = S^*$, and dN_t^g (dN_t^b) is a Poisson process with intensity $\nu p(I_t)$ ($\nu (1 - p(I_t))$). Differentiating (A.2) and taking expectations gives

$$e^{\gamma t} E [dv_t] = (\lambda - \alpha_t) E \left[dY_t - d\hat{Y}_t \right] + \alpha_t (I_t^* - I_t) dt + \nu \left(\beta_t^g - \beta_t^b \right) (p(I_t) - p(I_t^*)) dt.$$

Clearly, v_t is a martingale for $S = S^*$. For $S = S^*$ to be incentive compatible, the drift of v_t has to be non-positive for all possible deviations, i.e., v_t has to be a supermartingale for any feasible $S \neq S^*$. Consider, first, a deviation from truth-telling, i.e., underreporting for consumption $dY_t - d\hat{Y}_t > 0$. This deviation is suboptimal for the agent if

$$\alpha_t \geq \lambda. \tag{A.3}$$

36

Second, increasing I_t marginally above I_t^*, is suboptimal for the agent if

$$\alpha_t \geq \nu p'(I_t) \left(\beta_t^g - \beta_t^b\right), \tag{A.4}$$

while, third, decreasing I_t marginally below I_t^*, is suboptimal if

$$\alpha_t \leq \nu p'(I_t) \left(\beta_t^g - \beta_t^b\right). \tag{A.5}$$

So, for $I_t^* \in \left(0, \overline{I}\right)$, incentive compatibility requires (9) and (10) to hold. Clearly, if $I_t^* = 0$, the agent cannot underinvest such that incentive compatibility only requires (9) and (A.4), while the agent cannot overinvest if $I_t^* = \overline{I}$ and incentive compatibility is guaranteed if (A.5) holds next to (9). **Q.E.D.**

Proof of Proposition 1. For notational convenience, we will drop the dependence of the optimal choices of I^i, α^i, and $\beta^{j,i}$, $j \in \{g, b\}$, $i \in \{l, h\}$ on w for the remainder of this proof. Given the symmetric structure of the problem (cf. main text), we will focus on solutions where $f^h(w)$ and $f^l(w)$ differ only by an additive constant.[47] Then I^i, α^i, and $\beta^{i,j}$ are independent of the current state $i \in \{l, h\}$, implying that

$$f^h(w) - f^l(w) = \Delta \text{ for all } w \geq 0,$$

where Δ is the first-best value difference, as in (3). We then get the following set of first-order conditions characterizing, if interior, the optimal choices of I:

$$1 = \left[\begin{array}{c} \nu p'(I) \left(\beta^g - \beta^b\right) f_w(w) + \nu p'(I) \left[\Delta + f(w + \beta^g) - f(w + \beta^b)\right] \\ + \alpha p(I) \frac{p''(I)}{(p'(I))^2} \left[f_w(w) - f_w(w + \beta^g)\right] \end{array} \right], \tag{A.6}$$

and for β^j:

$$f_w(w) = p(I) f_w(w + \beta^g) + (1 - p(I)) f_w(w + \beta^b). \tag{A.7}$$

---

[47] Clearly, the equilibrium derived under this additional restriction also satisfies all necessary conditions of the original problem in (12).

We will now show concavity of $f(w)$ in w. Differentiating (14) we get

$$
\begin{aligned}
-(\gamma - r) f_w(w) \;=\; & I_w(w) \left[\begin{array}{c} -1 - \alpha f_w(w) + \frac{\alpha p(I(w)) p''(I(w))}{(p'(b(w)))^2} \left[f_w(w) - f_w(w + \beta^g(w)) \right] \\ + \nu p'(I(w)) \left[f(w + \beta^g(w)) + \Delta - f(w + \beta^b(w)) \right] \end{array} \right] \\
& + \left[\gamma w - \nu \left(\frac{\alpha p(I(w))}{\nu p'(I(w))} + \beta^b(w) \right) \right] f_{ww}(w) + \frac{1}{2} \sigma^2 \alpha^2 f_{www}(w) \\
& + \nu \left(1 + \beta_w^b(w) \right) \left[\begin{array}{c} p(I(w)) f_w(w + \beta^g(w)) \\ + \left(1 - p(I(w)) \right) f_w(w + \beta^b(w)) - f_w(w) \end{array} \right],
\end{aligned}
$$

which gives, using (A.6) and either (A.7), or, in case the limited liability constraint binds, $\beta_w^b = -1$,

$$
-(\gamma - r) f_w(w) = \left(\gamma w - \nu \left(\frac{\alpha p(I(w))}{\nu p'(I(w))} + \beta^b(w) \right) \right) f_{ww}(w) + \frac{1}{2} \sigma^2 \alpha^2 f_{www}(w). \quad \text{(A.8)}
$$

Together with the boundary conditions at $\overline{w}$ this implies

$$
(\gamma - r) = \frac{1}{2} \sigma^2 \alpha^2 f_{www}(\overline{w}),
$$

such that $\exists \varepsilon > 0$ with $f_{www}(\overline{w} - \varepsilon) > 0$ and $f_{ww}(\overline{w} - \varepsilon) < 0$. Now assume that $\exists \check{w} := \sup\{w < \overline{w} : f_{ww}(w) \geq 0\}$, where it holds by continuity that $f_{ww}(\check{w}) = 0$ and $f_{www}(\check{w}) < 0$, implying that

$$
f_w(\check{w}) = -f_{www}(\check{w}) \frac{1}{2} \frac{\sigma^2 \alpha^2}{\gamma - r} > 0.
$$

Then, consider two points $w^1 < \check{w} < w^2$ close to $\check{w}$, such that $f_{ww}(w^1) > 0 > f_{ww}(w^2)$ and $w^1 f_w(w^1) = w^2 f_w(w^2)$ and observe that $f(w)$ can be written as

$$
r f(w) = \gamma w f_w(w) + \frac{1}{2} \sigma^2 \alpha^2 f_{ww}(w) + g(w),
$$

with

$$
g(w) = \mu^l - I + \nu p(I) \Delta + \nu \left[\begin{array}{c} p(I) f(w + \frac{\alpha}{\nu p'(I)} + \beta^b) + (1 - p(I)) f(w + \beta^b) - f(w) \\ - \left(\frac{\alpha p(I)}{\nu p'(I)} + \beta^b \right) f_w(w) \end{array} \right].
$$

Next, compute the differential of $g(w)$ around $\check{w}$, i.e., $dg(w)|_{w=\check{w}} = g_w(\check{w})dw$, and observe

38

that

$$
\begin{aligned}
g_w(\breve{w}) &= I_w \left[\begin{array}{c} -1 + \nu p'(I)\Delta + \nu p'(I) \left[f(\breve{w} + \beta^g) - f(\breve{w} + \beta^b) \right] \\ -\alpha f_w(\breve{w}) + \frac{\alpha p(I)p''(I)}{(p'(I))^2} \left[f_w(\breve{w}) - f_w(\breve{w} + \beta^g) \right] \end{array} \right] \\
&\quad + \nu \left(1 + \beta_w^b \right) \left[p(I)f_w(\breve{w} + \beta^g) + (1 - p(I)) f_w(\breve{w} + \beta^b) - f_w(\breve{w}) \right] \\
&\quad - \nu \left(\frac{\alpha p(I)}{\nu p'(I)} + \beta^b \right) f_{ww}(\breve{w}) \\
&= 0,
\end{aligned}
$$

which follows from $f_{ww}(\breve{w}) = 0$ together with (A.6) and either (A.7), or, in case the limited liability constraint binds, $\beta_w^b = -1$. Thus, evaluating $f(w)$ in w^1 and w^2, we get

$$
r \left(f(w^1) - f(w^2) \right) = \frac{1}{2}\sigma^2\alpha^2 \left(f_{ww}(w^1) - f_{ww}(w^2) \right) > 0,
$$

where we have used that the effect of a change in $g(w)$ around $\breve{w}$ is of second order and will thus be dominated by the change in $f_{ww}(w)$. However, this directly contradicts $f_w(\breve{w}) > 0$.

It remains to show that the incentive constraint for α will be binding. From the respective first order condition

$$
0 = \sigma^2\alpha f_{ww}(w) + \frac{p(I)}{p'(I)} \left[f_w(w + \beta^g) - f_w(w) \right],
$$

together with concavity of $f(w)$, it follows that there can be no interior solution for α and, thus, $\alpha = \lambda$.

Finally, having established concavity of $f(w)$, the verification argument is standard (cf. DeMarzo and Sannikov 2006, Hoffmann and Pfeil 2010) and is therefore omitted. **Q.E.D.**

Proof of Proposition 2. Note first, that the f^i are strictly concave, which follows from the same arguments as in Hoffmann and Pfeil (2010). Further, the incentive constraint binds, i.e., $\alpha = \lambda$. Interior solutions for I^i are then given by

$$
\left[f^h(w + \beta^{g,i}) - f^l(w + \beta^{b,i}) \right] - \left(\beta^{g,i} - \beta^{b,i} \right) f_w^i(w) = \frac{1}{\nu p'(I^i)}, \tag{A.9}
$$

while the $\beta^{j,i}$, $j \in \{g, b\}$, $i \in \{h, l\}$ are determined from

$$
f_w^h(w + \beta^{g,i}) = f_w^i(w) = f_w^l(w + \beta^{b,i}), \tag{A.10}
$$

Due to the strict concavity of f^i this immediately implies that $\beta^{g,h} = 0$ and $\beta^{b,l} = 0$.

Let us show next that it is further optimal to set $\beta^{b,h} = \beta^{g,l} = 0$ and $I^h = I^l = I^{FB}$. To see this, observe that $\beta^{j,i} = 0$, $j \in \{g, b\}$, $i \in \{h, l\}$ implies from (A.10), that f^h and f^l can only differ by an additive constant. Then, subtracting the HJBs results in

$$f^h(w) - f^l(w) = \frac{1}{r + \nu} \left(\mu^h - \mu^l \right) = \Delta,$$

as in first-best. Together with (A.9), this implies

$$\nu p'(I)\Delta = 1,$$

which is the first-best investment level. So the proposed solution achieves efficient investment. **Q.E.D.**

Proof of Proposition 3. First, assuming that $\beta^b \geq -w$, substituting β^g from incentive constraint (10) in HJB (12) and taking the first derivative with respect to β^b yields first order condition (16). Let us now show that there exists $\widehat{w} \in (0, \overline{w})$ such that $\beta^b(w) = -w$ for $w \in [0, \widehat{w}]$. To see this, note that, for any $I(w) > 0$, we must have $\beta^g(w) - \beta^b(w) > 0$, which together with the strict concavity of $f(w)$ for $w \in [0, \overline{w})$ implies that the first-order condition in (16) cannot be satisfied in a neighborhood of $w = 0$. Thus, we get the stated result, with $\beta^b(w) = -w$ and $\beta^g(w) = \beta^b + \frac{\lambda}{\nu p'(I(w))} > 0$ on $w \in [0, \widehat{w}]$, where

$$\widehat{w} := \min \left\{ w > 0 : p(I(w)) f_w \left(\frac{\lambda}{\nu p'(I(w))} \right) + (1 - p(I(w))) f_w(0) = f_w(w) \right\}.$$

For $w \in (\widehat{w}, \overline{w})$ the strict concavity of $f(w)$ together with (16) directly implies $\beta^g(w) > 0$ and $\beta^b(w) < 0$. Finally, as $f(w)$ extends linearly for $w > \overline{w}$, we have that $f_w(\overline{w} + \beta^g(\overline{w})) = f_w(\overline{w})$ and (16) can only be satisfied if $\beta^b(\overline{w}) = 0$. If $I(w) = 0$, it is clearly optimal to set $\beta^b(w) = \beta^g(w) = 0$, while for $I(w) = \overline{I}$, we have $\beta^b(w) = -w$ and $\beta^g(w) = 0$ as long as this guarantees that $\nu p'(\overline{I}) \left(\beta^g(w) - \beta^b(w) \right) \geq \lambda$, and $\beta^b(w) = -w$ as well as $\beta^g(w) = \lambda / \left(\nu p'(\overline{I}) \right) - w$ else. **Q.E.D.**

Proof of Corollary 1. From (16) we have for $p(I) \in (0, 1)$

$$\frac{p(I(w))}{1 - p(I(w))} = \frac{f_w(w + \beta^b(w)) - f_w(w)}{f_w(w) - f_w(w + \beta^g(w))}, \tag{A.11}$$

and the monotonicity result trivially follows from the strict monotonicity of $p(I)/(1 - p(I))$. Further, if $p(I) \to 1$ and the limited liability constraint is slack[48], the strict concavity of

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[48] This requires $w \geq \lambda / \left(\nu p'(\overline{I}) \right)$.

$f(w)$ implies that (16) can only be satisfied for $\beta^g(w) \to 0$. In case $p(I) \to 0$, (16) similarly requires that $\beta^b(w) \to 0$, while, from incentive compatibility, $\beta^g(w) \to \frac{\lambda}{\nu p'(0)}$. For bounded $p'(0)$, it then trivially follows that $-\beta^b(w)/\beta^g(w)$ goes to zero as $I(w) \to 0$. Now, if $p'(I) \to \infty$ as $I \to 0$, both $\beta^g(w)$ and $\beta^b(w)$ go to zero as $I \to 0$. Then, up to a first-order approximation, we have

$$
\begin{aligned}
f_w(w + \beta^b(w)) &= f_w(w) + \beta^b(w) f_{ww}(w), \\
f_w(w + \beta^g(w)) &= f_w(w) + \beta^g(w) f_{ww}(w).
\end{aligned}
$$

Hence, there exists an $\varepsilon > 0$ such that, from (A.11), we have

$$
\frac{p(\varepsilon)}{1 - p(\varepsilon)} = \frac{-\beta^b(w)}{\beta^g(w)}.
$$

Q.E.D.

Proof of Proposition 4. Substituting β^g from incentive constraint (10) in HJB (12) and taking the first derivative with respect to I yields first order condition

$$
\begin{aligned}
1 &= \nu p'(I(w)) \left[\Delta + f(w + \beta^g(w)) - f(w + \beta^b(w)) \right] - \lambda f_w(w) \qquad \text{(A.12)} \\
&\quad + \lambda \frac{p(I(w)) p''(I(w))}{(p'(I(w)))^2} \left[f_w(w) - f_w(w + \beta^g(w)) \right] \\
&= \nu p'(I(w)) \left[\Delta + f(w + \beta^g(w)) - f(w + \beta^b(w)) - \left(\beta^g(w) - \beta^b(w) \right) f_w(w) \right] \\
&\quad + \lambda \frac{p(I(w)) p''(I(w))}{(p'(I(w)))^2} \left[1 - p(I(w)) \right] \left[f_w(w + \beta^b(w)) - f_w(w + \beta^g(w)) \right],
\end{aligned}
$$

where the last equality follows from (10) and (16). Rearranging then yields (18). **Q.E.D.**

Proof of Corollary 2. Consider a given $w \in (0, \overline{w})$ where the limited liability constraint does not bind for $0 \le p(I) \le 1$[49] and rewrite the marginal agency costs of delegated

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[49]A sufficient condition, that guarantees that the limited liability constraint does not bind at w is that $w \ge \frac{\lambda}{\nu p'(\overline{I})}$.

investment as

$$
\begin{aligned}
MAC(I,w) \; = \;\; & \nu\lambda\frac{-p''(I)}{[p'(I)]^2}p(I)\left(1-p(I)\right)\left[f_w(w+\beta^b(I,w))-f_w(w+\beta^g(I,w))\right] \quad\text{(A.13)} \\
& +\nu p'(I)\left(1-p(I)\right)\left[\begin{array}{c} f(w+\beta^b\,(I,w))-f(w+\beta^g\,(I,w)) \\ +\left(\beta^g\,(I,w)-\beta^b(I,w)\right)f_w(w+\beta^b(I,w)) \end{array}\right] \\
& -\nu p'(I)\,p(I)\left[\begin{array}{c} f(w+\beta^g\,(I,w))-f(w+\beta^b\,(I,w)) \\ -\left(\beta^g\,(I,w)-\beta^b(I,w)\right)f_w(w+\beta^g(I,w)) \end{array}\right].
\end{aligned}
$$

First, note that all the terms in square brackets are strictly positive for $p(I) > 0$ and bounded. Then, as $I \to \overline{I}$ and accordingly $p(I) \to 1$, we have

$$
MAC(I,w) \to \left(-\nu p'\left(\overline{I}\right)\left[f(w)-f(w-\frac{\lambda}{\nu p'(\overline{I})})-\frac{\lambda}{\nu p'(\overline{I})}f_w(w)\right]\right) < 0,
$$

where we have used concavity of $f(w)$ as well as Corollary 1 and that $p'(\overline{I}) > 0$, which follows from the assumption that the limited liability constraint does not bind.[50] For the limit where $I \to 0$ ($p(I) \to 0$) assume first that $p'(0)$ is bounded. Then, we can derive the following lower bound for the marginal agency costs:

$$
\lim_{I \to 0} MAC(I,w) \geq \left(\nu p'\left(0\right)\left[f(w)-f(w+\frac{\lambda}{\nu p'(0)})+\frac{\lambda}{\nu p'(0)}f_w(w)\right]\right) > 0,
$$

which again follows from $f_{ww}(w) < 0$ and Corollary 1. For the case where $p'(I)$ goes to infinity as $I \to 0$, we need the additional assumption that $\lim_{I\to 0}\left[p'(I)p(I)\right]$ remains bounded. The result then follows from the fact that there exists an $\varepsilon > 0$ such that the first two terms on the right-hand-side in (A.13) dominate the third term for $I \in (0,\varepsilon)$. As a final remark, let us note that all assumptions on $p(I)$ are satisfied for $p(I) = \sqrt{I}$, $0 \leq I \leq 1$. **Q.E.D.**

Proof of Proposition 5. Let us first show $I(\overline{w}) = I^{FB}$. We know from Proposition 3 that $\beta^g(\overline{w}) > 0$ and $\beta^b(\overline{w}) = 0$. Hence, all terms on the right-hand-side of (18) are equal to zero implying first-best investment. Next, let us show the underinvestment result for small w. With the limited liability constraint binding (cf. Proposition 3), (A.12) can be

---

[50]We further need the technical condition that $p''(I)\left(1-p(I)\right)$ goes to zero as $I \to \overline{I}$.

rewritten to obtain

$$1 - \nu p'(I(w))\Delta = \nu p'(I(w)) \left[f\left(\frac{\lambda}{\nu p'(I(w))}\right) - \left(f(0) + \frac{\lambda}{\nu p'(I(w))} f_w(w) \right) \right]$$
$$+ \lambda \frac{p(I(w))p''(I(w))}{(p'(I(w)))^2} \left[f_w(w) - f_w(w + \beta^g(w)) \right].$$

It remains to show that the right-hand-side of this expression is negative for small w. To see this, note first, that the expression in the second line is negative for all w by strict concavity of $f(w)$. As for the remaining right-hand-side expression, it also follows from concavity of $f(w)$ that this is negative for w small enough. **Q.E.D.**

Appendix B   Numerical Implementation

To solve numerically for the optimal contract of Proposition 1, we take the following iteration steps.

1. Solve for the principal's value function $f^{(0)}$ and the free boundary $\overline{w}^{(0)}$ without technology shocks. That is, we solve the ODE in (14) with $\nu = 0$ (thus the initial investment is $I^{(0)} = 0$ and the initial rewards and punishments are $\beta^{g^{(0)}} = \beta^{b^{(0)}} = 0$).

2. Given $f^{(0)}$, $\overline{w}^{(0)}$, and $\beta^{b^{(0)}}$, update the optimal investment scheme $I^{(1)}$ according to the first order condition (A.12) and subject to incentive compatibility, i.e., $\beta^g\left(I^{(1)}\right) = \beta^{b^{(0)}} + \lambda/\left(\nu p'(I^{(1)})\right)$.

3. Given $\beta^{g^{(0)}}$, $\beta^{b^{(0)}}$, and $I^{(1)}$, update the principal's value function $f^{(1)}$ and the free boundary $\overline{w}^{(1)}$.

4. Given $f^{(1)}$, $\overline{w}^{(1)}$, and $I^{(1)}$, update the optimal rewards and punishments $\beta^{g^{(1)}}$ and $\beta^{b^{(1)}}$ according to the first order condition (16), subject to the (binding) incentive constraint (10) and the limited liability constraint (11). That is, we solve (16) for $\beta^{b^{(1)}}$, such that $\beta^{g^{(1)}} = \beta^{b^{(1)}} + \lambda/\left(\nu p'(I^{(1)})\right)$ and $\beta^{b^{(1)}} \geq w$.

5. Repeat steps 2 to 4 until the problem converges. The convergence criterion is

$$\max\left[\sup_w \left|I^{(i+1)} - I^{(i)}\right|, \sup_w \left|f^{(i+1)} - f^{(i)}\right|\right] < 10^{-5}.$$