Rational Expectations Models with Unstable Dynamics and Multiplicative Sunspot Shocks*

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Abstract

In the present paper we propose a method to compute the complete set of solutions of a rational expectations model. There are two main contributions: (i) we admit the unstable equilibrium paths in the set of valid solutions, in addition to the stable ones usually considered; (ii) we demonstrate that sunspot shocks can be effective only when a fundamental error occurs. The sunspot disturbances that obey to this condition are called multiplicative.

The method successfully copes with identification problems underlined by the literature, and treats situations a priori excluded by rational expectations models, that is explosive dynamics for nominal variables.

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1 Introduction

In this paper we provide a simple method to compute the complete set of solutions of a rational expectations model. We consider the case in which all the solutions, included the unstable ones, are valid. In this framework indeterminacy is the big issue: we have multiple solutions and sunspot equilibria. We clarify the role of sunspots, introducing the idea that sunspot shocks can be effective only when a fundamental error hits the economy. We label this notion as multiplicative sunspots. In our framework all the sunspots are necessarily multiplicative, and this also represents a useful restriction to deal with the problem of identification.

The usual practice in solving systems with rational expectations consists in excluding explosive dynamics, and in general non local equilibrium paths (see, for example, Blanchard and Kahn, 1980; Sims, 2002; Lubik and Schorfheide, 2003), but is there any valid economic reason? First, have a look to the empirical evidence. In Argentina inflation reached the value of about 3000% in 1989; in Japan, on the other hand, deflation is present by the end of the nineties, with the exception of few periods. Hyperinflations and deflations are well known phenomena, moreover the inflation rate is not the unique variable that can follow unstable dynamics: we often have experience of financial bubbles and destabilizing sharp increases in nominal public debts.

Unless the case of Argentina and Japan are only few among many examples, the New-Keynesian theory, and modern DSGE (Dynamic Stochastic General Equilibrium) models exclude the possibility of these hyperinflationary or deflationary paths. In these frameworks the stability is viewed as a necessary condition implied by the rationality of economic agents, and requiring stability is also a mean to achieve determinacy, that is the uniqueness of the solution. This suggests that, despite the empirical evidence, the reasons to rule out non local equilibria are theoretical. Also under this point of view the stability requirement can be criticized: Cochrane (2011) argues that the rational expectations hypothesis rules out real explosions but not nominal ones, and he provides examples in which hyperinflationary and deflationary paths are perfectly valid equilibria. He also shows that a crucial identificability problem can arise in the New Keynesian model while imposing stability: the parameters that represent the systematic part of the monetary policy are not always identified.

In this paper we start from the remarks of Cochrane (2011) and we propose a simple mathematical instrument to consider all the solutions of a rational expectations model, included the explosive equilibrium paths. The method we develop consists in parametrizing all the equilibria. We generalize the procedure proposed by Blanchard and Kahn (1980), and their determinate solution is a particular case achievable under simple conditions.

We also give a simple criterion to select stable solutions. The criterion forces some parameters to have a particular value under stability, so it can be easily used also as a mean to run empirical tests.

The method we propose is alternative also in the way we introduce sunspots. We demonstrate that sunspot shocks can be effective only when a fundamental error hits the economy. Because it happens when sunspot disturbances are multiplied with the fundamental ones, we call them multiplicative. This notion clarifies the role of sunspots in rational expectations models, and it poses a limit in the way sunspot shocks can be introduced. There are many authoritative examples in which sunspot shocks do not respect this limit: non multiplicative sunspots are used, for example, by Farmer and Guo (1994), and by Lubik and Schorfheide (2003, 2004). We show how non multiplicative sunspot shocks lead to non valid solutions.

We underline another practical implication of the use of multiplicative sunspot disturbances: it can be considered also as a restriction to successfully cope with the identification problems underlined by Beyer and Farmer (2004, 2007) and by Cochrane (2011).

Finally, we show two examples: in the first one we focus only on stable dynamics, and we compare the method here presented with the one of Lubik and Schorfheide (2003) using the three equation New Keynesian model of that paper. In the second example we explore the possibility of explosive equilibrium paths through a simple model taken by Cochrane (2011).

2 Rational expectations, unstable dynamics and the role of sunspots

2.1 A simple model

We introduce the key concepts and the role of sunspots with the simplest example. Consider the following one equation model:

$$y_t = \frac{1}{\lambda} E_t y_{t+1} + \omega_t \qquad \omega_t \sim i.i.d.N(0, \sigma_\omega^2) \tag{1}$$

where E is the expectation operator, and we use the notation $E_t y_{t+1} = E(y_{t+1}|\Im_t)$, that is the expected value of y_{t+1} conditional to the information set $\Im_t$ available at time t (the information set is such that $\Im_t \supseteq \Im_{t-1}\ \forall t$). We follow Blanchard (1979) writing all the solutions for y_t as

$$y_t = \sum_{j=1}^{\infty} u_j \omega_{t-j} + b\omega_t + \sum_{j=1}^{\infty} c_j E_t \omega_{t+j} \tag{2}$$

where u_j, b and c_j are coefficients to be determined. In particular the coefficients u_j and c_j can be written as functions of b and λ (see the Appendix in which we treat the multivariate case):

$$\begin{aligned} u_1 &= \lambda(b-1) \qquad && u_{j+1} = \lambda u_j \quad j = 1, 2, ..., \infty \\ c_1 &= \frac{b}{\lambda} && c_{j+1} = \frac{c_j}{\lambda} \quad j = 1, 2, ..., \infty \end{aligned}$$

where $b \in (-\infty, +\infty)$. Equation (2) is a solution for model (1) if the above conditions about coefficients hold. In the last equalities there is a degree of freedom: while λ is a given parameters, b is free to vary.

This way of writing the solutions is very useful because it is easy to recognize two famous particular cases: the pure forward looking solution, obtainable when all the

u_j are equal to zero, that is when $b = 1$:

$$y_t^F = \omega_t + \sum_{j=1}^{\infty} \left(\frac{1}{\lambda}\right)^j E_t \omega_{t+j} = \omega_t \tag{3}$$

and the pure backward looking solution, when $b = 0$, so that the last two terms of the right hand side of equation (2) are null:

$$\begin{aligned} y_t^B &= -\sum_{j=1}^{\infty} \lambda^j \omega_{t-j} = -\lambda \omega_{t-1} - \lambda \sum_{j=1}^{\infty} \lambda^j \omega_{t-j-1} = \\ &= \lambda \left(y_{t-1}^B - y_{t-1}^F\right) \end{aligned} \tag{4}$$

All the solutions of equation (1) can be written as a linear combination of the forward and the backward one:

$$\begin{aligned} y_t &= (1-b) y_t^B + b y_t^F \\ y_t &= \lambda y_{t-1} - \lambda \omega_{t-1} + b \omega_t \end{aligned} \tag{5}$$

The last equation is obtained after some simple algebra, and it describes all the solutions for y_t parametrized by b. Then, model (1) has an infinite number of solutions (each one corresponding to a particular value of b) just because an expected value is present.

2.2 Unstable dynamics

To deal with the multiple solutions issue, the literature use to restrict the set of valid equilibrium paths to the stable dynamics. When λ is greater than one in absolute value, all the solutions are unstable but the forward looking one, corresponding to $b = 1$. In this case we have "determinacy", that is the uniqueness of the solution, but a problem of identification arise: equation (3) shows that in the forward looking solution λ does not appear, so it can not be identified. Conversely, if λ is, in absolute value, less than one, all the solutions are stable: we still have multiple solutions and this case is labeled as "indeterminacy". Then the hypothesis of stability leads to the uniqueness of the solution only if λ is outside the unit circle. In that case, however, this parameter is not identified. If we want to have some hope for the identification of λ, when it is greater than one in absolute value, we must abandon the stability requirement and accept the unstable paths in the set of valid solution.

The latter point can be very controversial. Cochrane (2011) argues that the rational expectations hypothesis rules out real explosions, not nominal ones, so if y_t is a nominal variable, for example the inflation rate, the case of $|\lambda| > 1$, and $b \neq 1$ can not be a priori excluded. Moreover, if y_t is a real variable, testing the hypothesis of $|\lambda| > 1$, and $b \neq 1$ can be an easy way to verify if the assumption of rational expectations is empirically valid.

2.3 Sunspot shocks

When there exists an infinite number of solutions, an additional source of disturbance due to exogenous belief shocks can change the equilibrium path, among the infinite possibilities. The literature refers to this phenomenon as sunspot equilibrium (for a comprehensive treatment see Benhabib and Farmer, 1999). Since in our framework indeterminacy is the natural status, we need to consider the possibility of sunspot equilibria.

We refer to the sunspot shock as a random variable ζ_t, orthogonal to the fundamental shocks, that can affect the expectation errors. Consider the variable y_t, the rational expectations hypothesis imply $y_t = E_{t-1}y_t + \eta_t$, where η_t is an expectation error such that $E_{t-1}\eta_t = 0\ \forall t$. Since in the rational expectations framework the economic agents perfectly know the model, the parameters, and the variables at the present time (they are all included in the information set), the expectation error is solely a function of the random disturbances, that is $\eta_t = \eta_t(\omega_t, \zeta_t)$. We now define the multiplicative sunspot shock:

Definition 1: *A sunspot shock ζ_t, for model (1), is multiplicative if*

$$\omega_t = 0 \Rightarrow \eta_t(\omega_t, \zeta_t) = 0 \qquad \forall \zeta_t \in supp(\zeta_t)$$

A multiplicative sunspot shock is a disturbance that can be effective only when also a fundamental error occurs. Before exploring the intuition behind this concept, we first introduce the following proposition:

Proposition 1: *If sunspot shocks hit model (1), they are multiplicative.*

proof: From equation (1) at $t-1$, we have $E_{t-1}y_t = \lambda y_{t-1} - \lambda\omega_{t-1}$, that subtracted from equation (5) give the expectation error:

$$\eta_t(\omega_t, \zeta_t) = b\omega_t \ .$$

Then we have $\omega_t = 0 \Rightarrow \eta_t(\omega_t, \zeta_t) = 0$.

Example 1 (Lubik and Schorfheide, 2003, 2004):

Define $\eta_t(\omega_t, \zeta_t) = M\omega_t + \zeta_t$. In that case sunspots are not multiplicative: if $\omega_t = 0$ then $\eta_t(\omega_t, \zeta_t) = \zeta_t \neq 0$. The rational expectations hypothesis imply

$$E_t y_{t+1} = y_{t+1} - M\omega_{t+1} - \zeta_{t+1}$$

Substitute in equation (1) to get

$$y_t = \lambda y_{t-1} - \lambda\omega_{t-1} + M\omega_t + \zeta_t \tag{6}$$

Equation (6) represents a valid solution for model (1) if it can be written in the form of equation (5). This corresponds on defining the coefficient b as

$$b_t = M + \frac{\zeta_t}{\omega_t}$$

that is not defined when $\omega_t = 0$. Then, when $\omega_t = 0$, equation (6) is not a valid solution for model (1).

Proposition 1 does not say how to introduce sunspots, but it imposes a limit on that choice: only multiplicative sunspots are valid. A practical suggestion comes from Benhabib and Farmer (1999): *"Sunspot equilibria can often be constructed by randomizing over multiple equilibria of a general equilibrium model, and models with indeterminacy are excellent candidates for the existence of sunspot equilibria since* *there are many equilibria over which to randomize."*[1] All the equilibria are parametrized by b, so it seems natural to introduce sunspots randomizing over it, that is posing

$$b = b(\zeta_t) \ . \qquad (7)$$

The key point is that when we randomize *over multiple equilibria of a general equilibrium model*, we intend the equilibria as the infinite equilibrium paths that the economy can follow when a fundamental error occurs. The existence of this infinite number of equilibrium paths is conditional to the realization of a fundamental error. This is the reason why a sunspot shock can only be multiplicative, and its role is precisely those of determine which is the path the economy will follow.

The following example can clarify the intuition behind Proposition 1. Suppose that an economy is in the steady state, and that only stable paths are allowed. Moreover, for the first four periods no fundamental errors hit this economy. The presence of a sunspot shock is usually justified referring to self fulfilling beliefs affecting the expected value, but why the agents, during these periods, should believe they could be in a different path that will bring them to the same steady state? They are already in the steady state, and they will reasonably remain there. Suppose that at time five a fundamental shock hits the economy. It is right then, that there exists an infinite number of stable paths bringing to the steady state, so only at that time a sunspot shock can be effective, having the power of determine which of the infinite paths will be taken by the economy.

We conclude this section showing an example of multiplicative sunspot.

Example 2 (multiplicative sunspots):

A simple choice is to treat b as Normally distributed, that is:

$$b_t = M + \zeta_t \quad \forall t, \qquad \zeta_t \sim i.i.d.N(0, \sigma_\varsigma^2),$$

where ζ_t is a sunspot shock uncorrelated with the fundamental error. In this case we have

$$\eta_t(\omega_t, \zeta_t) = (M + \zeta_t)\omega_t \ ,$$

[1] Benhabib and Farmer (1999), pag. 390.

and the sunspot is multiplicative. The complete set of solutions is represented by the following equation:

$$y_t = \lambda y_{t-1} - \lambda \omega_{t-1} + (M + \zeta_t)\omega_t$$

3 The solution of a system with rational expectations

The class of models we are considering can be written in the Blachard-Kahn form:

$$\begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \boldsymbol{\gamma} Z_t \tag{8}$$

X_t is a $(n \times 1)$ vector of predetermined variables and P_t is a $(m \times 1)$ vector of non predetermined variables, where "*a predetermined variable is a function only of variables known at time t*"[2](it holds $X_{t+1} = E_t X_{t+1}$). Z_t is a $(\kappa \times 1)$ vector of exogenous variables, and we suppose $Z_t \sim$ *i.i.d.* $N(\mathbf{0}, \Sigma)$. We will refer to Z_t as the vector of fundamental errors. Finally A and $\boldsymbol{\gamma}$ are matrices of appropriate dimensions.

As usual we apply the Jordan decomposition to rewrite A as

$$A = C^{-1} J C \ .$$

The matrix J collects all the eigenvalues of A, ordered by increasing absolute value, and in C^{-1} there are the corresponding eigenvectors. Adding the hypothesis that the first n eigenvalues in J are all inside the unit circle[3] we decompose the matrices

[2] Blanchard and Kahn (1980)

[3] This hypothesis ensures the existence of at least one stable solution for model (8). If the hypothesis does not hold all the solutions admitted by the model are unstable.

C^{-1}, C, J and $\boldsymbol{\gamma}$ in the following way:

$$C^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ (n \times n) & (n \times m) \\ B_{21} & B_{22} \\ (m \times n) & (m \times m) \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ (n \times n) & (n \times m) \\ C_{21} & C_{22} \\ (m \times n) & (m \times m) \end{bmatrix},$$

$$J = \begin{bmatrix} J_1 & \mathbf{0} \\ (n \times n) & (n \times m) \\ \mathbf{0} & J_2 \\ (m \times n) & (m \times m) \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 \\ (n \times \kappa) \\ \gamma_2 \\ (n \times \kappa) \end{bmatrix}.$$

There is a difference with the procedure of Blanchard and Kahn (1980) to be underlined: they put in J_1 all the eigenvalues inside the unit circle and in J_2 all the eigenvalues outside the unit circle. We decompose J collecting in J_1 the first n eigenvalues, where n is the number of predetermined variables. The last m eigenvalues are in J_2, where m is the number of non predetermined variables. Thanks to our previous hypothesis we are sure that the first n eigenvalues are less than one in absolute value, but nothing is said about the last m eigenvalues: they could be inside or outside the unit circle.

Premultiply equation (8) by C in both sides and take the last m disjoined equations

$$Q_t = J_2^{-1} E_t Q_{t+1} - \Omega_t \tag{9}$$

with $Q_t = C_{21}X_t + C_{22}P_t$ and $\Omega_t = J_2^{-1}(C_{21}\gamma_1 + C_{22}\gamma_2)Z_t$. Each equation in the system (9) has an infinite number of solutions, because of the presence of an expected value. Defining $q_{i,t}$ as the i^{th} element of Q_t, and $\omega_{i,t}$ the corresponding disturbance, we follow Blanchard (1979) writing all the solutions of the generic row of equation (9) as

$$q_{i,t} = \sum_{j=1}^{\infty} u_{i,j}\omega_{i,t-j} + b_i\omega_{i,t} + \sum_{j=1}^{\infty} c_{i,j} E_t \omega_{i,t+j} \tag{10}$$

where $u_{i,j}$, b_i and $c_{i,j}$ are coefficients to be determined. As in the simple example of Section 2, the coefficients $u_{i,j}$ and $c_{i,j}$ can be written as functions of b_i and of the corresponding eigenvalue in the J_2 matrix (see the Appendix), so the parameter b_i

indexes all the solutions: when $b_i = 0$ we have the pure backward looking solution, when $b_i = -1$ we have the pure forward looking solution, and any other solution can be written as a linear combination of the two.

The Appendix shows that, in a compact form, the solutions of equation (9), parametrized by $\mathbf{b}$, are

$$Q_t = J_2 Q_{t-1} + J_2 \Omega_{t-1} + \mathbf{b}\Omega_t \ , \tag{11}$$

where $\mathbf{b}$ is a $(m \times m)$ diagonal matrix:

$$\mathbf{b} = \begin{bmatrix} b_1 & & \\ & \ddots & \\ & & b_m \end{bmatrix} .$$

The system (11) has m equations, so for the complete solution of the model (8) we need other n equations. Using the first n rows of the model, the Jordan form and the definition of Q_t, we end with the equations (12) and (13):

$$X_t = \left(B_{11}J_1C_{11} + B_{12}J_2C_{21}\right) X_{t-1} + \left(B_{11}J_1C_{12} + B_{12}J_2C_{22}\right) P_{t-1} + \gamma_1 Z_{t-1} \tag{12}$$

$$\begin{aligned} C_{21}X_t + C_{22}P_t = J_2(C_{21}X_{t-1} + C_{22}P_{t-1}) \quad & +(C_{21}\gamma_1 + C_{22}\gamma_2)Z_{t-1} + \\ & +\mathbf{b}J_2^{-1}(C_{21}\gamma_1 + C_{22}\gamma_2)Z_t \end{aligned} \tag{13}$$

These equations tell us that, in general, a model with m non predetermined variables has an infinite to the power of m number of solutions. In fact there are m degrees of freedom, represented by the m elements in $\mathbf{b}$ that are free to vary. We will refer to two particular cases as the pure backward looking solution, when $\mathbf{b} = \mathbf{0}$, and the pure forward looking solution, when $\mathbf{b} = -I$, where I is the identity matrix.

3.1 Sunspot shocks

The sunspot shocks are defined as all the random variables, other than the fundamental error Z_t, that affect the expectation errors. They are collected in the random vector $\boldsymbol{\zeta}_t$ of dimension $(m \times 1)$. $\boldsymbol{\zeta}_t$ is uncorrelated with the fundamental errors, and it satisfies $E_{t-1}\boldsymbol{\zeta}_t = \mathbf{0}$

Definition 2: *A sunspot vector $\boldsymbol{\zeta}_t$, for model (8), is multiplicative if*

$$Z_t = \mathbf{0} \Rightarrow \boldsymbol{\eta}_t(Z_t, \boldsymbol{\zeta}_t) = \mathbf{0} \qquad \forall \boldsymbol{\zeta}_t \in supp(\boldsymbol{\zeta}_t)$$

where $\boldsymbol{\eta}_t(Z_t, \boldsymbol{\zeta}_t)$ is the vector of expectation errors.

In analogy with the univariate case, we also state the following proposition:

Proposition 2: *If sunspot shocks hit model (8), they are multiplicative.*

The proof is straightforward and it is analogous to the one of Proposition 1.

Practically, we introduce multiplicative sunspot shocks randomizing over $\mathbf{b}$, using the hypothesis that each element in the main diagonal depends on a sunspot shock, that is:

$$b_{i,t} = b_{i,t}(\zeta_{i,t}) \quad \forall t, \; i = 1...m \; , \tag{14}$$

where $\zeta_{i,t}$ is the i^{th} element of $\boldsymbol{\zeta}_t$. Then the solutions of model (8) are represented by the equations (12), (13) and (14).

3.2 The stable solutions

We characterize the set of stable solutions. There are cases, in fact, in which we are forced to require stability because some transversality conditions exclude real explosions. Otherwise we may focus on the stable equilibrium paths because the variables we are considering are described by stationary time series. In general a criterion is needed, not only to impose stability, but also to test for it, for example to verify if the hypothesis of rational expectations is empirically relevant.

Consider a generic row i of equation (11). If the corresponding eigenvalue in J_2 is outside the unit circle, all the solutions for $q_{i,t}$ are explosive but one: the forward looking solution. Then, the stability of that row is achieved imposing $b_{i,t} = -1\ \forall t$. This reasoning brings to the following *stability criterion*:

> *stability criterion*: for $i = 1...m$, if $|J_{2,i}| > 1$ put $b_{i,t} = -1\ \forall t$,
>
> where $J_{2,i}$ is the i^{th} element in the main diagonal of J_2, and $b_{i,t}$ is the i^{th} element in the main diagonal of $\mathbf{b}_t$.

The criterion says that if stability is imposed, the degrees of freedom in $\mathbf{b}_t$ are reduced by the number of eigenvalues outside the unit circle. In particular, if there are $r \leq m$ eigenvalues greater than one in absolute value, the model (8) admits an infinite to the power of $(m-r)$ number of solutions. The limiting case is represented by the Blanchard-Kahn condition: "*if the number of eigenvalues of A outside the unit circle is equal to the number of non-predetermined variables, then there exists a unique solution*".[4] In that case, in fact, $r = m$ and $\mathbf{b}_t = -I\ \forall t$.

4 Identification

We consider the unstable paths in the set of valid solutions, and we introduce the notion of multiplicative sunspots for the theorethical reasons described above. In this section we show how these two ingredients have the practical consequence of avoiding the identification problems underlined by Beyer and Farmer (2004, 2007), and by Cochrane (2011). There are two sources for lack of identification: first, when imposing stability it is necessary to offset the explosive eigenvalues, so they become unidentified; second, if sunspot shocks are not multiplicative (as in Lubik and Schorfheide, 2003, 2004), models with different lag length can be observationally equivalent.

To analyze these two points we use a simple example by Beyer and Farmer (2004, 2007). Consider the following model:

$$y_t = \frac{1}{\mu+\lambda} E_t y_{t+1} + \frac{\lambda\mu}{\mu+\lambda} y_{t-1} + \upsilon_t \qquad \upsilon_t \sim i.i.d.N(0, \sigma_\upsilon^2) \tag{15}$$

[4] Blanchard and Kahn (1980), Proposition 1, pag 1308.

with $\mu > 1 > \lambda > 0$. The Blanchard-Kahn form of model (15) is

$$\begin{bmatrix} y_t \\ E_t y_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\lambda\mu & \mu+\lambda \end{bmatrix} \begin{bmatrix} y_{t-1} \\ y_t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \upsilon_t$$

and, using our previous notation we have $n = m = \kappa = 1$. The eigenvalues of the matrix A are λ and μ, so the Blanchard-Kahn condition is satisfied and there exists only one stable solution. The complete set of solutions is found applying the method of Section 3:

$$y_t = (\mu+\lambda)y_{t-1} - \lambda\mu y_{t-2} - b\frac{\mu+\lambda}{\mu}\upsilon_t - (\mu+\lambda)\,\upsilon_{t-1} \qquad b \in (-\infty, +\infty)\ . \tag{16}$$

For simplicity we have assumed b to be constant. Both the eigenvalues λ and μ are identified because we are considering all the solutions included the unstable ones. If stability is required, following the *stability criterion* we fix $b = -1$, and equation (16) becomes

$$y_t = \lambda y_{t-1} + \frac{\mu+\lambda}{\mu}\upsilon_t\ . \tag{17}$$

Now, without knowledge of σ_υ^2, the unstable eigenvalue μ is not identified.

The other identification issue arises when models with different lag length are observationally equivalent, and it is not possible to infer whether we are under determinacy or under indeterminacy. This problem is relevant when the Lubik and Schorfheide (2003) method is applied. To understand this point consider again equation (1). We are imposing stability so, a part of the length structure, there is a further important difference between model (1) and model (15): the first is indeterminate while the second is determinate. In the Example 1 we showed that, following Lubik and Schorfheide (2003, 2004), the solutions of model (1) is equation (6). Note that, although the two models have different lag length, in both their solutions only one lag of y_t appears. To compare the two solutions consider their conditional likelihood. The conditional distribution of y_t, given the information until $t-1$ is, respectively

$$\begin{aligned} y_t|I_{t-1} &\sim N\left(\lambda y_{t-1}, \left(\frac{\mu+\lambda}{\mu}\right)^2 \sigma_\upsilon^2\right) && \text{for model (15)}, \\ y_t|I_{t-1} &\sim N\left(\lambda\left(y_{t-1} - \omega_{t-1}\right), M^2\sigma_\omega^2 + \sigma_\zeta^2\right) && \text{for model (1)}. \end{aligned}$$

When $\sigma_\omega^2 = 0$ and $\sigma_\zeta^2 = \left(\frac{\mu+\lambda}{\mu}\right)^2 \sigma_v^2$ the equations (17) and (6) are observationally equivalent and it is not possible to distinguish a determinate model from an indeterminate one.

If we use the hypothesis of multiplicative sunspots, the solution for model (1) is found combining equation (5) and equation (7):

$$y_t = \lambda y_{t-1} + b(\zeta_t)\omega_t - \lambda\omega_{t-1} \ . \tag{18}$$

The conditional distribution of y_t will not be Gaussian, and so the likelihood function will differ from the one implied by the equation (17). The conditional distribution of y_t is Gaussian only in one case: when we offset the sunspot disturbances and b_t is a constant equal to b. In this case we have

$$\begin{aligned} y_t|I_{t-1} &\sim N\left(\lambda y_{t-1}, \left(\frac{\mu+\lambda}{\mu}\right)^2 \sigma_v^2\right) && \text{for model (15)} \\ y_t|I_{t-1} &\sim N\left(\lambda\left(y_{t-1} - \omega_{t-1}\right), b^2\sigma_\omega^2\right) && \text{for model (1)} \end{aligned}$$

and the two distributions, albeit both Gaussian, are always different.

5 Examples

In this section we show two examples. In the first one we use the three equation New Keynesian model of Lubik and Schorfheide (2003) to compare the method explained in Section 4 to the one they introduced. To this aim we apply the *stability criterion* to focus on stable solutions. In the second example we use a simple model by Cochrane (2011) to analyze the behavior of inflation and nominal interest rate when described by explosive paths.

5.1 A three equation New Keynesian model

We apply the method of Section 3 to a simple New Keynesian model considered in Lubik and Schorfheide (2003). The model consists on the following three equations:

$$E_t y_{t+1} + \sigma E_t \pi_{t+1} = y_t + \sigma i_t \tag{19}$$

$$\beta E_t \pi_{t+1} = \pi_t - k y_t \tag{20}$$

$$i_t = \phi \pi_t + \varepsilon_t \tag{21}$$

where y_t is the output at time t, π_t is inflation and i_t is the nominal interest rate. These variables are considered in deviation from the steady state. The parameters are $\sigma, k, \phi > 0$, and $0 > \beta > 1$. Finally, there is one fundamental shock: $\varepsilon_t \sim i.i.d.N(0, \sigma_\varepsilon^2)$. Equation (19) is called the New Keynesian IS curve, it is derived from one of the first order conditions of the maximization problem by the representative household, and it relates negatively the output with the real interest rate. Inflation and output are positively related trough the New Keynesian Phillips curve, equation (20), and the Central Bank moves the nominal interest rate following a simple Taylor rule: equation (21).

It is convenient to substitute equation (21) in equation (19), so that the system becomes two-dimensional. The Blanchard-Kahn form is

$$\begin{bmatrix} E_t y_{t+1} \\ E_t \pi_{t+1} \end{bmatrix} = \begin{bmatrix} 1 + \frac{k\sigma}{\beta} & \sigma\left(\phi - \frac{1}{\beta}\right) \\ -\frac{k}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \end{bmatrix} + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} \varepsilon_t$$

and, recalling our previous notation, we have $n = 0$, $m = 2$ and $\kappa = 1$. Then, in a compact form the model is written as

$$E_t P_{t+1} = A P_t + \boldsymbol{\gamma} \varepsilon_t$$

where the matrix A has two eigenvalues:

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(1 + \frac{k\sigma + 1}{\beta}\right) \mp \frac{1}{2}\sqrt{\left(\frac{k\sigma + 1}{\beta} - 1\right)^2 + 4\frac{k\sigma}{\beta}(1 - \phi)}.$$

Thanks to a well known result in the literature, we are sure that λ_2 is always greater than one in absolute value. Moreover λ_1 is outside the unit circle when $\phi > 1$, and it is inside the unit circle when $0 < \phi < 1$, independently from the other parameters.[5]

[5] See Bullard and Mitra (2002).

The complete set of solutions is represented by the system (22)

$$P_t = AP_{t-1} + \boldsymbol{\gamma}\varepsilon_{t-1} + C^{-1}\mathbf{b}_t(\zeta_t)J^{-1}C\boldsymbol{\gamma}\varepsilon_t \tag{22}$$

in which the matrices C and J come from the Jordan decomposition. In this example we impose stability, and so we apply the criterion introduced in Section 3: the matrix $\mathbf{b}_t(\zeta_t)$ is

$$\mathbf{b}_t(\zeta_t) = \begin{bmatrix} b_{1,t} & 0 \\ 0 & -1 \end{bmatrix}$$

with

$$b_{1,t} = \begin{cases} -1 & \text{if } |\lambda_1| > 1 \\ M_1 + \zeta_{1,t} & \text{if } |\lambda_1| < 1 \ . \end{cases}$$

If the Central Bank reacts with the nominal interest rate more than proportionally to movements of inflation, it applies a so called active monetary policy. It means that $\phi > 1$, and so both the eigenvalues of A are outside the unit circle. In this case the Blanchard-Kahn condition is satisfied, and $\mathbf{b}_t = -I \ \forall t$. Since this is the unique solution allowed by the *stability criterion*, the problem is said determinate. Equation (22) becomes

$$P_t = -A^{-1}\boldsymbol{\gamma}\varepsilon_t \ \ \forall t. \tag{23}$$

Then, in case of determinacy, the variables in P_t do not display any persistence because we are considering the forward looking solution, and after a shock the economic agents expect to go back to the steady state immediately.

When the monetary policy is passive, that is $0 < \phi < 1$, the eigenvalue λ_1 is inside the unit circle, and $b_{1,t}$ is free to vary. In that case there exists an infinite number of solutions, corresponding to the infinite possible values of $b_{1,t}$, and the model is said indeterminate. In this case the solutions are represented by equation (22), with

$$C^{-1}\mathbf{b}_t(\zeta_t)J^{-1}C\boldsymbol{\gamma} = \frac{\sigma}{(1+k\sigma\phi)(\lambda_2-\lambda_1)} \begin{bmatrix} b_{1,t}\lambda_2 + \lambda_1 - (1+b_{1,t})(1+k\sigma\phi) \\ k(b_{1,t}\lambda_2 + \lambda_1) \end{bmatrix}.$$

Note that in case of indeterminacy the solutions are found as a linear combination of the backward and the forward looking solutions, and the response to fundamental shocks becomes more persistent. The persistency depends on the value of $b_{1,t}(\zeta_t)$, so the sunspot shock plays an important role. This is made clear in Figure 1. Suppose that at time $t = 1$ a fundamental shock $\varepsilon_1 = -0.25$ hits the economy. As in Lubik and Schorfheide (2003) we calibrate the model as follow: $\beta = 0.99$, $\sigma = 1$, $k = 0.5$, and $\phi = 0.95$. The two particular cases are plotted: the pure forward looking solution (the circled line) and the pure backward looking solution (the starred line). Suppose that in a certain time, say $t = 0$, $b_{1,0} = M_1 = -0.2$. The impulse response function is described by the continuous line: the interest rate and the inflation fall and than they head towards the steady state. The output initially increases, and before reaching the steady state it becomes slightly negative. As discussed before, a sunspot shock changes the value of $b_{1,t}$, and so it changes the equilibrium path that brings the economy to the steady state. Then, if a positive sunspot error comes contemporaneously with the fundamental disturbance the response path moves away from the forward looking case (the dotted line). A positive sunspot shock makes greater the distance of $b_{1,t}$ from minus one, amplifying the effect of the fundamental error. On the other hand, a negative sunspot shock moves the path towards the forward looking response (the dashed line). While the economy reaches the normality, other sunspot disturbances can change the value of $b_{1,t}$, but it will not affect the dynamics of the variables considered, unless a new fundamental error perturbs them. This is ensured by the definition of multiplicative sunspot.

In the language of Lubik and Schorfheide (2003) our forward looking solution goes under the name of "*continuity*", because in this case some properties of the model's behavior under determinacy continues to hold. Lubik and Schorfheide (2004) refers to the same solution, called the "*minimal state-variable*" solution, as a possible candidate for a value around which to center an eventual prior distribution. While in their case it would be difficult to find this solution, especially for larger models, in the method here presented the forward looking solution is simply obtained posing $\mathbf{b}_t = -I$.

Moreover, the solution proposed by Lubik and Shorfheide (2003) consists in the introduction of non-multiplicative sunspots, and therefore Proposition 2 is not re-

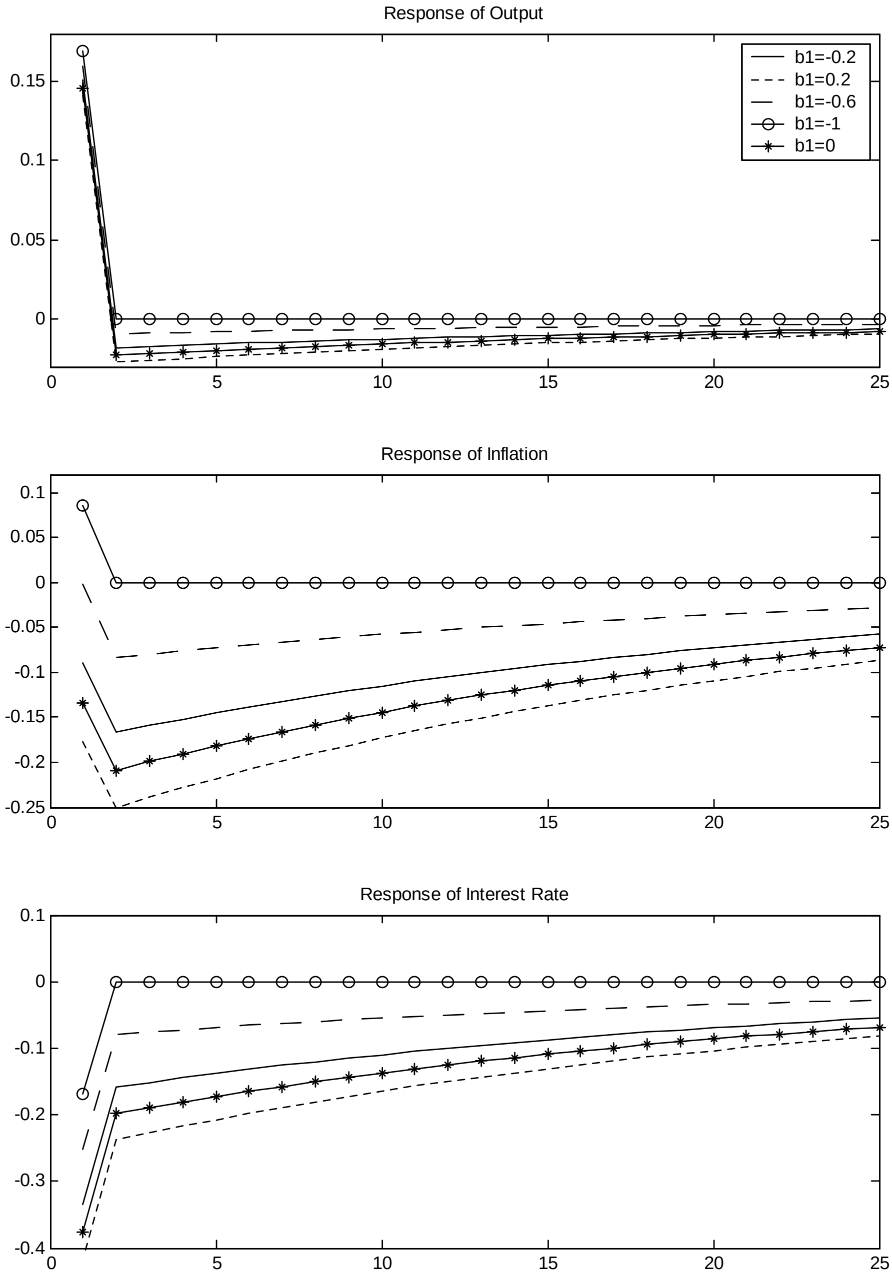

Figure 1: Example of Section 5.1. Impulse responses to a fundamental shock $\varepsilon_0 = -0.25$. The parameters are $\beta = 0.99$, $\sigma = 1$, $k = 0.5$, and $\phi = 0.95$.

spected. We show, in analogy with the Example 1, how non-multiplicative sunspots lead to non-valid solutions.[6] Applying the method of Lubik and Shorfheide (2003), when $0 < \phi < 1$, the expectation error is:

$$\boldsymbol{\eta}_t = -\frac{k\sigma}{d^2}\begin{bmatrix} k\lambda_2 \\ -(\lambda_2 - 1 - k\sigma\phi) \end{bmatrix}\varepsilon_t + \frac{1}{d}\begin{bmatrix} \lambda_2 - 1 - k\sigma\phi \\ k\lambda_2 \end{bmatrix}(M_1\varepsilon_t + \zeta_t) \tag{24}$$

where $d = \sqrt{(k\lambda_2)^2 + (\lambda_2 - 1 - k\sigma\phi)^2}$ and M_1 is a scalar. It is clear that the sunspot errors are not multiplicative. Equation (22) collects all the valid solutions for the New Keynesian model here presented, and it also defines the expectation error as

$$\begin{aligned} \boldsymbol{\eta}_t &= C^{-1}\mathbf{b}_t(\zeta_t)J^{-1}C\boldsymbol{\gamma}\varepsilon_t \\ &= \frac{\sigma}{(1+k\sigma\phi)(\lambda_2-\lambda_1)}\begin{bmatrix} b_{1,t}\lambda_2 + \lambda_1 - (1+b_{1,t})(1+k\sigma\phi) \\ k(b_{1,t}\lambda_2 + \lambda_1) \end{bmatrix}\varepsilon_t\ . \end{aligned} \tag{25}$$

Equating equations (24) and (25) we find that applying the Lubik and Shorfheide method correspond on defining $b_{1,t}$ as

$$b_{1,t} = h_1 + h_2\left(M_1 + \frac{\zeta_t}{\varepsilon_t}\right) \tag{26}$$

in which

$$\begin{aligned} h_1 &= \frac{(1+k\sigma\phi)(\lambda_2-\lambda_1)(\lambda_2 - 1 - k\sigma\phi) - \lambda_1 d^2}{\lambda_2 d^2} \\ h_2 &= \frac{(1+k\sigma\phi)(\lambda_2-\lambda_1)}{\sigma d}\ . \end{aligned}$$

When $\varepsilon_t = 0$, equation (26) shows that the method proposed by Lubik and Shorfheide does not lead to a valid solution.

The role of sunspots is not only to amplify the response to a fundamental disturbance: sunspot errors can also change the sign of the initial response, as the case of inflation in the previous example. This additional source of uncertainty for the monetary authority vanishes when there is a mechanism that ensure the existence of a unique equilibrium. The rational expectations hypothesis is called to represent this mechanism, but it may not be enough. "*Transversality conditions can rule out*

[6] Another famous and very authoritative example of non-multiplicative sunspot shock is in Farmer and Guo (1994).

real explosions, but not nominal explosions".[7] The next example makes clear this point.

5.2 Explosive dynamics for inflation and nominal interest rate

We use a simple model taken by Cochrane (2011):

$$i_t = r + E_t\pi_{t+1} \tag{27}$$

$$i_t = r + \phi\pi_t + x_t \tag{28}$$

$$x_t = \rho x_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim i.i.d.N(0, \sigma_\varepsilon^2) \ . \tag{29}$$

Equation (27) is the famous Fisher equation. It states that i_t, the nominal interest rate, is equal to the real interest rate r plus the expected value of tomorrow's inflation. For simplicity the real interest rate is supposed to be constant. Equation (28) is the Taylor rule, in which x_t is an autoregressive component of the monetary policy, whose dynamics is described by equation (29).

Substituting equation (28) in the Fisher equation trough the nominal interest rate we obtain the equation

$$\pi_t = \frac{1}{\phi}E_t\pi_{t+1} - \frac{1}{\phi}x_t \ . \tag{30}$$

The new Keynesian theory suggests the Central Banks to implement an active monetary policy: the parameter ϕ should be grater than one in absolute value. In this case, if you require stability, the model has only one stable solution that implies the following dynamics for the inflation rate:

$$\pi_t = \rho\pi_{t-1} + \frac{\varepsilon_t}{\rho - \phi} \ ,$$

and the expected value of tomorrow's inflation given the information available today is

$$E_t\pi_{t+1} = \rho\pi_t \ .$$

[7] Cochrane (2011), pag 6.

To find the equilibrium dynamics for the nominal interest rate, substitute the expected inflation in the Fisher rule:

$$i_t = r + \rho\pi_t.$$

Then if you estimate a Taylor rule, regressing the interest rate on the inflation rate, you make inference on ρ, not on ϕ! Under the two hypothesis of stability and active monetary policy the parameter ϕ is not identified.

We show that, if stability is not a priori required, we can identify ϕ only when the Central Bank implements an active monetary policy. There is no reason to rule out explosive paths. Consider, then, the full set of solutions of equation (30):

$$\pi_t = \phi\pi_{t-1} + x_{t-1} + b_t(\zeta_t)\frac{\varepsilon_t}{\phi - \rho} \ . \tag{31}$$

The covariance between fundamental and sunspot shocks is zero by hypothesis, so the expected value of tomorrow's inflation is

$$E_t\pi_{t+1} = \phi\pi_t + x_t$$

and the implied equilibrium for i_t is exactly the Taylor rule (28).

Consider, for simplicity, b_t as a constant equal to b. Substituting the Taylor rule in equation (31), we verify that present inflation is perfectly correlated with past inflation:

$$\pi_t = i_{t-1} + \frac{b}{\phi - \rho}\varepsilon_t \ .$$

Then, substituting this relation in the Taylor rule (28), we can write the equilibrium for i_t as an ARMA(2,1):

$$i_t = r(1-\rho)(1-\phi) + (\rho+\phi)i_{t-1} - \phi\rho i_{t-2} - \phi\rho b\frac{\varepsilon_{t-1}}{\phi-\rho} + \left(1 + \frac{\phi b}{\phi - \rho}\right)\varepsilon_t \ .$$

As pointed out by Cochrane (2011)[8], if you estimate the autoregressive coefficients of the above equation, you make inference on ρ and ϕ together, but, in general, you can not distinguish which parameter is ϕ. However we know that ρ in inside the

[8]See the online Appendix B of Cochrane (2011).

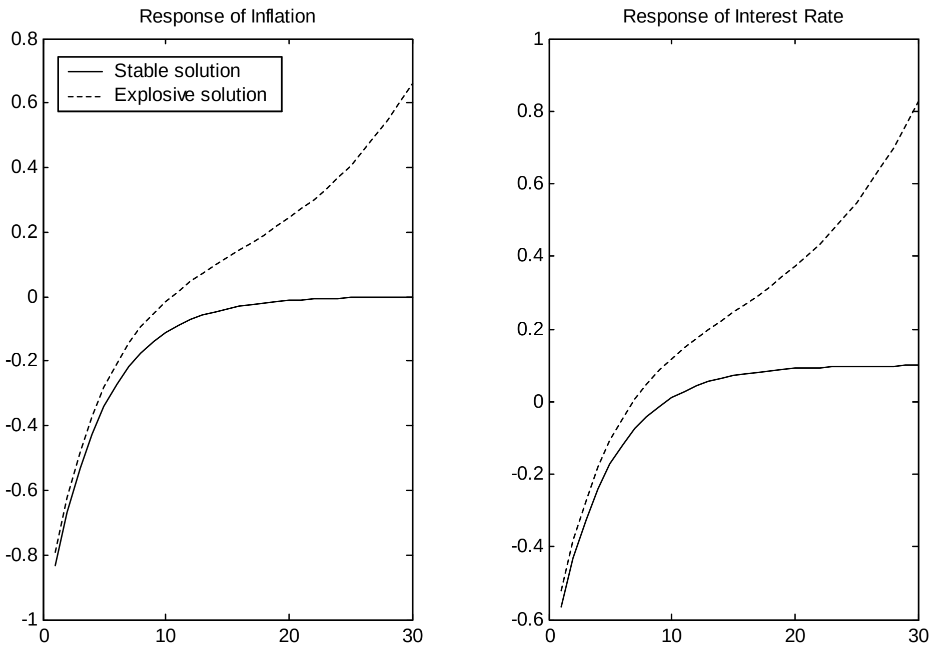

Figure 2: Example of Section 5.2. Impulse responses to a fundamental shock ε_0=0.25. The parameters are ρ=0.8, ϕ=1.1, r=0.1. For the stable solution b=-1; for the explosive solution b=-0.95.

unit circle. Then, we can identify ϕ if it is greater that one: this parameter, in this particular model, is identified only when inflation is described by unstable dynamics.

Figure (2) shows the impulse response functions of inflation and nominal interest rate when $\rho = 0.8$, $\phi = 1.1$, $r = 0.1$, and the economy is hit by a positive shock $\varepsilon_t = 0.25$. We consider two cases: in the first case $b_t = -1$ so the economy is on the unique stable path; in the second case $b_t = -0.95$, so both the inflation and the interest rate explode.

6 Conclusions

This work gives two contributions. First, we provide a simple mathematical tool to solve system with rational expectations. The method does not exclude, a priori, the unstable dynamics from the set of valid solutions. These solutions, in fact, are not always in contrast with the rational expectations hypothesis, as shown by Cochrane (2011).

If stability is not imposed, indeterminacy, that is the existence of more than one solution, is the natural status. Indeterminacy can arise also when stability is imposed, but the Blanchard-Kahn condition does not hold. Than, sunspot equilibria can characterize the economy. Our second contribution consist in clarifying the role of sunspots: we demonstrate that sunspot shocks can only be multiplicative. They can be effective only when the economy is hit also by a fundamental disturbance.

We understand that accepting the possibility of explosive dynamics can be very controversial for many reasons. In our opinion, a good practice for empirical research is to test for stability, before imposing it. The method we propose goes in this direction: testing for stability is very easy, because it reduces to verify if some parameters are significantly different from a given value.

The ambition of that paper is to provide a simple computational instrument that overcome limits that are present in the literature about rational expectations models. Our concerns is that these limits can be empirically relevant. The open issue is to demonstrate if our doubts are justified or not.

Appendix

We show how to compute the complete set of solutions of a system with rational expectations. In what follows we refer to Blanchard (1979) and to Blanchard and Kahn (1980).

Consider a system with rational expectations written in the form of Blanchard and Kahn (1980):

$$\begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \boldsymbol{\gamma} Z_t \tag{A1}$$

E is the expectation operator and, for any generic variable x $E_t x = E(x|\Im_t)$, that is the expected value of x conditioned to the information set $\Im_t$ available at time t.

X_t is a $(n \times 1)$ vector of predetermined variables and P_t is a $(m \times 1)$ vector of non predetermined variables. $Z_t \sim i.i.d.\ N(\mathbf{0}, \Sigma)$ is a $(\kappa \times 1)$ vector of exogenous random variables.

Use the Jordan form to rewrite A

$$A = C^{-1} J C.$$

In the main diagonal of J there are the eigenvalues of A, ordered by increasing absolute value. We decompose the matrices C^{-1}, J, C and $\boldsymbol{\gamma}$ as follows:

$$C^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ (n \times n) & (n \times m) \\ B_{21} & B_{22} \\ (m \times n) & (m \times m) \end{bmatrix}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ (n \times n) & (n \times m) \\ C_{21} & C_{22} \\ (m \times n) & (m \times m) \end{bmatrix},$$

$$J = \begin{bmatrix} J_1 & \mathbf{0} \\ (n \times n) & (n \times m) \\ \mathbf{0} & J_2 \\ (m \times n) & (m \times m) \end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix} \gamma_1 \\ (n \times \kappa) \\ \gamma_2 \\ (n \times \kappa) \end{bmatrix}.$$

Define

$$\begin{bmatrix} Y_t \\ Q_t \end{bmatrix} = C \begin{bmatrix} X_t \\ P_t \end{bmatrix},$$

and rewrite equation (A1) in terms of $\begin{bmatrix} Y_t \\ Q_t \end{bmatrix}$:

$$\begin{bmatrix} E_t Y_{t+1} \\ E_t Q_{t+1} \end{bmatrix} = \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix} \begin{bmatrix} Y_t \\ Q_t \end{bmatrix} + \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} Z_t \,. \tag{A2}$$

Now consider the second block of equation (A2),

$$Q_t = J_2^{-1} E_t Q_{t+1} - \Omega_t \tag{A3}$$

where $\Omega_t = J_2^{-1}(C_{21}\gamma_1 + C_{22}\gamma_2) Z_t$. The system (A3) has m disjoined equations, and each of them admits an infinite number of solutions because of the presence of an expected value. Defining $q_{i,t}$ as the i^{th} element of Q_t, and $\omega_{i,t}$ the corresponding disturbance, we follow Blanchard (1979) writing all the solutions of the generic row of equation (A3) as

$$q_{i,t} = \sum_{j=1}^{\infty} u_{i,j}\omega_{i,t-j} + b_i\omega_{i,t} + \sum_{j=1}^{\infty} c_{i,j}E_t\omega_{i,t+j} \ . \tag{A4}$$

Using matrices instead of scalars the solutions can be rewritten as

$$Q_t = \sum_{j=1}^{\infty} \mathbf{u}_j\Omega_{t-j} + \mathbf{b}\Omega_t + \sum_{j=1}^{\infty} \mathbf{c}_j E_t\Omega_{t+j} \tag{A5}$$

where $\mathbf{u}_j$, $\mathbf{b}$ and $\mathbf{c}_j$ are diagonal matrices of coefficients to be determined. Bring equation (A5) one step ahead

$$E_tQ_{t+1} = \sum_{j=1}^{\infty} \mathbf{u}_j\Omega_{t+1-j} + E_t\mathbf{b}\Omega_{t+1} + \sum_{j=1}^{\infty} \mathbf{c}_j E_t\Omega_{t+1+j}$$

and substitute in equation (A3)

$$Q_t = J_2^{-1}\sum_{j=2}^{\infty} \mathbf{u}_j\Omega_{t+1-j} + J_2^{-1}\mathbf{u}_1\Omega_t - \Omega_t + J_2^{-1}E_t\mathbf{b}\Omega_{t+1} + J_2^{-1}\sum_{j=1}^{\infty} \mathbf{c}_j E_t\Omega_{t+1+j} \ . \tag{A6}$$

We find the coefficients comparing the matrices of equation (A5) to the ones of equation (A6):

$$\begin{aligned}
&\mathbf{b} = J_2^{-1}\mathbf{u}_1 - I \Longrightarrow \boxed{\mathbf{u}_1 = J_2\mathbf{b} + J_2} \\
&\mathbf{u}_1 = J_2^{-1}\mathbf{u}_2 \Longrightarrow \boxed{\mathbf{u}_{j+1} = J_2\mathbf{u}_j} \quad j = 1...\infty \\
&\boxed{\mathbf{c}_1 = J_2^{-1}\mathbf{b}} \\
&\mathbf{c}_2 = J_2^{-1}\mathbf{c}_1 \Longrightarrow \boxed{\mathbf{c}_{j+1} = J_2^{-1}\mathbf{c}_j} \quad j = 1...\infty
\end{aligned}$$

The matrices $\mathbf{u}_j$ and $\mathbf{c}_j$ are functions of $\mathbf{b}$ and J_2, and since J_2 is given, the complete set of solutions is parametrized by $\mathbf{b}$. There are two particular cases: the pure backward looking solution, corresponding to $\mathbf{b} = \mathbf{0}$, that implies $\mathbf{c}_j = \mathbf{0}$ and $\mathbf{u}_j = J_2^j$, $j = 1...\infty$; the pure forward looking solution corresponding to $\mathbf{b} = -I$, that implies $\mathbf{u}_j = \mathbf{0}$ and $\mathbf{c}_j = -J_2^{-j}$, $j = 1...\infty$. The backward looking solution can be written as follows:

$$\begin{aligned}
Q_t^B &= \sum_{j=1}^{\infty} \mathbf{u}_j \Omega_{t-j} \\
Q_t^B &= \sum_{j=1}^{\infty} J_2^j \Omega_{t-j} = J_2 \Omega_{t-1} + J_2^2 \Omega_{t-2} + J_2^3 \Omega_{t-3} + ... \\
Q_t^B &= J_2 \Omega_{t-1} + J_2 \left[J_2 \Omega_{t-2} + J_2^2 \Omega_{t-3} + J_2^3 \Omega_{t-4} + ... \right] \\
Q_t^B &= J_2 Q_{t-1}^B + J_2 \Omega_{t-1} \qquad \text{(A7)}
\end{aligned}$$

The forward looking solution is

$$Q_t^F = b\Omega_t + \sum_{j=1}^{\infty} \mathbf{c}_j E_t \Omega_{t+j} = -I\Omega_t - J_2^{-1} E_t \Omega_{t+1} - J_2^{-2} E_t \Omega_{t+2} - ...$$

and since $E_t \Omega_{t+j} = \mathbf{0} \ \ \forall j \geq 1$, we obtain

$$Q_t^F = -\Omega_t \ . \qquad \text{(A8)}$$

Following Blanchard (1979) we write any other solution as a linear combination of the backward and the forward looking solutions. In compact form

$$Q_t = \boldsymbol{\lambda} Q_t^B + (I - \boldsymbol{\lambda}) Q_t^F \qquad \text{(A9)}$$

where $\boldsymbol{\lambda} = I + \mathbf{b}$ is a diagonal matrix. The elements in the main diagonal of $\mathbf{b}$ are such that $\mathbf{b} = \mathbf{0} \Rightarrow Q_t = Q_t^B$, and $\mathbf{b} = -I \Rightarrow Q_t = Q_t^F$.

Substitute the equations (A7) and (A8) in equation (A9)

$$\begin{aligned}
Q_t &= \boldsymbol{\lambda} \left(J_2 Q_{t-1}^B + J_2 \Omega_{t-1} \right) - (I - \boldsymbol{\lambda}) \Omega_t \\
&= \boldsymbol{\lambda} J_2 Q_{t-1}^B - \boldsymbol{\lambda} J_2 Q_{t-1}^F + J_2 Q_{t-1}^F - J_2 Q_{t-1}^F - (I - \boldsymbol{\lambda}) \Omega_t \ .
\end{aligned}$$

In the last passage we have added and subtracted $J_2 Q_{t-1}^F$. Since both J_2 and $\boldsymbol{\lambda}$ are diagonal matrices the commutative property holds and we can write

$$\begin{aligned}
Q_t &= J_2 \left(\boldsymbol{\lambda} Q_{t-1}^B + (I - \boldsymbol{\lambda}) Q_{t-1}^F \right) + J_2 \Omega_{t-1} - (I - \boldsymbol{\lambda}) \Omega_t \\
Q_t &= J_2 Q_{t-1} + J_2 \Omega_{t-1} + b \Omega_t \qquad \text{(A10)}
\end{aligned}$$

Equation (A10) represents the infinite number of solutions for Q_t parametrized by $\mathbf{b}$. The complete set of solutions for model (A1) is found using the definition of Q_t and the first n rows of the model written with the Jordan matrices:

$$X_t = (B_{11} J_1 C_{11} + B_{12} J_2 C_{21}) X_{t-1} + (B_{11} J_1 C_{12} + B_{12} J_2 C_{22}) P_{t-1} + \gamma_1 Z_{t-1} \qquad \text{(A11)}$$

$$C_{21}X_t + C_{22}P_t = J_2(C_{21}X_{t-1} + C_{22}P_{t-1}) \quad +(C_{21}\gamma_1 + C_{22}\gamma_2)Z_{t-1} + \\ +\mathbf{b}J_2^{-1}(C_{21}\gamma_1 + C_{22}\gamma_2)Z_t \quad \text{(A12)}$$

In the paper we focus on the case in which the matrix A has at least n eigenvalues inside the unit circle. This means that the model admits at least one stable solution. If this condition is not satisfied the equations (A11) and (A12) continue to represent the complete set of solutions that are all unstable.

References

Benhabib, J., and R. E. A. Farmer (1999): "Indeterminacy and sunspots in macroeconomics," in: J. B. Taylor & M. Woodford (ed.), *Handbook of Macroeconomics*, edition 1, volume 1, chapter 6, pages 387-448 Elsevier.

Beyer, A. and R. E. A. Farmer (2004): "On the Indeterminacy of New-Keynesian Economics," Computing in Economics and Finance 2004 152, Society for Computational Economics.

Beyer, A. and R. E. A. Farmer (2007): "Testing for Indeterminacy: An Application to U.S. Monetary Policy: Comment," *American Economic Review*, American Economic Association, vol. 97(1), pages 524-529, March.

Blanchard, O. J. (1979): "Backward and Forward Solutions for Economies with Rational Expectations," *American Economic Review*, American Economic Association, vol. 69(2), pages 114-18, May.

Blanchard, O. J. and C. M. Kahn (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, Econometric Society, vol. 48(5), pages 1305-11, July.

Bullard, J. and K. Mitra (2002): "Learning about monetary policy rules," *Journal of Monetary Economics*, Elsevier, vol. 49(6), pages 1105-1129, September.

Clarida, R., J. Galí and M. Gertler (2000): "Monetary Policy Rules And Macroeconomic Stability: Evidence And Some Theory," *The Quarterly Journal of Economics*, MIT Press, vol. 115(1), pages 147-180, February.

Cochrane, J. H. (2011): "Determinacy and identification with Taylor rules," *Journal of Political Economy*, The University of Chicago Press, Vol. 119(3), pages 565-615, June.

Farmer R. E. A. and J.-T. Guo (1994): "Real Business Cycles and the Animal Spirits Hypothesis," *Journal of Economic Theory*, Elsevier, vol. 63(1), pages 42-72, June.

Lubik, T. A. and F. Schorfheide (2003): "Computing sunspot equilibria in linear rational expectations models," *Journal of Economic Dynamics and Control*, Elsevier, vol. 28(2), pages 273-285, November.

Lubik, T. A. and F. Schorfheide (2004): "Testing for Indeterminacy: An Application to U.S. Monetary Policy," *American Economic Review*, American Economic Association, vol. 94(1), pages 190-217, March.

Sims, C. A. (2002): "Solving Linear Rational Expectations Models," *Computational Economics*, Springer, vol. 20(1-2), pages 1-20, October.