Assortative matching through signals*

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Abstract

In a model of sequential search with transferable utility, we allow heterogeneous agents to strategically choose a costless signal of their type. Search frictions are included as discounting and explicit search costs. Through signals, if only they are truthful, agents can avoid the inefficiencies of random search. Then the situation effectively approaches a setting without search frictions. We identify the condition under which signals are truthful and a unique separating equilibrium with perfect sorting arises despite frictions. We find that supermodularity of the match production function is a necessary and sufficient condition. This is a weaker condition than is needed for sorting in models without signals, which may explain why sorting is much more widespread in reality than existing models would suggest. Supermodularity functions here as both a sorting condition and a single-crossing property. The unique separating equilibrium in our model achieves nearly unconstrained efficiency despite frictions: agents successfully conclude their search after a single meeting, a stable matching results, and overall match output is maximised.

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Imagine a dark room filled with a number of married couples. The people in the room are busy attempting to find their mates. An economist would attempt to model the behavior in the room by assuming that everyone is searching for his better half in an optimal manner given the fact that the lights are out. Consequently, he would conclude that the behavior in the room is efficient. However, such a conclusion does not deny that the matching process would be improved if someone should happen to turn on the lights.

Dale T. Mortensen (1976), p. 195, echoing a parable told by Axel Leijonhufvud.

1 Introduction

A number of important markets, notably the labour market, bring trade partners together in pairs that form durable matches. The dominant approach in the analysis of decentralised interaction on such matching markets has been developed by Diamond, Mortensen, and Pissarides: agents engage in search for match partners, and frictions in the search process make this costly. While this approach and derived results have found ample empirical support, efforts to replicate important empirical regularities in search models continue. Alongside business cycle properties and price or wage dispersion, it is the pattern of sorting in matching markets that the literature has sought to explain. Indeed, that likes tend to match with likes is a pervasive phenomenon: more productive workers tend to be hired by more productive firms, more educated women tend to marry more educated men, more reliable tenants tend to secure nicer apartments.[1]

This sorting of likes along some dimensions, known as *positive assortative matching* (PAM), thus serves as a test on the validity of the search model: whenever a model of a decentralised matching market with heterogeneous agents cannot generate PAM, it appears to have missed something important. That apart, PAM is relevant because it may be more efficient than, say, random matching: highly skilled workers might be better placed in highly productive firms than unskilled workers, for example. If economists understand what drives assortative matching, they will be in a position to avoid mismatch in market design or to improve efficiency in existing markets by making PAM more pronounced.

The theoretical literature has identified some form or another of complementarity among the inputs into a match as the driver of PAM. Thinking of the value generated by a match as the match output, a match production function specifies how agents' inputs translate into match output. Then simple complementarity among the inputs is equivalent to *supermodularity* of the match production function: the marginal effect on output from a change in one input is increasing in the other input. A stronger concept of complementarity, log-supermodularity, requires that the marginal effect on output from a change in one input is also as a proportion increasing in the other input.

[1] As an exemplary reference for these stylised facts, see Mare (1991).

While supermodularity as such suffices for PAM to arise in the model whenever search is costless, it often does not suffice when search is costly, in particular when search takes valuable time, which it does in practice. The literature has therefore resorted to more demanding forms of complementarity, but it is doubtful that very demanding conditions hold across the numerous and diverse real matching markets that exhibit PAM. In this paper, however, we find that a model with more information in the search process, equivalent to light in the quote from Mortensen above, also only requires supermodularity to generate PAM: search with better information takes less time and can avoid costly but unsuccessful meetings, so that our model in effect approaches models with costless search. This paper therefore reconciles the search-theoretic approach with potentially very many decentralised markets that exhibit PAM in practice but do not meet the existing theoretical conditions for PAM.

State of the literature Following Smith (2006), we would classify the models in the literature by two criteria. One is whether or not utility is transferable between agents. The literature refers to *non-transferable utility* whenever agents divide the match output according to a pre-imposed split, like employer and employee do when the employee's wage is already determined by a union wage agreement. In cases of *transferable utility*, there is no pre-imposed split and agents have to bargain over the match output. The other criterion is which costs underlie the search frictions. The seminal article by Becker (1973) considers a frictionless setting (i.e. agents search costlessly) where utility is transferable and supermodularity as such turns out to suffice for PAM, in fact even perfect PAM: only equal types match.[2]

When search is costly, be it implicit costs through discounting or explicit additive costs, we have a setting with frictions. An influential contribution by Shimer and Smith (2000) examines a setting with discounting and transferable utility. They conclude that the match production function, the logarithm of its first derivative, and the logarithm of its cross-partial derivative all need to be supermodular for PAM to arise in this setting. For the same setting with non-transferable utility, Smith (2006) shows that PAM will only arise if the match production function is log-supermodular. While supermodularity as such is a very natural condition, the combination of three conditions required for PAM in Shimer and Smith (2000) is criticised by Atakan (2006) as "quite restrictive"; Goldmanis et al. (2009) find it "quite troubling" that supermodularity is not sufficient in Smith (2006) and point out that even situations in which there is hardly any or no sorting at all meet the formal criterion for PAM that these contributions employ.[3] In

[2]Using Becker's (1973) work, it is quickly found that a frictionless setting with non-transferable utility leads to PAM even without supermodularity: see Smith (2006), section II.

[3]See Atakan (2006), p. 667 and Goldmanis et al. (2009), p. 8 and p. 12. Eeckhout and Kircher (2010) show that the conditions in Shimer and Smith (2000) imply log-supermodularity whenever match production is non-decreasing in inputs, which establishes that these conditions are indeed considerably more restrictive than just supermodularity.

short, there is a paradox here: real-world agents in all sorts of matching markets both discount and sort into PAM, but theoretical models of these markets require improbably strong complementarities to generate even a very weak form of PAM.

Atakan's (2006) own model features transferable utility, as in Shimer and Smith (2000), but explicit costs instead of discounting. Morgan (1998) builds a model with non-transferable utility and explicit costs. In both models, supermodularity as such gives rise to PAM (while the results in Atakan (2006) crucially depend on search costs being identical for all agents). However, limiting oneself to explicit costs has not provided an answer to the paradox, since real-world agents do discount.

This paper We offer a solution to this paradox by appealing to another, equally pervasive phenomenon in matching markets: the use of signals. Signals come in the form of job advertisements and applications on the labour market, dress style and body language on the marriage market, online photographs of apartments and flat hunters on the market for rented housing, to name but a few. We argue that such signals, if only they are truthful, make all wasteful search obsolete: agents can target their search directly on the best offer they can possibly secure, and they thus never lose money or time on search that they regret. By cutting search short, signals can reduce the effect of search frictions and thus bring a setting with frictions closer to a frictionless setting, in which mild conditions suffice for PAM to arise.

Of course, signals will only have this beneficial effect if they are actually informative. In the famous signalling model by Spence (1973), years of education are an informative signal for ability because more able workers find it easier to acquire education than less able workers do. The model thus relies on signals being costly and on a single-crossing property. Unfortunately, when agents use applications or advertisements, the costs are typically small and a single-crossing property can in general not be expected to hold: writing a forged CV is as costly as writing a truthful CV, and painting an advertised job in unduly bright colours is as costly as honestly laying out its dull nature. Yet, as shown by contributions such as Crawford and Sobel (1982), signals will be informative even in an environment of such *cheap talk* if the interests of senders and receivers are sufficiently aligned. Then the costs of signals can just as well be normalised to zero.

In essence, the model we build introduces costless signals into the search model with transferable utility in Shimer and Smith (2000). We prove a unique separating equilibrium in which not only PAM but perfect PAM will result if and only if the match production function is supermodular. In this equilibrium, each agent finds it optimal to signal truthfully and to target her search on only one type. A low type will in fact not prefer to deviate to matches with higher types in equilibrium because the higher types' optimal behaviour in effect forces her not to renege on her signal. When reneging is not an option, the lower type has to conceal the difference between expected and actual match production by reducing her share accordingly. If the match production function is supermodular, this reduction will outweigh the gain from higher match production with a higher type. As signals will thus be truthful, perfect PAM is to be expected: with fully informative signals, agents can replace (almost all) costly search via meetings by costless search via signals and can therefore behave like in a frictionless setting. This paper thereby offers a resolution to the paradox created by the conditions in Shimer and Smith (2000).

The separating equilibrium of the model has a number of desirable efficiency properties. Above all, agents match at the very first opportunity, so that no time is wasted on unsuccessful search and search costs are minimised. In labour market terms, this would mean that frictional unemployment is reduced to a minimum. The signals allow agents to first locate their best feasible match partner at no cost, so that the only costs are created by the meeting with this agent. The source of mismatch in standard search models, the incentive to accept less than the best feasible match so as to avoid further search costs, is therefore absent here. As there is no mismatch, the equilibrium matching is stable and maximises aggregate production. While the matching in standard search models only achieves constrained efficiency due to frictions, the matching achieves unconstrained efficiency in this case because the frictions are in effect overcome. Only the costs associated with the single meeting that precedes each match distinguish the separating equilibrium from the first-best outcome imposed by a benevolent social planner.

Relation to the literature A few papers consider sorting in the context of a matching market with signals. Hoppe et al. (2009) and Hopkins (2010) build two similar models of a *matching tournament* with signalling: match partners are essentially prizes for ex-ante choices of costly signals. In both models, agents first select a costly signal of their unobservable type like in Spence (1973) and then, based on these signals, match roughly like in Becker (1973). Hopkins (2010) assumes a single-crossing property and Hoppe et al. (2009) assume a specific multiplicative production that satisfies supermodularity and even log-supermodularity. In the symmetric equilibrium, agents' signals are then strictly increasing in their types. This leads to perfect PAM at the matching stage - just as one would have expected, given Becker's (1973) findings. However, since search frictions do not exist in matching tournaments, neither of the two papers helps us resolve the paradox in Shimer and Smith (2000).

A search model built by Chade (2006) features discounting and noisy signals uncontrolled by the agents. Yet these signals are not observed before agents meet. Rather, when agents do meet, they do not observe each others' true types but only the noisy signal. Hence search is still random in this model, and the noisy signals in fact add information frictions to the search frictions. Assuming that the noisy signals exogenously carry some information, matching is shown to exhibit PAM in a very weak sense: the distribution of types a high type might match with first-order stochastically dominates this distribution for a low type. Chade (2006) further finds that an assumption of log-supermodularity of the match production function, while not necessary, reinforces these results.

Our set-up primarily differs from Chade's (2006) in that signals are observed before meetings and thus make search non-random, thereby tending to reduce search frictions. Moreover, signals in our model are not exogenously truthful signals with noise but are deliberately and strategically chosen by agents. We would argue that real-world agents will exert as much control over the signals as possible, given how important they can be for their payoffs. Hence, we do not assume that signals are necessarily informative.[4]

Further, a model by Eeckhout and Kircher (2010) features the key elements of models of *directed search*, including signals in the form of sellers' posted offers. Buyers observe the posted offers and simultaneously choose which seller to visit. The only way frictions enter this process is through congestion: buyers cannot coordinate, so that queues result and only some buyers manage to buy. It is shown that, for common meeting technologies, the matching of buyers and sellers will exhibit PAM if the square root of the match production function is supermodular. While 'root-supermodularity' is less restrictive than log-supermodularity, it is still more restrictive than supermodularity. Shimer (2005) looks at PAM in a directed search model of the labour market and finds, at least for the case of only two worker types, that there will be some stochastic form of PAM as long as workers of low type do not have a comparative advantage when working for employers of high type. While these models simply assume that signals are truthful (a criticism we would level against almost any directed search model), these models cannot resolve the paradox mainly because of their limitation to frictions from congestion, so that their results do not carry over to models with other frictions in any obvious way.

A contribution by Lentz (2010) does not feature any signals but allows for search on the job (or, more generally, search while matched) with endogenous search intensity in a model with discounting. While search is still random, higher types gain more than others from search on the job if the match production function is supermodular. Higher types thus search more intensively, which leads to PAM in terms of stochastic dominance as in Chade (2006). A related model by Goldmanis et al. (2009) features non-transferable utility, discounting, and search on the job. If only one agent in each match can switch to another match, PAM will result if the match output function is log-supermodular. If either agent can switch, a condition has to be met so that agents gain from switching to matches with higher types (at least in the equilibrium they identify, which might not be unique). Then the situation would evolve into perfect PAM over time, were it not always set back as matches randomly break up and agents begin climbing up the ladder anew. Therefore, perfect PAM is only achieved in the limit as the rate of random break-ups tends to zero. Moreover, agents in Lentz (2010) and Goldmanis et al. (2009) sort only over time. By contrast, the fundamentally different sorting mechanism in our model can, without invoking stronger conditions, explain sorting already among graduates in their

[4]Visschers (2005) makes another contribution that adds information frictions to the search frictions: upon meeting, agents only observe each other's signals, which are costly and strategically chosen. He argues that, with non-transferable utility and log-supermodularity as in Smith (2006), such signals can lead to block segregation rather than PAM, in contrast to the results in Smith (2006).

first job and achieve perfect PAM as a unique (separating) equilibrium despite random break-ups. Indeed, to the best of our knowledge, ours is the only model that generates perfect PAM in the context of discounting or explicit search costs or both.

Our model is finally related to Jacquet and Tan (2007). They consider an environment with discounting and a particular log-supermodular match production function: an agent's payoff equals her match partner's type (so that utility is non-transferable). For such an environment, Burdett and Coles (1997) found that types segregate into classes and match exclusively within these classes. Building on this, Jacquet and Tan (2007) let agents establish any number of marketplaces and show that each marketplace is populated by only one class in equilibrium (while perfect PAM cannot be achieved). By going to the appropriate marketplace, each agent can thus avoid meetings that do not lead to a match and can instead match after the first meeting. Agents can do the same in our model if signals are informative: they can use the signals to create any number of virtual marketplaces such as a website. While it is left open in Jacquet and Tan (2007) how marketplaces are established, virtual marketplaces are established simply by requiring certain signals. Apart from this difference, our environment will feature transferable utility and a general match production function.

Overview The paper proceeds as follows. Section 2 specifies a frictional matching market and the procedures of search. Section 3 defines equilibrium in the model and proposes a separating equilibrium in which supermodularity suffices for perfect PAM. Its existence is proven step by step through a series of lemmas in section 4. The separating equilibrium is found to be unique as well as efficient in section 5 before section 6 concludes.

2 Model

The matching market in our model consists of heterogeneous agents who match among themselves. Agents are indexed by a productivity type $x \in \Theta$, where $\Theta = [\underline{x}, \bar{x}] \subset \mathbb{R}_{++}$. For each type, there is a continuum of agents and the overall mass of agents is normalised to 1. The measure of agents with types weakly below $x \in \Theta$ is denoted $L(x)$, where $L(\cdot)$ is a cumulative distribution function assumed C^1. Its density function $l(\cdot)$ is required everywhere strictly positive and $l(x)$ may be thought of as the mass of agents of type x.

Agents can meet on N separate marketplaces indexed by n, where N may be countably infinite. Agents cannot be on several marketplaces simultaneously (i.e. their search activity is indivisible), but they can always switch between marketplaces without incurring any cost. When they match they immediately leave the marketplace. We denote the density function for unmatched agents on marketplace n by $u^n(\cdot) \leq l(\cdot)$. Then $u^n(x) \geq 0$ represents the mass of unmatched agents of type x on marketplace n, while $l(x) - \sum_{n=1}^{N} u^n(x)$ represents the total mass of matched agents of type x. All these quantities are determined endogenously.

Types are exogenously given, but only privately observable. To convey information about her type to other agents on the same marketplace, each agent can choose a costless signal $\tilde{x} \in \Theta$, which may or may not be an accurate signal of her true type x. Since agents can then condition meetings on signals, meetings are non-random. Agents can influence whom they meet through their choice of marketplace: each marketplace n is characterised by a set R^n of required signals, such that each agent who chooses this marketplace and sends a signal $\tilde{x} \in R^n$ can meet all other agents on the marketplace who also send a required signal. We let R^n be public information, as agents can in any case very quickly infer R^n from the signals they observe on marketplace n. Through her choice of marketplace, each agent then effectively selects the agents she meets (albeit not the types) by their signals.

Inside a marketplace, meetings are random and are described by a meeting function $m(\cdot)$. With a mass of agents

$$\lambda^n = \int_\Theta u^n(x)dx$$

the flow of meetings in marketplace n equals $m(\lambda^n) \leq \lambda^n$, and $m(0) = 0$. The meeting rate on this marketplace is

$$\eta^n = \begin{cases} \frac{m(\lambda^n)}{\lambda^n} & \text{if } \tilde{x} \in R^n \text{ and } \lambda^n > 0 \\ 0 & \text{if } \tilde{x} \notin R^n \text{ or } \lambda^n = 0 \end{cases} \tag{1}$$

We assume constant returns to scale in meeting, so that agent x faces the same meeting rate $\eta^n = \eta$ across all N marketplaces, provided she always chooses a required signal. Then x must choose her marketplace by the agents she wants to meet, as she would meet all agents equally quickly. When she is indifferent, she randomises over her most preferred marketplaces. In the appendix, we argue that all of our results will continue to hold if marketplaces are instead two-sided, so that the meeting function takes two arguments, and agents choose which side to join. Finally, any marketplace can be created at no cost but must attract a strictly positive mass of agents in order to last. The agent(s) creating marketplace n choose R^n, which cannot be changed thereafter.

Time is continuous with an infinite horizon. Each agent is always in one of three states: matched, searching for a match on a marketplace, or not participating. When indifferent whether to engage in search, whether to accept a match, and whether to stay in a marketplace or switch, an agent does respectively search, accept the match, and stay. An agent who engages in search pursues two activities that neither matched nor non-participating agents pursue. Firstly, she sends the signal $\tilde{x} \in \Theta$, which she can always instantly and costlessly change. Secondly, she seeks meetings: before x can match with some agent y, a meeting between the two will have to occur. To distinguish in the notation between two agents who meet, we will denote the type of one agent by x and the type of the other by y. Normalising also the flow output generated by an unmatched agent to zero, a match between types x and y generates a constant flow output $f(x, y)$. The match production function $f(\cdot,\cdot)$ is assumed to satisfy some regularity conditions:

Assumption 1 (Regularity conditions). *The match production function $f(\cdot,\cdot)$ is symmetric ($f(x,y) \equiv f(y,x)$), strictly increasing in both arguments, and, for any existing types, takes only positive values ($f : \Theta^2 \mapsto \mathbb{R}_{++}$).*

Because productivity types are scalars, there can only be gains from specialisation in production if one role in production rewards productivity more than the other role does. This specialisation will remain possible despite the assumption of symmetry if the more productive agent always assumes the role that rewards productivity more, so that the output of the match is maximised.

By observing a signal $\tilde{y}$ agent x can only form a belief about the true type y behind the signal. Agents never directly observe each other's actual types. Let h^x be the history of the interaction with some agent as observed by x, i.e. a set of actions such as the observed signal. We represent a belief as a probability distribution $\Psi(\cdot)$. Concretely, for each h^x, the belief held by agent x of the other agent's true type y is the probability distribution $\Psi(\cdot|h^x)$ over Θ. Then x, having observed h^x, believes that the other's type is y with probability density $\psi(y|h^x)$. All agents use Bayes' rule to form and update their beliefs. Note that match output must also be unobservable when types are unobservable: knowing $f(\cdot,\cdot)$, x could otherwise infer y from the observed output $f(x,y)$. To keep the notation simple, let $g(x,y|h^x)$ denote the match output that x expects based on her belief after observing h^x:

$$g(x,y|h^x) = \int_{y=\underline{x}}^{y=\bar{x}} f(x,y)\psi(y|h^x)dy$$

Since utility is transferable, agents in a meeting bargain over the division of the match output that they would produce between them. We model this using a strategic bargaining procedure with alternating offers. It is useful to imagine that $f(x,y)$ is contained in a pot that agents can only take from but cannot look into. When agents first meet, either of them is randomly selected with probability $\frac{1}{2}$ to move first. An agent x who moves first takes a share $\pi(x|y)$ for herself from the pot, which is not observed by y. Then y takes the remainder $f(x,y) - \pi(x|y)$ from the pot (which may be negative), unobserved by x. Now y has three options: she can accept the share left for her in the pot, reject this share, or walk away and immediately continue searching.[5] If y rejects, x can walk away; otherwise, the same two agents meet again for the next round of bargaining, in which one agent is again selected with probability $\frac{1}{2}$ to move first, and so on. Previously offered shares cannot be recalled, and if players never agree nor walk away, both will obtain 0. If y accepts, agents match immediately and obtain their respective share as a flow utility for the duration of the match.[6] Finally, matches dissolve exogenously at constant rate δ.

[5]The fact that agents have met implies that these agents prefer engaging in search to not participating. It is thus without loss of generality that non-participation is not a further outside option here.

[6]As each agent can assure herself flow utility 0 by not participating, negative shares will always be rejected and thus never arise as a flow utility.

In order to be more precise, let us formalise this bargaining game. The players are 'nature' Q and the agents x and y who meet. The history h^x records the actions that x has observed thus far, h^y records what y has observed, and we simply index histories in chronological order. When x and y first meet, they already know both signals, so that $h_1^x = h_1^y = \{\tilde{x}, \tilde{y}\}$. A player function $P(\cdot,\cdot)$ assigns to each history pair (h^x, h^y) (except any terminal history pairs) a player who moves at this history pair. $P(h_1^x, h_1^y) = Q$ selects x and y each with probability $\frac{1}{2}$ to move first. If x is selected, then $h_2^x = h_2^y = h_1^x \cup \{x\}$ and $P(h_2^x, h_2^y) = x$. Agent x now chooses an action according to her bargaining strategy $B(x)$ that assigns an action to every possible history pair for which $P(h^x, h^y) = x$. If $P(h_2^x, h_2^y) = x$, x chooses some $\pi(x|y)$. As y observes only the remainder, $h_3^y = h_2^y \cup \{f(x,y) - \pi(x|y)\}$ while $h_3^x = h_2^x \cup \{\pi(x|y)\}$, and y responds according to $B(y)$ by choosing an action from the set {"accept", "reject", "walk away"}. If she chooses "accept" or "walk away", then (h_4^x, h_4^y) will be a terminal history pair. If she chooses "reject", then $h_4^x = h_3^x \cup \{\text{"reject"}\}$ and $P(h_4^x, h_4^y) = x$ who chooses from {"continue", "walk away"}. If x does not walk away, bargaining will continue in the next meeting with $P(h_5^x, h_5^y) = Q$, and so on.

Assumption 2 (Common expected delay). *A further meeting with the same agent happens at the same rate as a new meeting with another agent.*

That is, a time $1/\eta$ elapses in expectation before another bargaining round, so that both a meeting to continue bargaining and a meeting with a different agent happen at rate η. While there is nothing in our model that would cause these meeting rates to differ systematically, assumption 2 is obviously a simplification. Agents who are currently waiting for another round of bargaining cannot engage in search at the same time, so that they do not send a signal while waiting. This avoids random meetings with other agents. Those between bargaining rounds therefore form a set of agents who cannot be encountered on any marketplace. The meeting rates on all marketplaces are unaffected due to constant returns to scale in meeting.

All agents are risk-neutral, discount future utility at discount rate r (with $0 < r < \infty$), and seek to maximise the present discounted value (pdv) of their expected utility. Throughout the paper, 'payoff' refers to the pdv, not to the flow utility. Because of discounting, the time that elapses before a meeting makes meetings costly. In addition, we allow for explicit costs $c \geq 0$ that an agent incurs each time she attends a meeting.

Assumption 3 (Gains from trade). *The output produced in a match between two agents of the lowest type, discounted at effective discount rate $r + \delta$, can reimburse both agents' explicit costs of one meeting:*

$$2c \leq \frac{f(\underline{x}, \underline{y})}{r + \delta}$$

While explicit costs always have to be limited relative to the available payoffs to ensure agents' participation, note that this assumption is particularly mild. For example, we do not assume that each agent is in fact reimbursed in the event of a match, nor that match output is sufficient to reimburse the costs of the expected number of meetings before a match. With the inclusion of explicit costs in addition to discounting, our model thus features two kinds of search frictions. Finally, agents know everything except the true type of any other agent and, by consequence, the actual match output $f(x, y)$ they produce with another agent.

3 Equilibrium

3.1 Definition of equilibrium

Define $U^n(x)$ as the expected present value to x of being unmatched and engaging in search in marketplace n, and $W(x|y)$ as the expected present value to x from a match with y. Let the set $A(h^x, h^y)$ comprise of all combinations of bargaining strategies $(B(x), B(y))$ that lead to a SPE of the bargaining game given history pair (h^x, h^y) so that an agreement is reached in the meeting in question and agents match. Let us define an indicator function $\alpha(\cdot, \cdot)$ such that $\alpha(B(x), B(y)) = 1$ if $(B(x), B(y)) \in A(h^x, h^y)$ and 0 otherwise. Then the following *asset equation* expresses, for one marketplace, the expected return on being unmatched as the expected gain from a meeting net of search cost c:

$$rU^n(x) = \eta^n \left(-c + \int_\Theta \alpha(B(x), B(y)) \left[W(x|y) - U^n(x) \right] \psi(y|h^x = \{\tilde{y} \in R^n\}) dy \right) \tag{2}$$

where $\psi(y|h^x = \{\tilde{y} \in R^n\})$ is the density of y that x believes conditional on meeting y in marketplace n (after observing a required signal $\tilde{y} \in R^n$). Whenever x does not send a required signal herself, $rU^n(x) = 0$.

Let us define $U(x)$ as the value of $U^n(x)$ that x obtains in equilibrium. As is natural when signals are involved, we look for a *perfect Bayesian equilibrium* (PBE) of our model. We focus our attention on separating equilibria.[7] Because signals are costless all PBE will necessarily be cheap-talk equilibria. A steady-state PBE of our model, separating or not, requires that the flows into and out of matches balance for every type (a pointwise steady state), that all agents choose all their strategies optimally, and that agents' beliefs are consistent with all agents' actual equilibrium behaviour.

Definition 1 (Search equilibrium with signals). *In a steady-state PBE of our model, each agent $x \in \Theta$*

(i) engages in search if and only if $U(x) \geq 0$

(ii) optimally chooses a marketplace such that $\forall n$, $U(x) \geq U^n(x)$ given $B(x), B(y)$, and $(R^n)_{n=1}^N$, where $U^n(x)$ is determined by equation (2)

[7] Kübler et al. (2008) report experimental evidence suggesting that pooling equilibria never arise when some types can benefit from the effective use of signals.

(iii) *chooses her signal optimally as* $\arg\max_{\tilde{x}} rU^n(x)$ *given* $B(x), B(y), R^n$*, noting that* η^n *depends on* $\tilde{x}$ *as specified by equation (1)*

(iv) *chooses a stationary subgame-perfect bargaining strategy as* $\arg\max_{B(x)} rU^n(x)$ *given* $B(y)$ *and* R^n*, noting that* $W(x|y)$ *depends on the share obtained in bargaining*

(v) *holds beliefs that are formed using Bayes' rule where possible and that are consistent with equilibrium play: given an equilibrium history* h^x*,* $\psi(y|h^x) = u^n(y|h^x)$ *where* $u^n(y|h^x)$ *is the true density of* y *conditional on history* h^x *in marketplace* n

and the matching market is in a pointwise steady state: given $B(x)$*,* $B(y)$*, and* $(R^n)_{n=1}^N$*, for all* $x \in \Theta$

$$\delta \left[l(x) - \sum_{n=1}^{N} u^n(x) \right] = \eta \sum_{n=1}^{N} u^n(x) \int_{\Theta} \alpha(B(x), B(y)) u^n(y|h^x = \{\tilde{y} \in R^n\}) dy \qquad (3)$$

The expression under the integral in equation (3) is simply the probability that a meeting on marketplace n leads to a match. We can also formalise the condition that always applies to the creation of new marketplaces. As agents do not switch when indifferent, attracting agents to a new marketplace n^0 requires that there is a strictly positive mass of agents for whom $U^{n^0}(x) > U^n(x)$, $\forall n \neq n^0$.

A PBE only requires agents' beliefs to be consistent with equilibrium play, not with actions out of equilibrium. As is well known, a PBE can therefore depend on unreasonable off-equilibrium beliefs because these beliefs are never tested in equilibrium. Since unreasonable beliefs are not needed for any of our results, we rule out beliefs that are unreasonable in the sense of the Intuitive Criterion. To do this formally, let us call the choices of n, $\tilde{x}$, and $B(x)$ the 'grand strategy' of agent x, denoted $GS(x) = (n, \tilde{x}, B(x))$. Also define $BR(x|h^x)$ as the set of continuation strategies $GS(x|h^x)$ that are best responses for x. To apply the Intuitive Criterion as an equilibrium refinement, we have to define the notion of equilibrium domination in our model:

Definition 2 (Equilibrium domination). *Given a PBE of the model, the continuation strategy* $GS(x|h^x)$ *is equilibrium-dominated at history pair* (h^x, h^y) *if*

$$U(x) > \max_{GS(y|h^y) \in BR(y|h^y)} U(x|GS(x|h^x))$$

where $U(x|GS(x))$ *is the present value to* x *of searching with strategy* $GS(x|h^x)$*.*

The Intuitive Criterion then demands that the beliefs of y about off-equilibrium actions place probability 0 on any x who would have to pursue equilibrium-dominated strategies to reach the off-equilibrium history: $\psi(x|h^y) = 0$ if, at a history up to h^y, x would have had to play an equilibrium-dominated strategy $GS(x|h^x)$.

3.2 Putative equilibrium

We next propose that a particular separating equilibrium exists under a simple condition on the match production function $f(\cdot,\cdot)$. All we need is a weak and intuitive form of complementarity known as strict supermodularity (or increasing differences): the marginal product of one agent in a match is strictly increasing in the type of the other agent.[8]

Definition 3 (Supermodularity). *The match production function $f(\cdot,\cdot)$ is strictly supermodular if, for all $x_H > x_L$ and $y_H > y_L$,*

$$f(x_H, y_H) - f(x_L, y_H) > f(x_H, y_L) - f(x_L, y_L)$$

Further, we refer to the sorting with $x = y$ in all matches as *perfect positive assortative matching* (PPAM). We can now propose existence of the following particular PBE in our model:

Proposition 1 (Existence). *Let agents' beliefs place probability 0 on the occurrence of equilibrium-dominated actions. Then for any type distribution $L(x)$, strict supermodularity of $f(\cdot,\cdot)$ is necessary and sufficient for the existence of a separating PBE in which each agent $x \in \Theta$*

(i) engages in search: $U(x) \geq 0$

(ii) chooses a marketplace where she meets exclusively agents of her own type

(iii) signals truthfully: $\tilde{x} = x$

(iv) reaches a bargaining agreement in the first meeting and thus matches: $\alpha(B(x), B(y)) = 1$ for $x = y$

(v) correctly believes all signals to be truthful: $\psi(y|h^x = \{\tilde{y}\}) = u^n(y|h^x = \{\tilde{y}\}) = 1$ for all $y = \tilde{y}$

and the market is in pointwise steady state. The equilibrium matching is PPAM.

A proof of proposition 1 will thus establish that not only PAM, but even PPAM arises in our model under the same weak condition as in a frictionless model, although our model allows for two kinds of frictions. In a model with frictions only from discounting, Shimer and Smith (2000) establish PAM, albeit not PPAM, under the condition that the match production function $f(\cdot,\cdot)$, the logarithm of its first derivative, and the logarithm of its cross-partial derivative are all supermodular. These conditions are directly comparable to our condition and are unambiguously more restrictive: proposition 1 claims that our

[8] A stronger form of complementarity is strict log-supermodularity, which is defined using $\ln f(\cdot,\cdot)$ instead of $f(\cdot,\cdot)$ in definition 3, so that the proportional marginal product of one agent is increasing in the other agent's type.

model achieves PPAM with just the first of these conditions, which is also the most intuitive.

The next section proves proposition 1 through a series of lemmas. Each time, we separately consider a component of proposition 1, taking as given that all other components are indeed as specified in proposition 1. We verify for the component in question, as applicable, that it is optimal for agents to behave as specified, that a steady state results, and that beliefs are consistent with equilibrium play.

4 Existence proof for the putative equilibrium

4.1 Bargaining

We first derive the expected present values that various states carry in the putative equilibrium situation. Given that beliefs are consistent with equilibrium play, we have

$$\psi(y|h^x = \{\tilde{y} \in R^n\}) = u^n(y|h^x = \{\tilde{y} \in R^n\})$$

If x only meets agents of her own type and matches with them, then

$$u^n(y|h^x = \{\tilde{y} \in R^n\}) = \alpha(B(x), B(y)) = 0 \quad \forall y \neq x \tag{4}$$

Since every meeting in the putative equilibrium thus leads to match, the rate of matches equals the rate of meetings, and an agent effectively incurs costs c each time she matches. For the marketplace chosen in the putative equilibrium, equation (2) thus simplifies to

$$rU(x) = \eta\,[W(x|y) - c - U(x)] \tag{5}$$

with $y = x$. Next, the expected return on being matched with y is the expected flow utility while matched and the loss from match dissolution at rate δ:

$$rW(x|y) = \sigma(x|y) - \delta[W(x|y) - U(x)] \tag{6}$$

where $\sigma(x|y)$ denotes the expected share that x obtains when bargaining with y over the flow of match output $f(x, y)$,

$$\sigma(x|y) = \frac{1}{2}\pi(x|y) + \frac{1}{2}[f(x, y) - \pi(y|x)] \tag{7}$$

One can solve equation (5) for $U(x)$ and equation (6) for $W(x|y)$, then use the latter to substitute for $W(x|y)$ in the former to obtain

$$rU(x) = \beta[\sigma(x|y) - (r + \delta)c] \tag{8}$$

where $\beta = \eta/(r + \delta + \eta)$. Now suppose y has been randomly selected to move first in the bargaining game. In response to the share left for her, x can walk out and continue searching, which carries the value $U(x)$, or she can reject this share and wait for another round of bargaining, which carries a value $V(x)$. Note that the first mover y cannot hope to attain a better position than she currently has: at best, she will find herself as first mover again in a later meeting, be it with the same agent x or another agent of the same type. As delay is costly, y seeks to seize the opportunity and to ensure that x accepts her offer. In turn, x will accept any implicitly offered payoff $W^O(x|y)$ that satisfies

$$W^O(x|y) \geq \max[V(x), U(x)] \tag{9}$$

as she would otherwise reject the offer or walk out. When x moves first, y requires

$$W^O(y|x) \geq \max[V(y), U(y)] \tag{10}$$

When an offer is rejected, the same logic as before implies that the first mover in the second meeting seeks to ensure agreement, so that the second meeting can be expected to result in a match. By assumption 2, the second meeting happens at rate η, so that

$$rV(x) = \eta\,[W(x|y) - c - V(x)] \tag{11}$$

in the putative equilibrium. Solving equation (11) for $V(x)$ and equation (5) for $U(x)$ establishes that $V(x) = U(x)$. Hence the outside option $U(x)$ is not binding. As we also require bargaining strategies to be stationary, the game reduces to a variant of Rubinstein's (1982) set-up, and we have the following result:

Lemma 1 (Bargaining equilibrium). *Given truthful signals and given marketplace choices as in the putative equilibrium situation, the following stationary strategies form the unique subgame-perfect equilibrium (SPE) of the bargaining game:*

(i) agent x always proposes

$$\pi^*(x|y) = \left(1 - \frac{\beta}{2}\right) f(x, y) + \beta(r + \delta)c \tag{12}$$

for herself; when y proposes $\pi(y|x)$ for herself, x always accepts if and only if $\pi(y|x) \leq \pi^*(y|x)$

(ii) y always proposes $\pi^*(y|x) = \pi^*(x|y)$ for herself; when x proposes $\pi(x|y)$ for herself, y always accepts if and only if $\pi(x|y) \leq \pi^*(x|y)$

Agreement is reached in the first round of bargaining.

Proof. Equations (9) and (10) uniquely determine equilibrium, as follows. Whenever y moves first, her optimisation problem is

$$\max \pi(y|x) \quad \text{s.t.} \quad W^O(x|y) \geq \max[V(x), U(x)]$$

With $V(x) = U(x)$, the constraint becomes $W^O(x|y) - U(x) \geq 0$. When match output is $f(x,y)$ and y takes a share $\pi(y|x)$ from the pot for herself, $f(x,y) - \pi(y|x)$ would be left for x. Therefore, in analogy to equation (6),

$$rW^O(x|y) = f(x,y) - \pi(y|x) - \delta[W^O(x|y) - U(x)]$$

Solving this for $W^O(x|y)$, we find that $W^O(x|y) - U(x) \geq 0$ if and only if

$$\begin{aligned} f(x,y) - \pi(y|x) - rU(x) &\geq 0 \\ \Leftrightarrow f(x,y) - \pi(y|x) &\geq \beta[\sigma(x|y) - (r+\delta)c] \end{aligned}$$

where the second line uses equation (8). Next substituting for $\sigma(x|y)$ from equation (7) and rearranging,

$$[f(x,y) - \pi(y|x)]\,\phi \geq \pi(x|y) - 2(r+\delta)c \tag{13}$$

where $\phi = (1 - \frac{\beta}{2})/\frac{\beta}{2}$. As y raises $\pi(y|x)$ the left-hand side of equation (13) linearly falls, while the right-hand side stays constant. Hence this constraint will hold with equality for the equilibrium value of $\pi(y|x)$. When x moves first, the constraint is analogously found as

$$[f(x,y) - \pi(x|y)]\,\phi \geq \pi(y|x) - 2(r+\delta)c \tag{14}$$

As binding constraints, equations (13) and (14) are two equations in two unknowns, so that they determine a unique equilibrium. Noting the symmetry of these equations, we can infer that $\pi(x|y) = \pi(y|x)$. When we make this substitution in either equation and solve for $\pi(y|x)$, we obtain

$$\begin{aligned} \pi(y|x) &= \frac{\phi}{1+\phi} f(x,y) + \frac{2}{1+\phi}(r+\delta)c \\ &= \left(1 - \frac{\beta}{2}\right) f(x,y) + \beta(r+\delta)c \end{aligned}$$

which also equals $\pi(x|y)$. Because both first-mover shares have been derived under the constraint that the second mover accepts, agreement is reached in the first round of bargaining. Finally, for a proof that such an equilibrium is subgame-perfect, see Rubinstein (1982). □

The essence of the bargaining SPE is that each agent makes offers that leave the other indifferent, and each agent accepts offers that make her indifferent or better off: the first-mover takes a share $\pi^*(x|y)$ such that the second-mover share

$$f(x,y) - \pi^*(x|y) = \frac{\beta}{2} f(x,y) - \beta(r+\delta)c$$

is just enough to prevent the second mover from taking her outside option and from rejecting. The two indifference conditions, depending on who moves first, then together pin down a unique SPE. Finally, expected shares in the SPE are

$$\sigma(x|y) = \sigma(y|x) = \frac{1}{2}\pi^*(x|y) + \frac{1}{2}\left[f(x,y) - \pi^*(x|y)\right] = \frac{1}{2} f(x,y) \tag{15}$$

as one would expect when both outside options do not bind.

4.2 Participation and steady state

To ensure that all agents engage in search, c must not be so high that $U(x)$ becomes negative for some x, since each agent can obtain a payoff 0 by refusing to search.

Lemma 2 (Participation). *Assumption 3 is necessary and sufficient for all agents to engage in search.*

Proof. As match output is the only source of utility in the model, agents who do not engage in search obtain payoff 0. Then agent x will only engage in search if $U(x) \geq 0$. By equation (8), this requires

$$c \leq \frac{\sigma(x|y)}{r+\delta} \quad \Leftrightarrow \quad 2c \leq \frac{f(x,y)}{r+\delta}$$

using equation (15). If this holds for $f(\underline{x}, \underline{y})$, as stated in assumption 3, then it will also hold for the output generated in any other match because $f(x,y)$ is strictly increasing in x and y by assumption 1. □

Hence all unmatched agents engage in search, so that $u(x) > 0$ and, by consequence, $l(x) > 0$ for all $x \in \Theta$. Combining equations (3) and (4), we obtain the pointwise steady state in the putative equilibrium:

$$\delta \left[l(x) - \sum_{\mathcal{N}(x)} u^n(x) \right] = \eta \sum_{\mathcal{N}(x)} u^n(x) \quad \forall x \in \Theta \tag{16}$$

where $\mathcal{N}(x) \equiv \{n | R^n = \{x\}\}$ is the set of all marketplaces on which x meets exclusively her own type when signals are truthful.

4.3 Signals and beliefs

In this section, we examine whether any one agent has an incentive to unilaterally deviate from the putative equilibrium by choosing another signal. Therefore, we take as given that all other agents signal truthfully, that all believe signals to be truthful, as well as the other components of the putative equilibrium. We proceed by identifying first the conditions under which every agent prefers her match in the putative equilibrium (henceforth the *equilibrium match*) to any other match available to her. From this, we infer under which conditions there will be no incentive to deviate from the truthful signal.

There are two reasons why we need to worry about false signals in the first place. First, because true types are only privately observable, agents can perfectly imitate agents of other types by sending their signal and then bargaining as these types would. Second, agents might just imitate another type's signal and then renege on it in the meeting. Since search frictions make switching to another meeting costly, the other agent in the meeting might still accept the match. For example, consider a rather high type y_H who matches with x_H in the putative equilibrium. If y_H finds she has been lured into a meeting with a type $x_L < x_H$ by a false signal, she will nevertheless grudgingly accept whenever her share of $f(x_L, y_H)$ is not so far below her expected share of $f(x_H, y_H)$ that the costs of another meeting would be justified. Therefore, there can in principle be an incentive to send false signals.

We first compare the equilibrium match to matches with lower types. Without loss of generality, let us take the perspective of some agent with a type $x_H > \underline{x}$, so that lower types necessarily exist. We thus want to compare being matched with $y_H = x_H$ to being matched with $y_L < x_H$. The expected present discounted values of these matches to x are $W(x_H|y_H)$ and $W(x_H|y_L)$, respectively. In the spirit of the one-deviation principle, x reverts to the putative equilibrium strategies after the deviation. Hence, the asset equations for both $rW(x_H|y_H)$ and $rW(x_H|y_L)$ in analogy to equation (6) depend on the same $U(x)$ and thus differ only in the expected shares. Solving these two asset equations respectively for $W(x_H|y_H)$ and $W(x_H|y_L)$, we therefore find that

$$W(x_H|y_H) > W(x_H|y_L) \quad \Leftrightarrow \quad \sigma(x_H|y_H) > \sigma(x_H|y_L)$$

where $\sigma(x_H|y_H)$ and $\sigma(x_H|y_L)$ denote the expected share obtained by x_H in a match with y_H and y_L, respectively.

Thus suppose a type $x_H > \underline{x}$ signals to be of type x_L in order to meet a type y_L. Further suppose that agent x_H continues to behave like a type x_L so as to conform to the beliefs of y_L given that all other agents signal truthfully. Recall from section 4.1 that neither agent's signal implies a binding outside option. Hence the bargaining equilibrium described by lemma 1 will be reached in the first round of bargaining. Then the expected flow utility for x_H in the match with y_L is

$$\begin{aligned}
\sigma(x_H|y_L) &= \frac{1}{2}\left[f(x_H,y_L)-\frac{\beta}{2}f(x_L,y_L)+\beta(r+\delta)c\right] \\
&\quad +\frac{1}{2}\left[f(x_H,y_L)-\left(1-\frac{\beta}{2}\right)f(x_L,y_L)-\beta(r+\delta)c\right] \\
&= f(x_H,y_L)-\frac{1}{2}f(x_L,y_L) \qquad (17)
\end{aligned}$$

If x_H moves first (with probability $\frac{1}{2}$), she leaves a second-mover share to y_L as if output was $f(x_L, y_L)$ and keeps the rest of the actual output $f(x_H, y_L)$. If y_L moves first, y_L takes the first-mover share of $f(x_L, y_L)$ for herself and x_H obtains the actual remainder. In an equilibrium match, by contrast, x_H would obtain

$$\begin{aligned}
\sigma(x_H|y_H) &= \frac{1}{2}\left[\left(1-\frac{\beta}{2}\right)f(x_H,y_H)+\beta(r+\delta)c\right]+\frac{1}{2}\left[\frac{\beta}{2}f(x_H,y_H)-\beta(r+\delta)c\right] \\
&= \frac{1}{2}f(x_H,y_H) \qquad (18)
\end{aligned}$$

Comparing $\sigma(x_H|y_L)$ and $\sigma(x_H|y_H)$, we find the following:

Lemma 3 (Matches with lower types). *In the putative equilibrium, strict supermodularity of $f(\cdot,\cdot)$ is necessary and sufficient for any agent $x \in \Theta$ to strictly prefer the equilibrium match to a match with a lower type in which she perfectly imitates the lower type.*

Proof. Any agent $x_H > \underline{x}$ will strictly prefer the equilibrium match to a match with a lower type y_L if $W(x_H|y_H) > W(x_H|y_L)$. As argued above, this is equivalent to

$$\begin{aligned}
\sigma(x_H|y_H) &> \sigma(x_H|y_L) \\
\Leftrightarrow f(x_H,y_H)-f(x_H,y_L) &> f(x_H,y_L)-f(x_L,y_L) \qquad (19)
\end{aligned}$$

using equations (17) and (18). Next, note that we can write

$$f(x_H,y_L)=f(y_H,x_L)=f(x_L,y_H) \qquad (20)$$

where the first equality holds because $x_H = y_H$ and $y_L = x_L$, while the second equality holds by symmetry of $f(\cdot,\cdot)$ (see assumption 1). We therefore substitute $f(x_L, y_H)$ for $f(x_H, y_L)$ on the right-hand side of equation (19) only and rewrite it as

$$f(x_H,y_H)-f(x_L,y_H)>f(x_H,y_L)-f(x_L,y_L)$$

By definition 3, strict supermodularity of $f(\cdot,\cdot)$ is necessary and sufficient for this to hold. Finally, the type $\underline{x}$ matches with $\underline{y}$ in the putative equilibrium, so that a lower type than in the equilibrium match does not exist in this case. $\square$

Now suppose that x_H has signalled to be of type x_L, has thus met a type y_L, but now wants to renege on the signal. We will find below that x_H has to let at least one round of bargaining fail to actually convince y_L of her true type. Here we ask whether reneging could possibly make the deviation to a match with a lower type worthwhile. By considering the hypothetical extreme case that y_L instantly observes the true type x_H, we obtain a negative answer:

Lemma 4 (Reneging in matches with lower types). *In the putative equilibrium, any agent $x \in \Theta$ would strictly prefer the equilibrium match to a match with a lower type if types were instantly observable in meetings.*

Proof. We want to prove that some $x_H > \underline{x}$ will strictly prefer the equilibrium match to a match with a type $y_L < x_H$ when the true type x_H is observed before bargaining begins. In analogy to equations (9) and (10), x_H and y_L would respectively accept if

$$W^O(x_H|y_L) \geq \max[V(x_H), U(x_H)], \quad W^O(y_L|x_H) \geq \max[V(y_L), U(y_L)]$$

where $V(x_H)$ and $U(x_H)$ are determined by

$$rV(x_H) = \eta[W(x_H|y_L) - c - V(x_H)], \quad rU(x_H) = \eta[W(x_H|y_H) - c - U(x_H)] \qquad (21)$$

and similarly for $V(y_L)$ and $U(y_L)$. For comparison with agents' equilibrium matches, we recall their expected equilibrium shares as

$$\sigma(x_H|y_H) = \frac{1}{2}f(x_H, y_H), \quad \sigma(y_L|x_L) = \frac{1}{2}f(x_L, y_L)$$

by equation (15), so that equation (8) implies

$$rU(x_H) = \beta\left[\frac{1}{2}f(x_H, y_H) - (r+\delta)c\right], \quad rU(y_L) = \beta\left[\frac{1}{2}f(x_L, y_L) - (r+\delta)c\right] \qquad (22)$$

First suppose that the outside option of x_H does not bind, $V(x_H) \geq U(x_H)$. With $U(x_H) > U(y_L)$ by equation (22), the outside option of y_L cannot bind when that of x_H does not bind, so that $V(y_L) > U(y_L)$. We then have the same situation as in lemma 1, except for labels. We thus know from lemma 1 that $\pi^*(x_H|y_L) = \pi^*(y_L|x_H)$ and therefore

$$\sigma(x_H|y_L) = \sigma(y_L|x_H) = \frac{1}{2}f(x_H, y_L)$$

When outside options do not bind, the bargaining game is thus still symmetric despite different types. Because $\sigma(x_H|y_H) > \sigma(x_H|y_L)$ we can conclude that x_H strictly prefers her equilibrium match when her outside option does not bind. Now suppose the outside option of x_H binds, $V(x_H) < U(x_H)$. Solving equation (21) respectively for $V(x_H)$ and $U(x_H)$, we obtain

$$V(x_H) = \frac{\eta[W(x_H|y_L) - c]}{r + \eta}, \quad U(x_H) = \frac{\eta[W(x_H|y_H) - c]}{r + \eta} \tag{23}$$

so that $V(x_H) < U(x_H)$ if and only if $W(x_H|y_L) < W(x_H|y_H)$. Hence x_H strictly prefers her equilibrium match also when her outside option binds. $\square$

Hence even if x_H could immediately convince y_L of her true type, x_H would strictly prefer the equilibrium match, as she does when she would have to imitate some lower type. Based on lemmas 3 and 4, we show below that higher types never have an incentive to deviate from the putative equilibrium to matches with lower types if $f(\cdot,\cdot)$ is supermodular, for any beliefs that lower types might hold about deviants. In turn, whenever a deviant causes bargaining to fail, the other agent thus knows that she faces a strictly lower type: for a weakly higher type, a deviation would be equilibrium-dominated. Next recall from assumption 2 that in expectation the same time $1/\eta$ elapses before another round of bargaining as before a meeting with a different agent. When bargaining fails due to a deviant, the other agent (whose type was observable from a truthful signal) now prefers by lemma 4 to meet a different agent: she simply chooses her equilibrium match rather than a match with some strictly lower type after the same expected delay.[9] As this insight is central to our argument, we formally state and prove it:

Lemma 5 (Equilibrium-dominated strategies). *Let agents' beliefs place probability 0 on the occurrence of equilibrium-dominated actions, let $f(\cdot,\cdot)$ be strictly supermodular, and consider a meeting between some x and y in the putative equilibrium. If x deviates such that bargaining fails, y will correctly believe to face a lower type and will choose to walk away.*

Proof. We first establish that any agent $x_H > \underline{x}$ always prefers, for any beliefs of $y_L \leq x_H$, her equilibrium match to a deviation such that she meets a weakly lower type with whom bargaining fails. We have to consider all possible beliefs held by y_L about the potential match output $f(x, y)$ when bargaining fails:

(i) $g(x, y_L|h^y) = f(x_H, y_L)$ so that y_L believes to face the true type x_H. By lemma 4, x_H then strictly prefers her equilibrium match.

(ii) $g(x, y_L|h^y) > f(x_H, y_L)$ so that y_L overestimates potential match output. By the same argument as in the proof of lemma 4, y_L does not believe the outside option of x to bind: if it did, x would have had to pursue an equilibrium-dominated

[9]This logic will also apply if a deviation is only detected after the start of the match: it can only be detected when agents' initial bargaining agreement breaks down, so that there is no basis for production when agents wait for the new round of bargaining required for renegotiation.

strategy. Given that outside options do not bind, lemma 1 applies (which does not require equal types): y_L optimally pursues a bargaining strategy with first-mover and second-mover shares respectively given by

$$\pi(y|x) = \left(1 - \frac{\beta}{2}\right) g(x, y_L|h^y) + \beta(r+\delta)c, \quad f(x,y) - \pi(x|y) = \frac{\beta}{2} g(x, y_L|h^y) - \beta(r+\delta)c$$

Since $0 < \beta < 1$, both shares are unambiguously increasing in $g(x, y_L|h^y)$. As $g(x, y_L|h^y) > f(x_H, y_L)$, y_L demands higher shares than under (i). Because x_H strictly prefers her equilibrium match under (i), she still prefers her equilibrium match when y_L is more demanding.

(iii) $f(x_H, y_L) > g(x, y_L|h^y) > f(x_L, y_L)$ so that y_L underestimates potential match output but still believes to face a higher type. By lemma 3, x_H will strictly prefer her equilibrium match if her type is underestimated and believed to be x_L, as she then either has to imitate x_L or cause bargaining to fail again. By the same argument as under (ii), x_H then still prefers her equilibrium match when y_L believes to face a higher type than x_L and thus demands higher shares.

(iv) $g(x, y_L|h^y) = f(x_L, y_L)$ so that y_L believes to face the same type as her own type. If $x_H > y_L$, the argument under (iii) applies. If $x_H = y_L$, lemma 1 implies that x_H would have preferred reaching a bargaining agreement with y_L (which coincides with the equilibrium match).

(v) $g(x, y_L|h^y) < f(x_L, y_L)$ so that y_L believes to face a strictly lower type. As we consider a unilateral deviation from the putative equilibrium by x_H, y_L has sent a truthful signal, so that her type has been disclosed to x_H. By lemma 4, y_L then strictly prefers her equilibrium match to a match with x_H who is perceived as a lower type. Hence y_L walks away to meet another agent (as the expected delay is the same), and x_H does not gain from the deviation.

Hence the deviation in question is equilibrium-dominated for weakly higher types than y_L. Now requiring that agents' beliefs place probability 0 on the occurrence of equilibrium-dominated actions, y_L must believe to face a strictly lower type when bargaining fails, $g(x, y_L|h^y) < f(x_L, y_L)$. By the argument under (v), y_L thus walks away when bargaining fails. As we supposed that $y_L \leq x_H$, the entire reasoning applies to any $y \in \Theta$. □

Let us now turn to the incentive for lower types to deviate to a match with a higher type. Without loss of generality, consider some agent with a type $x_L < \bar{x}$, so that higher types necessarily exist. Now we want to compare being matched with an exactly corresponding type $y_L = x_L$, as in the equilibrium match, to being matched with a higher type $y_H > x_L$. The lower type x_L has two possibilities: she can either perfectly imitate x_H, or she can signal to have type x_H in order to meet y_H but then renege on the signal. We have just shown that, if x_L reneges such that bargaining fails, y_H will walk away and x_L does not gain from the deviation. If x_L herself walks away, she does not gain from the deviation either. Yet x_L does not have a third way to renege on her signal. In particular, should x_L simply claim to have a lower type at some point during bargaining, y_H will have no reason to take such a claim seriously:[10]

Lemma 6 (Irrelevant communication). *Any claims by agents to have a lower type than signalled are not credible.*

Proof. We have to show that an agent x who does not have a lower type than y also has an incentive to make such claims. In particular, consider an agent of type $x = y$ with whom y matches in the putative equilibrium. From lemma 1, the first-mover and second-mover shares obtained by y in the equilibrium match are respectively

$$\pi^*(y|x) = \left(1 - \frac{\beta}{2}\right) f(x,y) + \beta(r+\delta)c, \quad f(x,y) - \pi^*(x|y) = \frac{\beta}{2} f(x,y) - \beta(r+\delta)c$$

As argued before, both shares are increasing in $f(x,y)$. By assumption 1, $f(\cdot,\cdot)$ is increasing in both arguments. Hence agent x has an incentive to downplay her type in order to make y propose and accept lower shares for herself. $\square$

In sum, any agent $x_L < \bar{x}$ strictly prefers her equilibrium match to a deviation such that she meets a higher type and reneges on her signal. The only way that x_L can match with y_H is therefore to perfectly imitate x_H. Then the expected flow utility for x_L is

$$\begin{aligned}
\sigma(x_L|y_H) &= \frac{1}{2}\left[f(x_L,y_H) - \frac{\beta}{2} f(x_H,y_H) + \beta(r+\delta)c\right] \\
&\quad + \frac{1}{2}\left[f(x_L,y_H) - \left(1 - \frac{\beta}{2}\right) f(x_H,y_H) - \beta(r+\delta)c\right] \\
&= f(x_L,y_H) - \frac{1}{2} f(x_H,y_H) \qquad (24)
\end{aligned}$$

If x_L moves first, she has to leave y_H the second-mover share of $f(x_H,y_H)$ to avoid being found out and can thus take whatever is left of the actual output $f(x_L,y_H)$. If y_H moves first, y_H takes the first-mover share of $f(x_H,y_H)$ for herself and x_L obtains the actual remainder. By contrast, the expected flow utility for x_L from her equilibrium match would be

$$\sigma(x_L|y_L) = \frac{1}{2} f(x_L,y_L) \qquad (25)$$

A comparison of $\sigma(x_L|y_H)$ and $\sigma(x_L|y_L)$ yields the following result:

[10] One can also show that any claims to have a higher type will not be taken seriously either if agents place probability 0 on the occurrence of equilibrium-dominated actions: by lemma 5, a type would thereby claim to have taken an equilibrium-dominated action.

Lemma 7 (Matches with higher types). *In the putative equilibrium, strict supermodularity of $f(\cdot,\cdot)$ is necessary and sufficient for any agent $x \in \Theta$ to strictly prefer the equilibrium match to a match with a higher type in which she perfectly imitates the higher type.*

Proof. Any agent $x_L < \bar{x}$ will strictly prefer the equilibrium match to a match with a higher type y_H if $W(x_L|y_L) > W(x_L|y_H)$, which is equivalent to

$$\begin{aligned} \sigma(x_L|y_L) &> \sigma(x_L|y_H) \\ \Leftrightarrow f(x_H, y_H) - f(x_L, y_H) &> f(x_L, y_H) - f(x_L, y_L) \end{aligned}$$

using equations (24) and (25). By equation (20), we can replace $f(x_L, y_H)$ on the right-hand side by $f(x_H, y_L)$. Hence strict supermodularity is necessary and sufficient for this equation to hold. Finally, for the type $\bar{x}$, a higher type than in the equilibrium match does not exist. □

We have thus identified conditions under which each agent $x \in \Theta$ strictly prefers her equilibrium match to a deviation to any other match. Also recall that matching rates are the same across marketplaces, so that matching rates do not reverse this preference. Corollary 1 collects the implications of this section for agents' signals and beliefs:

Corollary 1 (Truthful signals). *Let agents' beliefs place probability 0 on the occurrence of equilibrium-dominated actions. Then strict supermodularity of $f(\cdot,\cdot)$ is necessary and sufficient for each agent $x \in \Theta$ in the putative equilibrium to strictly prefer a truthful signal $\tilde{x} = x$. The only beliefs consistent with truthful signals are $\psi(y|h^x = \{\tilde{y}\}) = u^n(y|h^x = \{\tilde{y}\}) = 1$ for all $y = \tilde{y}$.*

Proof. Choose and fix some arbitrary unmatched agent with a type $x \in \Theta$ and call this type x_E. To this exemplary type, a type y_E exactly corresponds. Given the choices of marketplace and bargaining strategy in the putative equilibrium, an agent of type x_E matches with an agent of type y_E at rate η unless there is a deviation. Further, types x_E and y_E also meet unless there is a deviation. Hence, if x_E does not deviate, but sends a truthful signal $\tilde{x}_E = x_E$, it must be that $\tilde{x}_E \in R^n$. Given that agents meet exclusively their own type in the putative equilibrium, $|R^n| = 1$. Then we have $\tilde{x}'_E \notin R^n$ for any non-truthful signal $\tilde{x}'_E \neq x_E$. In words, a deviating x_E cannot meet y_E, and therefore cannot match with y_E. Hence x_E has to signal truthfully to obtain her equilibrium match, which she strictly prefers to a deviation by the preceding lemmas. Because type x_E was arbitrarily chosen, the reasoning extends to any type $x \in \Theta$. Finally, if all signals are truthful, then $u^n(y|h^x = \{\tilde{y}\}) = 1$ for all $y = \tilde{y}$, and agents' beliefs can only be consistent if $\psi(y|h^x = \{\tilde{y}\}) = 1$ for all $y = \tilde{y}$. □

Hence each agent $x \in \Theta$ finds it optimal to signal truthfully because this is the only way to obtain her equilibrium match, which she prefers to a deviation. As all agents therefore indeed signal truthfully, only beliefs that signals are truthful can be consistent with equilibrium play.

In conclusion, this section has presented an extensive but essentially simple reasoning. We found that higher types will never deviate from the putative equilibrium to match with lower types if $f(\cdot, \cdot)$ is supermodular. An agent who detects a deviation should therefore believe to face a lower type. When she can choose between continued bargaining with a lower type and her equilibrium match, she prefers the latter. Lower types can then only match with higher types by imitating them, but they will not gain from such a deviation if $f(\cdot, \cdot)$ is supermodular. Then each agent prefers not to deviate and consequently finds it optimal to signal truthfully.

4.4 Choice of marketplace

In this section, we take it as given that signals are truthful and concentrate on agents' optimal choice of marketplace. Consider three types x_L, x_M, and x_H, with $\underline{x} \leq x_L < x_M < x_H \leq \bar{x}$. Suppose these types search in the same marketplace, so that each of them can meet with y_L, y_M, or y_H. We know from lemma 4 that each x_H would prefer a match with y_H to a match with y_M or y_L. Using the truthful signals, the agents of type x_H can profitably set up a new marketplace where $R^n = \{x_H\}$ so that agents of type x_H exclusively meet each other. In the initial marketplace, they would also meet less preferred types, which is not offset by any advantage in meeting rates.[11] By setting up an exclusive marketplace, the congestion externality imposed by less preferred types is avoided (see Jacquet and Tan (2007) for details of this logic). Intuitively, it is best to attend the marketplace where one meets only one's most preferred type and where no other types stand in the way.

Given that signals are truthful, the remaining types x_M and x_L can no longer meet with y_H, as they would have to send a false signal to join the marketplace where y_H can be met. Among the possible matches, x_M prefers by lemma 4 the match with y_M, so that all agents of type x_M now set up an exclusive marketplace with $R^n = \{x_M\}$, leaving the initial marketplace to the agents of type x_L. This logic applies to any marketplace with different types, so that all types have their own exclusive marketplaces in equilibrium. (We will generalise this logic in section 5.3 to show that it does not only apply in the putative equilibrium, but always when signals are truthful.) There may be several exclusive marketplaces for the same type in equilibrium ($|\mathcal{N}(x)| \geq 1$), as none of our conclusions is affected by their exact number due to constant returns to scale in meeting. Formally, each agent x thus optimally chooses a marketplace $n \in \mathcal{N}(x)$ and thereby obtains the

[11] We argue in the appendix that the same conclusion would hold if marketplaces and meeting functions were two-sided, as long as agents can choose which side to join.

present value $U(x) \geq U^n(x)$, $\forall n$. Given the optimal bargaining strategies in lemma 1, every meeting in the putative equilibrium then leads to a match, as one would expect when truthful signals allow agents to know everything in advance.

4.5 Discussion

By way of summary, the preceding subsections have each shown a component of the putative equilibrium situation to hold, given the other components. We thus found the pointwise steady state in the PBE. Given a supermodular match production function and beliefs that rule out equilibrium-dominated actions, agents seek to meet only exactly corresponding types. All agents then signal their types truthfully and correctly believe that all other agents signal truthfully. They match only with exactly corresponding types, so that the resulting equilibrium matching of agents is PPAM. Our model thus leads to PPAM under the same weak condition as in Becker's (1973) frictionless model, despite two kinds of search frictions.

Let us clarify the role that supermodularity plays for our results. Since types are privately observable and nothing keeps agents from imitating other types, an agent may match incognito with any type she likes. However, because actual match output then differs from the match output suggested by the signals, the deviant will only remain incognito if she bears the necessary adjustment: she has to give up as much of her own share as is necessary to bridge the gap when actual output is lower (otherwise bargaining fails and the other agent walks out), and she quietly pockets the excess output when actual output is higher. To explain why a lower type x_L would then not match incognito with a higher type $y_H > x_L$, supermodularity is key: $f(x_H, y_H) - f(x_L, y_H)$ is the necessary adjustment when y_H otherwise matches with x_H in equilibrium, while $f(x_L, y_H) - f(x_L, y_L)$ is the extra output produced in comparison to the equilibrium match of x_L. With $f(x_L, y_H) = f(x_H, y_L)$ in the latter, as established by equation (20), the necessary adjustment will exceed the extra output if $f(\cdot, \cdot)$ is strictly supermodular. From the perspective of a lower type, any possible gains from higher output with a higher type are therefore more than outweighed by the costs from adjustment.

More generally, supermodularity has in effect assumed the role of a single-crossing property in our model. This way, we obtain a fully separating equilibrium even though signals are costless. Separation is therefore not driven by cost differences, but by differences in marginal productivity of the same agent over different matches. However, these differences in themselves do not deliver a separating equilibrium for a general type distribution and for unrestricted sets of signals that agents can choose from, as we found in previous versions of this paper. Yet under the realistic assumption that true types are always only privately observable, as in this model, supermodularity does deliver full separation for a general type distribution and unrestricted signal sets.

Sorting in the separating PBE is driven by a logic apparently new to the literature.

In intuitive terms, agents are effectively bound by their signal, so that a low type can only choose between "being herself" in a match with an equally low type and behaving like a high type in a match with a high type. The most desirable option, "being herself" in a match with a high type, is not available. When behaving like a high type implies disproportionate sacrifices due to supermodularity, low types prefer matches with equally low types.

To put this into a real-world labour market context, suppose a low-skilled worker faces the choice between working at McDonalds and working at McKinsey. While McKinsey would presumably pay a significantly higher wage, the low-skilled worker would have to perform at McKinsey like her high-skilled colleagues. It is easy to imagine that the sheer effort and the extra hours needed to reach this performance outweigh the benefit of a higher salary, so that the low-skilled worker actually prefers working at McDonalds. Whenever this is the case, McKinsey does not even have to check whether applications are truthful, but would still meet only those who claim to be high-skilled. Indeed, this seemingly paradoxical behaviour that our model rationalises appears to be widespread practice in recruiting. It also makes sense in practice to dismiss applicants who are known to have lied in their application: as our model suggests, it appears easier to find a replacement rather than to disentangle lies from truth for such applicants, thereby determine their actual qualifications, and then adjust the job requirements to fit these qualifications. The next section discusses key properties of the separating equilibrium.

5 Equilibrium properties

5.1 Efficiency

The separating equilibrium we have identified is efficient in a number of important respects. First and foremost, search costs are minimised, both for each agent individually and overall: in equilibrium, truthful signals allow each agent to ensure that no meeting is wasted, but that every meeting she attends results in a match. Hence, whenever an agent x searches, she attends a meeting and also matches at the first opportunity, that is, after an expected search time of $1/\eta$. This is the minimum delay because a meeting necessarily precedes a match. In a standard search model, each match would typically be preceded by a number of unsuccessful meetings, and only by chance will the first meeting of an agent result in a match. Therefore, search costs in standard search models are at least as high from the individual perspective as in our model with truthful signals, and much higher in expectation as well as on aggregate. Second, note that all agents match in equilibrium so that there is no unrealised surplus left in the form of unmatched agents, on the contrary:

Corollary 2 (Output efficiency). *If the match production function is supermodular, PPAM will maximise aggregate output.*

A proof of this can be found in the appendix of Becker (1973). Standard search models, be it with or without supermodularity of the match production function, do in general not maximise aggregate match output, as they lead to a certain degree of mismatch instead of PPAM. Finally, among the types with whom a match is mutually acceptable, agents in the equilibrium we found only meet the type they most prefer, and thus only match with this type. This again contrasts starkly with standard search models, where the type that an agent expects in a match is the expectation over the types with whom a match is mutually acceptable, not the most preferred one of them.

5.2 Stability

In this section, we examine whether the equilibrium matching we found is a *stable matching*. Associated with any matching is a vector $s = (\sigma_1, \ldots, \sigma_I)$ specifying for each matched agent i the expected flow utility σ_i obtained in this matching, where I is the total number of matched agents. (This is a slight abuse of notation because agents in our model are not countable.) The following definition is standard:

Definition 4 (Stable matching). *A matching is stable if the associated vector s of expected flow utilities satisfies $\sigma_i \geq 0$, $\forall i$ and there is no match between any agents x_i and y_j such that $\sigma(x_i|y_j) > \sigma_i$ and $\sigma(y_j|x_i) > \sigma_j$.*

While Becker (1973) used the concept of the *core* in his seminal work, we argue in the appendix that, if a standard definition of the core is adapted to our model, the set of matchings in the core and the set of stable matchings will coincide. A proof that PPAM is a stable matching would thus also prove that it is a matching in the core of our model. We find that supermodularity of the match production function is a sufficient condition here for PPAM to be a stable matching:

Corollary 3 (Stability of PPAM). *Given truthful signals and individual choices as in the putative equilibrium situation, strict supermodularity of $f(\cdot, \cdot)$ is necessary and sufficient for perfect positive assortative matching to be a stable matching.*

Proof. Suppose to the contrary that PPAM is not a stable matching. Then there must be a match between unequal types that is preferred by both types to matches with exactly corresponding types. However, by the proof of lemma 5, matching with a lower type is an equilibrium-dominated action for the higher type in any match between unequal types, provided $f(\cdot, \cdot)$ is supermodular. Hence the higher type would then always prefer a match with an exactly corresponding type so that a match between unequal types that is preferred by both does not exist. Finally, lemma 1 implies together with assumption 1 that $\sigma_i \geq 0$ $\forall i$ in PPAM, so that no agent prefers to be single either. $\square$

A stable matching is a most unusual result in a model with search frictions. In standard search models, agents cannot search selectively and might thus be matched with any type from a certain range of types. Of course, many of these types are only accepted because search frictions make continued search undesirable. A stable matching cannot be expected to arise under such circumstances and is very unlikely to arise by chance whenever the number of different types is not trivially small. Stable matchings normally only arise in frictionless models. We attribute the reason that a stable matching is achieved here despite search frictions to the signals: they allow agents to pursue their search almost as if there were no search frictions.

Adachi (2003) shows for a fairly general search model that the set of equilibria will reduce to the set of stable matchings in a model à la Gale and Shapley (1962) if search frictions become negligible. Our result in this section qualifies this finding in so far as search frictions remain in our model because agents do not meet immediately ($\eta < \infty$) and incur costs from meetings ($c \geq 0$), and yet a stable matching results. This suggests that frictions do not prevent a stable matching in a search model as long as they do not keep agents from meeting only specifically chosen types. Intuitively, arbitrarily high frictions do not have any effect when agents find ways to match like in a frictionless environment. If search costs are only incurred at the end of an otherwise costless search process, the matching will be as in the absence of any costs, provided agents still participate.

5.3 Uniqueness

While we have shown that the putative equilibrium situation exists as a separating equilibrium, there might also be other separating equilibria. However, we find in this section that this separating equilibrium is unique. We know that signals are truthful in any separating equilibrium.[12] We have also found that agents in the putative equilibrium join marketplaces where they meet exclusively their own type. It turns out that agents *always* divide themselves into exclusive marketplaces when signals are truthful:

Lemma 8 (Market segmentation). *Agents will meet only their own type in any separating equilibrium.*

Proof. Suppose there is at least one marketplace in which, with truthful signals, agents do not only meet their own type, so that two or more types meet. Focus on the lowest type y_L in this marketplace. This type must be the most preferred feasible type of some other type in the marketplace, otherwise the other types would exclude y_L from the marketplace to reduce congestion. Concretely, y_L must be the most preferred type of some higher type x_H, since y_L is the lowest type in the marketplace.

[12]We ignore separating equilibria where signals are not truthful yet still informative because they are linked by a one-to-one mapping to agents' true types, and this mapping forms the basis of agents' correct beliefs. Such equilibria would only be variants of equilibria with truthful signals.

We will show that this cannot be a separating equilibrium. When x_H and y_L bargain over $f(x_H, y_L)$ (known to both following truthful signals), we have $V(x_H) \geq U(x_H)$ (otherwise, this would imply that x_H does not prefer y_L most). We do not determine $U(y_L)$; suffice to note that $W^O(y_L|x_H) \geq \max[V(y_L), U(y_L)]$ for y_L to accept when x_H moves first. As the outside option of x_H does not bind, $W^O(y_L|x_H)$ (and thus the second-mover share for y_L) is non-decreasing in $U(y_L)$. Similarly, the first-mover share for y_L is non-decreasing in her outside option $U(y_L)$. $V(y_L)$ is given when x_H and y_L bargain. Hence, $U(y_L) \leq V(y_L)$ is the more favorable case for x_H. Focus on this case and note

$$rV(x_H) = \eta\left[W(x_H|y_L) - c - V(x_H)\right], \quad rV(y_L) = \eta\left[W(y_L|x_H) - c - V(y_L)\right]$$

which is the same as we found for $V(x)$ in the putative equilibrium above (see equation (11)), except for labels. Hence we can apply lemma 1, which only relied on outside options not being binding, to the more favorable case for x_H here. Therefore, $\pi^*(x_H|y_L) = \pi^*(y_L|x_H)$ and consequently

$$\sigma(x_H|y_L) = \frac{1}{2} f(x_H, y_L) \tag{26}$$

in this case (the more favorable case for x_H).

Now consider two agents x_H and y_H who match with y_L and x_L, respectively, and let us examine whether agents x_H and y_H both rather prefer the match with each other. First, suppose that the outside option of x_H does not bind, $V(x_H) \geq U(x_H)$, when x_H and y_H bargain over $f(x_H, y_H)$. Then also $V(y_H) \geq U(y_H)$ because $x_H = y_H$. This bargaining situation is therefore completely symmetric, implying $\pi^*(x_H|y_H) = \pi^*(y_H|x_H)$. Alternatively, one can write down the asset equations for $V(x_H)$ and $V(y_H)$ and note that lemma 1 again applies. We thus obtain $\sigma(x_H|y_H) = \frac{1}{2} f(x_H, y_H)$ when outside options are not binding. In this case, x_H strictly prefers the match with y_H to the match with y_L even compared to the more favorable case for x_H, as $\sigma(x_H|y_H) > \sigma(x_H|y_L)$ because $f(\cdot, \cdot)$ is increasing in its arguments by assumption 1. By the same reasoning, y_H also prefers the match with x_H to the match with x_L in this case.

Next, we show that the outside option of x_H never binds when x_H and y_H bargain. If it did, so that $V(x_H) < U(x_H)$, then this would imply $W(x_H|y_H) < W(x_H|y_L)$ as in the proof of lemma 4. We show this by contradiction: suppose it binds, $V(x_H) < U(x_H)$, then also $V(y_H) < U(y_H)$ and moreover $U(x_H) = U(y_H)$ because $x_H = y_H$. Provided there are gains from trade, this completely symmetric bargaining situation leads to $\sigma(x_H|y_H) = \frac{1}{2} f(x_H, y_H)$ (as can be verified by adapting the proof of lemma 1, maintaining $U(x_H)$ as an unknown). With this $\sigma(x_H|y_H)$ and $\sigma(x_H|y_L)$ from equation (26), we have

$$\begin{aligned} rW(x_H|y_H) &= \frac{1}{2} f(x_H, y_H) - \delta[W(x_H|y_H) - U(x_H)] \\ rW(x_H|y_L) &= \frac{1}{2} f(x_H, y_L) - \delta[W(x_H|y_L) - U(x_H)] \end{aligned}$$

in analogy to equation (6), so that $W(x_H|y_H) > W(x_H|y_L)$ because $f(x_H, y_H) > f(x_H, y_L)$. In analogy to equation (23) and noting that x_H meets y_L at best at rate η in a mixed marketplace, we can write

$$V(x_H) = \frac{\eta[W(x_H|y_H) - c]}{r + \eta}, \quad \sup U(x_H) = \frac{\eta[W(x_H|y_L) - c]}{r + \eta}$$

when x_H and y_H bargain. This implies $V(x_H) > \sup U(x_H)$, yielding the contradiction.

We have thus shown that x_H and y_H always prefer to match with each other than to match with y_L and x_L, respectively. To reduce congestion, y_L and x_L would therefore be excluded from the marketplace, which is possible because agents signal truthfully. Hence the proposed marketplace with two or more types cannot exist in a separating equilibrium. □

Since agents only meet their own type in any separating equilibrium, they can only match with their own type. Therefore, PPAM is the unique matching that may result in any separating equilibrium of our model. We can now conclude more comprehensively:

Proposition 2 (Uniqueness). *Whenever it exists, the separating equilibrium described by the putative equilibrium is unique up to off-equilibrium beliefs.*

No formal proof is needed, as this follows from our earlier results. We know from lemma 8 that any separating equilibrium would have to lead to PPAM, so that other separating equilibria would have to differ in agents' signals, their beliefs, their choice of marketplace, their bargaining strategy, or in the steady state. However, there is only one way for each agent to signal truthfully. When signals are truthful, lemma 8 means that the rule for choosing the marketplace determined in section 4.4 is uniquely optimal. Section 4.1 identified the unique bargaining SPE in this context. Then only one specification of beliefs about equilibrium actions will be compatible with these choices. Finally, the steady state conditions uniquely determine a density for the unmatched agents of each type. Hence, other separating equilibria can only differ in beliefs about off-equilibrium actions.

6 Conclusions

This paper has introduced costless signals into search with transferable utility and frictions. We find a unique separating equilibrium characterised by perfect positive assortative matching, minimised search duration and search costs, and maximised overall match output. These efficiency benchmarks are virtually never met by random search models because frictions lead to lengthy search and to some mismatch. In our model, signals allow agents to avoid this, so that frictions do not constrain efficiency in this way. The role of signals reflects the pervasive use of effective communication in real-world matching markets that facilitates search.

Positive assortative matching in the separating equilibrium depends just on supermodularity of the match production function. Supermodularity simultaneously ensures sufficient complementarity and replaces a single-crossing condition that is normally needed for truthful signals. Our model thereby proposes a solution to the paradox in Shimer and Smith (2000): supermodularity as such is unambiguously a weaker condition than the conditions they identified. In fact, our particularly mild condition does not merely ensure positive assortative matching despite discounting and explicit search costs, but even perfect positive assortative matching.

We conclude that positive assortative matching as an empirical regularity can be replicated by search models under plausible conditions. This is demonstrated by the model in this paper for the most extreme form of sorting; less pronounced sorting can presumably be obtained by adding random noise to various choices in the model. Compared to standard search models, a model with more information in the search process thus appears to generate sorting more easily. We have found this for the realistic case that agents control the additional information flows and may manipulate them strategically. Our results would therefore explain why sorting is much more frequent across many different real-world matching markets than one would expect, given the findings in previous models.

Sorting is likely to become more important as technological and societal progress favours specialisation. At the same time, many new means have appeared of effective and rapid communication that might, as in our paper, support sorting. The combination of these two developments offers ample scope for further research.

A Two-sided marketplaces

Suppose marketplaces have two sides, so that meetings are described by a symmetric function $m(\cdot,\cdot)$ with constant returns to scale and $m(0,\cdot) = m(\cdot,0) = 0$. Let λ_x^n denote the mass of agents on one side of marketplace n and λ_y^n the mass on the other. Then the flow of meetings equals

$$m\left(\lambda_x^n, \lambda_y^n\right) = m\left(\lambda_y^n, \lambda_x^n\right) \leq \min\left[\lambda_x^n, \lambda_y^n\right]$$

and the meeting rates for the two sides are, respectively,

$$\eta_x^n \equiv \frac{m\left(\lambda_x^n, \lambda_y^n\right)}{\lambda_x^n} \quad \text{and} \quad \eta_y^n \equiv \frac{m\left(\lambda_x^n, \lambda_y^n\right)}{\lambda_y^n} \tag{27}$$

Each agent may choose a marketplace where only agents of her own type are present. Call one such marketplace the benchmark marketplace $\bar{n}$. In equilibrium, agents have to be indifferent between joining either side of the benchmark marketplace. This implies $\eta_x^{\bar{n}} = \eta_y^{\bar{n}} \equiv \eta^{\bar{n}}$, the benchmark meeting rate, and by equation (27) also $\lambda_x^{\bar{n}} = \lambda_y^{\bar{n}}$. By constant returns to scale in meeting, $\eta^{\bar{n}}$ is then the same in any type's benchmark marketplace and in any of several benchmark marketplaces for the same type.

Now consider a marketplace n attended by more than one type. Suppose there is a relative abundance of high types on one side of the market, say in λ_x^n. Under truthful signalling, the marginal agent with a relatively high type who joins this side expects to meet relatively many lower types. Independently of whether these meetings lead to matches or not, she would strictly prefer meeting only her own type (by lemmas 3 through 5, assuming their conditions hold). She will only accept this rather than going to the benchmark marketplace if $\eta_x^n > \eta^{\bar{n}}$. However, a marginal agent in λ_y^n would also prefer to meet her own type rather than higher types and will only accept this marketplace if $\eta_y^n > \eta^{\bar{n}}$. With constant returns to scale in meeting, it is impossible that $\min[\eta_x^n, \eta_y^n] > \eta^{\bar{n}}$. Hence we can rule out marketplaces where the mix of types in equilibrium is not the same on both sides. When it is the same, agents will only be indifferent between joining either side if $\eta_x^n = \eta_y^n$, which leads to $\lambda_x^n = \lambda_y^n$. By constant returns to scale, we can conclude that then $\eta_x^n = \eta_y^n = \eta^{\bar{n}}$. Agents still meet various types in this marketplace, while they would meet only their most preferred type in the benchmark marketplace, at the same meeting rate. Hence each agent unambiguously prefers her benchmark marketplace. The result that only the benchmark marketplaces (with identical meeting rates) survive in equilibrium is the same as we found for one-sided marketplaces.

B Equivalence of core and stable matching

In Becker (1973), a matching is an equilibrium matching whenever it is in the core. A standard definition of the core is offered by Telser (1978):

Definition (Core). *Call the set C of agents a coalition, and let $Z(C)$ give the highest possible sum of flow utility the coalition can obtain under the most adverse conditions, with $Z(\emptyset) = 0$. Associated with any matching is a vector $s = (\sigma_1, \ldots, \sigma_I)$. The matching is said to be in the* *core only if, for all legal coalitions C,*

$$\sum_{i\in C}\sigma_i \geq Z(C)$$

In words, the core is the set of matchings such that no legal coalition can ensure more flow utility for all its members than obtained in the matching. However, if only the sum of utility obtained by the coalition counts, the possibility of side payments within the coalition is implicitly assumed. Side payments are crucial in Becker's (1973) reasoning: an individual agent then always prefers, among all matches available to her, the match generating the highest match output, since her partner in this match will use the extra output to outbid any other potential match partners. Yet where output is divided through bargaining, an agent's share in the match generating the highest output may fall short of her share in another match.

We thus have to modify the definition of the core to ensure that each agent's σ_i weakly exceeds the utility she obtains in any legal coalition available to her. In our model, the only legal coalitions are those with $|C| \in \{0, 1, 2\}$. Then such a modified definition reduces to two requirements: σ_i has to weakly exceed agent i's utility of being single ($|C| = 1$), and no match ($|C| = 2$) is available to i in which she obtains strictly more than σ_i. When agents go to perfectly segmented marketplaces under truthful signalling, we can identify a match that is available to i with a match where the match partner is better off than in any other available match. Definition 4 makes the same requirements.

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