Life Choices and Mortality Risk

Aihua Zhang [*†] Christian-Oliver Ewald[‡] David Newton[§] Charles Nolan [¶]

Abstract

In this paper we build up on the classical life cycle models presented by Merton (1969) and (1971) as well as Bodie, Merton and Samuelson (1992) and include mortality risk into our analysis. As such we study the consumption, labor supply, and portfolio decisions of a representative agent facing mortality risk, as represented in an actuarial life table. While working, the representative agent receives wage income as well as income from investment into one risky and one risk-free asset, depending on the current wage rate, the chosen labour supply and the chosen investment strategy. At any time prior to death, the agent can spend his wealth on consumption or further investment and is trying to maximize life time utility from consumption and leisure. Using martingale techniques instead of the Hamilton-Jacobi-Bellman approach allows us to consider general mortality risk. However, as in Blanchard (1985) we will assume the existence of life insurance markets. We derive closed-form solutions for optimal consumption, labor supply and investment strategy and will show that the inclusion of mortality risk, and in fact the shape of the mortality risk curve, significantly affect the level of consumption as well as the decomposition of the investment portfolio.

Keywords: Lifetime consumption and investment, mortality risk, martingale method

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[*]University of Munich (LMU), Institute for Risk Management and Insurance; Schackstrasse 4/III; 80539 Munich, Germany.

[†]Nottingham University Business School, China Campus; Administration Building, 199 Taikang East Road, Ningbo, 315100, China. E-mail: Aihua.Zhang@nottingham.edu.cn

[‡]Department of Economics, University of Glasgow, Glasgow, G12 8RT, United Kingdom. E-mail: Christian.Ewald@glasgow.ac.uk

[§]Nottingham University Business School, Jubilee Campus, Nottingham, NG8 1BB, United Kingdom. E-mail: David.Newton@nottingham.ac.uk

[¶]Department of Economics, University of Glasgow, Glasgow, G12 8RT, United Kingdom. E-mail: Charles.Nolan@glasgow.ac.uk

1 Introduction

Life time consumption and investment models have been considered by various authors, including Merton (1969) and (1971), Bodie, Merton and Samuelson (1992) as well as Bodie (2004). The setup in all these contributions is very similar, they all study the problem of maximizing expected discounted utility under consideration of a utility function which includes consumption and in some cases leisure, over the life time of a representative agent. Bodie, Merton and Samuelson (1992) consider an exogenously given retirement age and leave it as an open question, to determine the optimal retirement age within an optimal stopping context. This problem has now been considered by Dybvig and Liu (2010). Zhang (2010) considers retirement age as exogenously given, but allows for fully flexible labour supply, which in essence includes retirement as an option for the agent.

All of the above have in common that they include a stochastic investment asset and possibly stochastic wage income, but none of them takes mortality risk into account. The contribution of this article is the inclusion of time varying mortality risk into a continuous time stochastic life time consumption model, where a representative agent chooses consumption, labor supply and portfolio investment into riskless and risky assets, assuming a CRRA type of utility function measuring utility from consumption against disutility from labor. Mortality risk has been considered in continuous time overlapping generation models, such as Blanchard (1985), but in this case no risky investment assets had been included[1] and the utility function has been restricted to log-utility. Further the mortality rate has been chosen as constant in time, which is a rather unrealistic assumption, taking actual statistical life tables into account.

Admittingly, using classical techniques such as the Hamilton-Jacobi-Bellman framework or the Pontryagin maximum principle it is very difficult, if not impossible to allow for time varying mortality rates, in particular when real mortality curves as obtained from statistical life tables are supposed to be used. In this article we use a combination of martingale techniques to circumvent these problems. In particular we will avoid any form of partial or ordinary differential equation, and

[1] In fact, once Blanchard (1985) has averaged over the mortality risk, his model is completely deterministic.

will in effect be able to deal with completely arbitrary mortality curves. Nevertheless, even in this far more complex setup, we are able to derive analytic forms for the optimal consumption, labour supply and portfolio investment process in the presence of mortality risk. We are further able to derive a compact form for the Euler equation of consumption growth. We find that the effect of mortality risk on consumption and labour supply is through the Lagrange multiplier of the associated constrained optimization problem only, and as such it shifts consumption and labor supply, but has no effect on the Euler equation. Mortality risk however effects optimal portfolio investment in a more subtle way.

Under the assumption of constant mortality rate, we are able to derive a closed form expression for the elasticity of consumption with respect to the mortality rate. Using realistic parameters we find that this elasticity is negative, within the range of 0 (at zero mortality rate) to -0.53 (at mortality rate $0.002 \sim 39$ year old UK male). In the empirical part of the paper we use actual mortality curves as obtained from statistical life tables supplied by the UK's Government Actuary's Department covering the years from 1982 until 2006. Substituting these curves into our model we obtain that keeping all other parameters constant, a change in the mortality curves from 1982 to 2006 leads to a shift in consumption upwards of roughly 4%, contributing to approxiamtely 80% in real GDP growth in the UK over the same time period. We also observe that optimal labour supply in effect of the same change of the mortality curve is reduced by 2.5%, or about $1h$ from a $40h$ working week. As such the message of this article is that changes in mortalility risk do have a significant impact on consumption spending, labour supply and portfolio investment.

The remainder of the paper is organized as follows. In section 2 we set up our model and derive some basic equations, while in section 3 we consequently proceed by using martingale methods in order to transform the dynamic problem into a static problem, which we will solve. Section 4 contains both theoretical and empirically founded examples, while section 5 contains the conclusions.

2 Model

Let us consider a representative agent trying to maximize the following functional:

$$\max_{\pi,C,L} \mathbb{E}\left(\int_0^\tau e^{-\int_0^t \rho s ds} u(C_t, L_t) dt\right). \tag{1}$$

Here τ denotes the time of death, C_t denotes instantaneous consumption, L_t denotes instantaneous labour supply and π_t denotes the investment choice. $C_t \geq 0$, $L_t \geq 0$ and π_t are chosen by the agent depending on information contained in the sigma algebra $\mathcal{F}_t$ which will be introduced below. The investment assets available to the agent will also be introduced below. The time preference rate ρ_s of the agent is assumed to be a deterministic, positive function, while the time of death will be considered as a random time, with

$$\mathbb{P}\left(\tau \in [t, t+dt) | \tau \geq t\right) = \nu_t dt, \tag{2}$$

where ν_t is the time dependent instantaneous mortality rate. Intuitively, the mortality rate ν_t describes the likelihood of the agent aged t dying in the interval $[t, t+dt)$. This rate can be easily obtained from actuarial life tables and in general differs regionally and historically. We assume that ν_t is a deterministic function, but note that most of the following analysis could be carried out, if ν_t were a doubly stochastic process, compare Duffie (2001) page 276. Under this assumption, the agents likelihood of surviving until age t is given by

$$\mathbb{P}(\tau > t) = e^{-\int_0^t \nu_s ds}. \tag{3}$$

We assume that the random time τ is independent of any of the economic state variables, and hence using that $\mathbb{E}(\mathbf{1}_{\{t<\tau\}}) = \mathbb{P}(\tau > t)$ conclude that

$$\begin{aligned}\mathbb{E}\left(\int_0^\tau e^{-\int_0^t \rho s ds} u(C_t, L_t) dt\right) &= \mathbb{E}\left(\int_0^\infty e^{-\int_0^t \rho s ds} u(C_t, L_t) \cdot \mathbf{1}_{\{t<\tau\}} dt\right) \\ &= \mathbb{E}\left(\int_0^\infty e^{-\int_0^t \rho s ds} u(C_t, L_t) \cdot e^{-\int_0^t \nu_s ds} dt\right).\end{aligned}$$

Defining the mortality adjusted discount rate

$$\hat{\rho}_t = \rho_t + \nu_t \tag{4}$$

we may write (1) as

$$\max_{\pi,C,L} \mathbb{E}\left(\int_0^\infty e^{-\int_0^t \hat{\rho}_s ds} u(C_t, L_t)dt\right). \tag{5}$$

Let us now introduce the investment assets in our model. We assume that the economy features one risk-less asset modeled as

$$dB_t = B_t r_t dt \tag{6}$$

and one risky asset

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t). \tag{7}$$

Here W_t denotes a standard Brownian motion and we denote with $\mathcal{F}_t$ the filtration it generates. The parameters r_t, μ_t and σ_t are allowed to vary deterministically in time. As in Blanchard (1985) we assume the existence of fairly priced life insurance, which replaces a bequest motive: "In the absence of a bequest motive, and if negative bequests are prohibited, agents will contract to have their wealth (positive or negative) return to the life insurance company contingent on their death." The modeling framework in this article assumes that the representative agent represents "a large number of identical agents", and as such life insurance contracts can be offered risk-less by life insurance companies. We assume that the market for life insurance contracts is competitive, and hence free entry and exit will result in a zero profit condition, which in turn implies that the fair pricing of the insurance contract obliges/entitles the holder to payments

$$X_t \nu_t dt \tag{8}$$

per infinitesimal time interval dt, where X_t denotes the current wealth of the agent. Note that (8) represents a payment to be made by the agent to the insurance company, in case the agent has debt, i.e. $X_t < 0$ and otherwise presents an income, i.e. payment from the insurance company to the agent, in exchange for the agent giving up his wealth to the insurance company at the time of his death.

Denoting with π_t the fraction of wealth invested into the risky asset and with w_t the wage rate, the dynamics of the wealth process is described by

$$
\begin{aligned}
dX_t &= X_t \left\{ (r_t + \nu_t)dt + \pi_t \left[(\mu_t - r_t)dt + \sigma_t dW_t \right] \right\} \\
&\quad - C_t dt + w_t L_t dt,
\end{aligned} \tag{9}
$$

with $X_0 = x \geq 0$.

As the analysis above has shown, the problem of the finitely lived agent, problem (1) subject to constraint (9), is equivalent to the problem of the infinitely lived agent, problem (8) subject to constraint (9), where the discount rate as well as the drift of the wealth process have been adjusted to accommodate the mortality risk. We further define

$$
\hat{r}_t = r_t + \nu_t \tag{10}
$$

$$
\hat{\mu}_t = \mu_t + \nu_t \tag{11}
$$

and note that the market price of financial risk

$$
\theta_t = \frac{\mu_t - r_t}{\sigma_t} = \frac{\hat{\mu}_t - \hat{r}_t}{\sigma_t} \tag{12}
$$

is unaffected by mortality risk.

3 Martingale based approach

In order to apply martingale methods to solve the problem discussed in the previous section, we define the stochastic discount factor $\hat{H}_t$ via

$$
\begin{aligned}
d\hat{H}_t &= -\hat{H}_t \left(\hat{r}_t dt + \theta_t dW \right) \\
\hat{H}_0 &= 1.
\end{aligned} \tag{13}
$$

Note that the stochastic discount factor features the mortality adjusted rate $\hat{r}_t$ and the classical market price of risk θ_t in it. We can write $\hat{H}_t$ as

$$\hat{H}_t = e^{-\int_0^t \left(r_s+\nu_s+\frac{1}{2}\theta_s^2\right)ds - \int_0^t \theta_s dW_s} = e^{-\int_0^t \nu_s ds} H_t, \tag{14}$$

where H_t is the classical stochastic discount factor, see for example Korn (2000). Hence the stochastic discount factor $\hat{H}_t$ splits up into two components, $e^{-\int_0^t \nu_s ds}$ is adjusting for mortality risk and H_t is adjusting for financial risk.

Applying the Itô product rule, it is easy to verify that

$$d(\hat{H}_t X_t) = \hat{H}_t X_t \left(\pi_t \sigma_t - \theta_t\right) dW_t - \hat{H}_t C_t dt + \hat{H}_t w_t L_t dt. \tag{15}$$

Integrating (15) from t to ∞ and imposing the following transversality condition[2]

$$\lim_{u\to\infty} \mathbb{E}(\hat{H}_u X_u) = 0 \tag{16}$$

we obtain

$$-\hat{H}_t X_t = \int_t^\infty \hat{H}_s X_s (\pi_s \sigma_s - \theta_s) dW_s - \int_t^\infty \hat{H}_s C_s ds + \int_t^\infty \hat{H}_s w_s L_s ds. \tag{17}$$

Denoting the conditional expectation with respect to $\mathcal{F}_t$ as $\mathbb{E}_t$ we obtain

$$X_t = \mathbb{E}_t \left[\int_t^\infty \frac{\hat{H}_s}{\hat{H}_t} C_s ds \right] - \mathbb{E}_t \left[\int_t^\infty \frac{\hat{H}_s}{\hat{H}_t} w_s L_s ds \right]. \tag{18}$$

At time $t = 0$ we obtain the static budget constraint

$$\mathbb{E}\left(\int_0^\infty \hat{H}_s C_s ds \right) = x + \mathbb{E}\left(\int_0^\infty \hat{H}_s w_s L_s ds \right). \tag{19}$$

The intuition behind equation (19) is that expected stochastically discounted consumption need to be equal to initial wealth plus expected stochastically discounted wage income, where the discount factor takes both market risk and mortality risk into account.

[2]The corresponding deterministic version of the this transversality condition appears in Blanchard (1985) on page 227, and prevents the case where an agent takes up more and more debt, while being covered by life insurance.

We now obtain that problem (5) subject to the dynamic constraint (9) and transversality condition (16) is equivalent to problem (5) with the static budget constraint (19). In order to solve the latter problem we introduce the Lagrange function

$$\begin{aligned}\mathcal{L}(\lambda, C_t, L_t) &= \mathbb{E}\left(\int_0^\infty e^{-\int_0^t \hat{\rho}_s ds} u(C_t, L_t) dt\right) \\ &\quad + \lambda \left\{ x + \mathbb{E}\left(\int_0^\infty \hat{H}_s w_s L_s ds\right) - \mathbb{E}\left(\int_0^\infty \hat{H}_s C_s ds\right)\right\}.\end{aligned} \tag{20}$$

In order to proceed to a closed form solution, we need to specify the utility function $u(C_t, L_t)$ at this point. We define

$$u(C_t, L_t) := \frac{C_t^{1-\gamma}}{1-\gamma} - b_t \frac{L_t^{1+\eta}}{1+\eta} \tag{21}$$

where $b_t > 0$ is a deterministic function. The intuition behind (21) is to weigh up benefits from consumptions against costs from labour in constant relative risk aversion manner. The function b_t measures the relative cost of labour, which may vary between age classes.

Differentiating the Lagrange function (20), we obtain the following first order conditions

$$C_t^{-\gamma} = \frac{\partial u}{\partial C_t} = \lambda e^{\int_0^t \hat{\rho}_s ds} \hat{H}_t \tag{22}$$

$$-b_t L_t^{\eta} = \frac{\partial u}{\partial L_t} = -\lambda e^{\int_0^t \hat{\rho}_s ds} \hat{H}_t w_t. \tag{23}$$

We obtain from (22) and (23) that

$$C_t^{-\gamma} = \lambda e^{-\int_0^t \left(\hat{r}_s - \hat{\rho}_s + \frac{1}{2}\theta_s^2\right) ds - \int_0^t \theta_s dW_s} = \lambda e^{\int_0^t \rho_s ds} H_t \tag{24}$$

$$-b_t L_t^{\eta} = -\lambda e^{-\int_0^t \left(\hat{r}_s - \hat{\rho}_s + \frac{1}{2}\theta_s^2\right) ds - \int_0^t \theta_s dW_s} w_t = -\lambda e^{\int_0^t \rho_s ds} H_t w_t, \tag{25}$$

where we used in addition that $\hat{r}_s - \hat{\rho}_s = r_s - \rho_s$, see (4) and (10), as well as (14). The mortality component hence cancels out of the time dependent component of consumption and labour supply represented by H_t above. It can therefore be concluded that mortality risk as such will have no effect on the growth rate of consumption $\frac{d}{dt}\mathbb{E}_t\left(\frac{dC_t}{C_t}\right)$. However, as we will see below, it will affect the value of the Lagrange multiplier λ and hence shift consumption to a different level. These results are in line with the results in Blanchard (1985). The optimal consumption and labour supply can be easily derived from (24) and (25) as

$$
\begin{aligned}
C_t^* &= \lambda^{-\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\int_0^t \rho_s ds} H_t^{-\frac{1}{\gamma}} \qquad (26)\\
L_t^* &= \lambda^{\frac{1}{\eta}} e^{\frac{1}{\eta}\int_0^t \rho_s ds} (H_t w_t)^{\frac{1}{\eta}} b_t^{-\frac{1}{\eta}}. \qquad (27)
\end{aligned}
$$

We will now derive an analytic expression for the Lagrange multiplier λ and by doing this identify the mortality dependence in (26) and (27). Substitution into (19) and using (14) once more we obtain

$$
\begin{aligned}
&\lambda^{-\frac{1}{\gamma}}\mathbb{E}\left(\int_0^\infty \left(e^{-\int_0^t\left(\nu_s+\frac{1}{\gamma}\rho_s\right)ds}\right) H_t^{\frac{\gamma-1}{\gamma}} dt\right) \qquad (28)\\
&\qquad = x + \lambda^{\frac{1}{\eta}} \quad \mathbb{E}\left(\int_0^\infty \left(e^{-\int_0^t\left(\nu_s-\frac{1}{\eta}\rho_s\right)ds}\right) b_t^{-\frac{1}{\eta}} (H_t w_t)^{\frac{\eta+1}{\eta}} dt\right).
\end{aligned}
$$

Using that everything except H_t and and w_t is deterministic, we obtain

$$
\begin{aligned}
&\lambda^{-\frac{1}{\gamma}}\left(\int_0^\infty e^{-\int_0^t\left(\nu_s+\frac{1}{\gamma}\rho_s\right)ds}\mathbb{E}\left(H_t^{\frac{\gamma-1}{\gamma}}\right) dt\right) \qquad (29)\\
&\qquad = x + \lambda^{\frac{1}{\eta}} \quad \left(\int_0^\infty e^{-\int_0^t\left(\nu_s-\frac{1}{\eta}\rho_s\right)ds} b_t^{-\frac{1}{\eta}}\mathbb{E}\left((H_t w_t)^{\frac{\eta+1}{\eta}}\right) dt\right).
\end{aligned}
$$

In order to proceed, we need to make assumptions about the dynamics of the wage rate w_t. We assume that

$$
dw_t = w_t a_t dt, \qquad (30)
$$

with $a_t \geq 0$ a deterministic function.[3] Under (30) we compute

$$\mathbb{E}\left(H_t^{\frac{\gamma-1}{\gamma}}\right) = e^{-\int_0^t \frac{\gamma-1}{\gamma}\left(r_s+\frac{\theta_s^2}{2\gamma}\right)ds} \tag{31}$$

$$\mathbb{E}\left((H_t w_t)^{\frac{\eta+1}{\eta}}\right) = w_0^{\frac{\eta+1}{\eta}} e^{-\int_0^t \frac{\eta+1}{\eta}\left(r_s-a_s-\frac{\theta_s^2}{2\eta}\right)ds} \tag{32}$$

and substitution into (29) gives

$$\lambda^{-\frac{1}{\gamma}} \int_0^\infty e^{-\frac{1}{\gamma}\int_0^t\left(\rho_s+(\gamma-1)\left(r_s+\frac{\theta_s^2}{2\gamma}\right)\right)ds} \cdot e^{-\int_0^t \nu_s ds} dt \tag{33}$$
$$= x + \lambda^{\frac{1}{\eta}} w_0^{\frac{\eta+1}{\eta}} \int_0^\infty e^{\frac{1}{\eta}\int_0^t\left(\rho_s-(\eta+1)\left(r_s-a_s-\frac{\theta_s^2}{2\eta}\right)\right)ds} \cdot e^{-\int_0^t \nu_s ds} \cdot b_t^{-\frac{1}{\eta}} dt$$

Note that the computatability of the integrals above, depends on the deterministic functions ρ_s, r_s, θ_s, a_s, b_s and ν_s. If for example these are all constant, than it is a trivial exercise to compute all integrals in (33) explicitly. However, it will still not be possible to solve (33) explicitly for λ, as the equation $\lambda^a = x + \lambda^b$ can not be solved explicitly for λ in the generic case. On the other side, in the most general case, it is a simple exercise to compute the integrals and λ from (33) numerically.

The non-existence of a bequest motive on the other hand can also be interpreted as that the representative agent is born at time $t = 0$ without any means, i.e. $x = 0$. In this case we obtain an explicit solution for the Lagrange multiplier

$$\lambda = \left(\left(w_0^{\frac{\eta+1}{\eta}}\right) \frac{\int_0^\infty e^{\frac{1}{\eta}\int_0^t\left(\rho_s-(\eta+1)\left(r_s-a_s-\frac{\theta_s^2}{2\eta}\right)\right)ds} \cdot e^{-\int_0^t \nu_s ds} \cdot b_t^{-\frac{1}{\eta}} dt}{\int_0^\infty e^{-\frac{1}{\gamma}\int_0^t\left(\rho_s+(\gamma-1)\left(r_s+\frac{\theta_s^2}{2\gamma}\right)\right)ds} \cdot e^{-\int_0^t \nu_s ds} dt}\right)^{-\frac{\gamma\eta}{\gamma+\eta}} \tag{34}$$

We can see clearly in (34) the mortality dependence of the Lagrange multiplier. The classical model without mortality is obtained by setting $\nu_s \equiv 0$. Noticing once more that $\mathbb{P}(\tau > t) = e^{-\int_0^t \nu_s ds}$ we can see that compared to the classical case without mortality the integrands in the nominator and denominator in (33) are weighted by the probability of survival. We will later show that under realistic choices of

[3] With slightly more effort, the analysis below could also be carried out, by assuming that $dw_t = w_t(a_t dt + \sigma_t^w dB_t)$ with B_t a Brownian motion (possibly correlated to W_t) and σ_t^w a deterministic function, i.e. the wage rate volatility.

parameters, the inclusion of mortality risk will have significant quantitative effects.

Let us summarize the results so far in the following theorem:

Theorem 3.1. *The optimal consumption and labour supply strategy of the agent optimizing (1) under the dynamic constraint (9) and transversality condition (16) are given by*

$$
\begin{aligned}
C_t^* &= e^{-\frac{1}{\gamma}\int_0^t \rho_s ds} H_t^{-\frac{1}{\gamma}} \left(\left(w_0^{\frac{\eta+1}{\eta}} \right) \frac{\int_0^\infty e^{\frac{1}{\eta}\int_0^t \left(\rho_s-(\eta+1)\left(r_s-a_s-\frac{\theta_s^2}{2\eta}\right)\right)ds} \cdot e^{-\int_0^t \nu_s ds} \cdot b_t^{-\frac{1}{\eta}} dt}{\int_0^\infty e^{-\frac{1}{\gamma}\int_0^t \left(\rho_s+(\gamma-1)\left(r_s+\frac{\theta_s^2}{2\gamma}\right)\right)ds} \cdot e^{-\int_0^t \nu_s ds} dt} \right)^{\frac{\eta}{\gamma+\eta}} \\
L_t^* &= e^{\frac{1}{\eta}\int_0^t \rho_s ds} (H_t w_t)^{\frac{1}{\eta}} b_t^{-\frac{1}{\eta}} \left(\left(w_0^{\frac{\eta+1}{\eta}} \right) \frac{\int_0^\infty e^{\frac{1}{\eta}\int_0^t \left(\rho_s-(\eta+1)\left(r_s-a_s-\frac{\theta_s^2}{2\eta}\right)\right)ds} \cdot e^{-\int_0^t \nu_s ds} \cdot b_t^{-\frac{1}{\eta}} dt}{\int_0^\infty e^{-\frac{1}{\gamma}\int_0^t \left(\rho_s+(\gamma-1)\left(r_s+\frac{\theta_s^2}{2\gamma}\right)\right)ds} \cdot e^{-\int_0^t \nu_s ds} dt} \right)^{-\frac{\gamma}{\gamma+\eta}}
\end{aligned}
$$

We will now turn our attention to the optimal investment strategy π_t^* which is part of the solution. From (18) we obtain that the wealth process X_t^* under the optimal plan (π_t^*, C_t^*, L_t^*) is given by

$$X_t^* = A_t - B_t, \tag{35}$$

with

$$
\begin{aligned}
A_t &= C_t^* \mathbb{E}_t \left(\int_t^\infty \frac{\hat{H}_s}{\hat{H}_t} \frac{C_s^*}{C_t^*} ds \right) && (36) \\
B_t &= (w_t L_t^*) \mathbb{E}_t \left(\int_t^\infty \frac{\hat{H}_s}{\hat{H}_t} \frac{(w_s L_s^*)}{(w_t L_t^*)} \right) ds. && (37)
\end{aligned}
$$

In the following we will compute A_t and B_t. Substituting the expressions for C_t^*

and L_t^* from Theorem 3.1. gives

$$\begin{aligned} A_t &= C_t^* \mathbb{E}_t \left(\int_t^\infty \frac{H_s}{H_t} \frac{C_s^*}{C_t^*} \cdot e^{-\int_t^s \nu_u du} ds \right) \\ &= C_t^* \mathbb{E}_t \left(\int_t^\infty \left(\frac{H_s}{H_t} \right)^{\frac{\gamma-1}{\gamma}} \cdot e^{-\int_t^s \left(\nu_u + \frac{1}{\gamma}\rho_u\right) du} ds \right) \\ &= C_t^* \int_t^\infty \mathbb{E}_t \left(\left(\frac{H_s}{H_t} \right)^{\frac{\gamma-1}{\gamma}} \right) \cdot e^{-\int_t^s \left(\nu_u + \frac{1}{\gamma}\rho_u\right) du} ds. \end{aligned}$$

Now, using that $\frac{H_s}{H_t}$ is independent of $\mathcal{F}_t$ and distributed like a geometric Brownian motion with time varying drift term, we obtain

$$A_t = C_t^* \int_t^\infty e^{-\int_t^s \left(\frac{\gamma-1}{\gamma} \left(r_u + \frac{\theta_u^2}{2\gamma} \right) + \frac{1}{\gamma}\rho_u \right) du} \cdot e^{-\int_t^s \nu_u du} ds =: C_t^* f_t \tag{38}$$

Similarly we can compute

$$B_t = w_t L_t^* \int_t^\infty e^{-\int_t^s \left(\frac{\eta+1}{\eta} \left(r_u - a_u - \frac{\theta_u^2}{2\eta} \right) - \frac{1}{\eta}\rho_u \right) du} \left(\frac{b_s}{b_t} \right)^{-\frac{1}{\eta}} \cdot e^{-\int_t^s \nu_u du} ds =: w_t L_t^* g_t \tag{39}$$

Using (38) and (39) we can write

$$X_t^* = f_t C_t^* - g_t w_t L_t^*, \tag{40}$$

where f_t and g_t are deterministic functions. Let us also note at this point that the expression $e^{-\int_t^s \nu_u du} ds$ is equal to $\mathbb{P}(\tau > s | \tau > t)$, the probability of survival until s given that the agent is still alive at time $t < s$.

Using the representation (40) together with (13) we compute

$$d\left(\hat{H}_t X_t^*\right) = \hat{H}_t f_t dC_t^* - \hat{H}_t g_t w_t dL_t^* + X_t^* d\hat{H}_t + (\ldots)dt. \tag{41}$$

The terms indicated by ... in the brackets before dt will be irrelevant for the following, which is why we omit them. In fact we will only be interested in the diffusion term, i.e. the expression in front of dW_t, within the expression (41). To identify this term, we compute

$$dC_t^* = -\frac{1}{\gamma} C_t^* H_t^{-1} dH_t + (\ldots)dt \tag{42}$$

$$dL_t^* = \frac{1}{\eta} L_t^* H_t^{-1} dH_t + (\ldots)dt \tag{43}$$

Further, using (14) and

$$dH_t = -H_t(r_t dt + \theta_t dW_t) \tag{44}$$

we eventually obtain

$$\begin{aligned} d\left(\hat{H}_t X_t^*\right) &= \frac{1}{\gamma}\hat{H}_t f_t C_t^* \theta_t dW_t + \frac{1}{\eta}\hat{H}_t g_t w_t L_t^* \theta_t dW_t - \hat{H}_t X_t^* \theta_t dW_t + (\ldots)dt \\ &= \left\{ g_t \hat{H}_t w_t L_t^* \left(\frac{\eta+1}{\eta}\right) - f_t \hat{H}_t C_t^* \left(\frac{\gamma-1}{\gamma}\right)\right\} \theta_t dW_t (\ldots)dt, \end{aligned} \tag{45}$$

where for the second equality we used (40). Since the diffusion term in the representation (45) must coincide with the diffusion term in the representation (15) for $X_t = X_t^*$, we obtain by noticing (40) once more and solving for π_t:

Theorem 3.2. *The optimal investment strategy of the agent optimizing (1) under the dynamic constraint (9) and transversality condition (16) is given by*

$$\pi_t^* = \frac{1}{\gamma}\frac{\mu_t - r_t}{\sigma_t^2} + g_t \cdot \left(\frac{1}{\gamma} + \frac{1}{\eta}\right) \frac{\mu_t - r_t}{\sigma_t^2} \cdot \frac{w_t L_t^*}{X_t^*}. \tag{46}$$

Note that the function g_t in (46) depends on mortality risk. A comparative analysis of this expression will follow in the next section.

We have already indicated above, that consumption growth is not directly affected by the inclusion of mortality risk. Nevertheless we believe that it is interesting to derive the Euler equation for consumption growth at this point. Computing the term in front of dt in (41) explicitly, we obtain

$$dC_t = \frac{1}{\gamma}\theta_t C_t^* dW_t + \frac{1}{\gamma}\left(r_t - \rho_t + \frac{\gamma+1}{2\gamma}\theta_t^2\right) C_t^* dt. \tag{47}$$

Dividing (47) by C_t^* and taking expectations, we obtain

$$\frac{d}{dt}\mathbb{E}\left(\frac{dC_t^*}{C_t^*}\right) = \frac{1}{\gamma}\left(r_t - \rho_t + \frac{\gamma+1}{2\gamma}\theta_t^2\right). \tag{48}$$

Eq. (48) is the Euler equation of the intertemporal maximization problem.[4] Note that because of the separability in the utility of consumption and labor supply, the risk-aversion coefficient η with respect to labor supply does not affect the Euler equation.

From (48), it is easy to see that the growth rate of the expected consumption is strictly positive when $\rho < r + \frac{\gamma+1}{2\gamma}\theta^2$ and strictly negative if $\rho > r + \frac{\gamma+1}{2\gamma}\theta^2$, constant if $\rho = r + \frac{\gamma+1}{2\gamma}\theta^2$. A smaller discount rate implies that the consumer is more patient and therefore prefers less consumption today than tomorrow, implying that consumption must rise.

The positive term θ^2 in (48) captures the uncertainty of the financial market, indicating that a risky financial market induces the consumer to adjust consumption more frequently. Other things equal, a higher market price of risk θ leads to a steeper slope of the expected consumption. In the case where the asset does not pay a risk premium ($\mu = r$), all the wealth is be optimally invested into the risk-free bond to secure a fixed income. The Euler equation in this case is the same as the Euler equation for the case of certainty [5]

$$g_c = \frac{r-\rho}{\gamma}. \tag{49}$$

When there is no uncertainty, the growth of consumption is strictly decreasing in risk aversion γ, or equivalently, strictly increasing in the elasticity of substitution in consumptions $\frac{1}{\gamma}$: when γ is smaller, the less marginal utility changes as con-

[4] This sort of Euler equation is standard in models without labor supply. See the recent paper by Luo, Smith and Zou (2009), who derive the Euler equation for a CARA utility function and a Ornstein-Uhlenbeck process for wage income. Toche (2005) and Marson and Wright (2001) also find a similar structure of the Euler equation under uncertainty. In Toche (2005), the inclusion of an additional term to the Euler equation is due to the risk of permanent income loss while, in Mason/Wright (2001), the conclusion is drawn based on the approximation of a discrete-time problem.

[5] See e.g. Romer (2006) for more detailed discussions of the Euler equation when there is no uncertainty.

sumption changes, the more the individual is willing to substitute consumption across periods.

With uncertainty in asset returns ($\theta > 0$), the effects of γ on consumption are twofold: (i) it captures the individual's willingness to substitute consumption across time and risk, so consumption decreases as γ increases; (ii) it also governs the individual's risk aversion toward the uncertainty of financial market: the smaller the degree of risk aversion γ, the greater the proportion of wealth invested into the risky asset and the greater the fluctuation in financial wealth, and consequently the greater the frequency of adjustment in consumption.

4 Examples and Empirical Analysis

To begin with, we start with a toy example, in which all parameters, including the mortality rate ν_t, are assumed to be constant. Further assuming that the representative agents is born without any initial wealth, i.e. $x = 0$, we can compute the Lagrange multiplier $\lambda(\nu)$ in (34) as a function of the mortality rate ν explicitly:

$$\lambda(\nu) = w_0^{\left(\frac{\eta+1}{\eta}\right)} b^{\left(\frac{\gamma}{\gamma+\eta}\right)} \left(\frac{\nu + \frac{\rho}{\gamma} + \frac{\gamma-1}{\gamma}\left(r + \frac{\theta^2}{2\gamma}\right)}{\nu - \frac{\rho}{\eta} + \frac{\eta+1}{\eta}\left(r - a - \frac{\theta^2}{2\eta}\right)} \right)^{-\frac{\gamma\eta}{\gamma+\eta}}. \tag{50}$$

In the following we will compute the elasticity of consumption with respect to mortality. This elasticity represents the percentage change in consumption for each percentage change in the mortality rate. It is rather simple to verify that

$$\frac{\frac{dC_t^*(\nu)}{C^*(\nu)}}{\frac{d\nu}{\nu}} = -\frac{1}{\gamma} \frac{\frac{d\lambda(\nu)}{\lambda(\nu)}}{\frac{d\nu}{\nu}}, \tag{51}$$

i.e. the elasticity of consumption is a constant fraction of the elasticity of the Lagrange multiplier. The constant factors w_0 and b do not affect the elasticity with respect to mortality.

Using (50) it is then a tedious but straightforward exercise to verify that

$$\frac{dC_t^*(\nu)}{C_t^*(\nu)} / \frac{d\nu}{\nu} = \tag{52}$$

$$\frac{\eta\left(\frac{1}{\nu-\frac{\rho}{\eta}+\frac{(\eta+1)\left(r-a-\frac{\theta^2}{2\eta}\right)}{\eta}}-\frac{\nu+\frac{\rho}{\gamma}+\frac{(\gamma-1)\left(r+\frac{\theta^2}{2\gamma}\right)}{\gamma}}{\left((\nu-\frac{\rho}{\eta}+\frac{(\eta+1)(r-a-\frac{\theta^2}{2\eta}}{\eta}\right)^2}\right)\left(\nu-\frac{\rho}{\eta}+\frac{(\eta+1)\left(r-a-\frac{\theta^2}{2\eta}\right)}{\eta}\right)\nu}{(\gamma+\eta)\left(\nu+\frac{\rho}{\gamma}+\frac{(\gamma-1)\left(r+\frac{\theta^2}{2\gamma}\right)}{\gamma}\right)}$$

It can be verified by tedious but straightforward computation that for general parameters

$$\left.\frac{\frac{dC_t^*(\nu)}{C^*(\nu)}}{\frac{d\nu}{\nu}}\right|_{\nu=0} = 0. \tag{53}$$

This means that at mortality rate $\nu = 0$ there is no first order effect on consumption by increasing the mortality rate. The two effects of increasing current consumption because of fear of death in the future and decreasing consumption because of a decrease in human wealth exactly offset each other. This neutrality ceases to hold however when the mortality rate is positive, as the following numerical example shows.

For the analysis below we assume the following parameter values: $\rho = 0.06$; $\gamma = 2$; $r = 0.03$, $\mu = 0.09$, $\sigma = 0.35$; $a = 0.01$, $b = 0.5$ and $\eta = 3$. Figure 1 shows the elasticity of consumption depending on the level of the mortality rate, for mortality rates ranging from 0 to 0.025, the mortality rate corresponding to a 70 year old male living in the UK in 2006, according to current UK life expectancy tables, see UK-GAD (2011).

We observe that the elasticities are all negative, meaning that with increasing mortality consumption declines. Further, the effect of a change in the mortality rate is strongest at about $\nu = 0.001503$, which corresponds to the mortality rate of a 39 year old male living in the UK in 2006.At that age, the elasticity of consumption is approximately at -0.53, which can be loosely interpreted as saying that if the mortality rate of a 39 year old declines by 10%, then consumption will increase by about 5%. The mortality rate of a 39 year old male living in the UK in 1981 was 0.001682, and hence declined over the period of 25 years between 1981

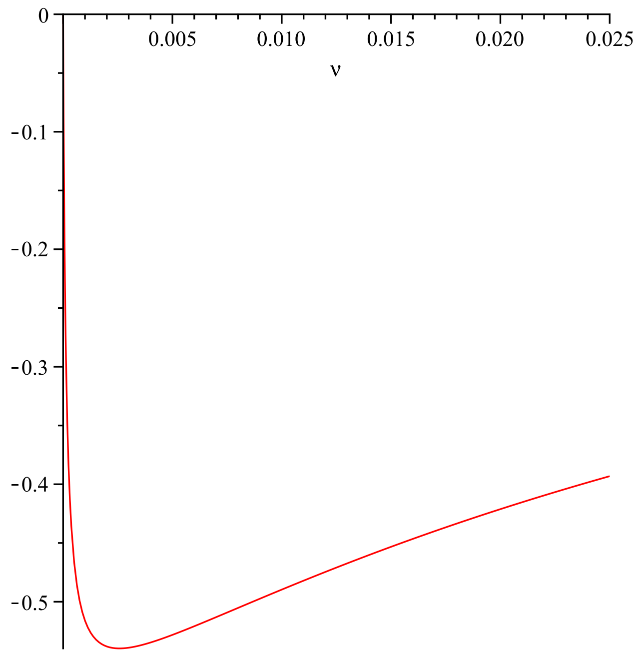

Figure 1:

and 2006 by 12% hence inducing a growth in consumption of about 6%. If we look further down, at around pension age of 66 the mortality rate in 1982 was 0.032541 while in 2006 the mortality rate for the same age group was 0.017108, which means that the mortality rate has been reduced over the period by roughly 50%. The elasticity in comsumption at that mortality rate is -0.44, so that the reduction in mortality of this age group effects in a growth of consumption by approximately 20%. Real GDP over that period in the UK grew by about 80%, which means that the simple analysis above, provides an indication that a reduction in mortality rate had a significant impact on real GDP growth, possibly explaining between $10\% - 25\%$ of it.

We are now considering the case of time dependent mortality rates. Figure 2 shows age dependent mortality rates for various years between 1982 and 2006 in the UK.

The figure clearly shows that mortality rates are on very similar levels until about age 40, but then diverge. The following Figure 3 represents the mortality rate of different age groups over the period 1982-2006.

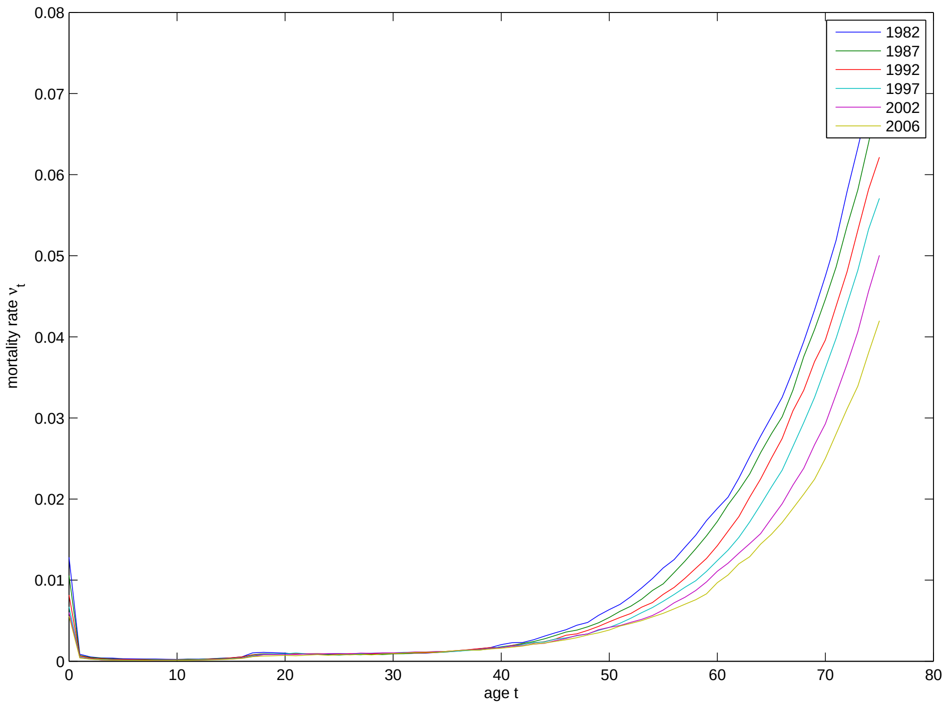
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
mortality rate ν_t
0
10
20
30
40
50
60
70
80
age t
1982
1987
1992
1997
2002
2006

Figure 2:

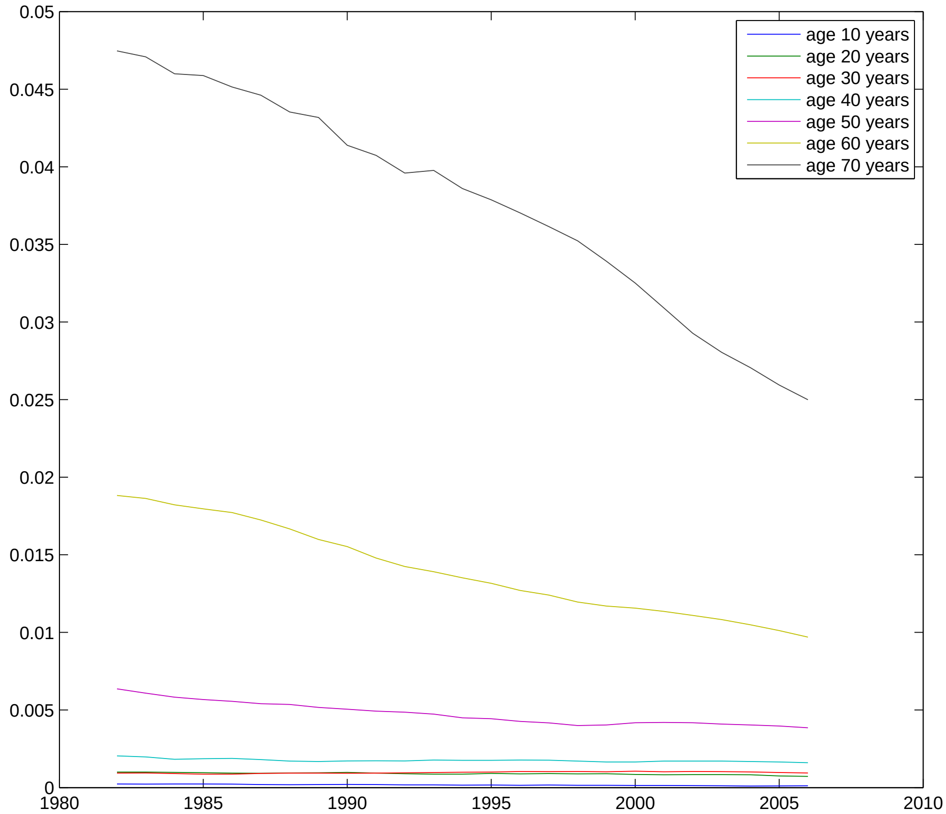
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
1980
1985
1990
1995
2000
2005
2010
age 10 years
age 20 years
age 30 years
age 40 years
age 50 years
age 60 years
age 70 years

Figure 3:

It can be seen that the mortality rate in the more senior age groups has decreased very significantly over the years, while in the more junior age groups up to age 30, the effect is far less significant.

In the following we assume that the initial wage rate is given by $w_0 = 60.000$. Figure 4 has been obtained by computing C_0^* in Theorem 1 with parameters as given before, but with time dependent mortality rates obtained from UK life expectancy tables for UK males from the years 1982 to 2006.

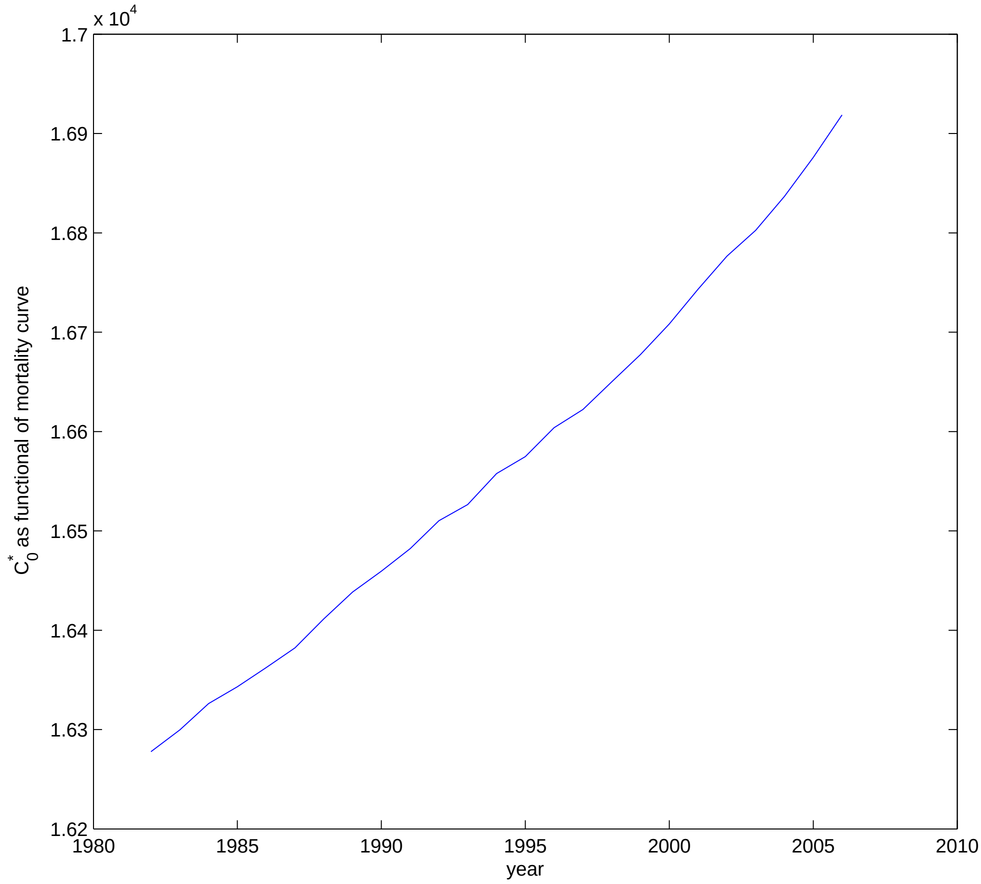

Figure 4:

The figure shows an upward trend, as expected. The over all growth in consumption caused by the changing mortality curves over the 25 year period in this case is about 4%. Compared with the aforementioned 80% in real GDP growth over the same period. Changes in the mortality curves in this setting still seem to have a significant impact on GDP.

Let us now have a look at the labour supply. With the same input data as before, we compute the expression L_0^* in Theorem 1 for the above historical mortality curves and obtain the following Figure 5.

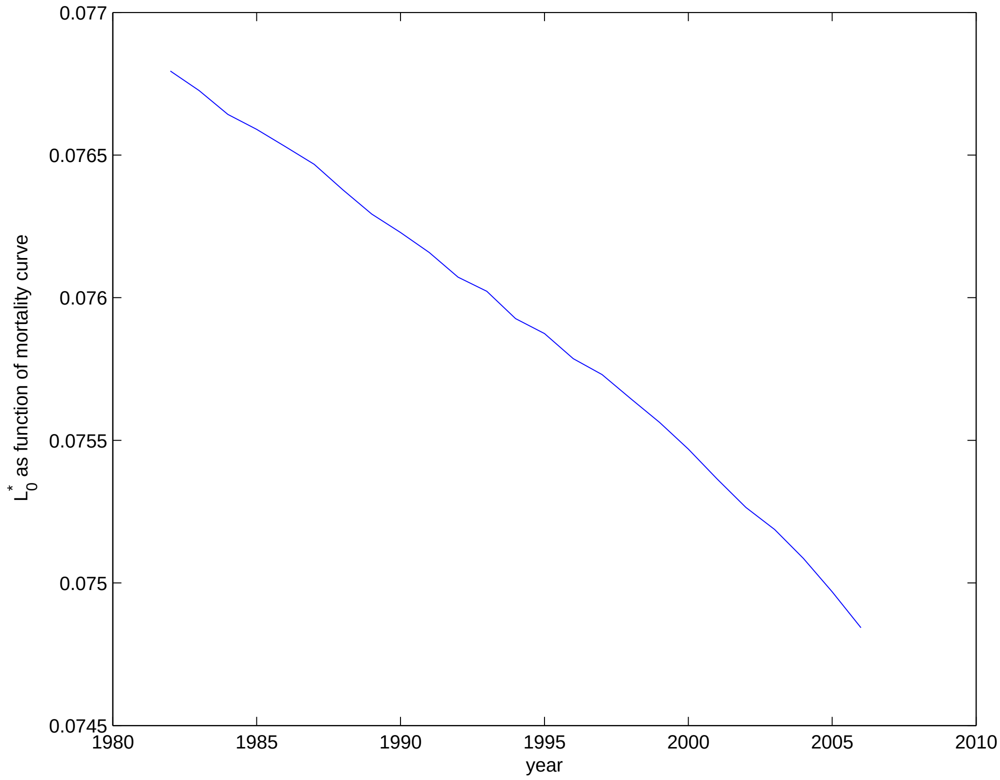

Figure 5:

Note that the absolute level of labour supply in the figure above is irrelevant, as we did not normalize labour supply in our model and allow for arbitrary large labour supply (we could also call it labour effort). Important on the other side, is the downward trend, and the fact that labour supply decreases by about 2.5% over the 25 year period due to changing mortality curves. This scraps about 1 hour from a 40 hours work week.

5 Work in Progress

- We also consider the case, where death is replaced by loss of employment. Life insurance is replaced by some kind of employment protection insurance. This has been considered in a far more simplistic setup by Toche (Economic Letters).

- The real challenge is to extend the framework to a continuous time overlapping generation framework, as in Blanchard (1985, Journal of Political Economy), and instead of working with a representative agent work with differently aged agents, and the integration over the age groups to obtain aggregate consumption. We are half-ways through.

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