Do Investors like to Diversify?
A Study of Markowitz Preferences,
with Discussions on Prospect Preference

Martin Egozcue
Department of Economics FCS, Universidad de la Republica del Uruguay

Wing-Keung Wong
Department of Economics and Institute for Computational Mathematics
Hong Kong Baptist University

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Corresponding author: Wing-Keung Wong, Department of Economics, Hong Kong Baptist University WLB, Shaw Campus, Kowloon Tong, Hong Kong; Telephone: (852)-3411-7542, Fax: (852)-3411-5580, Email: awong@hkbu.edu.hk

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Abstract

This paper characterizes the attitudes towards diversification for (Markowitz) investors with reverse S-shaped utility functions under some simple assumptions and under a variety of circumstances, including different mean values for the underlying assets’ distributions and different curvatures on the positive and negative portions of investors’ utility functions. In particular, our findings crucially induce that, different from risk-averse and risk-seeking investors whose preferences on diversification remain the same under different situations, Markowitz investors could be unanimously indifferent from any convex combination of the underlying assets in a circumstance. They could overwhelmingly prefer to diversify in another circumstance, and alternatively, prefer not to diversify in other circumstances. In addition, we discuss the diversification preferences for Markowitz investors when these assumptions do not hold, examine the attitudes of prospect investors towards risk, and study the relationships of diversification behaviors between Markowitz and prospect investors. Our findings could be used to substantially explain some financial puzzles such as diversification puzzle and momentum profits.

Keywords: Diversification, stochastic dominance, risk seeking, risk averse, S-shaped utility function, reverse S-shaped utility function

JEL Classification: D81, G11
1 Introduction

According to von Neuman and Morgenstern’s (1944) expected utility theory, the utility functions for risk averters and risk seekers are concave and convex, respectively, both are increasing functions. Examining the relative attractiveness of various forms of investments, Friedman and Savage (1948) claim that the strictly concave functions may not be able to explain why investors buy insurance or lottery tickets. Markowitz (1952a), the first to address Friedman and Savage’s concern, proposes a utility function that has convex and concave regions in both the positive and the negative domains. To support Markowitz’s proposed utility function, Williams (1966) documents data where a translation of outcomes produces a dramatic shift from risk aversion to risk seeking, whereas Fishburn and Kochenberger (1979) report the prevalence of risk seeking in choices between negative prospects. Kahneman and Tversky (1979) and Tversky and Kahneman (1992) claim that the utility function is concave for gains and convex for losses, yielding an S-shaped function. They also build up a formal theory of loss aversion called prospect theory in which investors can maximize the expectation of the S-shaped utility function. It is one of the prominent decision-making theories about risk-seeking and has gained much attention from economists and professionals in the financial sector.

Noticing the presence of risk seeking in preferences among positive as well as negative prospects, Markowitz (1952a) also proposes another type of utility function different from the pure S-shaped utility functions used in the prospect theory. He suggests a utility that is first concave, then convex, then concave, and finally convex to explain Friedman and Savage’s question about why investors buy insurance and lottery tickets. Using sequential gambling techniques, Thaler and Johnson (1990) obtain experimental evidence to show that prior outcomes affect subsequent behavior in a way that is contrary to the static version of the prospect theory. In particular, subjects are more risk seeking following

\[ \text{Ng (1965) and Machina (1982) also provide other explanations of Friedman and Savage’s paradox.} \]
gains and more risk averse following losses. This implies that in a dynamic context, a reverse S-shaped utility function may be more descriptive of actual behavior. Fong et al (2008) and others study the behaviors of investors in stock markets and find evidence to support Thaler and Johnson (1990)’s observations for investors in stock markets.

Levy and Wiener (1998) further develop the theory for investors with S-shaped and reverse S-shaped utility functions. Levy and Levy (2002, 2004) are the first to extend the work of Markowitz (1952a) and others to develop a new criterion called Markowitz stochastic dominance (MSD) to determine the dominance of one investment alternative over another for all reverse S-shaped utility functions, and another criterion called prospect stochastic dominance (PSD) to determine the dominance of one investment alternative over another for all S-shaped utility functions. Recently, Wong and Chan (2008) extend the PSD and MSD theory to the first three orders of SD and link the corresponding S-shaped and reverse S-shaped utility functions to the first three orders. For convenience, we call investors with S-shaped utility functions prospect investors or investors with prospect preference and investors with reverse S-shaped utility functions Markowitz investors or investors with Markowitz preference.

There are very few articles that address the portfolio diversification behaviors of agents with S-shaped or reverse S-shaped utility functions, especially reverse S-shaped utility functions, see for example, Berkelaar, Kouwenberg, and Post (2004) and Gomes (2005). Kaheman and Tversky develop prospect theory to study the behaviors of agents with S-shaped utility functions. It has been demonstrating to be extremely influential in explaining a range of phenomena. These include the disposition effect, asymmetric price elasticities, elasticities of labor supply that are inconsistent with standard models of labor supply and the excess sensitivity of consumption to income; see, for example, Camerer (2000). Levy and Levy (2004) show that mean-variance and prospect theory efficient sets almost coincide, under normal distribution of the assets returns. Further applications also include the explanation of financial anomalies (Thaler, 2005). However, the subsequent
literature in prospect theory mainly studies issues for single asset, not so much on the analysis of agents’ diversification behaviors when they are confronted with simultaneous series of assets, like in a portfolio selection environment.

In this paper, we fill in the gap in the literature to provide an axiomatic analysis for behavioral finance by developing properties for the attitudes of Markowitz investors towards risk under the assumptions that (a) the underlying assets being examined are independent and identically distributed (iid) with symmetric probability density functions, and (b) the utility functions $u$ for the investors being studied in this paper satisfy $u^{(2)}_-(x) < (\geq) - u^{(2)}_+(x)$ for any $x \geq 0$ where $u^{(i)}$ is the $i^{th}$ derivative of the utility function $u$, $u_+ = \max\{u, 0\}$ and $u_- = \min\{u, 0\}$.

We then discuss the diversification behaviors for Markowitz investors when these assumptions do not hold, study the attitudes towards diversification for prospect investors, and examine the relationships of diversification preferences between Markowitz and prospect investors.

In this vein, we first study the circumstance in which the distributions of the individual assets being examined are symmetric about zero. Under this circumstance, we first examine the situation in which $u^{(2)}_-(x) = - u^{(2)}_+(x)$. That is, the second derivative of the utility $u$ is “symmetrically reflective” from the origin, which mathematically speaking means that $u^{(2)}$ is an odd function. Remarkably, in this situation, we find that Markowitz investors are unanimously indifferent from any convex combination of the underlying assets. We then investigate the situations in which $u^{(2)}_-(x) \leq (\geq) - u^{(2)}_+(x)$. We call the positive domain of the asset “upside profit” or “gains” and the negative domain of the asset “downside risk” or “losses” of the asset. One could consider $u^{(2)}_-(x) \leq (\geq) - u^{(2)}_+(x)$ to be the situation in which the curvature of the corresponding utilities is greater (smaller) on the downside risk than in the upside profit. That is to say, investors are more sensitive to the loss (gain). We find that, when facing the iid, symmetric, and zero-mean

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2Without loss of generality and for notational convenience, we will also say “$u^{(i)}_-(x) < (\geq) - u^{(i)}_+(x)$” instead of “$u^{(i)}_-(x) < (\geq) - u^{(i)}_+(x)$” if no confusion occurs.
assets, Markowitz investors overwhelmingly prefer to invest in an individual asset than any partially diversified portfolio, which is preferred to the completely (equal-weighted) diversified portfolio if investors are more sensitive to the gain. On the other hand, they prefer to invest in the completely diversified portfolio than any partially diversified portfolio, which is preferred to any individual asset if investors are more sensitive to the loss. Analogously, our findings nevertheless conclude that unlike the behaviors of risk averters and risk seekers whose preferences of diversification remain the same, the diversification preferences for Markowitz investors could vary from preferring diversify to preferring not to diversify, depending on the sensitivity on gains and losses for their utility functions.

We next examine the preferences for Markowitz investors when the underlying assets are symmetric but their means are not zero. In this situation, we find that when the means of the underlying assets are positive, Markowitz investors whose utilities $u$ satisfy $u_-(2)(-x) = -u_+(2)(x)$ unanimously judge an individual asset to be superior to any partially diversified portfolio which, in turn, is preferable to the equal-weighted diversification of the assets. On the other hand, if the means of the underlying assets are negative, their preferences reversed. This implies that for any investor with reverse S-shaped utility function satisfying $u^{(2)}(-x) = -u^{(2)}(x)$, s/he will overwhelmingly prefer (not) to diversify when the means of the underlying assets are positive (negative).

At last, we discuss the diversification preferences of Markowitz investors when the assumptions we set in Section 3 do not hold. In this situation, the diversification preferences of Markowitz investors could become very complicated and their behaviors could be much different from the axiomatic theory we developed in this paper. In addition, we examine the diversification preferences of prospect investors. Their preferences behaviors could be very complicated too. In general, we find that, different from the behaviors of risk averters and risk seekers whose diversification behaviors remain the same and their preferences are always opposite when facing iid prospects, the diversification preferences for Markowitz and prospect investors could be the same in some circumstances but their
preferences could be different in other circumstances. We also study the situations in which Markowitz and prospect investors could prefer to invest in a partially diversified portfolio than both individual assets and the completely diversified portfolio. This provides a simple and elegant explanation to the widely-discussed diversification puzzle. The theory developed in this paper could also be used to explain other financial puzzles like momentum profits.

Our paper is organized as follows. We begin in Section 2 by introducing definitions and notations and stating some basic properties for the theory. Section 3 presents our findings on the diversification attitudes for Markowitz investors. We study in Section 4 the diversification preferences of Markowitz investors when the distributions of the assets are not symmetric, examine the diversification preferences of prospect investors, and explore the preferences behaviors relationship between Markowitz and prospect investors. Section 5 will round out the paper by summarizing the critical points that underlie the reasons for this paper and apply the findings in our paper to resolve some notably financial anomalies.

2 Definitions and Notations

In this section, we will state the definitions and notations used in this paper. Let $\mathbb{R}$ be the set of extended real numbers and $\Omega = [a, b]$ be a subset of $\mathbb{R}$ in which $a < 0$ and $b > 0$. Let $\mathcal{B}$ be the Borel $\sigma$-field of $\Omega$ and $\mu$ be a measure on $(\Omega, \mathcal{B})$. Let $(\Omega, \mathcal{F}; \mathbb{P})$ be a measure space and $\mathcal{X}$ be a set of random variables on $\Omega$, i.e., a set of functions $X : \Omega \to \mathbb{R}$, each $X \in \mathcal{X}$ represents the return or payoff of an asset. We first define the function $F$ and the measure $\mu$ on the support $\Omega$ as

\[
F(x) = \mu(a, x] \quad \text{for all} \quad x \in \Omega.
\] (1)
Function $F$ is a *cumulative distribution function* (CDF) or simply *distribution function* and $\mu$ is a *probability measure* if $\mu(\Omega) = 1$. We follow the basic probability theory that for any random variable $X$ and for any probability measure $P$, there exists a unique induced probability measure $\mu$ on $(\Omega, \mathcal{B})$ and a distribution function $F$ such that $F$ satisfies (1) and

$$\mu(B) = P(X^{-1}(B)) = P(X \in B) \quad \text{for any } B \in \mathcal{B}.$$  

An integral written in the form of $\int_A f(t) d\mu(t)$ or $\int_A f(t) dF(t)$ is a Lebesgue-Stieltjes integral for any integrable function $f(t)$. If the integral has the same value for any set $A$ which is equal to $(c, d]$, $[c, d)$ or $[c, d]$, then we use the notation $\int_c^d f(t) d\mu(t)$ instead. In addition, if $\mu$ is a Borel measure with $\mu(c, d] = d - c$, then we write the integral as $\int_c^d f(t) dt$.

Random variables, denoted by $X, Y \in \mathcal{X}$, defined on $\Omega$ are considered together with their corresponding distribution functions $F$ and $G$, their corresponding probability density functions $f$ and $g$, respectively, and their corresponding means defined in the following:

$$\mu_F = \mu_X = E(X) = \int_a^b \omega dF(\omega), \quad \text{and} \quad \mu_G = \mu_Y = E(Y) = \int_a^b \omega dG(\omega).$$

Considering an economic agent with unitary initial capital, in this paper we study the single period portfolio selection for investors to allocate their wealth to the $n$ ($n > 1$) risks without short selling in order to maximize their expected utilities from the resulting final wealth. Let random variable $X \in \mathcal{X}$ be an (excess) return of an asset or prospect. If there are $n$ assets $\bar{X}_n = (X_1, \cdots , X_n)'$, a portfolio of $\bar{X}_n$ without short selling is defined by a convex combination, $\overrightarrow{\lambda}_n \bar{X}_n$, of the $n$ assets $\bar{X}_n$ for any $\overrightarrow{\lambda}_n \in S_n$ where

$$S_n = \left\{ (s_1, s_2, \cdots , s_n)' \in \mathbb{R}^n : 0 \leq s_i \leq 1 \text{ for any } i , \sum_{i=1}^{n} s_i = 1 \right\}.$$
in which $\mathbb{R}$ is the set of real numbers. The $i^{th}$ element of $\overrightarrow{\lambda}$ is the weight of the portfolio allocation on the $i^{th}$ asset of return $X_i$. A portfolio will be equivalent to return on asset $i$ which we call a specialized portfolio or simply a specialized asset if $s_i = 1$ and $s_j = 0$ for all $j \neq i$. It is a partially diversified portfolio if there exists $i$ such that $0 < s_i < 1$ and is the completely diversified portfolio if $s_i = \frac{1}{n}$ for all $i = 1, 2, \cdots, n$.

Before developing the theory, we first state different types of utility functions as in the following definition:

**Definition 1**  

For $j = 1, 2, 3$; $U^A_j$, $U^D_j$, $U^S_j$, and $U^R_j$ are the sets of utility functions $u$ such that:

\[
U^A_j = \{ u : (-1)^i u^{(i)} \leq 0, \quad i = 1, \cdots, j \},
\]

\[
U^D_j = \{ u : \quad u^{(i)} \geq 0, \quad i = 1, \cdots, j \},
\]

\[
U^S_j = \{ u : u_+ \in U^A_j \quad \text{and} \quad u_- \in U^D_j \}, \quad \text{and}
\]

\[
U^R_j = \{ u : u_+ \in U^D_j \quad \text{and} \quad u_- \in U^A_j \},
\]

where $u^{(i)}$ is the $i^{th}$ derivative of the utility function $u$, $u_+ = \max\{u,0\}$, and $u_- = \min\{u,0\}$

We note that investors in $U^A_j$ are risk averse whereas investors in $U^D_j$ are risk seeking. We also note that without loss of generality in this definition, the reference point (status quo) for $U^S_j$ and $U^R_j$ is assumed to be zero. Thus, we refer to positive outcomes as gains and negative outcomes as losses. In this situation, investors in $U^R_j$ with reverse S-shaped utility functions are risk seeking for gains but risk averse for losses, while investors in

\[3\]We note that one could define “strictly increasing” and “increasing” situations for each of these sets of utility functions. In this paper, we combine both situations into one but, if without causing any confusion, we assume that for each utility function $u$, there is a portion in which $u^{(j)}$ is not equal to zero.

\[4\]One could easily extend the theory to include $U^S_j$ and $U^R_j$ with non-zero status quo.
$U^S_j$ with S-shaped utility functions are risk averse for gains but risk seeking for losses, see Figure 1 for the shape of utility functions in $U^A_2$, $U^D_2$, $U^R_2$, and $U^S_2$, and Figure 2 for the shape of the first derivatives of the utility functions in $U^A_3$, $U^D_3$, $U^R_3$, and $U^S_3$, respectively. For convenience, we call investors with utility functions in $U^S_j$ prospect investors or investors with prospect preference and investors with utility functions in $U^R_j$ Markowitz investors or investors with Markowitz preference. As the utilities for prospect investors are concave in the positive domain and convex for the negative domain, they show declining sensitivity in both gains and losses. On the other hand, as utilities for Markowitz investors are convex in the positive domain and concave in the negative domain, they show increasing sensitivity in both gains and losses.

In this paper, we mainly study the utility-maximizing representative investors with Markowitz preferences in the following subsets of $U^R_j$:

\begin{align*}
U^R_0 &= \{ u \in U^R_j : u^{(i)}(-x) = (-1)^{i+1} u^{(i)}(x) \text{ for any integer } i \leq j \} \quad j = 2, 3;
U^R_1 &= \{ u \in U^R_2 : u^{(2)}(-x) \leq -u^{(2)}(x) \} ; \quad \text{and} \\
U^R_2 &= \{ u \in U^R_2 : u^{(2)}(-x) \geq -u^{(2)}(x) \} .
\end{align*}

(2)

Tversky and Kahneman (1992) propose the following value (utility) function

\begin{equation}
    u(x) = \begin{cases} 
    x^{\gamma_G} & \text{if } x \geq 0 \text{ and } \gamma_G \in (0, 1) \\
    -\lambda x^{\gamma_L} & \text{if } x < 0, \lambda > 0, \text{ and } \gamma_L \in (0, 1) 
    \end{cases}
\end{equation}

(3)

In addition, al Nowaihi, Bradley and Dhami (2008) show that, under preference for homogeneity and loss aversion prospect theory, the value function, $u$, will have a power form with identical powers for gains and losses such that:

\begin{equation}
    u(x) = \begin{cases} 
    x^{\alpha} & \text{if } x \geq 0 \\
    -\lambda (-x)^{\alpha} & \text{if } x < 0 
    \end{cases},
\end{equation}

(4)
where $\lambda \geq 0$ and $\alpha \in (0, 1)$. These utility functions are important as they support the S-shaped types of utility functions for prospect investors. One could easily construct reverse S-shaped utility functions for Markowitz investors, for example, by setting $\gamma_G \geq 1$ and $\gamma_L \geq 1$ in (3) or by setting $\alpha \geq 1$ in (4).

Choosing between $F$ and $G$ in accordance with a consistent set of preferences will satisfy the von Neumann-Morgenstern (1944) consistency property. Accordingly, $F$ is (strictly) preferred to $G$, or equivalently, $X$ is (strictly) preferred to $Y$ if

$$\Delta E u \equiv u(F) - u(G) \equiv u(X) - u(Y) \geq 0 (> 0), \quad (5)$$

where $u(F) \equiv u(X) \equiv \int_a^b u(\omega)dF(\omega)$ and $u(G) \equiv u(Y) \equiv \int_a^b u(\omega)dG(\omega)$. We note that without loss of generality we restrict the theory developed in this paper to hold for finite expected utility functions. Throughout the paper, all integrations are assumed to be finite. We also consider a set of outcomes in $\mathbb{R}$ such that a random variable $X$ is distributed as finite probability distribution $P = (p_1, x_1; p_2, x_2; \cdots; p_n, x_n)$, denoted by $X \sim P$, meaning that probability $p_j$ is assigned to outcomes $x_j$ for $j = 1 \cdots n$ with $\sum_{j=1}^n p_j = 1$.

3 Theory

In this section, we will develop the properties of the diversification preferences for Markowitz investors. To distinguish the well-known results in the literature from the ones derived in this paper, all cited results will be called propositions and our derived results will be called theorems.

The properties of diversification preferences for risk averters and risk seekers have been well studied. For example, Hadar and Russell (1971), Tesfatsion (1976), Li and Wong (1999) and others have established the following property:
Proposition 1 Let $X_i, Y_i \in \mathcal{X}$ ($i = 1, \cdots, n$) be two sets of independent random variables. For $j = 1, 2$ and $3$, and for any $u \in U_j^A$ or $U_j^D$, $u(X_i) \geq u(Y_i)$ for $i = 1, \cdots, n$ if and only if $u \left( \sum_{i=1}^{n} \alpha_i X_i \right) \geq u \left( \sum_{i=1}^{n} \alpha_i Y_i \right)$ for any $\alpha_i \geq 0$.

In this paper, we first establish a similar property for investors with utility functions $u$ satisfying $u^{(2)}(-x) = -u^{(2)}(x)$ as stated in the following theorem:

Theorem 2 Let $X_i, Y_i \in \mathcal{X}$ ($i = 1, \cdots, n$) be two sets of independent and identically distributed random variables which are symmetric about zero. If $u$ satisfies $u^{(2)}(-x) = -u^{(2)}(x)$ for any $x$, then, for any $i = 1, \cdots, n$,

$$u(X_i) \geq u(Y_i) \quad \text{if and only if} \quad u \left( \sum_{i=1}^{n} \alpha_i X_i \right) \geq u \left( \sum_{i=1}^{n} \alpha_i Y_i \right)$$

for any $\alpha_i \geq 0$.

The investment preferences for Markowitz investors could be obtained immediately from the above theorem as stated in the following corollary:

Corollary 3 Let $X_i, Y_i \in \mathcal{X}$ ($i = 1, \cdots, n$) be two sets of iid random variables which are symmetric about zero. If $u \in U^{R_2}_2$, then, for any $i = 1, \cdots, n$,

$$u(X_i) \geq u(Y_i) \quad \text{if and only if} \quad u \left( \sum_{i=1}^{n} \alpha_i X_i \right) \geq u \left( \sum_{i=1}^{n} \alpha_i Y_i \right)$$

for any $\alpha_i \geq 0$.

We note that this result also holds for prospect investors. As this paper mainly studies the properties for Markowitz investors, we skip displaying the above relationship for prospect investors in this paper. One could easily construct examples to illustrate Theorem 2 and Corollary 3. We construct one as follows:

Example 1 Let $u(x) = x$ for $x \geq 0$ and $u(x) = 2x$ for $x < 0$. Consider $X_i$ to be iid a Bernoulli probability distribution $P_x = (\frac{-1}{2}, \frac{1}{2})$ and $Y_i$ to be iid

\[5\text{Readers may read Schmidt and Zank (2007) for the properties of this utility function.}\]
\( P_y = \left( -2, 1/2; 2, 1/2 \right) \) for \( i = 1, 2 \). Obviously, one could easily show that

\[
u(X_i) = -\frac{1}{2} > u(Y_i) = -1.
\]

Subsequently, one could also easily find that

\[
u \left( \sum_{i=1}^{n} \alpha_i X_i \right) \geq u \left( \sum_{i=1}^{n} \alpha_i Y_i \right)
\]

for any \( \alpha_i \geq 0 \). For example, consider a partially diversified portfolio \( Z_p \) and the completely diversified portfolio \( Z_c \) such that

\[
Z_p = \frac{1}{3} Z_1 + \frac{2}{3} Z_2 \quad \text{and} \quad Z_c = \frac{1}{2} Z_1 + \frac{1}{2} Z_2
\]

(6)

for \( Z = X \) and \( Y \). Then, it is obviously that

\[
u(X_p) = -\frac{1}{3} > u(Y_p) = -\frac{2}{3}
\]

and

\[
u(X_c) = -\frac{1}{4} > u(Y_c) = -\frac{1}{2}.
\]

In addition, there are many useful properties for diversification preferences of risk averters and risk seekers, see, for example, Hadar and Russell (1971), Tesfatsion (1976), and Li and Wong (1999). Among them, the following is one of the most important results:

**Proposition 4** For \( n \geq 2 \), if \( X_i \in \mathcal{X} (i = 1, \ldots, n) \) are independent and identically distributed, then, for any \( i = 1, \ldots, n \) and for any \( (\alpha_1, \ldots, \alpha_n)' \in S_n \),

a. if \( u \in U_2^A \), then

\[
u \left( \sum_{i=1}^{n} \frac{X_i}{n} \right) \geq u \left( \sum_{i=1}^{n} \alpha_i X_i \right) \geq u(X_i), \quad \text{and}
\]
b. if $u \in U_2^D$, then

$$u \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) \leq u \left( \sum_{i=1}^{n} \alpha_i X_i \right) \leq u(X_i).$$

The attractiveness of this proposition is that it could identify the preferences of risk averters and risk seekers among the completely diversified portfolio, specific diversified portfolios, and specialized assets. According to this proposition, risk averters overwhelmingly prefer to diversify while risk seekers prefer not to diversify when they face several iid assets. In particular, Proposition 4 induces the following properties for the diversification preferences of risk averters and risk seekers when they face simultaneous series of iid assets: (1) risk averters unanimously judge the completely diversified portfolio to be superior to any partially diversified portfolio which, in turn, is preferred to a specialized asset, (2) the preferences of risk seekers on diversification are always opposite to those of risk averters, they prefer to invest in a specialized asset, than invest in any partially diversified portfolio which, in turn, is preferred to the completely diversified portfolio; and (3) the preferences of both risk averters and risk seekers remain the same for different curvatures of their utilities and for different shapes of the distributions for the underlying assets being compared.

In this paper, we aim to study whether unanimous ranking of diversified portfolios and specialized assets for Markowitz investors. One may believe that their behaviors could be similar to those for risk-averse or risk-seeking investors. Nonetheless, the diversification behaviors of Markowitz investors are not straightforward. Analogously, our findings induce that, when facing iid, symmetric, and zero-mean assets, we will obtain the following properties: (1) Markowitz investors could be indifferent from any portfolio in a circumstance; (2) they could prefer to diversify in some situations but prefer not to diversify on other circumstances; and (3) their preferences could change when the shapes of the distributions for the underlying assets vary or the curvatures of their utilities change.

We first elucidate that different from the behaviors of risk averters and risk seekers,
some investors could be indifferent among all the diversified portfolios and the specialized assets as shown in the following theorem:

**Theorem 5** For \( n \geq 2 \), if \( X_i \in \mathcal{X} \) \( (i = 1, \cdots, n) \) are iid and symmetric about zero, and if \( u^{(2)}(-x) = -u^{(2)}(x) \), then, for any \( i = 1, \cdots, n \), and for any \( (\alpha_1, \cdots, \alpha_n)' \in S_n \), we have

\[
u\left(\sum_{i=1}^{n} \alpha_i X_i\right) = u(X_i).
\]

We note that the above results hold for both Markowitz and prospect investors. As we mainly study the behaviors of Markowitz investors, we only state their behaviors in the following corollary:

**Corollary 6** For \( n \geq 2 \), if \( X_i \in \mathcal{X} \) \( (i = 1, \cdots, n) \) are iid and symmetric about zero, and if \( u \in \mathcal{U}_R^0 \), then, for any \( i = 1, \cdots, n \) and for any \( (\alpha_1, \cdots, \alpha_n)' \in S_n \), we obtain

\[
u\left(\sum_{i=1}^{n} \alpha_i X_i\right) = u(X_i).
\]

We construct a simple example to demonstrate Theorem 5 and Corollary 6 as follows:

**Example 2** Let \( u(x) = x \) and consider a partially diversified portfolio \( X_p \) and the completely diversified portfolio \( X_c \) defined in (6) in which the assets \( X_i \) is iid a Bernoulli probability distribution \( P_x = \left( -1, 1/2; 1, 1/2 \right) \) \((i = 1, 2)\). Obviously,

\[
u(X_i) = u(X_p) = u(X_c) = 0.
\]

This example supports Corollary 6 that Markowitz investors could be indifferent from a diversified portfolio with the completely diversified portfolio and an individual asset when they face iid, symmetric, and zero-mean assets.

Next, we elucidate that some investors’ preferences towards risk could vary under different sensitivities in their utility functions for gains and losses as shown in the following theorem:
Theorem 7  For \( n \geq 2 \), if \( X_i \in \mathcal{X} \) (\( i = 1, \ldots, n \)) are iid and symmetric about zero, then, for any \( i = 1, \ldots, n \) and for any \((\alpha_1, \ldots, \alpha_n)' \in S_n\), if \( u \in U_2^R \),

a. if \( u^{(2)}(-x) \geq -u^{(2)}(x) \), then

\[
u \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) \leq u \left( \sum_{i=1}^{n} \alpha_i X_i \right) \leq u(X_i); \quad \text{and}
\]

b. if \( u^{(2)}(-x) \leq -u^{(2)}(x) \), then

\[
u \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) \geq u \left( \sum_{i=1}^{n} \alpha_i X_i \right) \geq u(X_i).
\]

We construct the following example to illustrate Theorem 7:

Example 3  Let \( u \) with \( u_+(x) = 2x^3 \) if \( x \geq 0 \) and \( u_-(x) = x^3 \) if \( x < 0 \). Consider a partially diversified portfolio \( X_p \) and the completely diversified portfolio \( X_c \) defined in (6) in which the assets \( X_i \) (\( i = 1, 2 \)) are iid \( P_x = (-1, 1/2; 1, 1/2) \). One could easily find that

\[
u(X_i) = \frac{1}{4}, \quad u(X_p) = \frac{7}{54}, \quad \text{and} \quad u(X_c) = \frac{1}{8};
\]

which yields

\[
u(X_c) < u(X_p) < u(X_i).
\]

On the other hand, for an \( u \) with \( u_+(x) = x^3 \) if \( x \geq 0 \) and \( u_-(x) = 2x^3 \) if \( x < 0 \), one could easily observe their reverse diversification preference as follows:

\[
u(X_c) > u(X_p) > u(X_i).
\]

From this example, we find that the inequalities in (7) support Part (a) of Theorem 7 that, when they face iid, symmetric, and zero-mean assets, Markowitz investors prefer to invest in an individual asset than any partially diversified portfolio, which is preferred
to the completely diversified portfolio if investors are more sensitive to the gain. Alternatively, the inequalities in (8) support Part (b) of Theorem 7 that, when they face iid, symmetric, and zero-mean assets, Markowitz investors prefer to invest in the completely diversified portfolio than any partially diversified portfolio, which is preferred to any individual asset if investors are more sensitive to the loss. Thus, we could tell from Theorem 7 that, when facing the iid, symmetric, and zero-mean assets, unlike the behaviors of risk averters and risk seekers whose preferences of diversification remain the same, the diversification preferences for Markowitz investors could vary from preferring diversify to preferring not to diversify, depending on the sensitivity on gains and losses for their utility functions.

We note that one could easily extend the above results to the situation in which the means of $X_i$ are not equal to zero but are all equal to the status quo of $u$. Now, we proceed to study the preferences of Markowitz investors when the status quo of their utilities is different from the means of the assets. In the situation, one could simply fix the status quo of their utilities to be zero and change the mean of $X_i$ to be non-zero as shown in the following theorem:

**Theorem 8** For $n \geq 2$, if $X_i \in \mathcal{X}$ ($i = 1, \cdots, n$) are iid and symmetric with mean $\mu$ and if utility $u$ satisfies $u^{(1)}(-x) = -u^{(1)}(x)$ and $u^{(3)} \geq 0$, then, for any $i = 1, \cdots, n$ and for any $(\alpha_1, \cdots, \alpha_n)' \in S_n$,

a. if $\mu \geq 0$, then

$$u\left(\sum_{i=1}^{n} \frac{X_i}{n}\right) \leq u\left(\sum_{i=1}^{n} \alpha_i X_i\right) \leq u(X_i) ; \text{ and}$$

b. if $\mu \leq 0$, then

$$u\left(\sum_{i=1}^{n} \frac{X_i}{n}\right) \geq u\left(\sum_{i=1}^{n} \alpha_i X_i\right) \geq u(X_i).$$

It is interesting that Theorem 8 gives rise to obtain the results of the attitude for Markowitz investors towards risk when the means of the assets are not zero as shown
in the following corollary:

**Corollary 9**  
For \( n \geq 2 \), if \( X_i \in \mathcal{X} \) \((i = 1, \cdots, n)\) are iid and symmetric with mean \( \mu \) and if \( u \in U_{3R} \), then, for any \( i = 1, \cdots, n \) and for any \((\alpha_1, \cdots, \alpha_n)' \in S_n\),

a. if \( \mu \geq 0 \), then
\[
 u \left( \sum_{i=1}^{n} \frac{X_i}{n} \right) \leq u \left( \sum_{i=1}^{n} \alpha_i X_i \right) \leq u(X_i) ; \quad \text{and}
\]

b. if \( \mu \leq 0 \), then
\[
 u \left( \sum_{i=1}^{n} \frac{X_i}{n} \right) \geq u \left( \sum_{i=1}^{n} \alpha_i X_i \right) \geq u(X_i).
\]

We note that Corollary 9 induces that, when facing iid and symmetric assets, if the means of the underlying assets are positive, than Markowitz investors will prefer not to diversify whereas if the means of the underlying assets are negative, then Markowitz investors will prefer to diversify. These results could be illustrated by the following example:

**Example 4**  
Let \( u(x) = x^3 \) and consider a partially diversified portfolio \( X_p \) and the completely diversified portfolio \( X_c \) defined in (6) in which the assets \( X_i \) \((i = 1, 2)\) are iid a Bernoulli probability distribution \( P_1 = (0, 1/2; 1/2, 1/2) \). Now, the mean of asset \( X_i \) is equal to \( \frac{1}{4} > 0 \). Obviously,
\[
 u(X_i) = \frac{1}{16}, \quad u(X_p) = \frac{1}{24}, \quad \text{and} \quad u(X_c) = \frac{5}{128};
\]
which yields
\[
 u(X_c) < u(X_p) < u(X_i). \tag{9}
\]

On the other hand, if \( X_i \) are iid \( P_2 = (-\frac{1}{2}, \frac{1}{2}; 0, \frac{1}{2}) \) \((i = 1, 2)\). One could easily obtain the reverse preference as follows:
\[
 u(X_c) > u(X_p) > u(X_i). \tag{10}
\]
Inequalities in (9) support Part (a) of Corollary 9 that, when they face iid and symmetric assets with positive means, Markowitz investors prefer to invest in any individual asset than any partially diversified portfolio, which is preferred to the completely diversified portfolio. On the other hand, inequalities in (10) support Part (b) of Corollary 9 that, when they face iid and symmetric assets with negative means, Markowitz investors prefer to invest in the completely diversified portfolio than any partially diversified portfolio, which is preferred to any individual asset.

We note that one could easily extend Corollary 9 to the case in which the \textit{status quo} of the utility functions of Markowitz investors is not zero. To achieve this, one could simply replace the mean in Corollary 9 to be the relative mean which is the difference of the mean of the underlying distribution of an asset and the \textit{status quo} of the utility function for the investor.

4 Further Research

In the preceding section, we develop the theory to study the diversification behaviors for Markowitz investors under the following simple assumptions: (a) when the distributions of the assets are symmetric, and (b) Markowitz investors with utility functions belong to any set of utility functions defined in (2). The diversification behaviors of Markowitz investors could be very complicated if these assumptions do not hold. In addition, this paper mainly develops properties to study the diversification behaviors for Markowitz investors, but not for prospect investors. It is because the behaviors of prospect investors are even more complicated. In this section, we demonstrate the complexity of their behaviors by illustrating some simple examples. If the assumptions set in our paper do not hold, the complexity of the behaviors of Markowitz and prospect investors suggest that this could be an intractable puzzle. The development of the corresponding theory could be a rich and challenging area for future research.
To demonstrate the complexity of Markowitz investors’ behaviors when the above-mentioned assumptions do not hold, we first construct a counterexample in which Corollary 9 does not work as follows:

**Example 5** Consider Markowitz investors whose utility functions $u_M$ satisfy:

$$u_M(x) = \begin{cases} -2x^2 & x < 0 \\ x^2 & x \geq 0 \end{cases}$$

invest in a partially diversified portfolio $X_p$ and the completely diversified portfolio $X_c$ defined in (6) in which $X_i (i = 1, 2)$ is iid $P_x = (-2, 0.25; -1, 0; 1, 0.15; 2, 0.6)$. One could easily show that $E(X) = 0.85 > 0$, $u_M(X_c) = 1.33$, $u_M(X_p) = 1.2307$, and $u_M(X_i) = 0.55$ for $i = 1, 2$. Consequently,

$$u_M(X_c) > u_M(X_p) > u_M(X_i) \quad i = 1, 2. \quad (11)$$

Part (a) of Corollary 9 tells us that if $\mu > 0$, Markowitz investors prefer not to diversify. However, the inequalities in (11) show that Markowitz investors in this example prefer to diversify. Does this finding contradict the results in Part (a) of Corollary 9? The answer is ‘no’ as Corollary 9 relies on the assumptions that the distributions of the assets are symmetric and the utility $u$ satisfies $u^{(2)}(-x) = -u^{(2)}(x)$. However, the distributions of the assets in the above example are asymmetric and the utility $u$ possesses the following relationship: $u^{(2)}(-x) = -4 \neq 2 = -u^{(2)}(x)$. Thus, it is not surprising that the result in the example contradicts Part (a) of Corollary 9. Similarly, one could easily construct a counterexample to show that the result in Part (b) of Corollary 9 does not hold when the distributions of the assets are not symmetric or the utility $u$ does not satisfy $u^{(2)}(-x) = -u^{(2)}(x)$.

So far, all results developed in the previous section show that the diversification preferences for Markowitz investors are consistent. That is, if they prefer to diversify, they will
prefer the completely diversified portfolio than any partially diversified portfolio which, in turn, is preferred to any individual asset and vice versa for their counterparts. Is it possible that their diversification preferences are not consistent? We show that it is possible that they could prefer a partially diversified portfolio than both the completely diversified portfolio and any individual asset as shown in the following striking example:

**Example 6**  Consider Markowitz investors whose utility functions $u_M$ satisfy:

\[
\begin{align*}
    u_M(x) &= \begin{cases} 
    -2x^3 & x < 0 \\
    x^3 & x \geq 0 
    \end{cases} 
\end{align*}
\]  \hspace{1cm} (12)

invest in a partially diversified portfolio $X_p$ and the completely diversified portfolio $X_c$ defined in (6) in which $X_i (i = 1, 2)$ is iid $P_x = (-2, 0.05; -1, 0.25; 1; 0.3; 2, 0.4)$. One could easily find that $u_M(X_c) = 1.047$, $u_M(X_p) = 1.07$, and $u_M(X_i) = 1$. Thus, we obtain

\[
    u_M(X_c) < u_M(X_p) > u_M(X_i) \quad i = 1, 2. \hspace{1cm} (13)
\]

In the above example, inequalities in (13) show that Markowitz investors could prefer a partially diversified portfolio than both the completely diversified portfolio and any individual asset. Does the finding in Example 6 contradict Theorem 7? The answer is ‘no’ because Theorem 7 relies on the assumption that the distributions of the assets are symmetric while the distributions of the assets in this example are not symmetric.

Subsequently, we turn to illustrate the complexity of the relationship between Markowitz and prospect preferences as shown in the following example:

**Example 7**  Consider a prospect investor whose utility function $u_P$ satisfies

\[
\begin{align*}
    u_P(x) &= \begin{cases} 
    -2\sqrt{-x} & x < 0 \\
    \sqrt{x} & x \geq 0 
    \end{cases} 
\end{align*}
\]  \hspace{1cm} (14)

invests in a partially diversified portfolio $X_p$ and the completely diversified portfolio $X_c$.
defined in (6) in which \( X_i \) \((i = 1, 2)\) is iid \( P_1 = (-1, 0.49; 0, 0.2; 1, 0.49) \). Subsequently, one could easily compute that

\[
u_P(X_c) = -0.252 > u_P(X_p) = -0.39 > -0.49 = u_P(X_i) \quad (i = 1, 2) \quad \text{for} \quad P_1. \quad (15)
\]

On the other hand, when considering \( X_i \) \((i = 1, 2)\) to be iid \( P_2 = (-1, 0.01; 0, 0.98; 1, 0.01) \), one could easily find that

\[
u_P(X_c) = -0.0139 < u_P(X_p) = -0.0138 < -0.01 = u_P(X_i) \quad (i = 1, 2) \quad \text{for} \quad P_2. \quad (16)
\]

As the distributions of both \( P_1 \) and \( P_2 \) are symmetric about zero, Theorem 7 tells us that, for any Markowitz investor with utility function \( u_{M1} \) satisfying \( u_{M1}^{(2)}(-x) \geq -u_{M1}^{(2)}(x) \), he/she will always prefer not to diversify. Alternatively, for any Markowitz investor with utility function \( u_{M2} \) in which \( u_{M2}^{(2)}(-x) \leq -u_{M2}^{(2)}(x) \), he/she will always prefer to diversify, no matter whether \( X_i \) \((i = 1, 2)\) is distributed as \( P_1 \) or \( P_2 \). Hence, based on the findings in the above example, one could conclude that, when facing portfolios of assets iid \( P_1 \), for any prospect investor with utility function \( u_P \) defined in (14), he/she will have the same diversification preference as Markowitz investors with utility functions \( u_{M2} \) but have reverse diversification preference to Markowitz investors with utility functions \( u_{M1} \). On the other hand, their diversification relationships will be reverse when they are facing portfolios of assets iid \( P_2 \).

This finding is very interesting. Li and Wong (1999) and others find that, when facing portfolios of iid assets, the diversification preferences of risk averters and risk seekers remain the same and their preferences are always opposite — risk averters always prefer to diversify while risk seekers always prefer not to diversify. One may expect Markowitz and prospect investors could have a similar property that Markowitz and prospect investors’ diversification preferences could always be opposite. Nonetheless, Example 7 demonstrates that this conjecture does not hold because, when facing several iid prospects, the
diversification preferences for Markowitz and prospect investors could be the same in some circumstances but their preferences could be reverse in other circumstances. Recall the findings we discussed in the preceding section that Markowitz investors could be unanimously indifferent from any convex combination of the underlying assets in a circumstance, they could overwhelmingly prefer to diversify in another circumstance, and alternatively, prefer not to diversify in other circumstances. As a consequence, these findings crucially show the interest of studying the behaviors of Markowitz and prospect investors’ diversification preferences.

Now, we turn to show the complexity of the diversification preferences for prospect investors as shown in the following example:

**Example 8**  Consider a prospect investor whose utility function $u_P$ follows

$$u_P(x) = \begin{cases} 
-2.25(-x)^{0.88} & x < 0 \\
-x^{0.88} & x \geq 0 
\end{cases} \quad (17)$$

invest in a partially diversified portfolio $X_p$ and the completely diversified portfolio $X_c$ defined in (6).

a. When $X_i$ is iid $P_1 = (-3, 0.02; -2, 0.96; 1, 0.02)$, one could easily obtain the following:

$$u_P(X_i) = -4.07, \quad u_P(X_p) = -4.05, \quad \text{and} \quad u_P(X_c) = -4.06 \quad \text{for} \ P_1 ;$$

which yields

$$u(X_i) < u(X_p) > u(X_c) \quad \text{for} \ P_1 . \quad (18)$$

b. On the other hand, when $X_i (i = 1, 2)$ is iid $P_2 = (-1, 0.15; 0, 0.7; 1, 0.15)$, one could easily find that

$$u_P(X_c) = -0.17 > u_P(X_p) = -0.18 < -0.15 = u_P(X_i) \quad (i = 1, 2) \quad \text{for} \ P_2 . \quad (19)$$
Our findings are striking. Analogously, our findings in Examples 6 and 8a show the tendency of incomplete diversification that investors prefer to invest in a partially diversified portfolio than both individual assets and the completely diversified portfolio. Thus, the theory developed in this paper provides a simple and elegant explanation to the widely-discussed diversification puzzle that investors could prefer not to completely diversified and solve the seemingly intractable diversification puzzle by our behavioral approach. In addition, we note that Example 8b exhibits that it is possible that prospect investors could also prefer to invest in both individual assets and the completely diversified portfolio than a partially diversified one.

5 Concluding Remarks

Our paper offers a unified approach to develop analytical properties to explain adequately the attitudes of Markowitz investors towards risk under some simple assumptions. Under these assumptions, our findings crucially induce Markowitz investors’ preferences on diversification that it depends on both the relative concavity/convexity of their utility functions and the means of the underlying assets. We also discuss the diversification preferences for Markowitz investors when these assumptions do not hold, study the complexity of prospect investors’ behaviors, and explore the relationships of the diversification preferences between Markowitz and prospect investors. For instance, our findings could be used to explain adequately various resolving financial anomalies and puzzles. We examine some here.

We first discuss the contribution of our theory to the fundamental theory in finance. The pioneer work of Markowitz (1952b) on the mean-variance (MV) portfolio optimization procedure to analyze how people make their choices concerning risky investments is

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6See, for example, Barberis and Huang (2007) for more information.
the milestone of modern finance theory for optimal portfolio construction, asset allocation, and investment diversification. In the procedure, portfolio optimizers respond to the uncertainty of an investment by selecting portfolios that maximize profit subject to achieving a specified level of calculated risk or, equivalently, minimize variance subject to obtaining a predetermined level of expected gain. The Markowitz efficient frontier also provides the basis for many important financial economics advances, including the Sharpe-Linter Capital Asset Pricing Model (Sharpe, 1964; Lintner, 1965), the well-known separation theorem, the mutual fund theorem, and the security market line theorem (Tobin, 1958; Sharpe, 1964) hold at any financial market equilibrium. The basic assumption in these theories is that investors are risk averse. With this assumption, the merits of diversification are established in the modern portfolio theory to allow investors to reduce portfolio systematic risk and maximize the rates of returns on their portfolios for a given risk that constitute efficient portfolios.

However, the theory with risk aversion alone could not be able to explain all financial phenomena. A good example is a notable anomaly of the momentum profits discovered by Jegadeesh and Titman (1993) that the winners outperform losers in all months except January but the losers significantly outperform the winners in January. Nonetheless, if winners are found to outperform losers and all investors are risk averse, all investors would buy winners and (short) sell losers in all months except January. This will continue, driving up the price of asset winners relative to losers, until the market price of winners relative to losers is high enough to make the marginal investor indifferent between winners and losers and the momentum profits should disappear. However, after many years several studies, for example, Jegadeesh and Titman (2001), still find momentum profits empirically, suggesting that the earlier finding was not a statistical fluke and the issue of

\[\textit{We note that recently Leung and Wong (2008) have developed a multivariate Sharpe ratio statistic to test the hypothesis of the equality of multiple Sharpe ratios whereas Bai et al. (2009a,b) have developed new bootstrap-corrected estimators of the optimal returns for the Markowitz mean-variance optimization.}\]
momentum profits is a financial anomaly.

In the conventional theories of market efficiency, if one is able to earn an abnormal return for a considerable length of time, the market is considered inefficient. If new information is either quickly made public or anticipated, the opportunity to use the new information to earn an abnormal return is of limited value. Thus, the existence of momentum profits for a long period could suggest that the market is not efficient or rational. A possible explanation of this anomaly is that there are not only risk-averse investors in the market. Since 1900s, academics do not believe that risk aversion alone could be able to explain investors behaviors.

Recently, Fong et al (2005), Wong et al (2006) and Sriboonchita et al (2009) find a resolution of momentum profits. They find that the winners and losers do not dominate one another in the sense of the first order stochastic dominance, inferring that there is no arbitrage opportunity between winners and losers. In addition, they find that risk averters prefer to invest in winners whereas risk seekers prefer to invest in losers. Thus, if the market consists of both risk averters and risk seekers, the former could prefer to invest in winners while the latter could prefer to invest in losers. Both parties could get what they want and there is no pressure to push up the prices of winners or pull down the prices of losers and thus “momentum profits” could be able to last for a long period.

The findings in this paper provide additional information to support the argument that besides risk averters and risk seekers, there could be Markowitz investors as well as prospect investors in the market. This will contribute to the equilibrium of stock prices by different groups of investors. Fong et al. (2008) and others investigate investors’ behaviors in stock markets and find evidence to support that there could have Markowitz and prospect investors in stock markets. Thus, the theory developed in this paper can nevertheless provide a simple and elegant resolution to the widely-discussed momentum profits why some investors prefer to buy winners and sell losers whereas others prefer to
buy losers and sell winners to sustain the “momentum profits” for a long period of time.

In addition, in this paper, we find that Markowitz investors and prospect investors could change their attitudes towards risk. Our findings induce that, different from risk-averse and risk-seeking investors whose preferences on diversification remain the same under different situations, Markowitz investors and prospect investors could be indifferent from any portfolio in a circumstance. On other circumstances they could have the same attitudes as well as different preferences towards risk and they could prefer to diversify sometimes and prefer not to diversify in other circumstances. This attitudes could be used to explain the complexity in the financial markets.

Similarly, the theory developed in this paper could also be used to explain other financial phenomena. Here, we discuss another important contribution of our findings that provide important insights to explain the diversification puzzle why investors prefer incomplete diversification. Though there are many theories attempting to explain the perplexing diversification puzzle, the answers are far from thoroughly convincing. So, the puzzle remains and it seems to be intractable. We note that, analogously, our findings in Examples 6 and 8a show the tendency of incomplete diversification that investors prefer to invest in a partially diversified portfolio than both individual assets and the completely diversified portfolio. This finding is striking. It provides an explanation for the diversification puzzle that the apparent tendency for portfolio investors do not like to completely diversify in their investment. Thus, the theory developed in this paper provides a simple and elegant explanation to the widely-discussed diversification puzzle.

In this paper we only develop properties for Markowitz investors under the assumption that the assets are iid and symmetric. Further extension could also include studying the diversification preferences for Markowitz investors and prospect investors when the distributions of the prospects being examined are not symmetric or they are not iid. Further research could also extend the results obtained in this paper to include non-
expected utility framework (Dekel, 1989; Wong and Ma, 2008). Recently, Egozcue and Wong (2009) develop a general theory and a unifying framework for the comparison of different partial diversified portfolios for risk averters. Further research in this area could extend their theory to compare the preferences of different partial diversified portfolios for Markowitz and prospect investors.

At last, in this paper we have shown that both Markowitz and prospect investors could be inconsistent on diversification. One may suggest to make probability transformation as Kahneman and Tversky suggested to make the diversification consistent. However, we doubt the possibility that applying transformation could make inconsistent diversification consistent. This could also be an interesting topic for further study. Since the theoretical results proved in this paper open the door for the development of real-life applications, another avenue for further research would apply the theory developed in this paper to different real-life applications in business, economics and finance. For example, one could incorporate the theory in this paper to explain some well-known financial phenomena or financial anomalies\(^8\) and to model investment risk.\(^9\)

\(^8\)See, for example, Post (2003), Post and Levy (2005), Fong and Wong (2006), Broll, et al. (2006), and Post (2008).

\(^9\)See, for example, Matsumura, et al. (1990), Wong and Miller (1990), Wong (2007), Gasbarro, et al. (2007), and Wong, et al. (2008).
Figure 1: Functions in $U^A_2$, $U^D_2$, $U^S_2$ and $U^R_2$

Note: Utility functions $U^A_2$, $U^D_2$, $U^S_2$ and $U^R_2$ are defined in Definition 1.

Figure 2: Derivatives of Utility Functions in $U^A_3$, $U^D_3$, $U^S_3$ and $U^R_3$

Note: Utility functions $U^A_3$, $U^D_3$, $U^S_3$ and $U^R_3$ are defined in Definition 1.
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