Green Spending Reforms, Growth and Welfare with Endogenous Subjective Discounting

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Abstract

This paper studies optimal fiscal policy, in the form of taxation and the allocation of tax revenues between infrastructure and environmental investment, in a general-equilibrium growth model with endogenous subjective discounting. A green spending reform, defined as a reallocation of government expenditures towards the environment, can procure a double dividend by raising growth and improving environmental conditions, although the environment does not impact the production technology. Also, endogenous Ramsey and growth-maximizing policies eliminate the possibility of an 'environmental and economic poverty trap'. Finally, we examine the optimal response of the Ramsey government to greener preferences.

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1 Introduction

The economic growth - environmental quality nexus has received considerable attention both in academic research and policy debates.[1] Economic activity contributes to environmental degradation by generating pollution, and thus understanding why richer economies may be more environmentally deteriorated is straightforward. Yet, there is evidence that many advanced economies perform better in terms of environmental quality than poorer countries.[2]

From a positive aspect, in an attempt to explain this stylized fact, a recent strand of the literature has derived, through various mechanisms, multiple equilibria in which environmental quality and income or growth are positively related (e.g. Ikefuji and Horii, 2007; Prieur, 2009; Mariani et al., 2010; Varvarigos, 2010a). Multiple equilibria in these studies may imply the existence of an 'environmental and economic poverty trap' characterized by economic stagnation and bad environmental conditions. In this context, the implications of endogenous government intervention when there are multiple paths over which income and environmental quality evolve, become important.

From a normative aspect, under unique equilibrium regimes, a large body of the literature has analyzed the impact of environmental policy reforms on growth and welfare by concentrating on the tax instruments (see e.g. Goulder, 1995; Bovenberg and Smulders, 1996; Bovenberg and Mooij, 1997). The main conclusion is that a rise in pollution taxes may provide a double dividend consisting in a simultaneous increase in the growth rate and environmental quality, provided the latter has a positive impact on the production technology. However, given the presence of binding fiscal constraints in the economy, alternative policy reforms on the expenditures side can be considered, which may yield positive effects on growth and environmental

[1] See Xepapadeas (2005) for a survey.

[2] For instance, using the Environmental Performance Index (YCELP, 2010) as a proxy for environmental quality, higher values in the range of 0-100 can be observed for a number of relatively rich nations, like Switzerland (89.1), Norway (81.1), France (78.2), and Austria (78.1), compared to countries with lower per capita GDP, such as Russia (61.2), Egypt (62.0), Thailand (62.2), Slovenia (65.0), and Lithuania (68.3). For many years, the dominant explanation for the possibility of a positive association between environmental quality and per capita GDP has been the environmental Kuznets curve (e.g. Grossman and Krueger, 1995), which depicts an inverse-U-shaped relationship between environmental degradation and income. However, several methodological pitfalls cast doubt on the validity of the empirical results obtained so far in the literature (Kijima et al., 2010). For a similar discussion see also Varvarigos (2010a).

quality in a fiscally neutral way.

The present paper (i) studies the role of optimal fiscal policy in eliminating an 'environmental and economic poverty trap' and (ii) focuses on green spending reforms, implying a shift in public spending from 'productive' towards environmental outlays, as a means to achieve a double dividend without the assumption of enviromental production externalities.[3]

Our main tool is a continuous-time growth model with renewable resources, in which the generating mechanism of multiple equilibria is the assumption of endogenous subjective discounting (time preference). Specifically, we incorporate an environmental externality into individuals' impatience by assuming that agents who experience a higher environmental quality are less myopic and tend to value the future more. Our approach captures the standard 'life expectancy effect' of environmental quality or pollution, through the impact exerted on human health.[4] In a similar spirit, Agénor (2010) has captured the 'life expectancy effect' of health services by endogenizing the degree of impatience to this variable. Recently, Yanase (2011) also incorporates an environmental externality by modeling the rate of time preference (RTP) as a negative function of total pollution and finds, in an exogenous policy setup, that multiple steady states may exist and that the dynamic equilibrium may display indeterminacy.[5]

Starting the analysis at the Decentralized Competitive Equilibrium (DCE) level, we show that global indeterminacy, in the form of multiple equilibria, may arise in the market economy. Intuitively, economies with the same fundamentals can end up in a 'bad' equilibrium, char-

[3]Public expenditures on environmental care may be thought of as 'cleanup' expenditures on pollution abatement or, more generally, as total spending on all environmental programs. Actually, the proportion of government spending in total abatement expenditures is high in most countries (see e.g. Haibara, 2009). Throughout the paper, we use interchangeably the terms 'public environmental maintenance/investment' and 'pollution abatement policies'.

[4]For recent contributions see Balestra and Dottori (2009), Mariani et al. (2010), Pautrel (2008), Jouvet et al. (2010), and Varvarigos (2010a,b). This strand of the literature is typically developed in an overlapping generations setup. We note that our positive results would hold in such a framework. Yet, the normative aspects of our second-best policies would be intractable in an overlapping generations setup.

[5]Behavioral evidence shows that people who are familiar to natural resources have low rates of discount. Viscusi et al. (2008) examine the relation between the RTP and water quality and find that recreational users of water bodies (lakes, rivers, and streams) have higher valuations of water quality and lower RTP than those who do not visit. For theoretical models see also Pittel (2002, Chapter 5) and Lines (2005). Mention should also be made of Ayong Le Kama and Schubert (2007) who consider a social planner economy, in which the future is discounted at a lower rate when environmental questions become more pressing (as indicated by a lower level of environmental quality) to reflect *social* motive of sustainability and intergenerational altruism.

acterized by high impatience, poor environmental conditions and low growth, or in a 'good' equilibrium, with lower impatience, better environmental conditions and higher growth. Our analysis reveals that these equilibrium regimes are associated with different policy prescriptions. We emphasize that in the 'good' equilibrium the economy can enjoy a double dividend in terms of higher growth and better environmental conditions if government spending is reallocated towards the environment because of the growth effect of environmental quality through patience and higher savings. Hence, a fiscally neutral shift in the spending mix, such as a green spending reform, can trigger a double dividend in our setup, although the environment does not impact the production technology.

Next, we take the analysis one step further by examining optimal fiscal policies aiming at maximizing long-run growth or welfare (Ramsey policy). By endogenizing government policy we can analyze the feedback effect of economic structure on the policy instruments. This becomes more interesting in the present setup, characterized by multiple equilibrium regimes, as it allows us to demonstrate how the government sets the fiscal instruments as a function of the long-run state of the economy, so as to implement a unique equilibrium, which corresponds to the 'good' DCE. Moreover, we study the implications of greener preferences for the second-best Ramsey policy.

Our results contribute to the existing literature in the following ways. First, we show that the endogenous choice of the policy instruments serves here as an equilibrium-selection device, resolving the global indeterminacy at the DCE level and implementing the 'good' equilibrium. Although there has been some investigation of the role of public policy in eliminating a poverty trap and selecting the 'good' equilibrium in models with multiple growth paths (but no environmental externalities), the focus has been on how government intervention can affect the set of equilibria that exist under laisser-faire, without explicitly specifying the government's objective (see Matsuyama, 1991; Boldrin, 1992; Rodrik, 1996). Public policy is also not endogenized in papers, like Grandmont (1986), Reichlin (1986), and Woodford (1986), which have made comparisons between the sets of equilibria that result from different policy choices. More recently, Agénor (2010) studies how an exogenous reallocation of government spending from unproductive expenditure to infrastructure can facilitate the shift from a low- to a high-growth equilibrium. The only study in which equilibrium indeterminacy is resolved through endogenous government intervention is Economides et al. (2007), which finds that the 'bad' equilibrium is implemented in a growth-maximizing setup. Instead, in the present paper the endogenous choices of the tax rate and the allocation of tax revenues implement the desirable equilibrium.[6]

Second, our results suggest that the growth-maximizing share of public environmental investment vis-à-vis infrastructure is positive, although environmental quality does not directly enter in the production process. This alters the typical finding in related setups with exogenous RTP that tax revenues should be devoted to 'productive' expenditures from a growth perspective and that public environmental investment is only justified by social welfare considerations due to the amenity value of the environment in the utility function (Ligthart and van der Ploeg, 1994; Pérez and Ruiz, 2007; Philippopoulos and Economides, 2008). Our result follows from the properties of the 'good' equilibrium, implemented in this second-best setup. Contrary to the case of the 'bad' equilibrium, in which a growth-enhancing strategy is to engage in pure 'productive' expenditures, in the 'good' equilibrium the more tax revenues the government allocates to environmental care vis-à-vis infrastructure above a critical value, the higher is the long-run growth rate. Only below this critical value the traditional recipe is obtained; namely that the more revenues the government allocates to infrastructure investment, the higher is long-run growth. Intuitively, this occurs because, in addition to the standard growth-promoting role of infrastructure investment, there is now a similar indirect role played by environmental spending by means of promoting patience and inducing higher savings, which in turn support capital accumulation. In the case of a 'good' equilibrium, in which the tax base is large enough for this effect to be relatively strong, a trade-off exists between the two types of public expenditures and

[6] In a different context, the government in Ennis and Keister (2005) sets its policy before observing the actions of private agents (and hence before knowing which of the equilibria will obtain) and imposes equilibrium selection rules based on the concept of risk dominance. Notice that the presence of public policy may also generate, rather than eliminate, multiple equilibria. Glomm and Ravikumar (1995) show in an overlapping-generations economy that there may be multiple equilibrium paths when public policy is endogenous. In Park and Philippopoulos (2004) and Park (2009) multiple equilibria are the outcome of policy indeterminacy in the form of multiple tax rates that are endogenously chosen. Similarly, Pérez and Ruiz (2007) find multiplicity of optimally chosen tax rates, which in turn determine multiple allocations of tax revenues between public abatement and 'productive' expenditures.

hence the relationship between long-run growth and the resource allocation to infrastructure vis-à-vis the environment becomes inverse-U shaped.

Finally, we analyze the optimal government response to changes in agents' preferences for the environment. We show that under endogenous subjective discounting the Ramsey planner has to pursue green spending reforms following an increase in environmental concern. The opposite government response of more growth-enhancing policies has been obtained by Philippopoulos and Economides (2008) for an economy with exogenous RTP and 'productive' government expenditures solely as the source of endogenous growth. In such an economy, the reallocation of revenues towards 'productive' spending promotes growth and yields larger tax bases and extra revenues for cleanup policy. In our model, such a shift in public spending allocation raises the RTP and can lead the economy to a vicious cycle of low growth, high impatience and poor environmental conditions. Instead, by increasing the share of environmental maintenance, the Ramsey government achieves a direct increase in welfare given the presence of environmental quality in the utility function (*'Static Amenity Channel'*) and additionally a reduction in subjective discounting, which impacts the growth dynamics positively (*'Dynamic Patience Channel'*).

A central policy implication of the paper is therefore that, even without considering a direct positive environmental externality in production, green spending reforms can yield a double dividend in fast-growing economies. Further, the stronger the agents' environmental concerns, the more a Ramsey government should engage in green spending reforms. The paper therefore suggests a channel for the impact of public environmental spending on long-run growth and welfare that has been left unnoticed in existing studies and adds to recent findings on the potentially favourable effect of environmental taxation on economic activity in the absence of environmental externalities in the production function (see Pautrel, forthcoming).

The rest of the paper is organized as follows. The model is described in Section 2. We then solve for a DCE for given policy in Section 3. Sections 4 considers the long-run growth impact of a change in resource allocation between government 'productive' spending and environmental maintenance and derives growth-maximizing policy. Section 5 examines welfare-maximizing policy by solving the Ramsey problem of the government. Finally, Section 6 concludes.

2 The model

This section presents the setup of our closed-economy model. The main features are as follows: (a) households derive utility from private consumption and environmental quality that has a public-good character; (b) the subjective RTP is a function of environmental quality and economy-wide average consumption, taken as external by the agents; (c) public infrastructure provides production externalities to firms; (d) production activities generate environmental pollution; (e) the government imposes a tax on polluting output and uses the collected tax revenues to finance infrastructure and environmental care.

2.1 Households

The economy is made up of a large number of identical, infinitely lived households, normalized to unity, and each of them seeks to maximize the present discounted value of the lifetime utility:

$$\int_0^\infty u(c_t, N_t) \exp\left[-\int_0^t \rho(C_v, N_v)dv\right] dt \tag{1a}$$

where $u(.)$ is the instantaneous utility function, which depends on the representative agent's consumption, c, and the stock of economy-wide natural resources, interpreted as an index for environmental quality, N. In particular, $u(.)$ takes the general form $u(c,N) = \left(c^\nu N^{1-\nu}\right)^{1-\sigma}/(1-\sigma)$, with $0 < \nu \leq 1$ measuring how much agents value c vis-à-vis N and $0 < \sigma \leq 1$ representing a degree of intertemporal substitution.[7]

In turn, $\rho(C,N)$ denotes the endogenous RTP, which is assumed to depend positively on aggregate (or equivalently, the economy-wide average) consumption, C, and negatively on environmental quality, i.e. $\rho_C \geq 0$ and $\rho_N \leq 0$.[8] The assumption that a higher level of the

[7]Note that positive felicity is guaranteed only for $0 < \sigma < 1$. We also include here the logarithmic-utility case ($\sigma = 1$) to allow for comparisons in our simulations with this extensively used specification.

[8]We retain the equality sign in our assumptions to allow, first, for comparisons with the case of constant RTP and, second, for the impatience function to be consistent with a balanced growth path along which the

economy-wide average consumption raises individual impatience follows a large strand of the literature that has linked the RTP to social factors taken as external by agents (see, among others, Shi, 1999; Schmitt-Grohé and Uribe, 2003; Choi et al., 2008; Dioikitopoulos and Kalyvitis, 2010).[9] The assumption that a higher level of environmental quality lowers individual impatience follows Yanase (2011) and captures the well-documented 'life expectancy effect' of environmental quality or pollution (see Introduction). For tractability, we will assume homogeneity of degree zero for the discount rate function, so that:

$$\rho(C, N) = \rho\left(\frac{C}{N}, 1\right) \equiv \rho\left(\frac{C}{N}\right) \tag{1b}$$

with $\frac{\vartheta\rho(.)}{\vartheta(C/N)} \equiv \rho'(.) \geq 0$ and $\rho''(.) \geq 0$.[10] Further, we assume that there exists a lower positive bound for the RTP, denoted by $\breve{\rho}$, i.e. $\lim\limits_{(C/N)\to 0} \rho\,(C/N) = \breve{\rho} > 0$.

Households save in the form of capital and supply inelastically one unit of labor services. The rental rate for capital and wage rate are r and w, respectively. Further, they receive dividends, π. The budget constraint of the household is given by:

$$\dot{A} + c = rA + w + \pi \tag{2a}$$

where a dot over a variable denotes a derivative with respect to time, A denotes financial assets and the initial asset endowment $A(0) > 0$ is given.

The household acts competitively by taking prices, policy, and environmental quality as given. The latter is justified by the open-access and public-good features of the environment. The control variables are the paths of c and A, so that the first-order conditions include the

time preference is constant (see below). Throughout the rest of the paper, the time subscript t is omitted for simplicity of notation and the terms 'average' and 'aggregate' are used interchangeably given the population of unit mass.

[9] Earlier literature has thoroughly investigated the connections between time preference and individual consumption (see e.g. Uzawa, 1968; Epstein, 1987; Obstfeld, 1990; Palivos et al., 1997).

[10] The homogeneity of the RTP to the ratio of aggregate consumption to environmental quality is required for the RTP to be bounded at the steady-state (see e.g. Palivos et al., 1997) and for the utility function to be consistent with balanced growth (Dolmas, 1996).

constraint (2a) and the Euler equation below:

$$\frac{\dot{c}}{c}=\frac{1}{1-\nu\left(1-\sigma\right)}\left[\left(1-\nu\right)\left(1-\sigma\right)\frac{\dot{N}}{N}+r-\rho\left(\frac{C}{N}\right)\right] \tag{2b}$$

Notice that environmental quality affects positively consumption growth through the RTP, and thus plays an implicit 'productive' role in the economy.

2.2 Firms

The production function of the single good in this economy is given by:

$$Y=K^{a}{K_{g}}^{1-a}L^{1-a} \tag{3}$$

where Y denotes output, $0<a<1$ denotes the share of physical capital, K, in the production function, K_g refers to the public capital stock (e.g. infrastructure), and L is labour.[11]

The law of motion for the public capital stock is given by:

$$\dot{K}_{g}=G-\delta_{K_{g}}K_{g} \tag{4a}$$

where δ_{K_g} denotes the depreciation rate and G is government investment in public capital (see below). The initial capital stock $K_g(0)>0$ is given.

The firm maximizes profits, π:

$$\pi=(1-\tau)Y-(r+\delta_{K})K-wL \tag{5}$$

where $0<\tau<1$ is a tax rate on output, δ_K is the depreciation rate of private capital, and its summation with r forms the rental cost of capital. The firm acts competitively by taking prices

[11] This specification follows the strand of endogenous growth theory assuming that the government can invest in productive public capital, which will stimulate aggregate productivity. The first model in which productive public spending leads to sustained per capita growth in the long run was presented by Barro (1990). Futagami et al. (1993) extended the Barro model by assuming that public capital as a stock variable exerts positive productivity effects.

and policy as given. The control variables are K and L and the standard first-order conditions are given by:

$$r = a(1-\tau)\left(\frac{K}{K_g}\right)^{a-1} L^{1-a} - \delta_k \tag{6a}$$

$$w = (1-a)(1-\tau)\left(\frac{K}{L}\right)^{a} K_g^{1-a} \tag{6b}$$

Equations (6a)-(6b) state that the marginal productivities of capital and labor have to equal factor prices.

2.3 Motion of environmental quality

We assume that the stock of environmental quality evolves over time according to:

$$\dot{N} = \theta E + \delta_N N - P \tag{7a}$$

where E is public environmental investment (specified in equation (8c) below), $\delta_N > 0$ and $0 < \theta \leq 1$ are parameters measuring respectively the regeneration rate of natural resources and how public spending is translated into actual units of renewable natural resources, and P is the pollution flow (see below).[12] Thus, natural resources can be renewed by regeneration and public policy. The initial stock $N(0) > 0$ is given.

We further assume that P occurs as a by-product of final output:

$$P = sY \tag{7b}$$

where $0 < s < 1$ is a technology parameter that quantifies the detrimental effect of economic

[12] This law of motion for the natural resources is borrowed from Economides and Philippopoulos (2008). The embodied assumption of a constant regeneration rate for the environment has also been used, among others, in Harrington et al. 2005); Valente (2005); Acemoglu et al. (2011). Although simplifying, it is necessary to ensure balanced growth (see below), given the structure of our model with endogeneity of the RTP to the ratio of aggregate consumption to environmental quality, public environmental investment financed by income taxation, and production activities as the source of pollution. For other papers with publicly financed abatement spending, see e.g. Ligthart and van der Ploeg (1994); Greiner (2005); Pérez and Ruiz (2007); Gupta and Barman (2010); Pautrel (forthcoming).

activity on the environment.[13] Production, Y, therefore impacts positively the evolution of environmental quality through providing a tax base for the finance of public environmental investment (see below) and negatively through the induced pollution.

2.4 Government budget constraint

On the revenue side, the government taxes the polluting firm's output at a rate $0 < \tau < 1$, while on the expenditure side, it spends G on infrastructure and E on environmental policy.[14] Assuming a balanced budget, we have:

$$G + E = \tau Y \tag{8a}$$

Equivalently, we can write (8a) as:

$$G = b\tau Y \tag{8b}$$

$$E = (1 - b)\tau Y \tag{8c}$$

where $0 < b \leq 1$ is the fraction of tax revenue used to finance infrastructure and $0 \leq (1-b) < 1$ is the fraction that finances environmental investment. Thus, at each instant, government policy can be summarized by the two policy instruments, τ and b.

3 Decentralized competitive equilibrium

In this section we solve for a DCE, which holds for any feasible policy and analyze its properties.

Definition 1 *The DCE of the economy is defined for the exogenous policy instruments τ and b, factor prices r and w, and aggregate allocations K, K_g, N, G, E, L, C such that:*

i) Individuals solve their intertemporal utility maximization problem by choosing c and A, given the policy instruments and factor prices.

[13] We consider a linear relationship between pollution flows and production for the sake of simplicity. Our results do not change if we assume that pollution occurs as a by-product of consumption.

[14] Given the setup of the model, a tax levied on output boils down to taxing pollution.

ii) Firms choose K, L in order to maximize their profits, given factor prices and aggregate allocations.

iii) All markets clear, which implies for the labor market $L = 1$ and for the capital market $A = K$ (assets held by agents equal the private capital stock).

iv) The government budget constraint holds.

Combining (1)-(8) and assuming for the rest of the paper, without loss of generality, that $\delta_K = \delta_{K_g} = \delta$, it is straightforward to show that the DCE is given by:

$$\frac{\dot{C}}{C} = \frac{(1-\nu)(1-\sigma)}{1-\nu(1-\sigma)}\frac{\dot{N}}{N} + \frac{1}{1-\nu(1-\sigma)}\left[a(1-\tau)\left(\frac{K}{K_g}\right)^{a-1} - \delta - \rho\left(\frac{C}{N}\right)\right] \tag{9a}$$

$$\frac{\dot{K}}{K} = (1-\tau)\left(\frac{K}{K_g}\right)^{a-1} - \frac{C}{K} - \delta \tag{9b}$$

$$\frac{\dot{K}_g}{K_g} = b\tau\left(\frac{K}{K_g}\right)^{a} - \delta \tag{9c}$$

$$\frac{\dot{N}}{N} = [\theta(1-b)\tau - s]\frac{K^{\alpha}K_g^{1-\alpha}}{N} + \delta_N \tag{9d}$$

Equations (9a)-(9d) summarize the dynamics of our economy. It should be underscored that owing to the presence of environmental quality in (9a), equations (9a)-(9c) cannot be solved independently of the environmental stock accumulation equation, (9d).

In line with Economides and Philippopoulos (2008), we assume for the rest of the paper that economic activity has a net damaging effect on the dynamics of environmental quality in (9d), i.e. $\Theta(\tau, b) \equiv \theta(1-b)\tau - s < 0$. The assumption implies that the environmental damage caused by one unit of production, s, is higher than the environmental benefit that arises from one unit of production (through providing a tax base for financing environmental investment), $\theta(1-b)\tau$, and is meant to describe a real world economy.

Finally, the transversality condition for this problem is given by:

$$\lim_{t\to\infty}\frac{K(t)}{C(t)}\exp\left[-\int_0^t \rho\left(\frac{C_s}{N_s}\right)ds\right] = 0 \tag{10}$$

The balanced growth path (BGP) is defined as a state where all the variables in the economy grow at a constant rate. At the BGP of this economy, consumption, private capital, public capital, and environmental quality have to grow necessarily at the same rate, i.e. $\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{K_g}}{K_g} = \frac{\dot{N}}{N} \equiv g$.[15] Following usual practice, we will reduce its dimensionality to facilitate analytical tractability. We thus proceed by defining the following auxiliary stationary variables, $\omega \equiv \frac{C}{K}$, $z \equiv \frac{K}{K_g}$, and $x \equiv \frac{K_g}{N}$. Then, it is straightforward to show that the dynamics of (9a)-(9d) are equivalent to the dynamics of the following system of equations:

$$\frac{\dot{\omega}}{\omega} = [1-\nu(1-\sigma)]^{-1}[(1-\nu)(1-\sigma)(\Theta(\tau,b)z^a x + \delta_N) - (1-\nu(1-\sigma)-a)(1-\tau)z^{a-1} - \rho(\omega z x) - \nu(1-\sigma)\delta] + \omega \quad \text{(11a)}$$

$$\frac{\dot{z}}{z} = (1-\tau)z^{a-1} - b\tau z^a - \omega \quad \text{(11b)}$$

$$\frac{\dot{x}}{x} = b\tau z^a - \Theta(\tau,b)z^a x - \delta - \delta_N \quad \text{(11c)}$$

It follows that at the BGP $\frac{\dot{\omega}}{\omega} = \frac{\dot{z}}{z} = \frac{\dot{x}}{x} = 0$. Then (11b) and (11c) imply that the long-run ratios of consumption to private capital and public capital to environmental quality, $\hat{\omega}$ and $\hat{x}$ respectively, are expressed as functions of the long-run ratio of private capital to public capital, $\hat{z}$, by:

$$\hat{\omega}(\hat{z}) = (1-\tau)\hat{z}^{a-1} - b\tau\hat{z}^a \quad \text{(12a)}$$

$$\hat{x}(\hat{z}) = (b\tau\hat{z}^a - \delta - \delta_N)\Theta(\tau,b)^{-1}\hat{z}^{-a} \quad \text{(12b)}$$

[15] This result can be easily obtained by investigating the equilibrium growth rates of these variables separately. In particular, as can be seen from (9c), for the long-run growth rate of public capital to be constant both K and K_g have to grow at the same constant rate. Then, by inspection of (9b), in order for the growth rate of private capital to be constant in the long-run we additionally need that C and K grow at the same constant rate. Finally, for the growth rate of natural resources, given by (9d), to be constant in the long-run we need that N, K and K_g grow at the same constant rate. It follows that the long-run growth rate of consumption will also be constant and given by $g_C = \frac{1}{\sigma}[r - \rho(\frac{C}{N})]$ and that the transversality condition (10) will be satisfied. Hence, the necessary condition for the existence of a BGP in this economy is that all variables grow at the same rate, g. Thus, we cannot have a case here in which C, K and K_g grow at the same positive rate, while N remains constant, but what is important is that the model can allow for positive or zero long-run growth without reducing the stock of natural resources, which is a sustainable equilibrium (see also Economides and Philippopoulos, 2008).

Finally, substituting (12a) and (12b) in (11a) we get that $\hat{z}$ is determined by:

$$\Phi(\hat{z}) \equiv -\sigma b\tau\hat{z}^{a} + a(1-\tau)\hat{z}^{a-1} - (1-\sigma)\delta - \rho(\hat{\omega}(\hat{z}) \cdot \hat{z} \cdot \hat{x}(\hat{z})) = 0 \tag{12c}$$

Provided that there exists a solution $\hat{z} > 0$ in (12c), the balanced growth rate, g, is then determined by (9c). Assuming equilibrium existence, equations (12a)-(12c) imply the following:

Proposition 1 *Under the assumptions of Section 2, the long-run equilibrium can be unique or multiple.*

Proof. See Appendix A. ▪

Proposition 1 states that endogenous impatience, determined by aggregate consumption and environmental quality, can lead to multiple solutions for $\hat{z}$, and, in turn, multiple Pareto-ranked DCE allocations. Hence, although the instantaneous utility and production technology functions satisfy the standard concavity assumptions, the existence of a unique positive balanced growth rate is not guaranteed here. The reason is that under the externalities introduced in individual discounting the degree of impatience becomes nonlinearly related to the stock of private capital (relative to public capital) in the long run. As a result of this nonlinearity, multiple equilibria may arise in the economy. Inspection of (9c), (12a), and (12b) reveals that an equilibrium with high $\hat{z}$ may be referred to as the 'good' equilibrium, since it is associated with a higher balanced growth rate, better environmental quality relative to public goods and private consumption (i.e. lower $\hat{x}(\hat{z})$ and $\frac{C}{N} \equiv \hat{\omega}(\hat{z}) \cdot \hat{z} \cdot \hat{x}(\hat{z})$), and lower impatience compared to an equilibrium with low $\hat{z}$ ('bad' equilibrium).[16] The 'bad' equilibrium can therefore be equivalently characterized as an 'ecological and economic poverty trap'.

For instance, consider that at the steady-state there is a shock that causes an increase in the physical capital stock. An increase in the capital stock results in a positive growth effect as it increases output. However, at the same time it increases consumption and pollution, which

[16] It can be easily verified that $\frac{\vartheta\hat{\omega}(\hat{z})}{\vartheta\hat{z}} < 0$, $\frac{\vartheta\hat{x}(\hat{z})}{\vartheta\hat{z}} < 0$, and $\frac{\vartheta[\hat{\omega}(\hat{z})\cdot\hat{z}\cdot\hat{x}(\hat{z})]}{\vartheta\hat{z}} < 0$.

is harmful to environmental quality. An increase in consumption and a decline in environmental quality make agents more impatient, causing a decrease in savings and, through the Euler equation, a decline in the balanced growth rate. A decline in the growth rate then reduces the tax base of the economy and, thus, the provision of public expenditures on infrastructure and abatement leading the economy to a vicious cycle of high impatience, low growth and bad environmental quality. If a threshold level of government expenditures is allocated to environmental investment so as to turn the initial increase in output to higher environmental quality, then the economy will experience a decline in subjective discounting and higher savings, resulting in a cycle of low impatience, high growth and high environmental quality. These mechanisms are more intense the higher is the elasticity of the physical capital stock in production (i.e. the higher is a), as the first-order effect on output is high. Also, the higher is the intertemporal substitutability of consumption (i.e. the lower σ), the more responsive is the balanced growth rate to changes in subjective discounting and, in turn, the more intense is the bidirectional relationship between growth and endogenous discounting.[17]

The nature of the long-run outcome in this economy may be further clarified numerically. To this end, a linear time preference function, $\rho(\frac{C}{N}) = \gamma \times (\frac{C}{N}) + \breve{\rho}$, is employed for computational tractability (see also Pittel, 2002). Table 1 displays the parameter values used and Table 2 reports the numerical findings. For sufficiently low values of σ ($0.1 \leq \sigma \leq 0.3$) there are two solutions for the growth rate, g_1 and g_2, while for values $0.4 \leq \sigma \leq 1$ the DCE allocation is unique. Under the first regime, the balanced growth rate is lower, impatience is higher, and environmental quality in relation to both consumption and public capital is lower.[18] This example therefore illustrates the possibility for some countries to be caught in a high-impatience, poor-environment, low-growth trap. The comparative statics properties are also opposite between the two regimes. As σ increases, z increases (falls) in the 'good' ('bad') equilibrium. Intuitively, a higher σ leads directly to lower consumption growth through the Euler equation, and hence more savings and greater capital accummulation, but at the same time it can also exert an

[17] Equation (9a) at the BGP implies $g = \frac{1}{\sigma}[r - \rho(\frac{C}{N})]$, from which it follows that the responsiveness of the long-run growth rate of consumption to changes in impatience is given by $-\frac{1}{\sigma}$.

[18] The analysis of local stability is performed in the Companion Appendix of the paper.

indirect positive impact through a decrease in the RTP in the presence of the consumption externalities, and hence may lead to higher balanced growth and provide a larger tax base for financing public infrastructure. Consequently, when z is high (low), as in the case of the 'good' ('bad') equilibrium, the first (second) channel dominates.

In contrast to recent studies that have attempted to explain the emergence of multiple equilibria for which the low (high) income equilibrium is associated with low (high) environmental quality without explicitly considering a government sector (e.g. Ikefuji and Horii, 2007; Prieur, 2009; Mariani et al., 2010; Varvarigos, 2010a), multiplicity here emerges in the presence of government policy. More importantly, the comparative statics exercises we perform in the next section demonstrate the existence of thresholds in the level of the public spending composition, b, that affect the properties of the DCE.

4 'Productive' versus environmental spending and long-run growth

Conventional wisdom argues that, in the absense of environmental externalities in the production function, public environmental maintenance will have an adverse effect on growth by diverting resources from the 'productive' sectors. A strand of the literature has formalized this notion in endogenous growth models by showing that public environmental investment has an unfavourable effect on long-run growth and is only justified by social welfare considerations due to the amenity value of environmental quality in the utility function (Ligthart and van der Ploeg, 1994; Pérez and Ruiz, 2007; Philippopoulos and Economides, 2008). In this section we demonstrate how shifting the allocation of resources from productive government spending to environmental care can promote long-run growth without the assumption of environmental externalities in production.

4.1 Exogenous policy prescriptions for each regime type

We first investigate how the balanced growth rate, g, reacts to exogenous changes in the share of public infrastructure spending vis-à-vis environmental expenditures, b, by taking the corresponding derivative in (9c):

$$\frac{\vartheta g}{\vartheta b} = \tau \hat{z}^{a-1}(\hat{z} + ab\hat{z}_b) \tag{13}$$

where $\hat{z}_b \equiv \frac{\vartheta \hat{z}}{\vartheta b} = -\frac{\vartheta \Phi(\hat{z})/\vartheta b}{\vartheta \Phi(\hat{z})/\vartheta \hat{z}}$ is derived from total differentiation of (12c), with:

$$\frac{\vartheta \Phi(\hat{z})}{\vartheta \hat{z}} = -a\sigma b\tau \hat{z}^{a-1} - a(1-\tau)(1-a)\hat{z}^{a-2} \underbrace{-\rho'(.)b\tau \frac{a\hat{\omega}(\hat{z}) - \Psi(\hat{z},\tau,b)}{\Theta(\tau,b)}}_{>0} \tag{14a}$$

$$\frac{\vartheta \Phi(\hat{z})}{\vartheta b} = -\sigma\tau\hat{z}^a \underbrace{-\rho'(.)\tau \frac{\theta\hat{x}(\hat{z}) + \hat{z}(\hat{\omega}(\hat{z}) - \Psi(\hat{z},\tau,b))}{\Theta(\tau,b)}}_{>0} \tag{14b}$$

where $\Psi(\hat{z},\tau,b) \equiv b\tau\hat{z}^a - \delta - \delta_N < 0$.

When the RTP is exogenous ($\rho'(.) = 0$), the standard result that the growth rate, g, monotonically increases with the share of tax revenue allocated to infrastructure, b, can be easily verified. In this case, because public expenditures for the environment contribute neither to the production technology nor to the savings rate, expenditures for infrastructure solely affect growth through the positive externality of the infrastructure stock in the production function. Hence, although there is a negative effect on g coming from the induced fall in the physical-to-public-capital ratio ($\hat{z}_b < 0$) when more tax revenues are allocated to infrastructure, this is outweighed by the direct positive effect from the increase in b, i.e. $\hat{z} + ab\hat{z}_b > 0$ in (13).

In the case of endogenous discounting ($\rho'(.) > 0$), not only the infrastructure stock but also the stock of environmental quality affects growth through the time preference which, in turn, affects positively the savings rate, and hencthe sign of (13) becomes ambiguous. Due to analytical intractability we resort to numerical simulations. We focus on the relatively rich case of multiplicity by setting $\sigma = 0.3$ and use for the rest of the parameters the values in Table 1.[19] The response of the DCE allocation is reported in Table 3 for the range of b in which a

[19] In the region of uniqueness the effect of b on g is monotonic, as described in the case of exogenous discounting.

well-defined solution exists.[20] As can be seen, there are threshold values of b that play a crucial role in the emergence of multiplicity, thus verifying that policy choices matter for the nature of the final outcome (uniqueness or multiplicity) in the economy. In particular, for sufficiently low levels of infrastructure investment vis-à-vis environmental maintenance ($b = 0.35$) the resulting equilibirum is unique, while for high levels ($0.4 \leq b \leq 0.7$) two equilibria arise. In addition, these two regimes exhibit different comparative statics properties. The standard monotonic effect of b on growth holds in the 'bad' equilibrium, but is altered in the 'good' regime, because now $\hat{z}$ is sufficiently high so that the positive direct effect from the increase in b does not always dominate the negative indirect one (i.e. $z_b < 0$), and a trade-off is in place. Specifically, for $b < 0.55$ as b rises g_2 rises, i.e. $\frac{\vartheta g_2}{\vartheta b} > 0$. By contrast, for $b \geq 0.55$ as b rises g_2 falls, i.e. $\frac{\vartheta g_2}{\vartheta b} < 0$. Consequently, the relationship between g and b appears inverse-U shaped in the 'good' equilibrium, as depicted in Figure 1, showing the different responses of the two equilibria.

Our findings thus imply that the two regimes are associated with different policy recipes. Specifically, when the economy is trapped in a 'bad' equilibrium with low growth, a growth-enhancing strategy is to engage in pure 'productive' expenditures by financing public infrastructure. However, when the economy is in the 'high-growth' regime this conventional policy recipe holds when the government allocates relatively little resources to infrastructure investment vis-à-vis public abatement (low values of b). Instead, for relatively high shares of productive expenditures vis-à-vis abatement, the more revenues the government allocates to environmental investment, the higher is the balanced growth rate. Hence, in the 'good' equilibrium, reallocating government spending towards the environment can procure a double dividend by raising growth and improving environmental conditions, although the environment does not impact the production technology.

The intuition behind these results is as follows. In addition to the standard growth-promoting role of infrastructure investment, there is also an indirect positive impact of public environmental spending on growth; by enhancing environmental quality, environmental spend-

The results are available upon request.

[20] For $b < 0.35$ or $b > 0.7$ at least one of the following: $\hat{\omega} > 0$, $\hat{x} > 0$, $\hat{\rho} > 0$, $g > 0$ is not satisfied.

ing promotes patience and induces higher savings, which support capital accumulation and fuel long-run growth. As a result, a trade-off exists between the two public spending components in the case of a 'good' equilibrium, in which the tax base is large enough for the effect of environmental expenditures to be relatively strong. Hence, the main conclusion derived is that, even without considering a direct positive environmental externality in production, environmental spending can be 'productive' in fast-growing economies.

Summarizing the above we get the following Results.

Result 1 *Under the assumptions of Section 2, in the DCE there is a critical value of* b*, denoted as* $b^{\dagger}$*, for which* $b < b^{\dagger}$ *implies a unique BGP and* $b > b^{\dagger}$ *implies two Pareto-ranked BGPs.*

Result 2 *Under the assumptions of Section 2, along the BGP of the 'good' regime in the DCE, there is a critical value of* b*, denoted as* b^{*}*, for which* $b > b^{*}$ *implies* $\frac{\vartheta g}{\vartheta b} < 0$ *and* $b < b^{*}$ *implies* $\frac{\vartheta g}{\vartheta b} > 0$*. Along the BGP of the 'bad' regime,* $\frac{\vartheta g}{\vartheta b} > 0$ *always holds.*

4.2 Endogenous policy choices

We now endogenize policy by assuming that the government's problem is to choose the tax rate, $0 < \tau < 1$, and the allocation of tax revenues between public capital investment and environmental spending, $0 < b \leq 1$, in order to maximize the long-run growth rate. In doing so, the government will try to correct the market imperfections (arising from externalities), raise tax revenue to finance public expenditures, and minimize the distorting effects of policy intervention on the economy. In this second-best setup, the government is benevolent and acts as a Stackelberg leader vis-à-vis private agents, i.e. takes into account their reaction function, given by (12c), when maximizing her objective function.

Definition 2 *A Growth-Maximizing Allocation (GMA) in the competitive equilibrium of the aggregate economy is given under Definition 1 when (i) the government chooses the tax rate,* τ*, and the allocation of tax revenues to infrastructure provision vis-à-vis environmental care,* b*, in order to maximize the long-run growth rate of the economy by taking into account the aggregate* *maximizing behavior of the competitive equilibrium, and (ii) the government budget constraints and the feasibility and technological conditions are met.*

Formally, the problem of the government is to choose τ and b to maximize g:

$$\max_{\tau,b} g = b\tau\hat{z}^{a} - \delta$$

by taking into account the (unique or multiple) solution for $\hat{z}$, denoted as $\hat{z}\,(\tau, b)$, from (12c). The first-order conditions with respect to τ and b are given by:

$$\hat{z} + a\tau^{*} \cdot \hat{z}_{\tau}(\hat{z}, \tau^{*}, b^{*}) = 0 \tag{15a}$$

$$\hat{z} + ab^{*} \cdot \hat{z}_{b}(\hat{z}, \tau^{*}, b^{*}) = 0 \tag{15b}$$

where an asterisk denotes growth-maximizing values and $\hat{z}_{\tau} \equiv \frac{\vartheta \hat{z}}{\vartheta \tau} = -\frac{\vartheta \Phi(\hat{z})/\vartheta \tau}{\vartheta \Phi(\hat{z})/\vartheta \hat{z}}$ is derived from total differentiation of (12c), with:

$$\frac{\vartheta \Phi(\hat{z})}{\vartheta \tau} = -\sigma b\hat{z}^{a} - a\hat{z}^{a-1} - \rho'(.)\frac{b\hat{z}\hat{\omega}(\hat{z}) - (1+b\hat{z})\Psi(\hat{z},\tau,b) - \theta\,(1-b)\,\hat{x}(\hat{z})}{\Theta(\tau,b)} \tag{16}$$

After some algebra we obtain from (15a) and (15b):

$$\Theta(\tau^{*},b^{*})\frac{\tau^{*}-(1-a)}{\tau^{*}}\hat{z}^{a-1} + \rho'(.)\Psi(\hat{z},\tau^{*},b^{*})\left[\frac{(1-a)}{a}b^{*}\hat{z} - 1 - (1-b^{*})\frac{\Omega(\hat{z},\tau^{*},b^{*})}{\Theta(\tau^{*},b^{*})}\right] = 0 \tag{17a}$$

$$\Theta(\tau^{*},b^{*})\frac{(1-a)(1-\tau^{*})}{b^{*}\tau^{*}}\hat{z}^{a-1} - \rho'(.)\Psi(\hat{z},\tau^{*},b^{*})\left[\frac{(1-a)}{a}\hat{z} + \frac{\Omega(\hat{z},\tau^{*},b^{*})}{\Theta(\tau^{*},b^{*})}\right] = 0 \tag{17b}$$

where $\Omega(\hat{z},\tau^{*},b^{*}) \equiv \theta(1-\tau^{*}-b^{*}\tau^{*}\hat{z})$.

When the RTP is exogenous ($\rho'(.) = 0$), (17a) yields the well-known Barro (1990) tax rule, which states that the government should impose a tax rate equal to the elasticity of the public capital in the production function, i.e. $\tau^{*} = (1-a)$. Since in this case the growth rate, g, increases monotonically with the share of tax revenue allocated to infrastructure, b, the growth-maximizing value of the latter is equal to one, $b^{*} = 1$, and (17b) is not part of the solution to the problem.

By contrast, under endogenous RTP ($\rho'(.) > 0$) the system of equations (17a) and (17b) offers the solutions for the optimal levels of the two policy instruments as implicit functions of $\hat{z}$, i.e. $\tau^*(\hat{z})$ and $b^*(\hat{z})$. Then, after policy has been chosen, the market economy reacts through (12c); that is, substituting the optimal policy rules, $\tau^*(\hat{z})$ and $b^*(\hat{z})$, in (12c) generates the solution for $\hat{z}$. The solution for $\hat{z}$ is then plugged back into (17a) and (17b) to get τ^* and b^* determined by parameter values and, in turn, equations (9c), (12a), and (12b) give the balanced growth rate, g, and the long-run ratios of consumption to private capital and public capital to environmental quality, $\hat{\omega}$ and $\hat{x}$.

The system of equations (17a), (17b), and (12c) is nonlinear and cannot be solved analytically. We therefore resort to numerical solutions, reported in Table 4, using the parameter values of Table 1.[21] The first result is that we obtain a unique GMA. By choosing its tax-spending policy optimally here, the government manages to resolve indeterminacy in the region of multiple DCE and, more specifically, implements the 'high-$\hat{z}$' equilibrium (characterized by a higher balanced growth rate), which is consistent with her objective.[22] This happens because, as a Stackelberg leader, the government is aware that multiple equilibria may arise in this economy due to the possibility of multiple solutions for $\hat{z}$ in (12c) and sets her instruments so as to impose additional restrictions on $\hat{z}$ and affect the market allocation rule, (12c), in a way that eliminates the possibility of an 'ecological and economic poverty trap'. By having set the policy instruments as functions of $\hat{z}$ the government manages to affect the solution for $\hat{z}$. For instance, notice that additional restrictions are now imposed on $\hat{z}$ so that $0 < \tau^*(\hat{z}) < 1$ and $0 < b^*(\hat{z}) \leq 1$ hold, thus reducing the set of equilibria.[23]

[21] Recall that for $\sigma > 0.3$, g is unique and rises monotonically with b at the DCE level. Solving the growth-maximization problem then requires to set $b^* = 1$ and hence drop equation (17b).

[22] Notice that if the growth-maximizing values of the policy instruments, τ^* and b^*, were plugged in the CDE equations, two regimes would arise for $\sigma = 0.1 - 0.3$. For instance, for $\tau^* = 0.493$ and $b^* = 0.507$ when $\sigma = 0.1$, we would get two solutions for $\hat{z}$ from equation (12c), $\hat{z}_1 = 0.073$ and $\hat{z}_2 = 1.492$, associated, in turn, with two balanced growth rates through equation (9c), $g_1 = 0.042$ and $g_2 = 0.280$. By contrast, $\hat{z}$ is determined simultaneously with τ^* and b^* under this second-best setting and is unique.

[23] Boldrin (1992) provides a preliminary discussion of the lines along which fiscal policy may be used to eliminate the multiplicity of equilibria by assuming that the government has the informational ability to apply a nonlinear tax scheme, dependent on the level of the capital stock in existence when the tax is actually levied.

The second finding is that the growth-maximizing share of infrastructure spending in the region of multiple DCE is less than one, i.e. $b^* < 1$, which implies that attaining the highest possible growth rate here requires public investment in the environment. This result directly follows from the comparative statics findings of the previous subsection with regard to the 'good equilibrium'. In contrast to the case of exogenous discounting, the growth-maximizing share of public environmental investment is not zero here. Hence, the key message is that governments aiming at growth maximization should promote pollution abatement policies, even if no environmental externalities are postulated in production.

In this setting, the growth-maximizing tax rate clearly differs from the Barro (1990) tax rule, $\tau^* = (1 - a)$, also encountered in Futagami et al. (1993) and Glomm and Ravikumar (1997). As shown in Table 4, optimal taxation in this economy depends not only on a, but also on demand-driven parameters, like the degree of intertemporal substitution, σ (see below). The main mechanism behind this result is that the endogeneity of the time preference changes the marginal cost of public funds. An increase in τ not only affects growth by increasing public capital expenditures and decreasing private capital, but also impacts on the steady-state RTP, which through the Euler equation affects directly the balanced growth rate and thus the tax base.

Finally, the comparative statics properties of the GMA resemble those of the 'good' equilibrium in the region of the DCE multiplicity ($0.1 \leq \sigma \leq 0.3$) and the 'bad' equilibrium in the region of the DCE uniqueness ($0.4 \leq \sigma \leq 1.0$), because in the former region the 'high-growth' regime is implemented, while in the latter region the unique equilibrium corresponds to the 'low-growth' regime (see Table 2). Hence, as σ increases, z increases (falls) for $0.1 \leq \sigma \leq 0.3$ ($0.4 \leq \sigma \leq 1.0$). The optimal tax rate, τ^*, which is influenced by changes in σ and z, reacts in the opposite direction when z falls (rises) so as to induce increased (reducted) tax revenues for the provision of public infrastructure, and hence the growth rate and resulting tax base tend to also fall (rise). The optimal share of productive expenditures, b^*, is constant in the DCE uniqueness region, but increases with a rise in σ in the DCE multiplicity region, since z falls, thus lowering growth and the tax base, which finances public infrastructure. The rest of the variables and the balanced growth rate (ω, x, ρ, g) are determined by z, τ^*, b^* and thus respond again in opposite way between the two regions.

The main findings of this subsection can be summarized as follows.

Result 3 *Under the assumptions of Section 2, the growth-maximizing policy rules imply $b^* < 1$ and $\tau^* \neq (1-a)$. These rules implement the 'good' equilibrium in the region of multiplicity in the DCE.*

5 Ramsey fiscal policy and green preferences

In this section, we endogenize policy, as summarized by the time paths of the two policy instruments, $0 < \tau < 1$ and $0 < b \leq 1$, by solving the Ramsey problem of the government. Given a welfare criterion that the government uses to evaluate different allocations, the Ramsey problem for the government is to pick the fiscal policy that generates the competitive equilibrium allocation with the highest value of this criterion.

Definition 3 *A Ramsey Allocation is given under Definition 1 when (i) the government chooses the tax rate, τ, and the levels of infrastructure and environmental investments, G and E, in order to maximize the welfare of the economy by taking into account the aggregate optimality conditions of the competitive equilibrium, and (ii) the government budget constraints and the feasibility and technological conditions are met.*

The government seeks to maximize welfare in the economy subject to the outcome of the decentralized equilibrium, summarized by (9a)-(9d), and the government budget constraint in (8a). Due to the variable RTP, Pontryagin's maximum principle cannot be applied directly. To solve the problem within the standard optimal control framework, we follow the procedure employed by Obstfeld (1990) and introduce an additional 'artificial' variable that accounts for the development of the accumulated discount rate, $\Delta(t) \equiv \int_0^t \rho(C_v, N_v)dv$. Then, the objective of the goverment is to maximize intertemporal utility

$$\max U^R = \int_0^\infty \frac{\left(C^\nu N^{1-\nu}\right)^{1-\sigma}}{1-\sigma} \exp\left[-\int_0^t \rho(C_v, N_v)dv\right] dt$$

constrained by the competitive equilibrium ((8a), (9b), (9c), (9d)) and the derivative of $\Delta(t)$ with respect to time which gives, $\dot{\Delta} = \rho(\cdot)$.

The first-order conditions of the Ramsey problem include the Euler equation, the growth rates of private capital, public capital and environmental quality, the resources constraint, the government budget constraint and the optimality conditions with respect to C, K_g, N, τ, G, E, Δ:

$$\nu C^{\nu(1-\sigma)-1} N^{(1-\nu)(1-\sigma)} e^{-\Delta} - \tilde{\lambda}_1 + \frac{1}{N}\tilde{\lambda}_5 \acute{\rho}\left(\frac{C}{N}\right) = 0 \tag{18a}$$

$$\tilde{\lambda}_1(1-a)(1-\tau)K^a K_g^{-a} - \tilde{\lambda}_2\delta - \tilde{\lambda}_3(1-a)sK^a K_g^{-a} + \tilde{\lambda}_4(1-a)\tau K^a K_g^{-a} = -\dot{\tilde{\lambda}}_2 \tag{18b}$$

$$(1-\nu)C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)-1} e^{-\Delta} + \tilde{\lambda}_3\delta_N - \tilde{\lambda}_5 \frac{C}{N^2}\acute{\rho}\left(\frac{C}{N}\right) = -\dot{\tilde{\lambda}}_3 \tag{18c}$$

$$\tilde{\lambda}_1 = \tilde{\lambda}_4 \tag{18d}$$

$$\tilde{\lambda}_2 = \tilde{\lambda}_4 \tag{18e}$$

$$\tilde{\lambda}_3 = \frac{1}{\theta}\tilde{\lambda}_4 \tag{18f}$$

$$\frac{\left(C^\nu N^{1-\nu}\right)^{1-\sigma}}{1-\sigma} e^{-\Delta} = \dot{\tilde{\lambda}}_5 \tag{18g}$$

where $\tilde{\lambda}_1$, $\tilde{\lambda}_2$, $\tilde{\lambda}_3$, $\tilde{\lambda}_4$, $\tilde{\lambda}_5$ are the dynamic multipliers associated with (8a), (9b), (9c), (9d), and the condition $\dot{\Delta} = \rho(\cdot)$, respectively. Equations (18a)-(18g), the optimality condition for the Hamiltonian $\lim_{t\to\infty} H^R = 0$ as given by:

$$\frac{C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)}}{1-\sigma} e^{-\Delta} + \tilde{\lambda}_1\dot{K} + \tilde{\lambda}_2\dot{K}_g + \tilde{\lambda}_3\dot{N} + \tilde{\lambda}_5\rho(\cdot) = 0 \tag{18h}$$

and equations (9a)-(9d) characterize the solution of the Ramsey problem.[24]

The methodology to derive the stationary Ramsey allocation is the following. Let us define $\lambda_j \equiv \tilde{\lambda}_j e^{\Delta(t)}$ where $j = 1, 2, 3, 4, 5$. We can now transform the variables by defining $\omega \equiv \frac{C}{K}$, $z \equiv \frac{K}{K_g}$, $x = \frac{K_g}{N}$, $\phi \equiv \lambda_3 N$, and $\chi \equiv \lambda_4 K_g$. Then, as demonstrated in detail in the Companion Appendix, the dynamics of (18a)-(18g) are equivalent to the dynamics of (11a)-(11c) together

[24]See the Companion Appendix of the paper for the detailed analysis and derivation of the Ramsey allocations.

with (19a)-(19d) presented below:

$$\frac{\dot{\phi}}{\phi} = \frac{\chi}{\theta x \phi}\left[\delta - \delta_N - (1-a)z^a(1-s)\right] + \Theta(\tau, b)z^a x + \rho(\omega z x) \tag{19a}$$

$$\frac{\dot{\chi}}{\chi} = b\tau z^a - \left(1 - \frac{s}{\theta}\right)(1-a)z^a + \rho\left(\omega z x\right) \tag{19b}$$

$$\frac{1}{\theta x}\left[\delta_N - \delta + \theta\omega z x + (1-a)(1-s)z^a\right]\left[\nu(1-\sigma)\rho\left(\omega z x\right) - \omega z x \rho'\left(\omega z x\right)\right] - (1-\sigma)\omega z \rho\left(\omega z x\right) = 0 \tag{19c}$$

$$(\chi z + \chi + \phi)(b\tau z^a - \delta) = 0 \tag{19d}$$

where (19a)-(19d) along with (11a)-(11c) constitute the dynamics of the Ramsey equilibrium.

In the long run we have that all stationary variables should grow at a common, zero rate. Then, after some algebra, the long-run allocation is given by:

$$b\tau = \rho\left(\omega z x\right) z^{-a} - \left(1 - \frac{s}{\theta}\right)(1-a) \tag{20}$$

$$\left[\nu(1-\sigma)\rho\left(\omega z x\right) - \omega z x \rho'\left(\omega z x\right)\right] = \frac{(1-\sigma)\omega z \rho\left(\omega z x\right)\theta x}{\left[\delta_N - \delta + \theta\omega z x + (1-a)(1-s)z^a\right]} \tag{21}$$

along with the three-equation system of the long-run DCE, (12a)-(12c). Thus, the long run Ramsey allocation is given by a five-equation system, (12a)-(12c) and (20)-(21) with five unknowns ω, z, x, τ, b, where τ and b are endogenous variables here. As this system is analytically intractable, we will present numerical solutions by using the same parameter values as before and experiment with different values of $1-\nu$, which measures how much agents value environmental quality vis-à-vis consumption.

Table 5 reports results for varying values of $1-\nu$ in the region $0.1 \leq 1-\nu \leq 0.9$. First it should be pointed out that the Ramsey allocation is unique and implements the 'good' equilibrium.[25] This outcome is feasible because the Ramsey planner has two policy instruments,

[25] Notice that if the welfare-maximizing values of the policy instruments displayed in Table 5 were plugged in the DCE equations, two regimes would arise. For instance, for $\tau^* = 0.426$ and $b^* = 0.507$ when $\nu = 0.5$, we would get two solutions for $\hat{z}$ from equation (12c), $\hat{z}_1 = 0.087$ and $\hat{z}_2 = 2.555$, associated, in turn, with two balanced growth rates through equation (9c), $g_1 = 0.038$ and $g_2 = 0.314$. By contrast, $\hat{z}$ is determined simultaneously with τ^* and b^* under this second-best setting and is unique.

namely taxation and the allocation of expenditures, to impact the dynamics of the economy, thus eliminating the possibility of a trap, and to attain welfare maximation. This is reflected in the additional equations under this second-best setting in comparison to the competitive equilibrium, (20) and (21), which give the values of the policy instruments as functions of the state of the economy (described by the ω, z, and x allocations).

Inspection of the results then reveals that when agents care more about the environment relative to private consumption ($1-\nu$ increases), it is optimal to allocate more tax revenues to environmental care vis-à-vis infrastructure provision (b falls) and to tax more (τ rises). These findings are in contrast with those reported by Economides and Philippopoulos (2008) who use a similar setup but assume exogenous RTP and the resulting market equilibrium is unique. The authors have found that the more the representative agent cares about the environment (i.e. the higher is $1-\nu$), the more growth-enhancing policies the Ramsey government finds it optimal to choose (b rises), along with lower tax rates (τ falls), in order to achieve higher growth, which will yield larger tax bases and extra revenues for cleanup policy.[26]

The intuition behind these results is as follows. An increase in environmental concern implies a stronger utility effect of environmental quality and thus agents can directly benefit in terms of welfare if the government increases environmental investment by raising taxation and by altering the allocation of existing revenues from infrastructure towards abatement *('Static Amenity Channel')*. When only pure 'productive' expenditures impact the growth process in the economy, as is the case in Economides and Philippopoulos (2008), the opposite policy mix of using lower taxes and shifting the allocation of tax revenues towards 'productive' spending forms an optimal government response to greener preferences through dynamically creating a higher tax base that finances both types of expenditures *('Dynamic Supply-Side Channel')*. In their case, the 'static' effect on utility is negative because of a lower environmental quality resulting from the reallocation of spending towards infrastructure, but is outweighted by the induced higher growth, which results in higher intertemporal utility. In our model the initial decline in environmental quality from such a shift in public spending allocation towards in-

[26]These results are obtained in our setting by assuming exogenous RTP and are available upon request.

frastructure impacts also on subjective discounting, making agents more impatient, and can lead the economy to a vicious cycle of low growth and poor environmental quality, as shown in the analysis of the market economy. Hence, when environmental quality not only enters the utility function, but also exerts a positive externality on impatience, greener preferences lead the Ramsey planner to engage in green spending reforms by diverting resources from infrastructure to the environment. This raises directly welfare via the 'static' channel and additionally impacts the growth dynamics positively, given the implicit productive role of environmental quality through subjective discounting *('Dynamic Patience Channel')*.[27]

The main findings of this section are summarized as follows.

Result 4 *The long-run Ramsey allocation is unique. In this allocation, the more the agents care about the environment, the more green spending reforms the Ramsey government finds it optimal to choose.*

6 Concluding remarks

This paper studied optimal policies in a general equilibrium model of growth and natural resources, in which the endogeneity of time preference to aggregate consumption and environmental quality gives rise to multiple equilibria in the market economy. Analyzing the different policy prescriptions for each regime type, we showed that fast-growing economies can achieve a double dividend in terms of higher environmental quality and economic growth through green spending reforms. When we endogenized fiscal choices we obtained the following results. First, second-best fiscal policy, aiming at growth- or welfare-maximization, eliminates the possibility of an 'environmental and economic poverty trap' and leads the economy to the desired BGP. Second, we derived a positive growth-maximizing share of public abatement expenditures vis-à-vis 'productive' spending, without assuming environmental externalities in production. Finally, we demonstrated that, in contrast to the case of exogenous RTP, greener preferences are not

[27]Notice that the response of the growth rate will depend on which of the two dynamic channels (i.e. the 'Supply-Side' or the 'Patience' one) dominates. For low levels of environmental concern in Table 5, the growth rate increases, while for higher levels, the growth rate falls.

associated with more infrastructure-enhancing policies for the Ramsey government.

Given that countries with similar structural characteristics often seem to display divergent economic behavior and environmental performance, our results suggest an additional generating mechanism of multiple equilibria corresponding to this observed divergence. This stems from the linkage between endogenous subjective discounting and environmental quality, with the latter now operating through the demand, rather than the supply, side of the economy by promoting the patience of agents. In turn, our results on the role of second-best optimal policy in driving the economy to a 'high-growth' path, albeit highly stylized, indicate the importance of active policymaking in determining long-run growth and environmental performance. Moreover, to the extent that environmental quality affects individual patience, our findings suggest a channel for the impact of public environmental spending on long-run growth that has been left unnoticed in existing studies. Finally, an interesting avenue for further investigation would be, instead of focusing on fiscal policy, to examine a different set of environmental policy instruments, like pollution permits and numerical targets for cutting emissions. This extension is left for future research.

A Appendix: Proof of Proposition 1

Let us first investigate the conditions for a well-defined equilibrium in the long run. In order for the balanced growth rate to be positive, we must have $\hat{z} > \left(\frac{\delta}{b\tau}\right)^{\frac{1}{a}}$ from (9c). Also, in order for $\hat{\omega}(\hat{z}) > 0$ and $\hat{x}(\hat{z}) > 0$ to hold, we must have $\hat{z} < \frac{1-\tau}{b\tau}$ from (12a) and $\hat{z} < \left(\frac{\delta+\delta_N}{b\tau}\right)^{\frac{1}{a}}$ from (12b), since we are assuming that $\Theta(\tau, b) \equiv \theta\,(1-b)\,\tau - s < 0$. Combining all the above we get the following for the domain of $\hat{z}$:

(i) if $\delta + \delta_N \geq (1-\tau)^a (b\tau)^{1-a}$, then $\left(\frac{\delta}{b\tau}\right)^{\frac{1}{a}} < \hat{z} < \frac{1-\tau}{b\tau}$

(ii) if $\delta + \delta_N \leq (1-\tau)^a (b\tau)^{1-a}$, then $\left(\frac{\delta}{b\tau}\right)^{\frac{1}{a}} < \hat{z} < \left(\frac{\delta+\delta_N}{b\tau}\right)^{\frac{1}{a}}$.

The next step is to solve (12c) by separating function $\Phi(z)$ in two parts and finding their intersection. We thus define $\Gamma(z) \equiv -\sigma b\tau z^a + a(1-\tau)z^{a-1} - (1-\sigma)\delta$ and $\Lambda(z) \equiv \rho(z \cdot \omega(z) \cdot x(z))$. $\Gamma(z)$ has the following properties:

1. $\Gamma(z)$ is continuous in z.
2. $\lim_{z \to \left(\frac{\delta}{b\tau}\right)^{\frac{1}{a}}} \Gamma(z) = a(1-\tau)\left(\frac{\delta}{b\tau}\right)^{-\frac{(1-a)}{a}} - \delta$.
3. $\lim_{z \to \frac{1-\tau}{b\tau}} \Gamma(z) = (a-\sigma)(1-\tau)^a (b\tau)^{1-a} - (1-\sigma)\delta$.
4. $\lim_{z \to \left(\frac{\delta+\delta_N}{b\tau}\right)^{\frac{1}{a}}} \Gamma(z) = -\sigma\delta_N - \delta + a(1-\tau)\left(\frac{\delta+\delta_N}{b\tau}\right)^{-\frac{(1-a)}{a}}$.
5. $\frac{\partial \Gamma(z)}{\partial z} = -a\sigma b\tau z^{a-1} - (1-a)a(1-\tau)z^{a-2} < 0$.
6. $\frac{\partial^2 \Gamma(z)}{\partial z^2} = (1-a)a\sigma b\tau z^{a-2} + (2-a)(1-a)a(1-\tau)z^{a-3} > 0$.

In turn, $\Lambda(z)$ has the following properties:

1. $\Lambda(z)$ is continuous in z.
2. $\lim_{z \to \left(\frac{\delta}{b\tau}\right)^{\frac{1}{a}}} \Lambda(z) = \rho\left(\frac{-\delta_N\left[1-\tau-\delta^{\frac{1}{a}}(b\tau)^{\frac{-(1-a)}{a}}\right]}{\Theta(\tau,b)}\right)$.
3. $\lim_{z \to \frac{1-\tau}{b\tau}} \Lambda(z) = \rho(0) = \check{\rho}$.
4. $\lim_{z \to \left(\frac{\delta+\delta_N}{b\tau}\right)^{\frac{1}{a}}} \Lambda(z) = \rho(0) = \check{\rho}$.
5. $\frac{\vartheta \Lambda(z)}{\vartheta z} = \underbrace{\frac{\rho'(.)b\tau}{\Theta(\tau,b)}}_{<0}[\underbrace{-(b\tau z^a - \delta - \delta_N)}_{>0} + \underbrace{(1-\tau-b\tau z)az^{a-1}}_{>0}] < 0$.
6. $\frac{\vartheta^2 \Lambda(z)}{\vartheta z^2} = \underbrace{\frac{b\tau}{\Theta(\tau,b)}}_{<0}\{\underbrace{\rho''(.)\frac{b\tau}{\Theta(\tau,b)}[-(b\tau z^a - \delta - \delta_N) + (1-\tau-b\tau z)az^{a-1}]^{\mathbf{2}}}_{<0}$
$\underbrace{-\rho'(.)a[(1+a)b\tau z^{a-1} + (1-a)(1-\tau)z^{a-2}]}_{<0}\} > 0$.

Therefore, from 5 and 6 of $\Gamma(z)$ and $\Lambda(z)$ it follows that they both are strictly decreasing and convex functions. This implies that if an intersection exists, it can be unique or multiple. Then, assuming equilibrium existence, we have from 2-4 of $\Gamma(z)$ and $\Lambda(z)$ that if $a(1-\tau)\left(\frac{\delta}{b\tau}\right)^{-\frac{(1-a)}{a}} - \delta > \lim_{z \to \left(\frac{\delta}{b\tau}\right)^{\frac{1}{a}}} \Lambda(z)$, then a sufficient condition for more than one intersections is $(a-\sigma)(1-$

$\tau)^a(b\tau)^{1-a} - (1-\sigma)\delta > \check{\rho}$ under (i), or $-(\sigma\delta_N + \delta) + \alpha(1-\tau)\left(\frac{b\tau}{\delta+\delta_N}\right)^{\frac{1-\alpha}{\alpha}} > \check{\rho}$ under (ii). By contrast, if $a(1-\tau)\left(\frac{\delta}{b\tau}\right)^{-\frac{(1-a)}{a}} - \delta < \lim_{z \to \left(\frac{\delta}{b\tau}\right)^{\frac{1}{a}}} \Lambda(z)$, then a sufficient condition for more than one intersections is $(a-\sigma)(1-\tau)^a(b\tau)^{1-a} - (1-\sigma)\delta < \check{\rho}$ under (i), or $-(\sigma\delta_N + \delta) + \alpha(1-\tau)\left(\frac{b\tau}{\delta+\delta_N}\right)^{\frac{1-\alpha}{\alpha}} < \check{\rho}$ under (ii). That is, if $\Gamma(z)$ starts above (below) $\Lambda(z)$, more than one intersections can exist when $\Gamma(z)$ also ends above (below) $\Lambda(z)$.

References

Acemoglu D., P. Aghion, L. Bursztyn, and D. Hemous, 2011, 'The environment and directed technical change', CEPR Discussion Paper 8660.

Agénor P.-R., 2010, 'A theory of infrastructure-led development', *Journal of Economic Dynamics and Control*, 34, 932-950.

Ayong Le Kama A. and K. Schubert, 2007, 'A note on the consequences of an endogenous discounting depending on the environmental quality', *Macroeconomic Dynamics*, 11, 272-289.

Balestra C. and D. Dottori, 2009, 'Aging society, health and the environment', Center for Operation Research and Econometrics (CORE) Discussion Paper, Université catholique de Louvain, No. 37.

Barro R.J., 1990, 'Government spending in a simple model of endogenous growth', *Journal of Political Economy* 98, 103-125.

Boldrin M., 1992, 'Dynamic externalities, multiple equilibria, and growth', *Journal of Economic Theory*, 58, 198-218.

Bovenberg A. and R. de Mooij, 1997, 'Environmental tax reform and endogenous growth', *Journal of Public Economics*, 63, 207-237.

Bovenberg A. and S. Smulders, 1996, 'Transitional impacts of environmental policy in an endogenous growth', *International Economic Review*, 37(4), 861-893.

Choi H., N.C. Mark and D. Sul, 2008, 'Endogenous discounting, the world saving glut and the U.S. current account', *Journal of International Economics*, 75, 30-53.

Dioikitopoulos E. and S. Kalyvitis, 2010, 'Endogenous time preference and public policy: Growth and fiscal implications', *Macroeconomics Dynamics*, 14, 243-257.

Dolmas J., 1996, 'Balanced-growth-consistent recursive utility', *Journal of Economic Dynamics and Control*, 20, 657-680.

Economides G., H. Park and A. Philippopoulos, 2007, 'Optimal protection of property rights in a general equilbrium model of growth', *Scandinavian Journal of Economics*, 109(1), 153-175.

Economides G. and A. Philippopoulos, 2008, 'Growth enhancing policy is the means to sustain the environment', *Review of Economic Dynamics*, 11, 207-219.

Ennis H.M. and T. Keister, 2005, 'Optimal fiscal policy under multiple equilibria', *Journal of Monetary Economics*, 52, 1359- 1377.

Epstein L.G., 1987, 'A simple dynamic general equilibrium model', *Journal of Economic Theory*, 41, 68-95.

Futagami K., Y. Morita and A. Shibata, 1993, 'Dynamic analysis of an endogenous growth model with public capital', *Scandinavian Journal of Economics*, 95, 607-625.

Glomm G. and B. Ravikumar, 1995, 'Endogenous public policy and multiple equilibria', *European Journal of Political Economy*, 11, 653-662.

Glomm G. and B. Ravikumar, 1997, 'Productive government expenditures and long-run growth', *Journal of Economic Dynamics and Control*, 21, 183-204.

Goulder L.H., 1995, 'Environmental taxation and the double dividend: A reader's guide', *International Tax and Public Finance*, 2, 157-183.

Gradus R. and S. Smulders, 1993, 'The trade-off between environmental care and long-term growth - Pollution in three prototype growth models', *Journal of Economics*, 58(1), 25-51.

Grandmont J.M., 1986, 'Stabilizing competitive business cycles', *Journal of Economic Theory*, 40, 57-76.

Greiner A., 2005, 'Fiscal policy in an endogenous growth model with public capital and pollution', *Japanese Economic Review*, 56(1), 67-84.

Grossman G.M. and A.B. Krueger, 1995, 'Economic growth and the environment', *Quarterly Journal of Economics*, 110, 353-377.

Haibara T, 2009, 'Environmental funds, public abatement and welfare', *Environmental and Resource Economics*, 44, 167-177.

Ikefuji M. and R. Horii, 2007, 'Wealth heterogeneity and escape from the poverty-environment trap', *Journal of Public Economic Theory*, 9(6), 1041-1068.

Jouvet P.A., P. Pestieau and G. Ponthière, 2010, 'Longevity and environmental quality in an OLG model', *Journal of Economics*, 100, 191-216.

Kijima M., K. Nishide and A. Ohyama, 2010, 'Economic models for the Environmental Kuznets Curve: A survey', *Journal of Economic Dynamics and Control*, 34, 1187-1201.

Ligthart J.E. and F. van der Ploeg, 1994, 'Pollution, the cost of public funds and endogenous growth', *Economics Letters*, 46, 339-349.

Lines M., 2005, 'Intertemporal equilibrium dynamics with a pollution externality', *Journal of Economic Behavior and Organization*, 56, 349-364.

Mariani F., A. Pérez-Barahona and N. Raffin, 2010, 'Life expectancy and the environment', *Journal of Economic Dynamics and Control*, 34(4), 798-815.

Matsuyama K., 1991, 'Increasing returns, industrialization, and indeterminacy of equilibrium', *Quarterly Journal of Economics*, 106, 617-650.

Obstfeld M., 1990, 'Intertermporal dependence, impatience, and dynamics', *Journal of Monetary Economics*, 26, 45-75.

Palivos T., P. Wang and J. Zhang, 1997, 'On the existence of balanced growth equilibrium', *International Economic Review*, 38(1), 205-223.

Park H. and A. Philippopoulos, 2004, 'Indeterminacy and fiscal policies in a growing economy', *Journal of Economic Dynamics and Control,* 28, 645-660.

Pautrel X., 2008, 'Reconsidering the impact of the environment on long-run growth when pollution influences health and agents have a finite-lifetime', *Environmental and Resource Economics*, 40(1), 37-52.

Pautrel X., forthcoming, 'Pollution, private investment in healthcare, and environmental policy', *Scandinavian Journal of Economics.*

Pérez R. and J. Ruiz, 2007, 'Global and local indeterminacy and optimal environmental public policies in an economy with public abatement activities', *Economic Modelling*, 24, 431-452.

Pittel K., 2002, *Sustainability and Endogenous Growth*, Cheltenham, UK: Edward Elgar.

Prieur F., 2009, 'The environmental Kuznets curve in a world of irreversibility', *Economic Theory*, 40, 57-90.

Ramirez Harrington D., M. Khanna and D. Zilberman, 2005, 'Conservation capital and sustainable growth', *Oxford Economic Papers*, 57, 336-359.

Reichlin P., 1986, 'Equilibrium cycles in an overlapping generations economy with production', *Journal of Economic Theory*, 40, 89-102.

Rodrik D., 1996, 'Coordination failures and government policy: A model with applications to East Asia and Eastern Europe', *Journal of International Economics,* 40, 1-22.

Schmitt-Grohé S. and M. Uribe, 2003, 'Closing small open economy models', *Journal of International Economics,* 61, 163-185.

Shi S., 1999, 'Fashion and wealth accumulation', *Economic Theory*, 14, 439-461.

Smulders S. and R. Gradus, 1996, 'Pollution abatement and long-term growth', *European Journal of Political Economy*, 12, 505-532.

Uzawa H., 1968, *Time preference, the consumption function, and the optimum asset holdings*, In: Wolfe, J.N. (Ed.), Value, Capital, and Growth: Papers in Honour of Sir John Hicks, University of Edinburgh Press, Edinburgh.

Valente S., 2005, 'Sustainable development, renewable resources and technological progress', *Environmental and Resource Economics*, 30, 115-125.

Varvarigos D., 2010a, 'On the joint dynamics of pollution and capital accumulation', University of Leicester, Department of Economics, Working Paper No. 11/05.

Varvarigos D., 2010b, 'Environmental degradation, longevity, and the dynamics of economic development', *Environment and Resource Economics*, 46, 59-73.

Viscusi W.K., J. Huber and J. Bell, 2008, 'Estimating discount rates for environmental quality from utility-based choice experiments', *Journal of Risk Uncertainty*, 37 (2), 199-220.

Woodford M., 1986, 'Stationary sunspot equilibria in a finance constrained economy, *Journal of Economic Theory*, 40, 128-137.

Xepapadeas A., 2005, 'Economic growth and the environment', in *Handbook of Environmental Economics* volume 3, edited by K.G. Mäler and J.R. Vincent, North-Holland, Amsterdam.

Yale Center for Environmental Law and Policy, 2010, Environmental Performance Index. [Data available online at http://epi.yale.edu].

Yanase A., 2011, 'Impatience, pollution and indeterminacy', *Journal of Economic Dynamics and Control,* 35, 1789-1799.

Table 1. Values for parameters and exogenous policy instruments

Parameters and policy instruments	Description	Value
α	share of private capital in the production function	0.5
δ	depreciation rate of private and public capital	0.025
δ_N	regeneration rate of natural resources	0.5
γ	slope in the impatience function	1.0
$\breve{\rho}$	lowest bound for the impatience function	0.04
θ	transformation of environmental spending in natural stock	1.0
s	polluting effect of economic activity	0.5
τ	income tax rate	0.6
b	share of tax revenues allocated to public infrastructure	0.5

Table 2. Decentralized competitive equilibrium (DCE)

σ	$\hat{z}_1$	$\hat{z}_2$	$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{x}_1$	$\hat{x}_2$	$\hat{\rho}_1$	$\hat{\rho}_2$	g_1	g_2
					Multiplicity					
0.1	0.0430	1.0170	1.8667	0.0941	11.1583	1.1030	0.9357	0.1456	0.0372	0.2775
0.2	0.0426	1.1238	1.8760	0.0593	11.2174	0.9762	0.9366	0.1051	0.0369	0.2930
0.3	0.0422	1.2657	1.8851	0.0180	11.2754	0.8333	0.9374	0.0590	0.0366	0.3125
					Uniqueness					
0.4	0.0418	-	1.8941	-	11.3325	-	0.9382	-	0.0364	-
0.5	0.0415	-	1.9029	-	11.3885	-	0.9389	-	0.0361	-
0.6	0.0411	-	1.9115	-	11.4437	-	0.9397	-	0.0358	-
0.7	0.0408	-	1.9201	-	11.4980	-	0.9404	-	0.0356	-
0.8	0.0405	-	1.9285	-	11.5515	-	0.9411	-	0.0353	-
0.9	0.0401	-	1.9367	-	11.6042	-	0.9418	-	0.0351	-
1.0	0.0398	-	1.9449	-	11.6560	-	0.9425	-	0.0349	-

Note: $a = 0.5$, $\delta = 0.025$, $\delta_N = 0.5$, $\theta = 1$, $s = 0.5$, $\gamma = 1$, $\breve{\rho} = 0.04$, $\tau = 0.6$, $b = 0.5$.

Table 3. Effects of b on DCE

b	$\hat{z}_1$	$\hat{z}_2$	$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{x}_1$	$\hat{x}_2$	$\hat{\rho}_1$	$\hat{\rho}_2$	g_1	g_2
					Uniqueness					
0.35	0.011	1.904	-	0.0002	-	1.550	-	0.041	-	0.265
					Multiplicity					
0.4	0.019	1.652	2.882	0.003	25.612	1.204	1.430	0.046	0.008	0.283
0.45	0.029	1.445	2.307	0.008	16.580	0.980	1.145	0.051	0.021	0.300
0.5	0.042	1.266	1.885	0.018	11.275	0.833	0.937	0.059	0.037	0.313
0.55	0.060	1.096	1.551	0.037	7.876	0.745	0.774	0.070	0.056	0.321
0.6	0.085	0.921	1.266	0.071	5.538	0.719	0.637	0.087	0.080	0.320
0.65	0.123	0.727	1.002	0.137	3.809	0.778	0.511	0.117	0.112	0.308
0.7	0.201	0.492	0.703	0.276	2.345	1.027	0.372	0.179	0.163	0.270

Note: $a = 0.5$, $\delta = 0.025, \delta_N = 0.5$, $\theta = 1$, $s = 0.5$, $\gamma = 1$, $\check{\rho} = 0.04$, $\tau = 0.6$, $\sigma = 0.3$.

Table 4. Growth-maximizing allocations

σ	τ	b	z	ω	x	ρ	g
			DCE Multiplicity				
0.1	0.493	0.507	1.492	0.110	0.700	0.154	0.280
0.2	0.452	0.552	1.678	0.099	0.523	0.126	0.299
0.3	0.401	0.623	2.006	0.069	0.346	0.088	0.329
			DCE Uniqueness				
0.4	0.501	1.000	0.971	0.013	0.064	0.041	0.469
0.5	0.528	1.000	0.658	0.153	0.238	0.064	0.404
0.6	0.565	1.000	0.430	0.293	0.471	0.099	0.346
0.7	0.588	1.000	0.315	0.404	0.694	0.128	0.305
0.8	0.601	1.000	0.253	0.492	0.886	0.150	0.277
0.9	0.608	1.000	0.215	0.565	1.051	0.168	0.257
1.0	0.612	1.000	0.188	0.629	1.196	0.182	0.241

Note: $a = 0.5$, $\delta = 0.025$, $\delta_N = 0.5$, $\theta = 1$, $s = 0.5$, $\gamma = 1$, $\check{\rho} = 0.04$.

Table 5. Ramsey allocations and green preferences

$1-\nu$	τ	b	z	ω	x	ρ	g
0.1	0.210	0.931	3.217	0.089	0.200	0.097	0.3261
0.2	0.273	0.737	3.055	0.063	0.231	0.085	0.327
0.3	0.329	0.627	2.893	0.044	0.272	0.075	0.3256
0.4	0.379	0.553	2.726	0.030	0.328	0.067	0.321
0.5	0.426	0.498	2.555	0.019	0.406	0.060	0.314
0.6	0.471	0.456	2.380	0.012	0.516	0.055	0.306
0.7	0.513	0.421	2.205	0.007	0.676	0.050	0.296
0.8	0.553	0.394	2.034	0.003	0.914	0.046	0.285
0.9	0.590	0.371	1.870	0.001	1.281	0.043	0.274

Note: $a = 0.5$, $\delta = 0.025$, $\delta_N = 0.5$, $\theta = 1$, $s = 0.5$, $\gamma = 1$, $\check{\rho} = 0.04$, $\sigma = 0.3$.

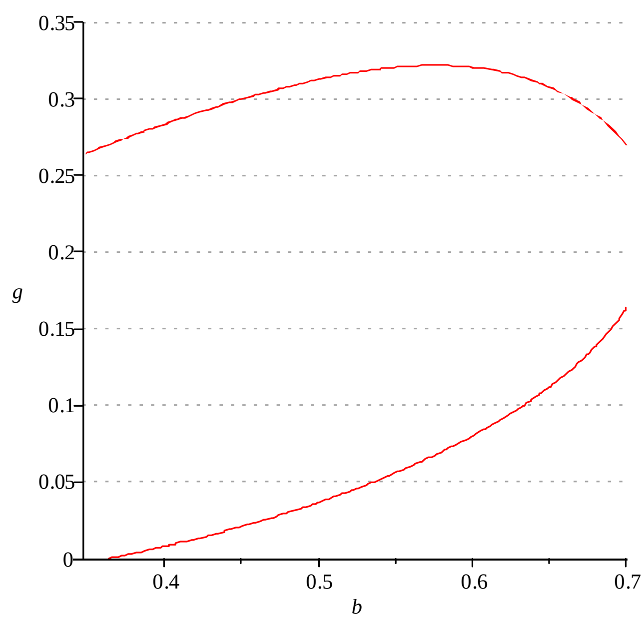

Figure 1: Long-run growth and resource allocation to infrastructure vis-à-vis environmental care

Notes: See Table 3

Companion Appendix to
"Green Spending Reforms, Growth and Welfare with Endogenous Subjective Discounting"
(Not for Publication)

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1 Transitional dynamics and stability analysis

Linearizing (11a)-(11c) around (12a)-(12c) implies that the local dynamics are approximated by the linear system:

$$\begin{bmatrix} \dot{\omega} \\ \dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} J_{\omega\omega} & J_{z\omega} & J_{x\omega} \\ J_{\omega z} & J_{zz} & J_{xz} \\ J_{\omega x} & J_{zx} & J_{xx} \end{bmatrix} \begin{bmatrix} \omega - \hat{\omega} \\ z - \hat{z} \\ x - \hat{x} \end{bmatrix}$$

where the elements of the Jacobian matrix, J, evaluated at the long run are:

$J_{\omega\omega} \equiv \frac{\vartheta\dot{\omega}}{\vartheta\omega} = \hat{\omega}\left[1 - \frac{\rho'(.)\hat{z}\hat{x}}{1-\nu(1-\sigma)}\right] \gtreqless 0,$

$J_{z\omega} \equiv \frac{\vartheta\dot{\omega}}{\vartheta z} = \frac{\hat{\omega}}{[1-\nu(1-\sigma)]\hat{z}}[a(1-\nu)(1-\sigma)\Theta(\tau,b)\hat{z}^a\hat{x} + (1-a)(1-\nu(1-\sigma)-a)(1-\tau)\hat{z}^{a-1} - \rho'(.)\hat{\omega}\hat{z}\hat{x}] \gtreqless 0,$

$J_{x\omega} \equiv \frac{\vartheta\dot{\omega}}{\vartheta x} = \frac{\hat{\omega}}{1-\nu(1-\sigma)}[(1-\nu)(1-\sigma)\Theta(\tau,b)\hat{z}^a - \rho'(.)\hat{\omega}\hat{z}] < 0,$

$J_{\omega z} \equiv \frac{\vartheta\dot{z}}{\vartheta\omega} = -\hat{z} < 0,$

$J_{zz} \equiv \frac{\vartheta\dot{z}}{\vartheta z} = -(1-a)(1-\tau)\hat{z}^{a-1} - ab\tau\hat{z}^a < 0,$

$J_{xz} \equiv \frac{\vartheta\dot{z}}{\vartheta x} = 0,$

$J_{\omega x} \equiv \frac{\vartheta\dot{x}}{\vartheta\omega} = 0,$

$J_{zx} \equiv \frac{\vartheta\dot{x}}{\vartheta z} = a\frac{\hat{x}}{\hat{z}}(\delta + \delta_N) > 0,$

$J_{xx} \equiv \frac{\vartheta\dot{x}}{\vartheta x} = -\Theta(\tau,b)\hat{z}^a\hat{x} > 0.$

The trace and the determinant of J, $trace(J) = J_{\omega\omega} + J_{zz} + J_{xx}$ and $det(J) = J_{\omega\omega}J_{zz}J_{xx} - J_{z\omega}J_{\omega z}J_{xx} + J_{x\omega}J_{\omega z}J_{zx}$ have ambiguous signs. Due to the complexity for the computation of these signs, we provide numerical results for the eigenvalues of J, denoted by ε. First, as a benchmark we perform the computations in the case of exogenous RTP ($\rho'(.) = 0$). In this case, the long-run equilibrium is unique and the dynamic system has two positive and one negative eigenvalues (results available upon request). Hence it follows that there exist locally a one-dimensional stable and a two-dimensional unstable manifolds, since we have one jump variable (ω) and two state/predetermined variables (z, x).

Next, we perform the computations in the case of endogenous RTP ($\rho'(.) > 0$) with regard to the long-run equilibria displayed in Table 2. The findings, reported in Table A1, are similar (i.e. one negative and two positive eigenvalues) for the 'bad' equilibrium, which corresponds to the unique equilibrium of the exogenous RTP case. However, in the 'good' equilibrium there are two negative and one positive eigenvalues, which implies that this regime is saddle-path stable.

2 Equations (19a)-(19d) in the Ramsey allocation

The Hamiltonian of the problem is given by:

$$H^R = \frac{(C^\nu N^{1-\nu})^{1-\sigma}}{1-\sigma} e^{-\Delta} + \tilde{\lambda}_1 \left[(1-\tau)K^a K_g^{1-a} - C - \delta K\right] + \tilde{\lambda}_2 [G - \delta K_g] + \tilde{\lambda}_3 \left[\delta_N N - s K^a K_g^{1-a} + \theta E\right] + \tilde{\lambda}_4 \left[\tau K^a K_g^{1-a} - G - E\right] + \tilde{\lambda}_5 \rho \left(\frac{C}{N}\right)$$

The optimality conditions, as given by equations (18a)-(18h), and the competitive-equilibrium growth rates, given by (9a)-(9d), completely characterize the solution of the Ramsey problem.

2.1 Derivation of (19a)

From (18a):

$$\nu C^{\nu(1-\sigma)-1} N^{(1-\nu)(1-\sigma)} - \lambda_1 + \frac{1}{N}\lambda_5 \rho'(\frac{C}{N}) = 0$$

$$\overset{(18d)}{\Rightarrow} \frac{C}{N}\lambda_5 \rho'(\frac{C}{N}) = \lambda_4 K_g \frac{K}{K_g}\frac{C}{K} - \nu C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)}$$

$$\Rightarrow \frac{C}{N}\lambda_5 \rho'(\frac{C}{N}) = \chi\omega z - \nu C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} \tag{A1}$$

Also, from (18c):

$$\Rightarrow (1-\nu) C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)-1} + \lambda_3 \delta_N - \lambda_5 \frac{C}{N^2}\rho'(\frac{C}{N}) = -\dot{\lambda}_3 + \lambda_3 \rho(\frac{C}{N})$$

$$\Rightarrow \dot{\lambda}_3 N = -(1-\nu) C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} - \lambda_3 N \delta_N + \lambda_5 \frac{C}{N}\rho'(\frac{C}{N}) + \lambda_3 N \rho(\frac{C}{N})$$

$$\overset{(A1)}{\Rightarrow} \dot{\lambda}_3 N = -(1-\nu) C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} - \lambda_3 N \delta_N + \chi\omega z - \nu C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} + \lambda_3 N \rho(\frac{C}{N})$$

$$\Rightarrow \dot{\lambda}_3 N = -C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} - \phi\delta_N + \chi\omega z + \phi\rho(\frac{C}{N}) \tag{A2}$$

From (18d)-(18f) we have $\lambda_1 = \lambda_2 = \lambda_4$, $\dot{\lambda}_1 = \dot{\lambda}_2 = \dot{\lambda}_4 = \theta\dot{\lambda}_3$. Then (18b) implies:

$$\dot{\lambda}_2 = -\lambda_1(1-a)(1-\tau)K^a K_g^{-a} + \lambda_2\delta + \lambda_2(1-a)sK^a K_g^{-a} - \lambda_4(1-a)\tau K^a K_g^{-a} + \lambda_2\rho(\frac{C}{N})$$

and (18c) implies:

$$\dot{\lambda}_3 = -(1-\nu)C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)-1} - \lambda_3\delta_N + \lambda_5\frac{C}{N^2}\rho'(\frac{C}{N}) + \lambda_3\rho$$

$$\Rightarrow \theta\dot{\lambda}_3 = -\theta(1-\nu)C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)-1} - \theta\lambda_3\delta_N + \theta\lambda_5\frac{C}{N^2}\rho'(\frac{C}{N}) + \theta\lambda_3\rho(\frac{C}{N})$$

Then, equating $\theta\dot{\lambda}_3 = \dot{\lambda}_2$ we get:

$$\Rightarrow -\theta(1-\nu)C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} - \theta\lambda_3\delta_N N + \theta\lambda_5\frac{C}{N}\rho'(\frac{C}{N}) + \theta\lambda_3 N\rho(\frac{C}{N}) = -\lambda_1 N(1-a)(1-\tau)K^a K_g^{-a}$$
$$+ \lambda_2 N(1-a)sK^a K_g^{-a} - \lambda_4 N(1-a)\tau K^a K_g^{-a} + \lambda_2 N\delta + \lambda_2 N\rho(\frac{C}{N})$$

$$\overset{\lambda_1=\lambda_2=\lambda_4,\ \dot{\lambda}_1=\dot{\lambda}_2=\dot{\lambda}_4=\theta\dot{\lambda}_3}{\Rightarrow} -\theta(1-\nu)C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} - \lambda_4 N\delta_N + \theta\lambda_5\frac{C}{N}\rho'(\frac{C}{N}) = -\lambda_4 N(1-a)K^a K_g^{-a}$$
$$+ \lambda_4 N(1-a)sK^a K_g^{-a} + \lambda_4 N\delta$$

$$\overset{\chi N \equiv \lambda_4 K_g N \Rightarrow \lambda_4 N = \chi\frac{N}{K_g} \Rightarrow \lambda_4 N = \frac{\chi}{x}}{\Rightarrow} -\theta(1-\nu)C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} + \frac{\chi}{x}\delta_N + \theta\lambda_5\frac{C}{N}\rho'(\frac{C}{N}) = -\frac{\chi}{x}(1-a)z^a(1-s) + \frac{\chi}{x}\delta$$

$$\overset{(A1)}{\Rightarrow} -\theta(1-\nu)C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} + \frac{\chi}{x}\delta_N + \theta\chi\omega z - \theta\nu C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} = -\frac{\chi}{x}(1-a)z^a(1-s) + \frac{\chi}{x}\delta$$

$$\Rightarrow C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} = \frac{\chi}{\theta x}[\delta_N - \delta + (1-a)z^a(1-s)] + \chi\omega z \tag{A3}$$

Then substituting (A2) and (9d) in $\dot{\phi} \equiv \dot{\lambda}_3 N + \lambda_3\dot{N}$ we get that:

$$\Rightarrow \dot{\phi} = -C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} + \chi\omega z + \rho(\omega z x)\phi + \Theta(\tau, b)z^a x\phi$$

$$\overset{(A3)}{\Rightarrow} \frac{\dot{\phi}}{\phi} = \frac{\chi}{\theta x \phi}\left[-\delta_N + \delta - (1-a)z^a(1-s)\right] + \rho(\omega z x) + \Theta(\tau, b) z^a x \tag{19a}$$

2.2 Derivation of (19b)

Using $\lambda_1 = \lambda_2 = \lambda_4$ and $\dot{\lambda}_1 = \dot{\lambda}_2 = \dot{\lambda}_4 = \dot{\lambda}_3\theta$ in (18b) we get:

$$\dot{\lambda}_4 = -\lambda_4(1-a)(1-\tau)K^a K_g^{-a} + \lambda_4\delta + \frac{1}{\theta}\lambda_4(1-a)sK^a K_g^{-a} - \lambda_4(1-a)\tau K^a K_g^{-a} + \lambda_4\rho(\frac{C}{N})$$

$$\Rightarrow \dot{\lambda}_4 = -\lambda_4(1-a)K^a K_g^{-a} + \lambda_4\delta + \frac{1}{\theta}\lambda_4(1-a)sK^a K_g^{-a} + \lambda_4\rho(\frac{C}{N}) \tag{A4}$$

Then from $\dot{\chi} = \dot{\lambda}_4 K_g + \lambda_4 \dot{K}_g$ it follows that:

$$\overset{(A4),\,(9c)}{\Rightarrow} \frac{\dot{\chi}}{\chi} = -\left(1 - \frac{s}{\theta}\right)(1-a)z^a + b\tau z^a + \rho(\omega z x) \tag{19b}$$

2.3 Derivation of (19c)

From (18g) we have in the long run:

$$\frac{\dot{\lambda}_5}{\lambda_5} = \frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{(1-\sigma)\lambda_5} + \rho(\frac{C}{N}) = 0 \Rightarrow \lambda_5 = -\frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{(1-\sigma)\rho(\frac{C}{N})} \tag{A5}$$

Substituting (A5) in (A1) we get that:

$$-\frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{(1-\sigma)\rho(\omega z x)}\frac{C}{N}\rho'(\frac{C}{N}) = \chi\omega z - \nu C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}$$

$$\Longrightarrow -C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}x\omega z\rho'(x\omega z) = (1-\sigma)\rho(\omega z x)\chi\omega z - \nu(1-\sigma)\rho(\omega z x)C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}$$

$$\Longrightarrow C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}\left[\nu(1-\sigma)\rho(\omega z x) - x\omega z\rho'(x\omega z)\right] - (1-\sigma)\rho(\omega z x)\chi\omega z = 0$$

$$\overset{(A3)}{\Longrightarrow} \left[\frac{1}{\theta x}(\delta_N - \delta) + \omega z + \frac{1}{\theta x}(1-a)(1-s)z^a\right]\left[\nu(1-\sigma)\rho(\omega z x) - x\omega z\rho'(x\omega z)\right] - (1-\sigma)\rho(\omega z x)\omega z = 0 \tag{19c}$$

2.4 Derivation of (19d)

From (18h) we have:

$$\frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{1-\sigma} + \lambda_1 \dot{K} + \lambda_2 \dot{K}_g + \lambda_3 \dot{N} + \lambda_5 \rho(\cdot) = 0$$

$$\overset{(18d),\,(18e)}{\Rightarrow} \frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{1-\sigma} + \lambda_4 K_g \frac{K}{K_g}\frac{\dot{K}}{K} + \lambda_4 K_g \frac{\dot{K}_g}{K_g} + \lambda_3 N \frac{\dot{N}}{N} + \lambda_5 \rho(\cdot) = 0$$

$$\overset{(A5)}{\Rightarrow} \frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{1-\sigma} + \chi z \frac{\dot{K}}{K} + \chi \frac{\dot{K}_g}{K_g} + \phi \frac{\dot{N}}{N} - \frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{(1-\sigma)} = 0$$

$$\Rightarrow (\chi z + \chi + \phi)\frac{\dot{K}_g}{K_g} = 0 \overset{(9c)}{\Rightarrow} (\chi z + \chi + \phi)(b\tau z^a - \delta) = 0 \qquad \text{(19d)}$$

Table A1. Eigenvalues of the Jacobian matrix

	'Bad' Equilibrium			'Good' Equilibrium		
σ	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3
0.1	-24.7649	1.7677	0.0158	-1.2587	0.3327	-0.2556
0.2	-24.8602	1.6197	0.0161	-0.9518	0.3134	-0.2875
0.3	-24.9850	1.4922	0.0163	-0.5946	-0.3683	0.2916
0.4	-25.1329	1.3812	0.0166		-	
0.5	-25.2992	1.2837	0.0169		-	
0.6	-25.4802	1.1973	0.0171		-	
0.7	-25.6733	1.1204	0.0174		-	
0.8	-25.8761	1.0513	0.0176		-	
0.9	-26.0869	0.9890	0.0179		-	
1.0	-26.3044	0.9325	0.0182		-	

Note: $a = 0.5$, $\delta = 0.025$, $\delta_N = 0.5$, $\theta = 1$, $s = 0.5$, $\gamma = 1$, $\check{\rho} = 0.04$, $\tau = 0.6$, $b = 0.5$, $\nu = 0.5$.