If buyer one exits after period 1, otherwise conduct an auction.

**Theorem: Incentive Compatibility of the Relaxed Solution**

**Bunching:** The relaxed solution is not incentive compatible if
- the virtual valuation of buyer 1 is non-increasing in \( d \), for all \( v_i \). (effect 1)
- and strictly concave in \( v_i \) for some \( d_i \). (effect 2)

**Separation:** Converse conditions imply incentive compatibility.

**Characterization of Incentive Compatibility (IC)**

General problem has two-dimensional private information.

**Solution Strategy:**
- It is optimal to allocate only at the deadline.
- Then, the 2-dimensional IC constraint is equivalent to IC for the valuation and downward IC for the deadline.
- If the seller does not use lotteries in the first period, the downward IC constraint for the deadline binds only for the highest valuation.

**Structure of the optimal mechanism**

<table>
<thead>
<tr>
<th>Bidder 1</th>
<th>Bidder 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_1 )</td>
<td>( r_2 )</td>
</tr>
</tbody>
</table>

**Time of allocation:** period 1 period 2 \( v_i \in [0, \bar{v}] \)

**Benchmark: Relaxed Solution with Known Deadline**

- If \( d = 1 \): One (short-lived) buyer in each period.
  - Post prices \( r_i^{\text{fx}} \) in period 1 and 2.
- If \( d = 2 \): Bidder one can wait for period 2.
  - Optimal auction with both buyers in period 2 (Myerson, 1981).

**Methods Used to Solve the Auction Problem (\( d = 2 \))**

Myerson’s (1981) solution method by point-wise maximization is not feasible. \( \Rightarrow \) A control problem has to be solved.

**Complications:**
- Monotonicity of \( J_i^{\text{px}} \): Usual regularity conditions not sufficient.
- Winning probability has jumps.
- Non-standard feasibility constraint.

**Solution method (Reid, 1968):** Solve the problem for Lipschitz continuous winning probabilities and take limit as the Lipschitz constant approaches infinity. (+ Ironing)

This method is new to the mechanism design literature and may be useful in auction problems with budget constrained bidders or bidders that demand different capacities.