Learnability of Heterogeneous Misspecification Equilibrium[1]

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Abstract

This paper investigates the learnability of an equilibrium where agents formulate their forecasts under adaptive learning with heterogeneously misspecified econometric models; the equilibrium is called a *Heterogeneous Misspecification Equilibrium* (HME). The paper finds that the learnability condition of the HME is equal to or less than the condition of the equilibrium under learning with a correctly specified model; that is, heterogeneous misspecification in adaptive learning never makes an equilibrium less learnable. Thus, in a basic New Keynesian model with a Taylor-type monetary policy rule, the central bank should follow the Taylor principle to ensure the learnability of an equilibrium, regardless of whether heterogeneous misspecification exists or not.

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1 Introduction

Rational expectations are based on the standard but unlikely assumption that agents have full information about the economy. Agents are considered to understand perfectly the structure of the economy and economic variables so that they are able to make an optimal response to fundamental shocks. In reality, we face an environment in which the information for formulating rational expectations is limited. We always struggle with our understanding of macroeconomic dynamics. Identifying fundamental shocks might be an even more challenging task because these shocks cannot be measured unless the structure of the economy is correctly identified.

The concept of Adaptive Learning has been incorporated into recent macroeconomic studies as an alternative framework in which agents are assumed to formulate their forecasts with imperfect—rather than the usual assumption of perfect—information. Agents are considered to behave as econometricians. Agents specify econometric models (i.e., perceived laws of motion, PLMs) and estimate parameters through least-squares techniques. Using specified models and estimated parameters, agents formulate their forecasts (see Evans and Honkapohja, 2001, 2008).

One of the important issues in this area is whether an equilibrium is attainable under adaptive learning. If estimated parameters in PLMs (and hence, agents' forecasts) converge, an equilibrium will be attained under adaptive learning; that is, the equilibrium is considered to be *learnable*. The concept of learnability provides a criterion for selecting a stable equilibrium among nonexplosive multiple equilibria (see Honkapohja and Mitra, 2004). The literature investigates necessary conditions imposed on macroeconomic policies to ensure the learnability of an equilibrium (see Bullard and Mitra, 2002).

The learnability of an equilibrium has been investigated in a variety of frameworks with respect to information sets held by agents. Evans and Honkapohja (2001) assume that any economic variables are observable so that agents are able to form a correctly specified PLM that includes all variables. They also consider a Restricted Perceptions Equilibrium (RPE) where some variables are unobservable so that agents are constrained to form an underparameterized PLM. Branch and Evans (2006) consider an environment where agents must choose among a list of underparameterized PLMs. Recent studies consider that information sets held by agents are mutually different so that agents form heterogeneous PLMs. Guse (2005) considers that some agents form a correctly specified PLM, whereas other agents form an overparameterized PLM. Berardi (2007) focuses on an economy where a fraction of agents forms an underparameterized PLM, whereas others form a correctly specified PLM.[1]

However, the literature has not considered a situation where information sets of economic variables held by agents are imperfect and mutually different. Such a situation is likely to occur on a daily basis,

[1]These frameworks are applied to the analysis of optimal monetary policies. Evans and Honkapohja (2003a,b, 2006) investigate optimal monetary policies under adaptive learning with a correctly specified PLM. Adam (2005) and Branch and McGough (2009) examine the learnability conditions of RPEs in New Keynesian (NK) macroeconomic models. Honkapohja and Mitra (2005) focus on a situation in which the private sector and the central bank form different forecasts using different PLMs. Berardi (2009) examines the learnability conditions of an equilibrium considered by Berardi (2007) in an NK model.

because there exists an economic variable that is observable for a specific agent and unobservable for other agents. For example, a preference shock possessed by a household is likely to be observable only for this household, and unobservable for other households. In financial markets, the profitability of a borrower tends to be observable only for this borrower, and unobservable for lenders. If such privately observable variables exist in the economy, information sets held by agents become imperfect and mutually different. Hence, agents' PLMs in adaptive learning must be not merely misspecified, but *heterogeneously misspecified* (HM).[2]

This paper investigates the learnability of an equilibrium under adaptive learning with HM PLMs (hereafter, HM learning). The economy consists of a given number of different types of agents. Agents of each type have their own private information about economic variables and form an HM PLM that includes only observable variables. Under HM learning, there can exist an extended version of RPE, which is called a *Heterogeneous Misspecification Equilibrium* (HME). By investigating the learnability of the HME, the paper examines whether heterogeneous misspecification in learning has an impact on the learnability of an equilibrium. Next, the paper compares the learnability condition of the HME with the determinacy condition of an equilibrium; the determinacy of an equilibrium is another necessary condition for the stability of an equilibrium (see Woodford, 1994). Finally, the paper examines the learnability condition of the HME in a basic NK macroeconomic model with a Taylor-type monetary policy rule.[3]

The paper finds that the learnability conditions of the HME are equal to or less than the conditions of an equilibrium under learning with a correctly specified (CS) PLM (hereafter, CS learning). If the steady state of the economy is observable, the magnitude of heterogeneous misspecification weakly increases the learnability of an equilibrium. Thus, heterogeneous misspecification in adaptive learning never makes an equilibrium less learnable. Next, the learnability conditions are equal to or less than the determinacy condition, regardless of whether heterogeneous misspecification exists or not. Finally, in the NK model, even if information sets held by agents conducting adaptive learning become imperfect and mutually different, the central bank should follow the *Taylor principle*—to raise nominal interest rates by more than one-for-one in response to an increase in inflation to secure the determinacy of the rational expectations equilibrium (REE) (see Taylor, 1993; Bernanke and Woodford, 1997)—to ensure the learnability of an equilibrium.

Each of these results offers a specific contribution to the literature on the learnability of an equilibrium. The first result complements the findings of studies of heterogeneity in adaptive learning. Honkapohja and Mitra (2006) find that the learnability of an equilibrium under CS learning is largely

[2]We note that this framework is suggested as a future work by Honkapohja and Mitra (2006, section 7).

[3]Our framework might be considered a special case of the *Misspecification Equilibrium* investigated by Branch and Evans (2006), where agents are able to choose among a list of HM PLMs. However, in their framework, information sets held by agents are assumed to be homogeneous. In addition, analytical investigation of the learnability of an equilibrium under HM learning, which is our main topic, is beyond of the scope of their research.

sufficient for the learnability of equilibria attained under various forms of heterogeneous learning.[4] Our result implies that the same principle holds under heterogeneous misspecification in PLMs. The second result also complements the finding by McCallum (2007) and Ellison and Pearlman (2011), who show that the determinacy of a fundamental REE is a sufficient condition for the learnability of an equilibrium under CS learning. Our result implies that even if heterogeneous misspecification exists, the determinacy condition of the REE continues to be the sufficient condition for the learnability. The final result reinforces the importance of the Taylor principle in monetary policy. Bullard and Mitra (2002) find that the Taylor principle is a necessary and sufficient condition for not only the determinacy, but also the learnability of an equilibrium under CS learning. Our result emphasizes that their result is robust under HM learning.

The paper is structured as follows. The next section provides our model. Section 3 shows the dynamics of the HME and obtains learnability conditions of the HME. Section 4 shows the impact of heterogeneous misspecification on the learnability of an equilibrium. Section 5 examines the learnability condition of the HME in a basic NK model. Finally, we present our conclusions.

2 Model

We consider the general form of a multivariate linear expectational model and confirm its fundamental REE.

2.1 Setup

The economy is organized by two vector equations:

$$y_t = A + By^e_{t+1} + Cw_t, \tag{1}$$

$$w_t = \Phi_n w_{t-1} + v_t. \tag{2}$$

The equations represent the dynamics of endogenous variables and the evolution of exogenous variables. The economy has m endogenous variables and n exogenous variables. y_t is a $m \times 1$ vector of endogenous variables. $w_t = (w_{1t}, \ldots, w_{nt})'$ is a $n \times 1$ vector of autoregressive exogenous variables. The standard deviation of w_i for each i is defined by $\sigma_{ii} > 0$, and the correlation matrix of w_t is defined by $\Gamma_n \equiv \left(\rho_{ij}\right)_{n\times n}$, where $\rho_{ij} \in [0,1]$ denotes the correlation between w_i and w_j for each $i, j \in \{1, \ldots, n\}$. v_t is a $n \times 1$ vector of fundamental shocks with mean zero that drive the stochastic processes of w_t. A is

[4] Specifically, Evans and Honkapohja (1996) and Giannitsarou (2003) investigate an equilibrium where agents' initial beliefs on parameters in their homogeneous PLM are mutually different. Evans, Honkapohja, and Marimon (2001) and Honkapohja and Mitra (2005) consider a situation where agents' learning algorithms (e.g., updating functions or gain parameters) are heterogeneous. Honkapohja and Mitra (2006) introduce *structural heterogeneity*, in which the forecasts of different agents have different effects on the dynamics of the economy.

a $m \times 1$ vector of constant terms. B is a $m \times m$ coefficient matrix of endogenous variables. C is a $m \times n$ coefficient matrix of exogenous variables. Φ_n is a $n \times n$ matrix of autoregressive coefficients of exogenous variables. The superscript e is the operator of expectation in time t, which is not necessarily rational under adaptive learning. The model is purely forward-looking, but it can be applied to the analysis in models with lagged endogenous variables.

For ease of calculation, we impose regularity assumptions on these parameters in Appendix A. In particular, Φ_n is assumed to be a nonzero, diagonal, and nonnegative matrix whose diagonal elements exist in the interval $[0, 1)$: $\Phi_n \equiv diag(\varphi_1, \ldots, \varphi_n)$, where $0 \leq \varphi_i < 1$ for each i and $\varphi_i > 0$ for some i. However, these assumptions never affect our analytical results

2.2 REE

If agents formulate rational expectations, the system (1)–(2) can have a nonexplosive fundamental REE and nonexplosive sunspot REEs that are arbitrarily determined by sunspot variables.

If parameters in the system satisfy the conditions for the economy to be determined at the fundamental REE, the REE takes the form of a minimal state variables (MSV) solution:

$$y_t = a + cw_t, \tag{3}$$

where a is a $m \times 1$ vector of constant terms, and c is a $m \times n$ coefficient matrix for w_t.[5] Using the method of undetermined coefficients, the solution $(\bar{a}, \bar{c})$ is uniquely obtained as:

$$\bar{a} = (I_m - B)^{-1} A, \tag{4}$$

$$\bar{c} = B\bar{c}\Phi_n + C, \tag{5}$$

where I_m is a m-dimensional identity matrix. The constant terms vector $\bar{a}$ corresponds to the steady state $\bar{y}$ of the fundamental REE.

When agents have no information about the structure of the economy, agents may formulate their forecast y^e_{t+1} through adaptive learning. If all economic variables are observable, agents estimate a CM PLM that has the same form of the MSV solution (3). Evans and Honkapohja (2001) show that parameter estimates of (a, c) can converge to the REE (4)–(5); that is, the REE is attainable under CS learning.[6]

[5] McCallum (2004) shows that in the model without lagged endogenous variables y_{t-1}, the fundamental REE must take the form of the MSV solution (3). The determinacy condition of the REE will be briefly shown in footnote 19.

[6] See the *Supplementary Material* to clarify an equilibrium under CS learning; the equilibrium will be found to be a special case of an equilibrium under HM learning.

3 Heterogeneous Misspecification

In what follows, we investigate an equilibrium attainable under HM learning. We consider the existence of private information about exogenous variables. If each exogenous variable is observable only for specific agents while being unobservable for other agents, then information sets held by agents must be imperfect and mutually different so that agents are constrained to form HM PLMs.

Our framework incorporates an equilibrium attainable under CS learning that is conducted when all economic variables are observable. This section intends to describe the common characteristics of equilibria under HM and CS learning, and Section 4 will clarify the impact of heterogeneous misspecification in learning by comparing the two equilibria.

3.1 Information Sets

First, we make an assumption about information sets held by agents conducting HM learning. Under CS learning, the literature assumes that any economic variables $\{y_s, w_s\}_{s=1}^t$ are observable in time t.[7] Accordingly, we assume the existence of private information in the information sets under HM learning, as follows:

Assumption 1 *An exogenous variable w_{it} in $w_t = \{w_{it} | i = 1, ..., n\}$ is observable for agents of type $i \in \{1, ..., n\}$ and unobservable for agents of other types.*

Thus, agents of any type know neither the number n of exogenous variables nor the stochastic processes of unobservable variables.

Note that if all exogenous variables are perfectly correlated such that $\rho_{ij} = 1$ (and hence, $\varphi_i = \varphi_j$ and $\frac{w_{it}}{\sigma_{ii}} = \frac{w_{jt}}{\sigma_{jj}}$) for all i, j, the information sets in the assumption are essentially the same as the information sets under CS learning, which include no private information. When the existence of private information is allowed, the decrease in the correlation of exogenous variables increases the magnitude of heterogeneous misspecification in HM learning.[8]

For reasons of analytical tractability, the population of each type is assumed to be fixed and symmetrically distributed by $\frac{1}{n}$. This environment may be likely in the existence of private information. A preference shock in a household, for example, seems to continue to be privately observable only by

[7] Evans and Honkapohja (2001, chapter 10) name the forecast formulated with the information of y_t the *t dating of expectations*, which is also assumed by Honkapohja and Mitra (2006), McCallum (2007), and Ellison and Pearlman (2011). One thing to note is that Adam (2003) and Adam, Evans, and Honkapohja (2006) indicate that different assumptions about information sets can lead to different conclusions about the learnability of an equilibrium. For example, the $t-1$ *dating of expectations* is considered in which forecasts are formed without the information of y_t. The analysis in that framework is omitted here because of limited space, but it will be one of our future researches.

[8] One might consider the case where information sets held by different types of agents are mutually overlapping. Such a case is also incorporated in Assumption 1 as the case where exogenous variables are highly correlated.

specific agents. If so, the population of agents observing each preference shock should be constant. Further, the observability of different preference shocks seems to be at an equal level.[9]

3.2 HME

With the information set $\{y_s, w_{is}\}_{s=1}^{t}$, agents i formulate their forecast $y_{i,t+1}^e$ like econometricians. First, they are constrained to form an HM PLM that underparameterizes the MSV solution (3):

$$y_t = a_i + c_i w_{it} + \varepsilon_{it}, \tag{6}$$

where a_i and c_i are $m \times 1$ vectors of constant terms and coefficients, and ε_{it} is a $m \times 1$ vector of error terms that are believed to be white noise.[10] Next, agents estimate the parameter matrix $\phi_i' \equiv (a_i, c_i)$ using a recursive least-squares (RLS) method. Those estimates satisfy the orthogonality condition between w_{it} and ε_{it}, and hence agents i never recognize the misspecification in the PLM (6). Finally, using the HM PLM (6) and the estimated ϕ_i, agents i formulate the forecast $y_{i,t+1}^e$ as:

$$y_{i,t+1}^e = a_i + c_i \varphi_i w_{it}. \tag{7}$$

In the same manner, agents j for any $j \neq i$ form an HM PLM and a forecast $y_{j,t+1}^e$.

The aggregate PLM and the aggregate forecast y_{t+1}^e are determined by aggregating Eqs. (6) and (7) for all types. If we assume the *structural homogeneity* that forecasts of different types $\left\{y_{i,t+1}^e\right\}_{i=1}^{n}$ have an equal contribution to the dynamics of the economy, the aggregate PLM is given by the average of PLMs of all types:

$$y_t = a + c w_t + \frac{1}{n}\sum_{i=1}^{n} \varepsilon_{it}, \tag{8}$$

where $a \equiv \frac{1}{n}\sum_{i=1}^{n} a_i$ is the average of constant term vectors for all types, and $c \equiv \left(\frac{1}{n}c_1, ..., \frac{1}{n}c_n\right)$ is a $m \times n$ matrix that combines the coefficients in the HM PLMs (6) for all types multiplied by $\frac{1}{n}$.[11] With the aggregate PLM (8), the aggregate forecast y_{t+1}^e is formulated by:

$$y_{t+1}^e = a + c\Phi_n w_t. \tag{9}$$

Meanwhile, an actual law of motion (ALM) of the economy depends upon y_{t+1}^e in Eq. (9). Then, substituting Eq. (9) into Eqs. (1)–(2), the ALM is determined by:

$$y_t = (A + Ba) + (Bc\Phi_n + C)\, w_t. \tag{10}$$

[9] More general cases are considered in the related literature, although analytical results on the learnability are not given. Berardi (2007) considers that the proportion of agents observing a specific variable is different from that of agents observing other variables. Branch and Evans (2006) allow the proportion of agents choosing different PLMs to be endogenously determined.

[10] We do not consider that agents include a sunspot variable in a PLM. Evans and Honkapohja (2001, chapter 10.5.1) show that under the information sets that include current endogenous variables like Assumption 1, even if agents form such a PLM, the economy converges to not a sunspot equilibrium, but a fundamental REE; that is, sunspot equilibria are not learnable.

[11] Honkapohja and Mitra (2006) and Berardi (2007) consider the effect of *structural heterogeneity*, where heterogeneous forecasts contribute to the dynamics of the economy to different degrees.

The stability of the economy is subject to whether the aggregate parameters $\phi' \equiv (a, c)$ converge to the coefficient matrices $(A + Ba, Bc\Phi_n + C)$ in the ALM; the dynamics of ϕ are established by the real-time learning processes of agents of all types for $\{\phi_i\}_{i=1}^n$. If the PLM (6) were correctly specified, the *E-stability principle* found by Evans and Honkapohja (2001, chapter 2) would hold: the convergence of ϕ_i for each i in real-time learning would be characterized by the ordinary differential equation (ODE) that could be made by the mapping from the PLM (6) to the ALM (10). In this paper, the principle does not necessarily hold, and the convergence should be found by analyzing the stochastic recursive algorithms (SRAs) of ϕ_i formulated by the PLM (6) and the ALM (10) (see Appendix B).

As a result, the global convergence of $\{\phi_i\}_{i=1}^n$ is found to be governed by the ODEs of aggregate parameters a and c, as follows.

Definition 1 *The economy is defined as the stationary stochastic process of y_t following the system (1)–(2). Under HM learning, the PLM of agents i for each $i \in \{1, \cdots, n\}$ and the aggregate PLM are formed by Eqs. (6) and (8), respectively. Then, the global convergence of the estimates $\left\{\phi_i' = (a_i, c_i)\right\}_{i=1}^n$ of all types is subject to the associated ODEs for the aggregate parameters $\phi' = (a, c)$:*

$$\frac{da}{d\tau} = T_a(a) - a, \tag{11}$$

$$\frac{dc}{d\tau} = T_c(c) - c, \tag{12}$$

where

$$\begin{aligned} T_a(a) &\equiv A + Ba, \\ T_c(c) &\equiv (Bc\Phi_n + C)\left(\frac{1}{n}\Psi_n\right), \\ \Psi_n &\equiv diag(\sigma_{11}, \ldots, \sigma_{nn})\,\Gamma_n diag(\sigma_{11}, \ldots, \sigma_{nn})^{-1}. \end{aligned}$$

τ denotes notional time. Note that the ODEs of $\{a_i\}_{i=1}^n$ are equivalent to that of a in Eq. (11), and that the ODEs of $\{c_i\}_{i=1}^n$ are combined into c's ODE in Eq. (12).

If and only if the ODEs are globally asymptotically stable, the economy is stable with ϕ converging to the fixed point of the ODEs:

$$\bar{a} = (I_m - B)^{-1} A, \tag{13}$$

$$\bar{c} = (B\bar{c}\Phi_n + C)\left(\frac{1}{n}\Psi_n\right). \tag{14}$$

The economy with the coefficients (13)–(14) is considered an equilibrium to be attainable under HM learning, which is called a *Heterogeneous Misspecification Equilibrium* (HME). The fixed point $\bar{a}$ in Eq. (13) is the steady state $\bar{y}$ of the HME, which is equal to the steady state of the REE in Eq. (4).

3.3 Observable Steady State

Let us also consider the case where the steady state $\bar{y} = \bar{a}$ is observable for all agents. The assumption of an observable steady state is seen in recent macroeconomic analysis. In particular, a nonlinear macroeconomic model tends to be log-linearized around a steady state by assuming the steady state to be observable.[12]

If the steady state $\bar{a}$ of the HME is observable for agents of all types, agents i for each i specify an HM PLM with a_i fixed at $\bar{a}$:

$$y_t = \bar{a} + c_i w_{it} + \varepsilon_{it}. \tag{15}$$

The global convergence of $\left\{\phi_i' = (\bar{a}, c_i)\right\}_{i=1}^{n}$ of all types is subject to solely the ODE for c in Eq. (12). If and only if the ODE is globally asymptotically stable, matrix c converges to the fixed point given by Eq. (14).

Thus, regardless of whether the steady state $\bar{a}$ is observable or not, the HME (13)–(14) is a unique equilibrium attainable under HM learning.

3.4 Learnability Conditions

Finally, we show the learnability conditions of the HME (13)–(14). For simplicity of exposition, we introduce a notation $\lambda[X]$ as the largest value of the real parts of the eigenvalues of a matrix X.

If the steady state $\bar{a}$ is unobservable, the learnability of the HME is ensured by the stability of the ODEs (11)–(12). According to Evans and Honkapohja (2001, Proposition 5.6), the ODEs are globally asymptotically stable if and only if their Jacobians:

$$D\left(T_a(a) - a\right) = B - I_m, \tag{16}$$

$$D\left(T_c(c) - c\right) = \left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B - I_{mn}, \tag{17}$$

have all negative real parts of eigenvalues. In other words, the sufficient and necessary conditions for the stability of the ODEs are that the largest values of the real parts of the eigenvalues of B and $\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B$ are less than unity; that is, $\lambda[B] < 1$ and $\lambda\left[\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B\right] < 1$, respectively. Note that $\lambda\left[\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B\right] < \lambda[B]$ when $\lambda[B] > 0$, because eigenvalues of $\Phi_n\left(\frac{1}{n}\Psi_n\right)$ are real and exist in the interval $[0, 1)$ (see Appendix C).[13] Thus, the stability conditions are solely represented by $\lambda[B] < 1$.

If the steady state $\bar{a}$ is observable, the learnability is governed by solely the ODE for c in Eq. (12), which is globally asymptotically stable if and only if Eq. (17) has all negative real parts of eigenvalues. The stability condition is given by $\lambda\left[\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B\right] < 1$.

[12] See, for example, Slobodyan and Wouters (2011), who incorporate adaptive learning into an NK dynamic stochastic general equilibrium (DSGE) model log-linearized around its steady state.

[13] We may focus on the case of $\lambda[B] > 0$, because when $\lambda[B] \leq 0$, then $\lambda\left[\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B\right] \leq 0$ as well, and hence the stability conditions are arbitrarily satisfied.

Hence, the learnability conditions of the HME are summarized as follows:

Proposition 1 *In the system (1)–(2), there uniquely exists the equilibrium (13)–(14) attainable under HM learning, which is called the Heterogeneous Misspecification Equilibrium (HME). The HME is globally asymptotically stable if and only if* $\lambda[B] < 1$ *(respectively,* $\lambda\left[\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B\right] < 1$*) when the steady state* $\bar{a}$ *is unobservable (observable).*

The proposition also provides the learnability conditions under CS learning. If all exogenous variables are perfectly correlated, the effect of heterogeneous misspecification vanishes so that the HME corresponds to the equilibrium under CS learning, and $\lambda\left[\Phi_n\left(\frac{1}{n}\Psi_n\right)\right] = \lambda[\Phi_n]$ (see Appendix D).[14] Thus:

Corollary 1 *An equilibrium under CS learning is globally asymptotically stable if and only if* $\lambda[B] < 1$ *(respectively,* $\lambda[\Phi_n' \otimes B] < 1$*) when the steady state* $\bar{a}$ *is unobservable (observable).*[15]

The common feature of equilibria under HM and CS learning is found in terms of the effect of the observability of the steady state $\bar{a}$. As $\lambda\left[\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B\right] < \lambda[B]$ and $\lambda[\Phi_n' \otimes B] < \lambda[B]$ when $\lambda[B] > 0$:

Corollary 2 *The learnability condition of an equilibrium with the observable steady state* $\bar{a}$ *is less stringent than the learnability condition with the unobservable steady state, regardless of whether heterogeneous misspecification exists or not.*

That is, the steady state $\bar{a}$ to be observable entirely makes an equilibrium more learnable.

The corollary is consistent with the literature's finding that if constant terms are excluded from PLMs, learnability conditions are relaxed (e.g., Bullard and Mitra, 2002). When the steady state $\bar{a}$ is observable in Section 3.3, the constant terms in PLMs (15) are not estimated, in the same manner as when constant terms are excluded from PLMs. The corollary reinforces the traditional fact to hold under HM learning as well.

4 Impact of Heterogeneous Misspecification

Let us find the impact of heterogeneous misspecification on the learnability of an equilibrium.

[14] If $\rho_{ij} = 1$ for all i, j, then $\left(\frac{1}{n}\Psi_n\right) w_t = w_t$ so that Eq. (14) gives the same solution $\bar{c}$ as in Eq. (5).

[15] These conditions correspond to Proposition 10.3 in Evans and Honkapohja (2001).

4.1 Learnability

The impact of heterogeneous misspecification is identified by comparing the learnability conditions under HM and CS learning obtained in Proposition 1 and Corollary 1, respectively.

If the steady state $\bar{a}$ is unobservable, the learnability conditions under both types of learning are equivalent to $\lambda[B] < 1$. If $\bar{a}$ is observable, the relationship of both conditions is subject to the sizes of $\lambda[\Phi_n' \otimes B]$ and $\lambda\left[\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B\right]$. If $\rho_{ij} < 1$ for some i, j, $0 \leq \lambda\left[\Phi_n\left(\frac{1}{n}\Psi_n\right)\right] < \lambda[\Phi_n] < 1$ (see Appendix D); then, $\lambda\left[\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B\right] < \lambda[\Phi_n' \otimes B]$ when $\lambda[B] > 0$.[16] Therefore:

Proposition 2 *In the system (1)–(2), the learnability condition of the HME is equivalent to (respectively, less stringent than) the condition of the equilibrium under CS learning when the steady state $\bar{a}$ is unobservable (observable).*

That is, the HME is not less learnable than the equilibrium under CS learning, and heterogeneous misspecification never makes an equilibrium less learnable.

For macroeconomic policy, the proposition implies that even if informational heterogeneity among agents emerges in the economy, the heterogeneous misspecification in learning gives no additional constraint on macroeconomic policy to ensure the learnability of an equilibrium. On a daily basis, idiosyncratic shocks among agents tend to make people's information sets imperfect and mutually different. Nevertheless, the government should continue to follow the existing policy under CS learning.

The reason for the strong learnability of the HME is because the aggregate forecast y^e_{t+1} is not updated in response to the exogenous variable w_t as much as under CS learning. Under HM learning, if, for example, an exogenous variable w_{it} changes, only agents of type i will update their forecast $y^e_{i,t+1}$, while the other types of agents will not update their forecasts $\left\{y^e_{j,t+1}\right\}_{j\neq i}$. In contrast, under CS learning, all agents will update their forecasts, and hence, the aggregate forecast y^e_{t+1} is updated to the same degree. Hence, the strong learnability of the HME stems from the inactivity of the aggregate forecast y^e_{t+1}.

One might be interested in an equilibrium where the proportion of agents observing each variable is modified, while in this paper, the proportion is fixed at $\frac{1}{n}$. Proposition 2 seems to hold under a different proportion of agents, because the inactivity of y^e_{t+1} under HM learning sustains as long as individual forecasts $\left\{y^e_{i,t+1}\right\}_{i=1}^{n}$ are not updated simultaneously due to the existence of private information.[17]

[16] If $\lambda[B] \leq 0$, both conditions under CS and HM learning are arbitrarily satisfied because $\lambda\left[\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B\right] \leq 0$ and $\lambda[\Phi_n' \otimes B] \leq 0$.

[17] Different proportions of different agents in a different framework are considered by Berardi (2007, Fig. 2), whose numerical analysis suggests the possibility that heterogeneity in learning makes an equilibrium more learnable.

4.2 Magnitude of Heterogeneous Misspecification

We also consider the relationship between the magnitude of heterogeneous misspecification and the learnability of an equilibrium.

The relationship is taken from Proposition 1. When the steady state $\bar{a}$ is observable, the learnability condition depends upon the eigenvalues of $\lambda\left[\Phi_n\left(\frac{1}{n}\Psi_n\right)\right]$, which is characterized by $\left\{\rho_{ij}, \varphi_i, \sigma_{ii}\right\}_{i,j=1}^{n}$. The magnitude of heterogeneous misspecification is negatively related with $\left\{\rho_{ij}\right\}_{i,j=1}^{n}$. Thus, the relationship in question is characterized by the relation between eigenvalues of $\lambda\left[\Phi_n\left(\frac{1}{n}\Psi_n\right)\right]$ and ρ_{ij}; $\frac{d\lambda\left[\Phi_n\left(\frac{1}{n}\Psi_n\right)\right]}{d\rho_{ij}} \geq 0$.[18] When $\lambda[B] > 0$, $\lambda\left[\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)' \otimes B\right]$ has the same relationship with ρ_{ij}. Hence:

Corollary 3 *If the steady state $\bar{a}$ is observable, the magnitude of heterogeneous misspecification weakly increases the learnability of an equilibrium.*

4.3 Comparison with Determinacy of REE

Finally, let us consider the relationship between the learnability conditions of the HME and the determinacy condition of the fundamental REE in Section 2.2. The determinacy of an equilibrium is another necessary condition for the stability of an equilibrium. A government pursuing the stability of an equilibrium should consider which condition is the more stringent.

The relationship between the two conditions is clarified by Proposition 2 and the findings of the related literature. McCallum (2007) and Ellison and Pearlman (2011) show that the learnability conditions of a fundamental REE under CS learning are equal to or less than the determinacy condition of the REE. Accordingly, Proposition 2 immediately suggests:

Corollary 4 *In the system (1)–(2), the learnability conditions of the HME (13)–(14) are equal to or less than the determinacy condition of the fundamental REE (4)–(5), regardless of whether heterogeneous misspecification in learning exists or not.*

Hence, the determinacy continues to be the sufficient condition for the learnability of an equilibrium under HM learning.[19]

[18]Kolotilina (Kolotilina, L. Y., "Bounds for the Perron root, Singularity/Nonsingularity Conditions, and Eigenvalue Inclusion Sets," *Numerical Algorithms*, Vol. 42, No. 3–4, 2006, pp. 247–280) shows in her Theorem 2.1: "Let A and B be nonnegative matrices of order $n \geq 1$ and let $B \leq A$. Then, $\lambda[B] \leq \lambda[A]$." This means that the increase in ρ_{ij} consisting of the elements of nonnegative matrix $\Phi_n\left(\frac{1}{n}\Psi_n\right)$ weakly increases $\lambda\left[\Phi_n\left(\frac{1}{n}\Psi_n\right)\right]$.

[19]The sufficiency of the determinacy of the REE is confirmed in our model. The determinacy condition is that all eigenvalues of matrix B lie inside the unit circle (see Blanchard and Kahn, 1980). The sufficient condition for the learnability of the HME is solely represented by $\lambda[B] < 1$ (see Proposition 1). Notice that if eigenvalues of matrix B lie inside the unit circle, then the real parts of the eigenvalues are less than unity. Thus, the determinacy condition of the REE is sufficient for the learnability of the HME.

This result also provides an implication for macroeconomic policy. The above literature implies that even if agents conduct adaptive learning instead of rational expectations, the government should satisfy the determinacy condition to ensure the learnability of an equilibrium. Our result further implies that even if information sets held by agents become imperfect and mutually different, the government should follow the existing policy.

5 Application

In this section, we examine the learnability of the HME in a basic NK macroeconomic model. The NK model has recently become a benchmark macroeconomic framework for establishing DSGE models and analyzing optimal monetary policy to stabilize the business cycle.[20] We find constraints imposed on monetary policy rules to ensure the learnability of the HME in the NK model.

5.1 NK Model

We consider a basic NK model with an aggregate demand shock g_t:

$$x_t = -\alpha\left(i_t - \pi^e_{t+1}\right) + x^e_{t+1} + g_t, \quad (18)$$

$$\pi_t = \kappa x_t + \beta \pi^e_{t+1}. \quad (19)$$

The model has three endogenous variables: output gap x_t, inflation rate π_t, and nominal interest rate i_t. The first equation is a log-linearized intertemporal Euler equation that is derived from the households' optimal choice of consumption. The second equation is a forward-looking Phillips curve that is derived from the optimizing behavior of monopolistically competitive firms under Calvo price setting. $\alpha > 0$, $\kappa > 0$, and $0 \leq \beta < 1$ are assumed.

The central bank adopts a Taylor-type nominal interest rate rule:

$$i_t = \phi_\pi \pi_t + \phi_x x_t, \quad (20)$$

where ϕ_π and ϕ_x are the policy parameters controlled by the central bank, and they are assumed to be nonnegative.

To consider the dynamics of an HME in the NK model, we assume that g_t is the aggregation of idiosyncratic shocks: $g_t \equiv \sum_{i=1}^n g_{it}$. The shock g_{it} for each i follows an AR(1) process: $g_{it} = \varphi_i g_{i,t-1} + v_{it}$, where $0 \leq \varphi_i < 1$ and a disturbance term v_{it} has a zero mean. The correlation of g_{it} and g_{jt} is ρ_{ij} for each i, j. Under CS learning, g_{it} for each i is observable for all agents, while under HM

[20] See Bullard and Mitra (2002), Evans and Honkapohja (2003a,b, 2006), Honkapohja and Mitra (2004, 2005), Adam (2005), Berardi (2009), and Branch and McGough (2009) for the analysis on optimal monetary policy to ensure the learnability of an equilibrium.

learning, the shock is observable for $1/n$ of agents and unobservable for other agents. The aggregate forecasts $\left(x_{t+1}^e, \pi_{t+1}^e\right)$ are the averages of the forecasts $\left\{\left(x_{i,t+1}^e, \pi_{i,t+1}^e\right)\right\}_{i=1}^{n}$ of all types.[21]

5.2 Determinacy and Learnability Conditions

The determinacy condition of the fundamental REE is provided by Bullard and Mitra (2002, Proposition 1):

$$\kappa\left(\phi_\pi - 1\right) + \phi_x\left(1 - \beta\right) > 0, \tag{21}$$

which is called the *Taylor principle* that the policy parameters (ϕ_π, ϕ_x) must follow for the determinacy of the REE.[22]

The learnability conditions of equilibria under CS and HM learning are provided in Table 1, the derivation of which is shown in Appendix E. The conditions under the unobservable and observable steady states are given for each type of learning, respectively. The table confirms that if the steady state is unobservable, the learnability condition under CS learning is equivalent to the Taylor principle (21); this result has been found by Bullard and Mitra (2002, Proposition 2). A new finding is that the learnability condition under HM learning is also equivalent to the Taylor principle. Thus, if the steady state is unobservable, the Taylor principle is a sufficient and necessary condition for the learnability of an equilibrium, regardless of whether heterogeneous misspecification in learning exists or not.

If the steady state is observable, the learnability conditions of equilibria under CS and HM learning are less stringent than the Taylor principle. For ease of comparison, Figure 1 illustrates the domains of (ϕ_π, ϕ_x) that satisfy the Taylor principle and the learnability conditions in Table 1. Parameter values in the upper right region of a line (e.g., "Learnable under CS learning") satisfies the learnability condition under each type of learning (CS learning). As $0 \leq \lambda^h < \lambda^c < 1$, the learnability domain under HM learning is larger than the domain under CS learning. These relationships illustrate Proposition 2 that the learnability condition under HM learning is less stringent than the condition under CS learning. In addition, the Taylor principle is found to be the most stringent of all conditions. Therefore:

Proposition 3 *In the NK model (18)–(19) with a Taylor-type nominal interest rate rule (20), the Taylor principle (21) is the sufficient condition for the learnability of an equilibrium, regardless of whether heterogeneous misspecification in learning exists or not.*

Our results reinforce the importance of the Taylor principle in monetary policy. Bullard and Mitra (2002) find that the Taylor principle is a sufficient condition for the learnability of an equilibrium under

[21] To focus on investigating the properties of the HME, we follow the methodology given by Branch and McGough (2009) for incorporating heterogeneous forecasts at the individual level within the NK model. As a result, we can use the original form of the NK model (18)–(19) by aggregating individual forecasts.

[22] As $\phi_x \geq 0$, the seminal form of the Taylor principle is provided by $\phi_\pi > 1$: the central bank should raise the nominal interest rate i_t by more than one-for-one in response to an increase in the inflation rate π_t.

CS learning. Guse (2008) confirms the robustness of their results when agents' learning is misspecified. Our result emphasizes that the sufficiency of the Taylor principle holds even if agents' learning is heterogeneously misspecified.

5.3 Calibrations

Finally, to evaluate the impact of heterogeneous misspecification on the learnability of an equilibrium, we calibrate the parameter domain of (ϕ_π, ϕ_x) satisfying the learnability condition under the observable steady state. We consider the existence of two idiosyncratic shocks ($n = 2$) and, for robustness, the two cases of structural parameters: (a) $\alpha = 1$, $\kappa = 0.3$, and $\beta = 0.99$ (Clarida, Gali, and Gertler, 2000); (b) $\alpha = 1/0.157$, $\kappa = 0.024$, and $\beta = 0.99$ (Woodford, 1999). We set the autocorrelations $\{\varphi_i\}_{i=1}^2$ of the shocks to be equally 0.9 (Milani, 2008).

Figure 2 shows the parameter domains satisfying the learnability conditions under different values of the correlation ρ of the shocks, which affects the magnitude of heterogeneous misspecification (Corollary 3). Parameter values in the upper right region of a line (e.g., "$\rho = 0.9$") satisfies the condition under each value of ρ ($\rho = 0.9$). Note that the domain under $\rho = 1.0$ satisfies the learnability condition under CS learning.

In both examples, we find that the learnability condition under HM learning is significantly weaker than the condition under CS learning. The former condition is weakened as the correlation ρ decreases.[23] In particular, when $\rho < 0.7$, almost any positive values of (ϕ_π, ϕ_x) are permitted under HM learning.[24]

Therefore, when there exists private information of idiosyncratic shocks, the central bank might be considerably relieved of learnability conditions. When those shocks are mutually independent, the central bank might not have to address the learnability of an equilibrium.

6 Conclusions

This paper has investigated the learnability of an equilibrium in an environment where information sets held by agents conducting adaptive learning are imperfect and mutually different. In the real economy, there might exist economic variables observable by some agents and unobservable by other agents. In the existence of such private information, agents are constrained to form not merely misspecified, but heterogeneously misspecified PLMs.

[23] We note that this result is not affected by the values of the autocorrelations $\{\varphi_i\}_{i=1}^2$ of the shocks.

[24] Although few papers estimate the correlations of idiosyncratic shocks at the individual agent level, Fidrmuc and Korhonen (2003, Table 2) find the correlations of supply and demand shocks between euro countries in the 1990s to be 0.65 at the highest. Those values might be seen as the proxies for the average of the correlations of idiosyncratic shocks at the individual level.

Under HM learning, the economy converges to a Heterogeneous Misspecification Equilibrium (HME). The paper finds that the learnability conditions of the HME are equal to or less than the conditions of the equilibrium under CS learning. If the steady state of the economy is observable, the magnitude of heterogeneous misspecification weakly increases the learnability of an equilibrium. Thus, heterogeneous misspecification never makes an equilibrium less learnable. In addition, the learnability conditions are equal to or less than the determinacy condition, regardless of whether heterogeneous misspecification in learning exists or not. As a result, in a basic NK model, the central bank should follow the Taylor principle for the learnability of an equilibrium even if information sets held by agents are imperfect and mutually different.

A few potential issues remain for future research. The heterogeneous misspecification in adaptive learning may be investigated as a possible ingredient of the persistence of economic fluctuations. Adam (2005), Milani (2008), and Slobodyan and Wouters (2011) find that adaptive learning prolongs the response of an economy to fundamental shocks in NK DSGE models. The present paper finds that heterogeneous misspecification makes an aggregate forecast inactive to exogenous variables. Thus, the responses of the HME to fundamental shocks are expected to be more persistent than the responses of an equilibrium under CS learning.

The heterogeneous misspecification may be considered as a possible reason for the *Great Moderation* (i.e., the remarkable decline in macroeconomic volatility experienced by the US economy since the mid-1980s). In addition, Sims and Zha (2006) show that the Great Moderation is explained by the variation in the sources of economic disturbances. Canova (2009) shows that changes in the covariance matrix of structural shocks is one of the most important components in accounting for the Great Moderation. The present paper shows that the learnability of an equilibrium depends upon the correlations of exogenous variables. Hence, those changes might make an equilibrium more stable under adaptive learning.

Appendix

A Regularity Assumptions

Assumption 2

1. $\det(I_m - B) \neq 0$ *and* $\det(I_{mn} - \Phi_n \otimes B) \neq 0$.
2. Φ_n *is a nonzero, diagonal, and nonnegative matrix whose diagonal elements exist in the interval* $[0, 1)$.
3. Φ_n *is observable.*

Assumption 2.1 avoids the possibility that a nonexplosive fundamental REE could be indeterminate (see Honkapohja and Mitra, 2006, Proposition 1). $\Phi_n \neq 0$ in Assumption 2.2 is common in the literature. The diagonal representation of Φ_n is to simplify the analysis by equating the eigenvalues of Φ_n with its diagonal elements existing in the interval $[0, 1)$. Note that this assumption has no effect on our analysis, because even if Φ_n were originally nondiagonal, Eq. (2) could be transformed to the equation that includes a diagonal autoregressive matrix by premultiplying Eq. (2) by the $n \times n$ matrix formed from the eigenvectors of Φ_n. The diagonal elements in the interval $[0, 1)$ ensure the stationarity of w_t. Assumption 2.3 is common in the literature, and enables agents to formulate their forecasts using the information of Φ_n.

Assumption 3 Γ_n *is a nonnegative matrix, in which* $0 \leq \rho_{ij} \leq 1$ *for each* $i, j \in \{1, ..., n\}$.

Assumption 3 has no effect on our analytical results because any model can be transformed to the system with $\Gamma_n \geq 0$. For example, if any ρ_{ij} is negative in an original model, this negative correlation can be made positive by changing the sign of w_i (or w_j) and redefining the correlation between $-w_i$ and w_j as $\rho_{ij} \geq 0$. Applying this transformation to any negative correlation, the original model is transformed to the system with $\Gamma_n \geq 0$.

B Derivation of Definition 1

According to Evans and Honkapohja (2001, chapter 13), the ODEs under misspecified PLMs are obtained as follows.

Agents i for each $i \in \{1, \ldots, n\}$ form $y^e_{i,t+1}$ by real-time learning with the HM PLM (6) and the information set $\{y_s, w_{is}\}^t_{s=1}$. The estimates of coefficient parameters $\phi'_{it} = (a_{it}, c_{it})$ are given by the least-squares projection of y_t on $z'_{it} = (1, w_{it})$. Then, the updating rule of ϕ_{it} is shown by the RLS representation:

$$\phi_{it} = \phi_{i,t-1} + t^{-1} R^{-1}_{it} z_{i,t-1} \left(y_{t-1} - \phi'_{i,t-1} z_{i,t-1} \right)', \tag{22}$$

$$R_{it} = R_{i,t-1} + t^{-1} \left(z_{i,t-1} z'_{i,t-1} - R_{i,t-1} \right), \tag{23}$$

where $R_{it} = t^{-1} \sum^t_{s=1} z_{it} z'_{it}$, which is the updating of the matrix of the second moment of z_{it}. See Evans and Honkapohja (2001, section 10.3) for the detail of obtaining RLS equations to satisfy the orthogonality condition.

The SRA for ϕ_{it} is obtained by substituting the ALM (10) into Eq. (22):

$$\phi_{it} = \phi_{i,t-1} + t^{-1} R^{-1}_{it} z_{i,t-1} \begin{pmatrix} 1 & w_{1,t-1} & \cdots & w_{n,t-1} \end{pmatrix} \left[\begin{pmatrix} D_{0,t-1} & D_{1,t-1} & \cdots & D_{n,t-1} \end{pmatrix} - \begin{pmatrix} a_{i,t-1} & c^+_{i,t-1} \end{pmatrix} \right]',$$

where we denote $D_{0t} \equiv A + Ba_t$, $a_t \equiv \frac{1}{n}\sum_{j=1}^{n} a_{jt}$ as the constant term of the aggregate PLM (8), $D_{jt} \equiv B\left(\frac{1}{n}c_{jt}\right)\varphi_j + C_j$ for each $j \in \{1,...,n\}$, C_j as the j th column of matrix C in Eq. (1), and $c_{it}^{+} \equiv (\mathbf{0},...,\mathbf{0},c_{it},\mathbf{0},...,\mathbf{0})$ as a $m \times n$ matrix in which the columns, except c_{it}, are zero vectors.

To obtain the ODEs for ϕ_i associated with the SRA, we have to calculate the unconditional expectations of the updating terms in the SRA. The convergence of the SRA is analyzed by Marcet and Sargent (1989) as the stochastic approximation approach, which is also introduced by Evans and Honkapohja (2001, chapter 6). Denote the operator E as the expectation of variables, for ϕ_i fixed, taken over the invariant distributions of w_t. Then, by letting $Ez_iz_j' = \lim_{t\to\infty} Ez_{it}z_{jt}'$ for any $i,j \in \{1,...,n\}$, the unconditional expectation of the updating term in Eq. (22) is transformed to:

$$\begin{aligned}
& ER_i^{-1}z_{i,t-1}\begin{pmatrix} 1 & w_{1,t-1} & \cdots & w_{n,t-1}\end{pmatrix}\left[\begin{pmatrix} D_0 & D_1 & \cdots & D_n\end{pmatrix} - \begin{pmatrix} a_i & c_i^{+}\end{pmatrix}\right]' \\
= \; & R_i^{-1}\left(\left(Ez_{i,t-1}z_{i,t-1}'\right)\left[\begin{pmatrix} D_0 & D_i\end{pmatrix} - \begin{pmatrix} a_i & c_i\end{pmatrix}\right]' + E\begin{pmatrix} 1 \\ w_{i,t-1}\end{pmatrix}\left(\sum_{j=1}^{n} w_{j,t-1}D_j' - w_{i,t-1}D_i'\right)\right) \\
= \; & R_i^{-1}\begin{pmatrix} \left(Ez_{i,t-1}z_{i,t-1}'\right)\left[\begin{pmatrix} D_0 & D_i\end{pmatrix} - \begin{pmatrix} a_i & c_i\end{pmatrix}\right]' \\ +\begin{pmatrix} \mathbf{0} \\ \sum_{j=1}^{n}\left(Ew_{i,t-1}w_{j,t-1}\right)D_j' - \left(Ew_{i,t-1}w_{i,t-1}\right)D_i' \end{pmatrix}\end{pmatrix} \\
= \; & R_i^{-1}\left(Ez_{i,t-1}z_{i,t-1}'\right)\begin{pmatrix} \left[\begin{pmatrix} D_0 & D_i\end{pmatrix} - \begin{pmatrix} a_i & c_i\end{pmatrix}\right]' \\ +\begin{pmatrix} \mathbf{0} \\ \sum_{j=1}^{n}\left(Ew_{i,t-1}w_{i,t-1}\right)^{-1}\left(Ew_{i,t-1}w_{j,t-1}\right)D_j' - D_i' \end{pmatrix}\end{pmatrix}.
\end{aligned}$$

In addition, the expectation of the updating term in Eq. (23) is given by:

$$Ez_iz_i' - R_i.$$

Hence, the ODEs for ϕ_i and R_i associated with the SRA are obtained as:

$$\frac{d\phi_i}{d\tau} = R_i^{-1}\left(Ez_iz_i'\right)\left(T_i\left(a_i,c_i\right) - \phi_i'\right)', \tag{24}$$

$$\frac{dR_i}{d\tau} = Ez_iz_i' - R_i, \tag{25}$$

where

$$T_i\left(a_i,c_i\right) \equiv \begin{pmatrix} D_0 & \sum_{j=1}^{n} D_j\omega_{ij}\omega_{ii}^{-1}\end{pmatrix}.$$

A scalar ω_{ij} denotes the covariance of w_i and w_j; $\omega_{ij} \equiv \sigma_{ii}\rho_{ij}\sigma_{jj}$ for each $i,j \in \{1,...,n\}$. Further, because R_i and Ez_iz_i' in Eq. (25) are asymptotically equal, $R_i^{-1}\left(Ez_iz_i'\right)$ in Eq. (24) globally converges to unity. Hence, the stability of the ODE for $\phi_i' = (a_i,c_i)$ in Eq. (24) is determined by smaller differential equations:

$$\frac{da_i}{d\tau} = D_0 - a_i, \tag{26}$$

$$\frac{dc_i}{d\tau} = \sum_{j=1}^{n} D_j\omega_{ij}\omega_{ii}^{-1} - c_i. \tag{27}$$

In the same manner, the smaller ODEs for the parameters $\{\phi_j\}_{j\neq i}$ are obtained.

The ODEs (26)–(27) for all i are represented by the ODEs for the aggregate parameters a and c in Eq. (8). First, the ODEs for all a_is have the same form. In addition, a is an arithmetic average of all a_is. Then, the convergence property of a is equivalent to that of a_i for any i; the ODEs for all a_is are represented by a single ODE for a that has the same form as that for a_i:

$$\frac{da}{d\tau} = T_a(a) - a,$$

where

$$T_a(a) \equiv D_0 = A + Ba.$$

Next, the ODEs for all c_is are represented by a single ODE for the aggregate parameter c. If the ODEs (27) for all i are multiplied by $\frac{1}{n}$ and combined in a single $m \times n$ matrix, the single ODE for c is obtained by:

$$\frac{dc}{d\tau} = T_c(c) - c,$$

where

$$\begin{aligned} T_c(c) &\equiv \begin{pmatrix} \frac{1}{n}\sum_{j=1}^{n} D_j \omega_{1j}\omega_{11}^{-1} & \cdots & \frac{1}{n}\sum_{j=1}^{n} D_j \omega_{nj}\omega_{nn}^{-1} \end{pmatrix} \\ &= (Bc\Phi_n + C)\left(\frac{1}{n}\Psi_n\right), \end{aligned}$$

and

$$\begin{aligned} \Psi_n &\equiv \begin{pmatrix} 1 & \omega_{12}\omega_{22}^{-1} & \cdots & \omega_{1n}\omega_{nn}^{-1} \\ \omega_{21}\omega_{11}^{-1} & 1 & \cdots & \omega_{2n}\omega_{nn}^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega_{n1}\omega_{11}^{-1} & \omega_{n2}\omega_{22}^{-1} & \cdots & 1 \end{pmatrix} \\ &= diag(\sigma_{11}, \ldots, \sigma_{nn})\Gamma_n diag(\sigma_{11}, \ldots, \sigma_{nn})^{-1}. \end{aligned}$$

The above results are summarized in Definition 1.

C Eigenvalues of $\Phi_n\left(\frac{1}{n}\Psi_n\right)$

To find that eigenvalues of $\Phi_n\left(\frac{1}{n}\Psi_n\right)$ are real and exist in the interval $[0, 1)$, we use a lemma given by Trenkler (1995).

Lemma 1 *If X and Y are nonnegative definite matrices of the same dimension, then eigenvalues of XY are real and nonnegative, and they are zero if and only if $XY = 0$.*

Proof. If X is nonnegative definite, it may be written as $X = PP'$ for some matrix P. Then, the eigenvalues of $XY = PP'Y$ are those of $P'YP$ (plus possibly some zeros). $P'YP$ is obviously a nonnegative definite matrix. It follows that all eigenvalues of XY are nonnegative, and that they are zero iff $XY = 0$. The proof is complete. ■

As Φ_n and $\frac{1}{n}\Psi_n$ are nonnegative definite matrices by Assumptions 2.2 and 3, Lemma 1 yields eigenvalues of $\Phi_n\left(\frac{1}{n}\Psi_n\right)$ that are real and nonnegative. Notice that the diagonal elements of $\Phi_n\Psi_n$ are equal to the diagonal elements of Φ_n; then, $tr\left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right) < 1$. Therefore, all eigenvalues of $\Phi_n\left(\frac{1}{n}\Psi_n\right)$ are real and exist in the interval $[0, 1)$.

D Proof of $0 \leq \lambda\left[\Phi_n\left(\frac{1}{n}\Psi_n\right)\right] < \lambda\left[\Phi_n\right] < 1$

As the correlation matrix Γ_n is a nonzero and nonnegative definite matrix, the eigenvalues of $\frac{1}{n}\Gamma_n$ are all nonnegative, and at least one of them is positive. Then, $\lambda\left[\frac{1}{n}\Gamma_n\right] > 0$. In addition, the diagonal elements of $\frac{1}{n}\Gamma_n$ are all $\frac{1}{n}$, so $tr\left(\frac{1}{n}\Gamma_n\right) = 1$. Then, $\lambda\left[\frac{1}{n}\Gamma_n\right] \leq 1$ with equality iff $\rho_{ij} = 1$ for all i, j. Thus:

$$0 < \lambda\left[\frac{1}{n}\Gamma_n\right] \leq 1. \tag{28}$$

Next, as Φ_n is a nonzero, diagonal, and nonnegative matrix by Assumption 2.2 and $\frac{1}{n}\Gamma_n \geq \mathbf{0}$ by Assumption 3, matrix $\Phi_n\left(\frac{1}{n}\Gamma_n\right)$ satisfies $\mathbf{0} \leq \Phi_n\left(\frac{1}{n}\Gamma_n\right) \leq \lambda\left[\Phi_n\right]\left(\frac{1}{n}\Gamma_n\right)$ with $\Phi_n\left(\frac{1}{n}\Gamma_n\right) = \lambda\left[\Phi_n\right]\left(\frac{1}{n}\Gamma_n\right)$ iff $\rho_{ij} = 1$ such that $\varphi_i = \varphi_j$ for all i, j. According to the *Perron–Frobenius Theorem* (see Peter Berck and Knut Sydsaeter, *Economists' Mathematical Manual: Second Edition*, Springer-Verlag, 1993, p. 110), these inequalities give $0 \leq \lambda\left[\Phi_n\left(\frac{1}{n}\Gamma_n\right)\right] \leq \lambda\left[\lambda\left[\Phi_n\right]\left(\frac{1}{n}\Gamma_n\right)\right] \left(= \lambda\left[\Phi_n\right]\lambda\left[\frac{1}{n}\Gamma_n\right]\right)$. By Assumption 2.2, $0 < \lambda\left[\Phi_n\right] < 1$; then, Eq. (28) gives $0 < \lambda\left[\Phi_n\right]\lambda\left[\frac{1}{n}\Gamma_n\right] \leq \lambda\left[\Phi_n\right] < 1$. Summarizing these results:

$$0 \leq \lambda\left[\Phi_n\left(\frac{1}{n}\Gamma_n\right)\right] \leq \lambda\left[\Phi_n\right] < 1, \tag{29}$$

with $\lambda\left[\Phi_n\left(\frac{1}{n}\Gamma_n\right)\right] = \lambda\left[\Phi_n\right]$ iff $\rho_{ij} = 1$ for all i, j.

Finally, notice that the eigenvalues of $\Phi_n\left(\frac{1}{n}\Psi_n\right)$ are equivalent to those of $\Phi_n\left(\frac{1}{n}\Gamma_n\right)$ because:

$$\begin{aligned} \Phi_n\left(\frac{1}{n}\Psi_n\right) &= \Phi_n\left(diag\left(\sigma_{11}, \ldots, \sigma_{nn}\right)\left(\frac{1}{n}\Gamma_n\right)diag\left(\sigma_{11}, \ldots, \sigma_{nn}\right)^{-1}\right) \\ &= diag\left(\sigma_{11}, \ldots, \sigma_{nn}\right)\left(\Phi_n\left(\frac{1}{n}\Gamma_n\right)\right)diag\left(\sigma_{11}, \ldots, \sigma_{nn}\right)^{-1}. \end{aligned}$$

Therefore, Eq. (29) yields:

$$0 \leq \lambda\left[\Phi_n\left(\frac{1}{n}\Psi_n\right)\right] \leq \lambda\left[\Phi_n\right] < 1,$$

with $\lambda\left[\Phi_n\left(\frac{1}{n}\Gamma_n\right)\right] = \lambda\left[\Phi_n\right]$ iff $\rho_{ij} = 1$ for all i, j. The proof is complete.■

E Derivation of Learnability Conditions

Before proceeding, we provide a lemma that will be used to obtain learnability conditions of equilibria under different learning rules.

Lemma 2 *Define a $n \times n$ matrix X whose eigenvalues are all real and exist in the interval* $[0,1)$. *Given $\alpha \geq 0$, $\kappa \geq 0$, $0 \leq \beta < 1$, $\phi_\pi \geq 0$, $\phi_x \geq 0$, and $B = \begin{pmatrix} 1+\alpha\phi_x & \alpha\phi_\pi \\ -\kappa & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}$, then the real parts of eigenvalues of $X \otimes B - I_{2n}$ are all negative if and only if:*

$$\kappa\left(\phi_\pi - \lambda[X]\right) + \phi_x\left(1 - \beta\lambda[X]\right) > -\frac{\left(1-\lambda[X]\right)\left(1-\beta\lambda[X]\right)}{\alpha}. \tag{30}$$

Proof. Define the eigenvalues of X as $0 \leq \chi_i < 1$ for each $i \in \{1,...,n\}$ and the eigenvalues of B as δ_j for each $j \in \{1,2\}$. Then, $0 \leq \lambda[X] < 1$, and the eigenvalues of $X \otimes B$ are given by $\chi_i\delta_j$ for each i, j. First, we show $\lambda[B] \geq 0$ by calculating the characteristic equation of B: $q(x) = x^2 + p_1 x + p_0$, where $p_0 = \frac{\beta}{1+\kappa\alpha\phi_\pi+\alpha\phi_x} > 0$ and $p_1 = -\frac{1+\beta+\kappa\alpha+\alpha\beta\phi_x}{1+\kappa\alpha\phi_\pi+\alpha\phi_x} < 0$. According to the *Routh Theorem* (see Alpha C. Chiang, *Fundamental Methods of Mathematical Economics: Second Edition*, McGraw-Hill, 1974), eigenvalues of B have all negative real parts; that is, $\lambda[B] < 0$, if and only if $|p_1|$ and $\begin{vmatrix} p_1 & 0 \\ 1 & p_0 \end{vmatrix}$ are all positive, that is, $p_1 > 0$ and $p_1 p_0 > 0$. The above $q(x)$ violates these conditions; therefore, $\lambda[B] \geq 0$. Here, let us prove Lemma 2. Because $\lambda[B] \geq 0$ and $\lambda[X] \geq 0$, $\lambda[X \otimes B] = \lambda[X]\lambda[B] = \lambda[\lambda[X]B]$, and hence, $\lambda[X \otimes B - I_{2n}] = \lambda[\lambda[X]B - I_2]$. Thus, the real parts of eigenvalues of $X \otimes B - I_{2n}$ are all negative if and only if eigenvalues of $\lambda[X]B - I_2$ have all negative real parts. The characteristic equation of $\lambda[X]B - I_2$ is $q(x) = x^2 + p_1 x + p_0$, where:

$$\begin{aligned} p_0 &= \frac{\left(1-\lambda[X]\right)\left(1-\beta\lambda[X]\right) + \alpha\left(\kappa\left(\phi_\pi - \lambda[X]\right) + \phi_x\left(1-\beta\lambda[X]\right)\right)}{1+\kappa\alpha\phi_\pi+\alpha\phi_x}, \\ p_1 &= \frac{\left(1-\lambda[X]\right) + \left(1-\beta\lambda[X]\right) + \kappa\alpha\left(2\phi_\pi - \lambda[X]\right) + \alpha\phi_x\left(2-\beta\lambda[X]\right)}{1+\kappa\alpha\phi_\pi+\alpha\phi_x}. \end{aligned}$$

Note that $p_1 = p_0 + \frac{(1-\beta\lambda[X])+\alpha(\kappa\phi_\pi+\phi_x)}{1+\kappa\alpha\phi_\pi+\alpha\phi_x}$; then $p_1 > p_0$. Eigenvalues of $\lambda[X]B - I_2$ have all negative real parts if and only if $p_1 > 0$ and $p_1 p_0 > 0$. As $p_1 > p_0$, the necessary and sufficient condition is solely given by $p_0 > 0$, that is, Eq. (30). The proof is complete. ■

Substituting Eq. (20) into Eqs. (18)–(19), the NK model is transformed to the form of the system (1)–(2) with $y_t = (x_t, \pi_t)'$, $w_t = (g_{1t}, ..., g_{nt})'$, and $B = \begin{pmatrix} 1+\alpha\phi_x & \alpha\phi_\pi \\ -\kappa & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & \alpha \\ 0 & \beta \end{pmatrix}$.

Under the unobservable steady state $\bar{a}$, the HME (13)–(14) is globally asymptotically stable if and only if the Jacobian (16) has the negative real parts of eigenvalues. In the NK model of the above form, $m = 2$; eigenvalues of the Jacobian are equal to those of $I_n \otimes B - I_{2n}$. This case corresponds to $X = I_n$ in Lemma 2. As $\lambda[X] = 1$, the sufficient and necessary condition for the stability of the equilibrium is obtained by $\kappa(\phi_\pi - 1) + \phi_x(1-\beta) > 0$, which is also the learnability condition under CS learning.

In the same manner, under the observable steady state, the HME is stable if and only if the Jacobian (17) has the negative real parts of eigenvalues. This case corresponds to $X = \left(\Phi_n\left(\frac{1}{n}\Psi_n\right)\right)'$ in Lemma 2. If we define $\lambda^h \equiv \lambda\left[\Phi_n\left(\frac{1}{n}\Psi_n\right)\right]$, the stability condition is provided by $\kappa\left(\phi_\pi - \lambda^h\right) + \phi_x\left(1-\beta\lambda^h\right) > -\frac{\left(1-\lambda^h\right)\left(1-\beta\lambda^h\right)}{\alpha}$. Under CS learning, $X = \Phi_n'$. Then, by defining $\lambda^c \equiv \lambda[\Phi_n]$, the stability condition is obtained by $\kappa(\phi_\pi - \lambda^c) + \phi_x(1-\beta\lambda^c) > -\frac{(1-\lambda^c)(1-\beta\lambda^c)}{\alpha}$. Note that $0 \leq \lambda^h < \lambda^c < 1$ by Appendix D.

References

Adam, K. (2003): "Learning and Equilibrium Selection in a Monetary Overlapping Generations Model with Sticky Prices," *Review of Economic Studies*, 70, 887–907.

——— (2005): "Experimental Evidence on the Persistence of Output and Inflation," *Economic Journal*, 117, 603–636.

Adam, K., G. W. Evans, and S. Honkapohja (2006): "Are Hyperinflation Paths Learnable?" *Journal of Economic Dynamics and Control*, 30, 2725–2748.

Berardi, M. (2007): "Heterogeneity and Misspecifications in Learning," *Journal of Economic Dynamics and Control*, 31, 3203–3227.

——— (2009): "Monetary Policy with Heterogeneous and Misspecified Expectations," *Journal of Money, Credit, and Banking*, 41, 79–100.

Bernanke, B. S. and M. Woodford (1997): "Inflation Forecasts and Monetary Policy," *Journal of Money, Credit, and Banking*, 29, 653–684.

Blanchard, O. J. and C. M. Kahn (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48, 1305–1311.

Branch, W. A. and G. W. Evans (2006): "Intrinsic Heterogeneity in Expectation Formation," *Journal of Economic Theory*, 127, 264–295.

Branch, W. A. and B. McGough (2009): "A New Keynesian Model with Heterogeneous Expectations," *Journal of Economic Dynamics and Control*, 33, 1036–1051.

Bullard, J. and K. Mitra (2002): "Learning about Monetary Policy Rules," *Journal of Monetary Economics*, 49, 1105–1129.

Canova, F. (2009): "What Explains the Great Moderation in the U.S.? A Structural Analysis," *Journal of the European Economic Association*, 7, 697–721.

Clarida, R., J. Gali, and M. Gertler (2000): "Monetary Policy Rules and Macroeconomic Stability: Evidence and Some Theory," *Quarterly Journal of Economics*, 115, 147–180.

Ellison, M. and J. Pearlman (2011): "Saddlepath Learning," *Journal of Economic Theory*, 146, 1500–1519.

Evans, G. W. and S. Honkapohja (1996): "Least Squares Learning with Heterogeneous Expectations," *Economics Letters*, 53, 197–201.

——— (2001): *Learning and Expectations in Macroeconomics*, Princeton University Press.

——— (2003a): "Adaptive Learning and Monetary Policy Design," *Journal of Money, Credit, and Banking*, 35, 1045–1072.

——— (2003b): "Expectations and the Stability Problem for Optimal Monetary Policies," *Review of Economic Studies*, 70, 807–824.

——— (2006): "Monetary Policy, Expectations and Commitment," *Scandinavian Journal of Economics*, 108, 15–38.

——— (2008): "Expectations, Learning and Monetary Policy: An Overview of Recent Research," *Centre for Dynamic Macroeconomic Analysis Working Paper Series*, 08/02.

Evans, G. W., S. Honkapohja, and R. Marimon (2001): "Convergence in Monetary Inflation Models with Heterogeneous Learning Rules," *Macroeconomic Dynamics*, 5, 1–31.

Fidrmuc, J. and I. Korhonen (2003): "Similarity of Supply and Demand Shocks between the Euro Area and the CEECs," *Economic Systems*, 27, 313–334.

Giannitsarou, C. (2003): "Heterogeneous Learning," *Review of Economic Dynamics*, 6, 885–906.

Guse, E. A. (2005): "Stability Properties for Learning with Heterogeneous Expectations and Multiple Equilibria," *Journal of Economic Dynamics and Control*, 29, 1623–1642.

——— (2008): "Learning in a Misspecified Multivariate Self-referential Linear Stochastic Model," *Journal of Economic Dynamics and Control*, 32, 1517–1542.

Honkapohja, S. and K. Mitra (2004): "Are Non-fundamental Equilibria Learnable in Models of Monetary Policy," *Journal of Monetary Economics*, 51, 1743–1770.

——— (2005): "Performance of Monetary Policy with Internal Central Bank Forecasting," *Journal of Economic Dynamics and Control*, 29, 627–658.

——— (2006): "Learning Stability in Economies with Heterogeneous Agents," *Review of Economic Dynamics*, 9, 284–309.

Marcet, A. and T. J. Sargent (1989): "Convergence of Least-Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337–368.

McCallum, B. T. (2004): "On the Relationship between Determinate and MSV Solutions in Linear RE Models," *Economics Letters*, 84, 55–60.

——— (2007): "E-stability vis-a-vis Determinacy Results for a Broad Class of Linear Rational Expectations Models," *Journal of Economic Dynamics and Control*, 31, 1376–1391.

Milani, F. (2008): "Learning, Monetary Policy Rules, and Macroeconomic Stability," *Journal of Economic Dynamics and Control*, 32, 3148–3165.

Sims, C. A. and T. Zha (2006): "Were There Regime Switches in U.S. Monetary Policy?" *American Economic Review*, 96, 54–81.

Slobodyan, S. and R. Wouters (2011): "Learning in an Estimated Medium-scale DSGE Model," *Journal of Economic Dynamics and Control*, 36, 26–46.

Taylor, J. B. (1993): "Discretion versus policy rules in practice," *Carnegie-Rochester Conference Series on Public Policy*, 39, 195–214.

Trenkler, G. (1995): "Eigenvalues of the Product of Nonnegative Definite Matrices," *Econometric Theory*, 11, p.808.

Woodford, M. (1994): "Determinacy of Equilibrium under Alternative Policy Regimes," *Economic Theory*, 4, 323–326.

——— (1999): "Optimal Monetary Policy Inertia," *Manchester School, Supplement*, 61, 1–35.

Table 1: Learnability Conditions under CS and HM Learning

	Steady State	
	Unobservable	Observable
CS	$\kappa(\phi_\pi - 1) + \phi_x(1-\beta) > 0$	$\kappa(\phi_\pi - \lambda^c) + \phi_x(1-\beta\lambda^c) > -\frac{(1-\lambda^c)(1-\beta\lambda^c)}{\alpha}$
HM	$\kappa(\phi_\pi - 1) + \phi_x(1-\beta) > 0$	$\kappa\left(\phi_\pi - \lambda^h\right) + \phi_x\left(1-\beta\lambda^h\right) > -\frac{\left(1-\lambda^h\right)\left(1-\beta\lambda^h\right)}{\alpha}$

Note: The derivation of learnability conditions is summarized in Appendix E. $\lambda^c \equiv \lambda[\Phi_n]$, $\lambda^h \equiv \lambda\left[\Phi_n\left(\frac{1}{n}\Psi_n\right)\right]$, and $0 \leq \lambda^h \leq \lambda^c < 1$.

Figure 1 Parameter Domains of Determinacy and Learnability

(with the observable steady state)

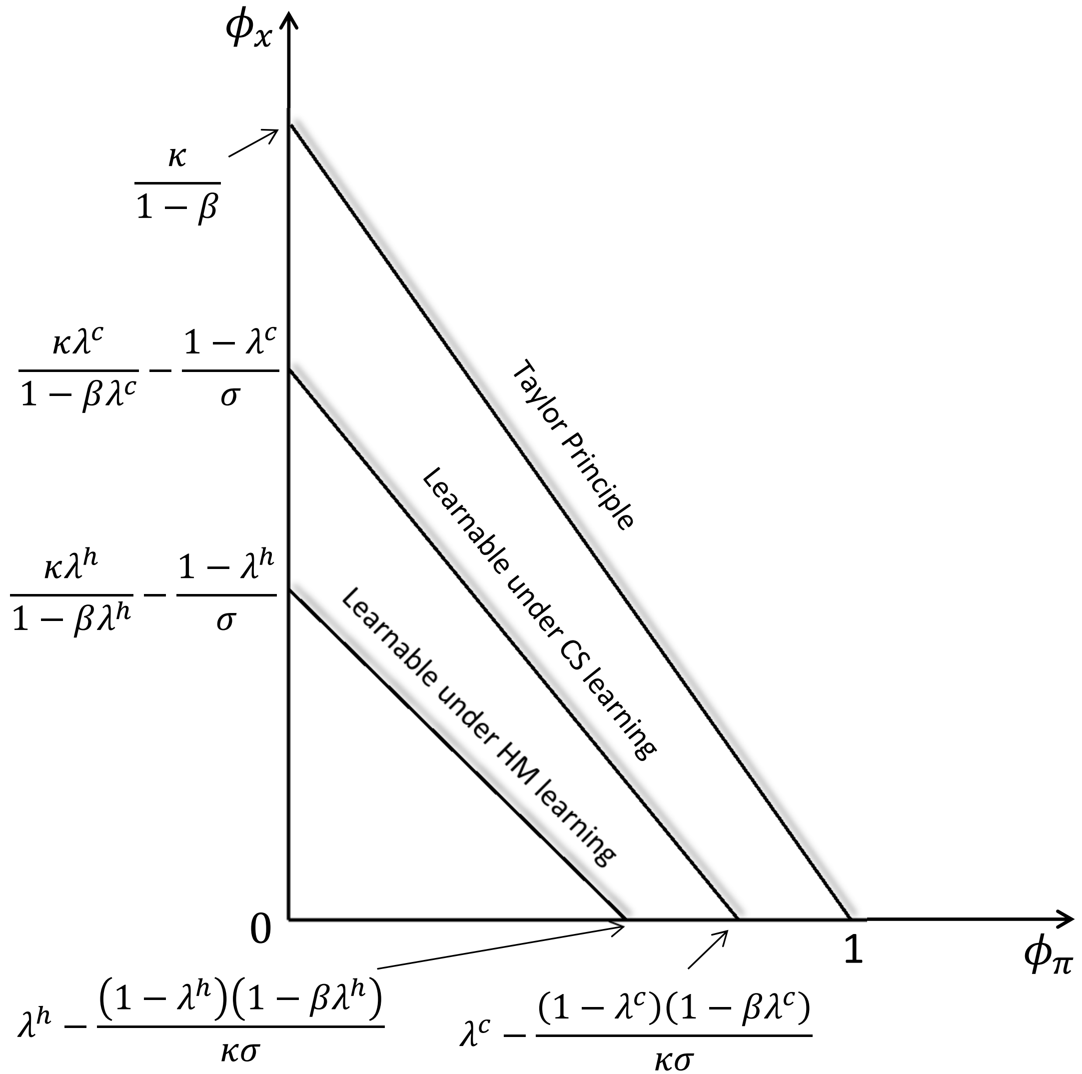

Figure 2 (a) Magnitude of HM and Learnability Conditions
(with the observable steady state, sigma=1, kappa=0.3, beta=0.99, psi=0.9)

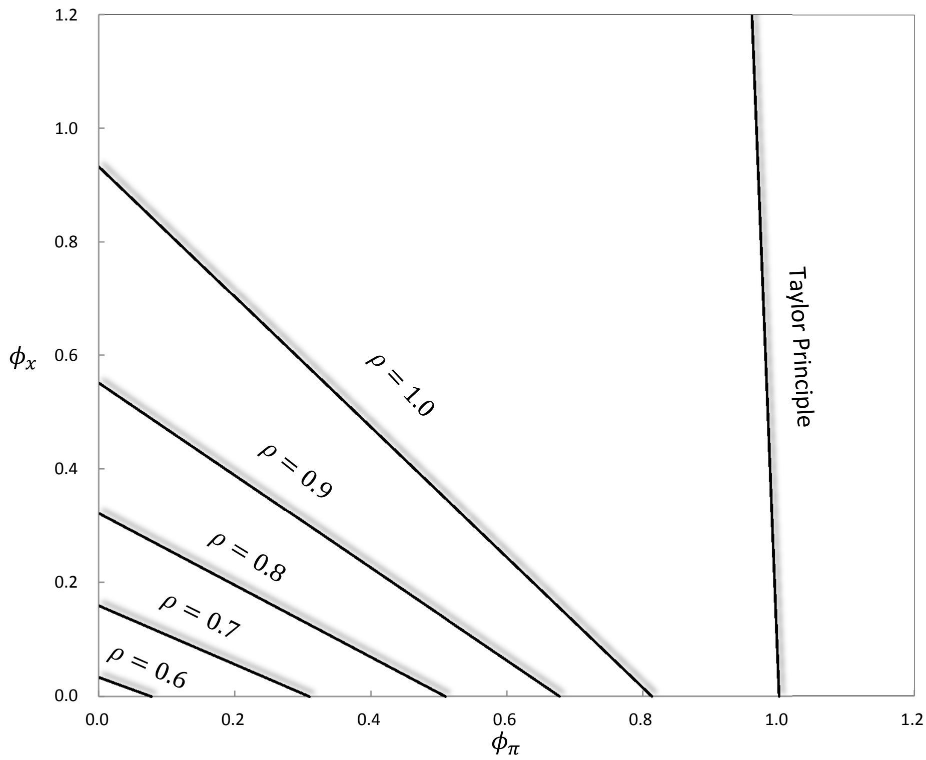

Figure 2 (b) Magnitude of HM and Learnability Conditions
(with the observable steady state, sigma=1/0.157, kappa=0.024, beta=0.99, psi=0.9)