Completeness, interconnectedness and distribution of interbank exposures - a parameterized analysis of the stability of financial networks

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Abstract

This paper assesses the impact of a certain structure of interbank exposures on the stability of a stylized financial system. Given a certain balance sheet structure of financial institutions, a large number of valid matrices of interbank exposures is created by a random generator. Assuming a certain loss given default, domino effects are simulated. The main results are, first, that financial stability depends not only on the completeness and interconnectedness of the network but also on the distribution of interbank exposures within the system (measured by entropy). Second, looking at random graphs, the sign of the correlation between the degree of equality of the distribution of claims and financial stability depends on the connectivity of the financial system as well as on additional parameters that affect the vulnerability of the system to interbank contagion. Third, the more concentrated assets are within a money center model, the less stable it is. Fourth, a money center model with asset concentration among core banks is less stable than a random graph with banks of homogeneous size. Results obtained in this paper extend existing theoretical literature that exclusively focuses on completeness and interconnectedness of the network as well as empirical literature that exclusively focuses on one particular financial network.

Keywords: domino effects, interbank lending, financial stability, contagion

JEL Classification: C63, G21, G28

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1 Introduction

Research on financial stability is an important topic in order to assess certain risks and dangers within financial systems that potentially lead to huge losses for the overall economy. Especially the investigation of various channels of interbank contagion has been in the focus of recent research. This is also the aim of this paper.

In this context, interbank contagion means that the failure of one financial institution leads to the failure of other financial institutions. To be more precise, this paper examines pure domino effects between banks. Thus, it is investigated what happens if one bank fails and therefore a part of other banks' claims to that bank also fail. It is then possible that creditor banks lose all their capital and therefore fail as well. In the worst case, there are subsequent rounds of failures until the whole system defaults. Of course, this is just one channel through which interbank contagion can occur. Further channels can be contagion due to liquidity problems because of correlated asset portfolios among banks, contagion due to refinancing problems affecting banks or contagion due to information spillovers. As a starting point, however, to be able to exclusively focus on the effect of the structure of the liability matrix on the stability of the financial network, only domino effects are considered in this paper.

The main contributions of this paper are, first, that there is an explicit investigation of the impact of the structure of the matrix of interbank liabilities on the stability of the interbank network. In this context, for given balance sheets of a hypothetical banking system, a large number of valid matrices of interbank exposures is created and characterized by the degree of inequality of the distribution of exposures (measured by entropy). Second, this paper examines, how the impact of the structure of the matrix of interbank exposures on the stability of the financial system interacts with other parameters like banks' capitalization or the loss given default. Third, this paper provides a comparison of the stability of the financial system between differen network topologies like a complete network, a random graph and a money center system with a core-periphery structure.

The main results are, first, that not only the topology of the network (eg its completeness and interconnectedness) determines its stability but also how equally interbank exposures are distributed. The second result is that the sign of the correlation between the degree of equality of the distribution of interbank exposures and the average number of bank failures depends on the number of interbank links within the financial system as well as on banks' equity ratio and the loss given default. Additionally, by assuming reasonable parameter values concerning the amount of bilateral interbank exposures, money center systems with asset concentration among core banks are more unstable than networks with banks of homogeneous size that form their links randomly.

The paper is organized as follows. Section 2 gives an overview of the related literature as well as the papers' main contributions to this literature. In Section 3 the basic structure of the financial system is defined. Section 4 explains in more detail how interbank liability matrices are created and characterized. Simulations of domino effects are run and results are presented in Section 5 which is divided into the investigation of complete networks (Section 5.1), random graphs (Section 5.2) and money center models (Section 5.3). Section 6 summarizes the main findings.

2 Literature

Various fields of studies have been developed to capture the numerous facets of this comprehensive topic (for a literature survey, see Allen and Babus (2009)). From the theoretical point of view, Allen and Gale (2000) show that interbank connections can be useful in order to provide an insurance against liquidity shocks. Because of these interbank linkages, however, the bankruptcy of one bank can lead to the bankruptcy of other banks. In this context, Allen and Gale show that a financial system with a complete network structure is less prone to contagion than a financial system with an incomplete network structure. In addition, they state that a disconnected system is useful to limit contagion. Freixas et al. (2000) implement, among other things, a theoretical analysis of contagion within a "money center system", where banks in the "periphery" are linked to one "core bank" but not to each other. They show that there are parameter constellations under which the failure of a periphery bank does not lead to contagion, whereas the failure of the "core bank" does.

Another part of the literature that investigates financial stability are empirical studies that use supervisory data to analyze the danger of domino effects within a banking system (for example van Lelyveld and Liedorp (2006), Upper and Worms (2004), Wells (2004), Furfine (2003), Sheldon and Maurer (1998)). As a lot of detailed data on interbank exposures are necessary but often not available, assumptions such as maximum entropy are made concerning the structure of these exposures. This means that banks are assumed to spread their interbank claims as equally as possible among their counterparties. However, it is likely that, under the maximum entropy assumption, results are biased. In his summary of the analysis of interbank contagion, Upper (2007) states that maximum entropy assumptions tend to underestimate the incidence but overestimate the severity of contagion. Mistrulli (2007) investigates interbank contagion using actual Italian interbank data and compares his findings with an analysis using the maximum entropy assumption. He finds that, for most parameter constellations, the maximum entropy assumption tends to underestimate the extent of contagion. There are, however, also some parameter constellations (in particular a high loss given default) where the maximum entropy assumption overestimates the scope of contagion.

Cifuentes et al. (2005) extend a contagion model of domino effects by simulations that include contagion due to liquidity problems. Within their simulations they use a clearing algorithm developed by Eisenberg and Noe (2001).[1] The analysis of pure domino effects is also extended by Chan-Lau (2010) and Espinosa-Vega and Solé (2010) by additionally considering contagion due to banks' refinancing problems and due to risk transfers. Also, the impact of market and funding liquidity risk on the stability of a financial network is investigated by Aikman et al. (2009).

In recent years there has been a growing literature which uses theory of complex networks to describe real-world financial systems and simulate the effects of potentially dangerous events. For example, Boss et al. (2004) analyze the network topology of the Austrian interbank market. Iori et al. (2008) apply network theory to describe the Italian overnight money market. Haldane (2009) provides a characterization of the world's financial network. Additionally, Gai and Kapadia (2010), as well as Nier et al. (2008), use random graphs to analyze the danger of contagion dependent on certain characteristics of the financial system by simulation. Gai and Kapadia (2010) find that for a high connectivity of the network, the probability that contagion occurs is low but the impact if contagion occurs is high. Nier et al. (2008) find out by parameterized simulation some non-linearities between certain parameter values and financial stability. Contrary to most of the empirical literature, where one special financial system is considered to test the danger of contagion, simulation-based work instead tries to find out the main characteristics that make a financial system especially vulnerable to contagion.

As, up to now, only few studies exist about the detailed structure of real-world financial networks, this paper considers several stylized structures and investigates their impact on financial stability by simulation. This paper builds on the empirical literature that uses entropy methods to construct and characterize interbank linkages as well as on literature that tries to simulate the danger of contagion according to certain characteristics of the financial system, in particular the

[1] Eisenberg and Noe showed that under mild regularity conditions (strong connectivity and at least one node has positive equity value or all nodes have positive operating cash flows) there exists a unique fixed point that describes the clearing payment vector of the financial system.

matrix of interbank exposures. For example, it extends the work of Mistrulli (2007) in such a way that not only one matrix is compared to the maximum entropy solution but a great variety of randomly generated matrices with different network structures. For the first time, a large set of valid interbank matrices is constructed by a random generator and then characterized according to certain properties, such as entropy, relative entropy to the maximum entropy solution or connectivity.

Additionally, this paper differs from Nier et al. (2008) in the sense that, first, balance sheets are constructed and, as a second step, the liability matrix is generated, which is, besides row and column sums of the matrix, independent from banks' balance sheets. Thus, stability results obtained in this paper can be attributed purely to changes in the liability matrix.

Furthermore, results can be interpreted as an extension to the theoretical literature about the impact of certain network patterns on contagion. Up to now the focus has been exclusively on the completeness and interconnectedness of interbank networks (see Allen and Gale (2000)). In this context, banks are modeled as completely homogeneous, especially with all interbank exposures being the same size. This work, however, investigates a large number of possible matrices with various possible specifications of interbank exposures and can thus have an additional focus on the *distribution* of claims within the network for given completeness and interconnectedness. Results of this paper show that the distribution of interbank claims within the network is an important parameter affecting the stability of the network.

3 Structure of the financial system

The financial system is modeled as a network of N nodes where nodes 1 to $N-1$ are financial institutions (referred to as banks in the following) and node N constitutes the external (non-banking) sector (such as households or non-financial companies). These nodes are linked by directed edges that depict direct claims/obligations between the financial institutions and the external sector. For some of the subsequent financial networks modeled, it is assumed that there are two different types of banks, core banks and periphery banks, that are equal within their groups but differ across groups with regard to their connectivity and size, respectively. The distribution of assets among the two types of banks is given by a concentration ratio CR, that denotes the share of total bank assets that core banks hold.

Bank i's balance sheet has the following structure:

$$A_i^{IB} + A_i^E = L_i^{IB} + L_i^E + E_i \tag{1}$$

$\forall\ i\ \epsilon\ \{1, ..., N-1\}$.

Interbank assets A_i^{IB} (liabilities L_i^{IB}) are claims (obligations) between banks. External assets A_i^E are interpreted as credit to the external sector. External liabilities L_i^E denote obligations of banks to the external sector such as customer deposits. The balance sheet is completed by equity E_i that is given by the difference of bank i's total assets A_i $(= A_i^{IB} + A_i^E)$ and liabilities L_i $(= L_i^{IB} + L_i^E)$.

The (risk-unweighted) equity ratio, which is presumed to be equal across banks, is given by:[2]

$$r = \frac{E_i}{A_i^{IB} + A_i^E} \tag{2}$$

$\forall\ i\ \epsilon\ \{1, ..., N-1\}$.

The financial system is characterized by the total amount of banks' assets A^{banks}, as well as the total amount of banks' interbank assets A^{IB}. The ratio of interbank to total assets in the financial system is defined as:

$$\phi = \frac{A^{IB}}{A^{banks}} \tag{3}$$

Banks' total assets, total liabilities and equity of this stylized financial system can be perfectly described by the total amount of banks' assets A^{banks}, the total number of banks $N-1$, the number of core banks n_{core}, the concentration ratio CR and banks' equity ratio r.

The direct connections between the nodes can be illustrated by a liability matrix:

$$\mathbf{L} = \begin{array}{c} \\ L_1 \\ L_2 \\ \vdots \\ L_{N-1} \\ L_N \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & \dots & A_{N-1} & A_N \end{array} \\ \begin{pmatrix} 0 & L_{1,2} & \dots & L_{1,N-1} & L_{1,N} \\ L_{2,1} & 0 & \dots & L_{2,N-1} & L_{2,N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ L_{N-1,1} & L_{N-1,2} & \dots & 0 & L_{N-1,N} \\ L_{N,1} & L_{N,2} & \dots & L_{N,N-1} & 0 \end{pmatrix} \end{array}$$

[2]More general models including banks with heterogeneous equity ratios can be implemented in a straightforward way.

with $L_{i,j}$ being the obligation of bank i to bank j ($i,\ j\ \epsilon\ \{1,...,N-1\}$). Because banks and the external sector do not lend to themselves, $L_{i,i} = 0\ \forall\ i\ \epsilon\ \{1,..,N\}$. Additionally, as banks are linked on both sides of their balance sheets, it is easy to interpret row sums (= total liabilities) and column sums (= total assets) of the matrix. The elements of the last row, $L_{N,i}$ ($\forall\ i\ \epsilon\ \{1,..,N-1\}$), are equal to banks' external assets A_i^E. Thus, the sum of the elements in the last row of the matrix is equivalent to banks' total external assets, which are given by $(1-\phi)\cdot A^{banks}$. The elements of the last column, $L_{i,N}$ ($\forall\ i\ \epsilon\ \{1,..,N-1\}$), are equal to banks' external liabilities L_i^E. Hence, the sum of the elements in the last column of the matrix (A_N) is equivalent to the total amount of external liabilities of banks. Furthermore, it is assumed that the system is closed, i.e. there is no lending / borrowing to somewhere outside the network. Technically, this means that the sum of row sums of the liability matrix has to be equal to the sum of column sums. Thus, total external liabilities of banks (or total assets of the external sector A_N) can be calculated by the difference between total liabilities in the system (the external sector included) and total assets of banks.[3]

4 Creation and characterization of liability matrices

Regulators often face the problem of limited data. Sometimes only the row sums and column sums of the liability matrix are observable. At least it is quite common for some elements of the liability matrix to be missing. As already mentioned, this problem is often surrounded by using the assumption that banks spread their exposures as evenly as possible, which is equivalent to maximizing the entropy of the (normalized) liability matrix.[4] However, using matrices under the maximum entropy assumption tends to bias the results.

The approach of this paper is to abstract from generating only one matrix using the maximum entropy assumption but to create, for given row and column sums, a large number of valid liability matrices by a random generator. This is done in two steps: first, a random number L_{ij}^{rand}, that does not exceed the number of total liabilities in the system L^{total} (or total assets in the system A^{total}, respectively), is assigned to each off-diagonal element. This random number is drawn from a uniform distribution with $L_{ij}^{rand}\ \epsilon\ \left[0, L^{total}\right]$, $\forall\ i\ \neq\ j$, where $RSgoal(i)$ is the aspired

[3]In the aggregate A_N, it is not considered that the external sector might be the owner of the banks. Thus, A_N only comprises the amount of banks' liabilities that is provided by the external sector.

[4]For the calculation of the maximum entropy solution of a matrix with given row and column sums, see Appendix 1.

row sum and $CSgoal(j)$ is the aspired column sum associated to this element. The interval the random number is drawn from seems at first more restrictive than it is. A reduction/expansion of the interval of the uniform distribution to some smaller/higher upper bound does not change the simulation results. To make the matrix fit exactly, the RAS algorithm is applied:[5] In a first step, each element of the matrix is multiplied by the ratio of the aspired row sum ($RSgoal(i)$) and the actual row sum ($rs(i)$).

$$L_{ij} = L_{ij} \cdot \frac{RSgoal(i)}{rs(i)} \tag{4}$$

In a second step, each element of the matrix is multiplied by the ratio of the aspired column sum ($CSgoal(j)$) and the actual column sum ($cs(j)$).

$$L_{ij} = L_{ij} \cdot \frac{CSgoal(j)}{cs(j)} \tag{5}$$

By repeating these two steps sufficiently often, a matrix with elements that fit to the aspired row and column sums will be generated.
The RAS algorithm shows some interesting features. First, restrictions to connectivity can be imposed by setting certain elements equal to zero. These elements will remain zero after running the algorithm. Second, given certain random starting values within the matrix, the RAS algorithm yields a unique solution, independent of the "position" of a certain bank within the matrix. The algorithm is also robust to a transposition of the matrix. Third, the randomly generated starting values determine a certain correlation structure within the matrix. The RAS algorithm determines a unique solution that matches the given correlation structure as well as possible and that fulfills row and column sum restrictions. However, there are cases where the RAS algorithm does not provide a valid solution. This happens especially when too many zero restrictions are imposed. Within the simulations, randomly generated matrices that do not fit are dropped.

After liability matrices are generated, they have to be characterized. As the aim of this paper is to investigate the stability of the financial system dependent on the matrix of *interbank exposures*, the focus is, for the following characterizations, on the $(N-1 \times N-1)$ matrix that covers the interbank market. It is created by deleting the last row and the last column of the $(N \times N)$ liability matrix L. As a next step, there has to be some normalization of matrices because entropy measures have to be applied on probability fields. This is done by dividing

[5] For a detailed description of the RAS algorithm, see Blien and Graef (1991).

all elements by the total amount of interbank liabilities. As a result, elements of normalized matrices add up to 1 and thus can be treated as probabilities. In the following, all normalized elements are marked with a superscript p and written in lower case letters.

After normalization, the next step is to characterize matrices according to the following measures:

- **Entropy:** In information theory, entropy is a measure for information and can, for example, be explained in the context of search problems. To be more precise, entropy is a lower bound of the average path length from the root to the leaves of a binary search tree. Thus, entropy is a lower bound to the average number of yes/no questions that is needed to obtain full information. The more equal the probability distribution of the elements in the search space, the more questions are on average needed to obtain the desired element and, hence, the higher entropy is. The more unequal the probability distribution, the lower entropy is. The lowest entropy (equal to zero) can be obtained when one element in the search space occurs with probability 1 and the other elements with probability 0, i.e. the most unequal distribution of elements occurs.

  This entropy measure can be reinterpreted to quantify the inequality of the distribution of claims of a liability matrix. Using the normalization mentioned above, the elements of the matrix can be seen as realizations of a probability distribution of elements within a search space that need not be defined more specifically. Entropy measures the amount of information inherent in these realizations and is maximal if banks spread their claims / obligations as equally as possible. The higher the entropy, the more equally interbank claims are distributed for given row and column sums. The entropy is calculated by:[6]

  $$ENT = -\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} l_{ij}^{p} \cdot ln\left(l_{ij}^{p}\right) \qquad (6)$$

  with $0 \cdot ln(0) := 0$.

- **Relative entropy (Kullback-Leibler divergence)** to maximum entropy solution: The relative entropy is a measure for the difference between two probability distributions. Given two normalized liability matrices X^p (in this case the maximum entropy solution X^*, see Appendix 1, with last row and last column deleted and normalized by the total amount of interbank liabilities) and L^p (in this case a valid normalized liability matrix generated by

[6]When applying the entropy measure in the context of binary search trees, $log2$ is used. However, in economics literature it is more common to use the natural logarithm. This is equivalent to multiplying a constant factor.

random generator), the relative entropy is given by:

$$RE = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} l_{ij}^{p} ln \left(\frac{l_{ij}^{p}}{x_{ij}^{p}} \right) \tag{7}$$

with $0 \cdot ln(0) := 0$ and $0 \cdot ln(\frac{0}{0}) := 0$.

A higher value of RE denotes a greater difference between the two distributions. In the financial system modeled here, a higher relative entropy means a greater distance to the probability distribution of the maximum entropy solution and thus a more unequal distribution of claims among banks. As long as the relative entropy to the *maximum entropy solution* is considered and banks are assumed to be of equal size, there is a negative linear relationship between the entropy of a matrix and its relative entropy to the maximum entropy solution.[7]

- **Connectivity:** The connectivity of the financial system can be described by the probability that a directed link between two banks exists. While constructing the liability matrix of a random graph, each off-diagonal interbank element is (independently) given a certain positive real number with probability p and 0 with probability $1-p$. This probability p is called Erdös-Rényi probability. However, during implementation one has to be careful that, for given starting values (including zeros with a certain probability), the RAS algorithm is able to find a valid solution of matrix entries. This problem increases with decreasing connectivity. The algorithm used in this paper simply drops matrices that are not valid.

5 Simulation of domino effects

Within this paper, pure domino effects are modeled dependent on characteristics of the interbank liability structure.[8] As a trigger event, one bank fails.[9] Assuming a certain loss given default (LGD), creditor banks lose a share of their claims to the defaulting bank.[10] If this lost share is

[7]The general derivation of this linear relationship is provided in Appendix 2. Simulation results confirm this theoretical finding.

[8]For advantages and disadvantages of modelling domino effects, see Upper (2007).

[9]This is a rather simple way to model a shock on the financial system. A more sophisticated approach is, for example, used by Elsinger et al. (2006), who apply aggregate macroeconomic shocks to test for the resilience of the Austrian interbank market.

[10]In this paper, a constant, exogenously given LGD is assumed. An obvious extension is to endogenize the LGD as, for example, in Degryse and Nguyen (2007).

larger than the creditor bank's equity, the creditor bank also fails. If one or more banks fail due to the first failure, the next round starts with banks losing additional shares of their claims to failing banks. Thus, a bank fails if:

$$\sum \text{Interbank exposures to failed banks} * \text{LGD} > \text{Equity}$$

For a large number of randomly generated matrices, it is investigated how many banks fail on average, after the failure of one bank, dependent on the characteristics of the liability matrix mentioned in Section 4. To be more precise, it is calculated which percentage of total assets of the banking system belongs to failing banks, i.e. which percentage of total bank assets is affected by bank failure.[11] Note, however, that this does not mean that all assets affected by bank failure actually default. This depends on the value of the loss given default.

To depict the results graphically, value intervals of characteristics have to be defined. One possibility to do this is to adjust interval size according to the number of observations. After the random generation of matrices, they are sorted according to their characterization values, and then intervals are defined with each interval having the same number of observations.

The network simulations are run several times and for different banks failing initially to check how robust these results are with respect to sample changes and to changes in the trigger event.

5.1 Complete networks

To begin with, simulations are run for complete networks, i.e. it is assumed that there exists a directed link from each node to all other nodes. The parameter values used for subsequent simulations are $A^{banks} = 1.000$, $N = 11$, $n_{core} = 10$, $CR = 1$ (i.e. all banks are the same size), $\phi = 0.3$, $r = 0.06$ and $LGD = 0.5$. The following figures show, for $50,000$ randomly generated matrices of interbank exposures, the average percentage of total assets of the banking system affected by bank failures in a network with 11 nodes (10 banks and the external sector) dependent on entropy (Figure 1) and relative entropy to the maximum entropy solution (Figure 2), each color representing a randomly generated sample.

From Figure 1 it can be seen that an increase in entropy leads to a lower average percentage of banks' assets affected. Under the assumption that all banks are of equal size, the average number of bank defaults dependent on entropy can be derived easily by multiplying the average

[11] An alternative target value to measure the harm of interbank contagion is the loss of the external sector, which can be computed easily within this model.

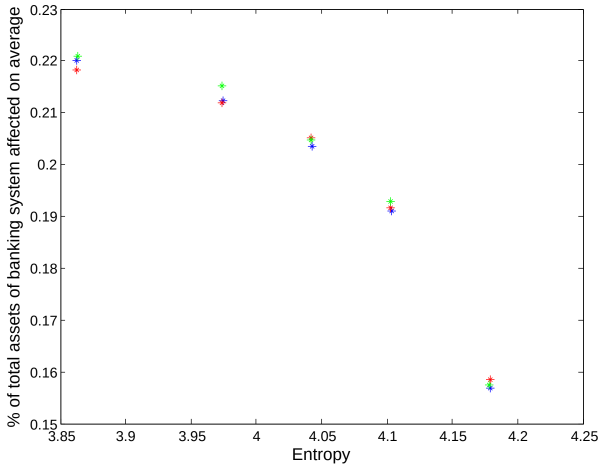

Figure 1: % of total assets of banking system affected on average dependent on entropy

percentage of banks' assets affected by failure by the total number of banks (in this case, 10). Hence, the more equally banks spread their claims, the fewer institutions default on average. These results suggest that, within a complete network and for the parameter values given above, shocks are absorbed best if banks diversify their (credit) risk exposures well. The results, as well as all subsequent simulation results, are robust to changes in the sample and to changes in the bank that fails first.

Figure 2, which shows the relation between relative entropy to the maximum entropy solution

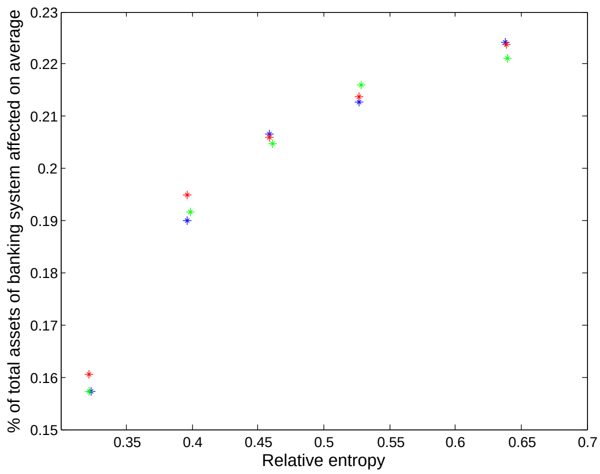

Figure 2: % of total assets of banking system affected on average dependent on relative entropy

and average percentage of banks' assets affected, yields an inverse result compared to Figure 1:

The higher the relative entropy, the higher the percentage of banks' assets affected and therefore the higher the average number of bank failures. Thus, because of the negative linear relationship between entropy and relative entropy shown in Appendix 2, Figure 2 also confirms that a more equal distribution of claims leads to a more stable system.

Up to now, the impact of the distribution of claims on financial stability can be summed up as follows:

Result 1: In a complete network, for the parameter values given above,[12] a liability matrix with an equal distribution of interbank exposures (a high entropy or a low relative entropy to the maximum entropy solution, respectively) leads to a more stable system than a liability matrix with an unequal distribution of interbank exposures.

5.2 Random graphs

A connectivity of 100% is rather unrealistic. Thus, some network has to be designed that omits some directed links within the financial system. One option in this context is to model *random graphs.*

Concerning completeness and interconnectedness of the network only and assuming that banks are completely homogeneous, especially with a completely homogeneous asset / liability structure, subsequent results should be expected according to the theoretical findings of Allen and Gale (2000). They examine three types of networks that are displayed in Figure 3. The complete and perfectly interconnected network (Figure 1 in Allen and Gale) is equivalent to a random graph with an Erdös-Rényi probability of 100%. In this case, the possibility that contagion occurs is rather low because the more complete a financial system, the greater is the potential for risk diversification. With decreasing connectivity, the network structure moves towards systems that are still highly interconnected but also incomplete (equivalent to Figure 2 in Allen and Gale). Allen and Gale show that these systems are more vulnerable to contagion. With connectivity decreasing further, the network structure becomes equivalent to the disconnected system in Figure 3 in Allen and Gale. This disconnection can limit the extent of contagion. Hence there is a non-monotonic relationship between completeness of the network and financial stability.

[12] In the sections below, it is specified in more detail for which parameter values these results hold.

In the following simulations, the degree of disconnection is measured by the average number of strongly connected components across all matrices in a sample. Within a strongly connected component, every bank can be reached by every other bank. This does not mean that there are direct links between all banks as in a complete network. It is sufficient that there exists a directed path between all nodes. If the graph contains only one strongly connected component, the failure of one bank can (potentially after several rounds of contagion) cause the failure of all other banks. If there is more than one strongly connected component, however, it is possible that the failure of one bank cannot cause the failure of all other banks because not all banks can be reached by the failing bank. Hence, the higher the average number of strongly connected components for a given Erdös-Rényi probability, the more disconnected is the system.

As mentioned above, the analysis of Allen and Gale is based on a banking system with completely homogeneous banks with a completely equal asset / liability structure. Within this simulation, in addition to completeness and interconnectedness, a *third* aspect is introduced into the analysis: the distribution of claims within the system.

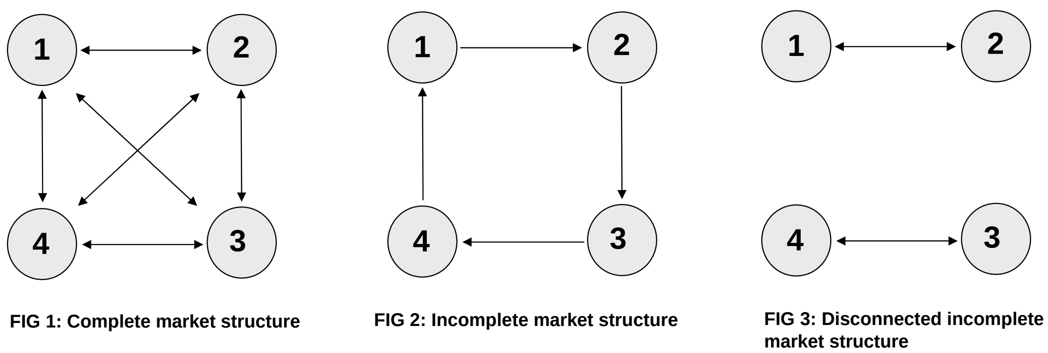

Figure 3: Types of networks investigated by Allen and Gale (2000)

5.2.1 Varying connectivity

In the following, the default algorithm is run for different values of the Erdös-Rényi probability, all other parameter values kept equal to those of Section 5.1. Each off-diagonal element of the liability matrix that denotes an *interbank* claim / obligation is set equal to zero with this

characteristic	p	ENT	RE
p	1	0.94	-0.94
ENT		1	-1
RE			1

Table 1: Correlation coefficients between characteristics in a random graph (generation of $25,000$ matrices for each 10%-step from $p = 10\%$ to $p = 100\%$); p = Erdös-Rényi probability, RE = relative entropy, ENT = entropy

probability.[13] As for certain zero constellations, the RAS algorithm is not able to find a valid solution; matrices that do not fit are dropped. Furthermore, matrices where the actual share of existing links to total possible links deviates more than 0.02 from the desired connectivity are also dropped. Thus, only matrices that fit exactly to desired row and column sums and to desired connectivity are used for the analysis.

To capture the degree of disconnection of the randomly generated network, the number of strongly connected components is computed for each graph using an algorithm developed by Tarjan (1972). After generating a large number of matrices, the average number of strongly connected components for a given Erdös-Rényi probability is calculated. It turns out that the system starts to become disconnected for $p = 0.5$ with an average number of strongly connected components of around 1.03. For $p = 0.3$, more randomly generated graphs are not perfectly interconnected any more, which yields an average number of strongly connected components of about 1.80. The degree of disconnection jumps up for $p = 0.1$, where the average number of strongly connected components is around 9.07.

All the following simulations are implemented by generating $50,000$ matrices for $p = 100\%$, 90%, 70%, 50%, 30% and 10%, respectively. Table 1 shows overall correlations for an Erdös-Rényi probability of 10% to 100% (with 10%-steps) using the same parameter values as in the previous section.

As a first step, to capture the effect of completeness and interconnectedness on financial stability, the average percentage of bank assets affected by failures dependent on the connectivity

[13] Alternatively, the whole graph (including the node that denotes the external sector) can be modeled as a random graph. The aim of this paper is, however, to investigate the impact of the network topology of the interbank market on financial stability. Furthermore, it is certainly more realistic to assume that all banks have connections to the external sector.

of the financial system is calculated. Figure 4 shows the average percentage of banks' assets affected by failures dependent on the median entropy for a given Erdös-Rényi probability. It can be seen that a complete network (i.e. with $p = 100\%$) leads on average to matrices that are characterized by high entropy. With decreasing connectivity, entropy also decreases, meaning that claims are distributed more unequally (according to Table 1, there is a high correlation of 0.94 between entropy and the Erdös-Rényi probability). Furthermore, Figure 4 shows that with decreasing completeness (i.e. a decreasing Erdös-Rényi probability) the average percentage of assets affected by bank failure rises. This is in line with the finding of Allen and Gale that an incomplete but perfectly interconnected network leads to a less stable financial system than a complete network. The effect appearing in Allen and Gale's disconnected network can be observed for $p = 10\%$. For $p = 10\%$, the average percentage of assets affected (and therefore the average number of bank failures) is much lower than for $p = 30\%$ which can be explained by the large rise in the average number of strongly connected components from around 1.80 to around 9.07.

These results shown in Figure 4 can be obtained *on average* if networks are exclusively charac-

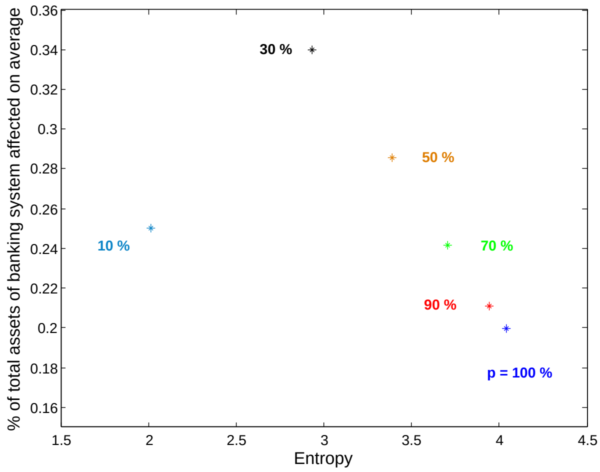

Figure 4: % of total assets of banking system affected on average dependent on entropy and connectivity

terized by their completeness and interconnectedness. However, the effect of the structure of the financial system on financial stability can be analyzed in more detail by additionally considering the effect of the distribution of claims for a given connectivity. Intervals of characteristics are, as in Section 5.1, defined in a way that the number of observations is the same within all intervals.

Simulations show that results are still not dependent on which bank failed first and the sample generated.

In Figure 5 it can be seen that for $p = 100\%$ the same result is obtained as in Figure 1. A

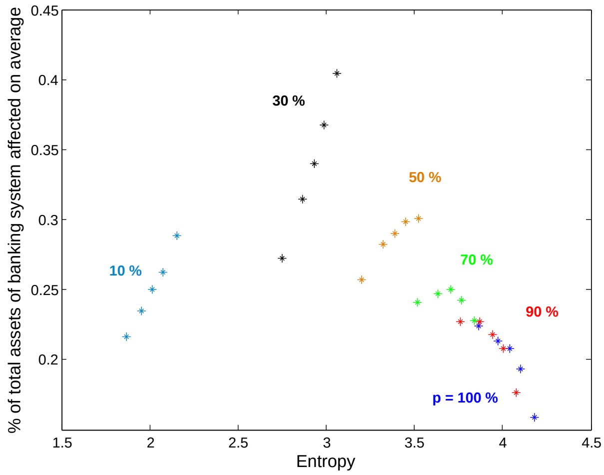

Figure 5: % of total assets of banking system affected on average dependent on entropy and connectivity

liability matrix with a higher entropy leads to a lower average percentage of assets affected by bank failure and therefore a lower average number of bank failures. However, with the 100% connectivity assumption, only rather high values for entropy can be generated. Inserting zero off-diagonal "interbank" elements into the matrix with 10% probability, which is equivalent to $p = 90\%$, generates matrices with a lower entropy. The negative correlation between bank failures and entropy still holds for $p = 90\%$. With decreasing connectivity, matrices with an even lower entropy can be created. The negative correlation between entropy and average bank failures, however, becomes weaker and turns into a positive correlation. This means that for a given low Erdös-Rényi probability (for example $p = 50\%$, $p = 30\%$ or $p = 10\%$), a comparatively high entropy leads on average to more banks defaulting than a comparatively low entropy.

An interpretation for this observation is that, for a high connectivity, an equal distribution of interbank claims is the best shock absorber due to credit risk diversification, whereas an unequal distribution of claims increases the probability that there is a second-round effect after the failure of one bank. On the contrary, when connectivity is low, the failure of one bank is very likely to cause second-round effects because the average amount of interbank exposures to the few connected banks is very high. Hence, the more equal the distribution to the few other banks, the higher the probability that all these banks fail because there are not enough counterparties to diversify the losses induced by the shock. On the other hand, the more unequal the distribution to the few other banks, the higher the probability that not all of these banks fail and therefore the average number of failures is smaller in this case.

Thus, a change in the average percentage of assets in the banking sector affected by failure (i.e. a change in the average number of bank failures) is not just due to a change in connectivity but can also be due to a change in the distribution of interbank claims. For example, though the overall average number of banks failing is higher for a connectivity of 30% compared to a connectivity of 50% (see Figure 4), a system with a very unequal distribution of interbank claims (low entropy) and 30% connectivity is more stable than a system with a rather equal distribution of interbank claims (high entropy) and 50% connectivity (see Figure 5). Also, a system with a very equal distribution of interbank claims (high entropy) and 90% connectivity is more stable than a network with 100% connectivity and a very unequal distribution of interbank claims (low entropy).

Thus, by additionally considering the distribution of claims within the system, it can be seen that a complete network can be more unstable than an incomplete but perfectly interconnected network. This finding extends the work of Allen and Gale in a way that results could change if interbank claims are allowed to be heterogeneous.

Figure 6 and 7 show the average number of bank failures dependent on the relative entropy to the maximum entropy solution. As entropy and relative entropy are exactly negatively correlated (see Appendix 2 and Table 1), these two figures can be regarded as the mirror image of Figures 4 and 5.

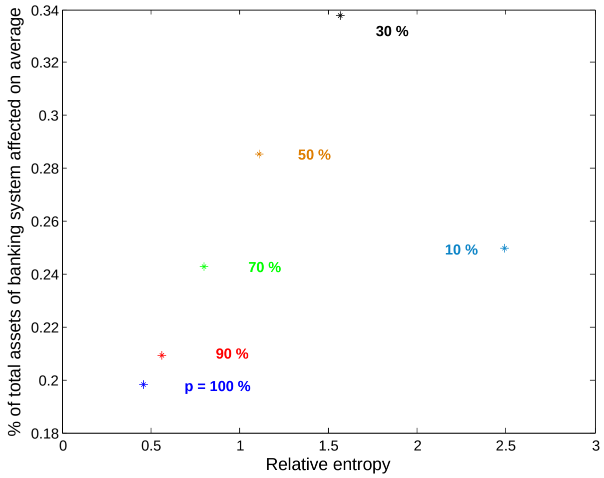

Figure 6: % of total assets of banking system affected on average dependent on relative entropy and connectivity

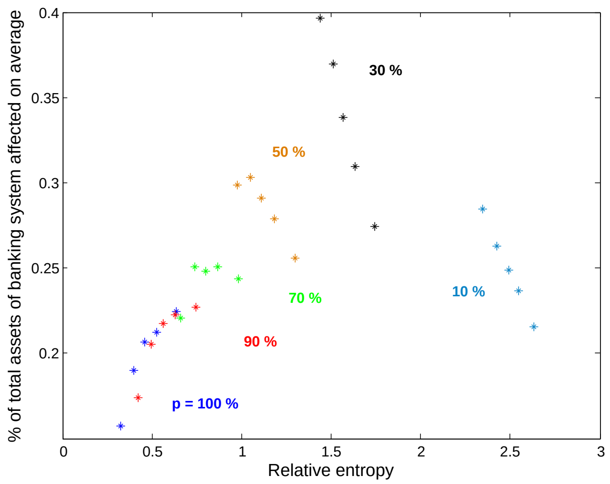

Figure 7: % of total assets of banking system affected on average dependent on relative entropy and connectivity

The results of this paragraph can be summarized as follows:

Result 2: Financial stability does not only depend on the completeness and interconnectedness of the network but also on the *distribution of claims* within the system.

Result 3: For the parameter values given above,[14] the sign of the correlation between the equality of the distribution of claims (measured by entropy and relative entropy) and financial stability changes with decreasing completeness of the network. For high completeness (and high interconnectedness) an equal distribution of claims leads to the most stable system. For lower completeness (but still high interconnectedness) the positive correlation between entropy and number of banks failing weakens. For very low completeness (and low interconnectedness) a more unequal distribution of interbank claims leads to a more stable system.

As long as banks are assumed to be of equal size, the characterization of matrices by entropy and relative entropy yields exactly the same results. Thus, all the following investigations are only made dependent on entropy. As a next step, some sensitivity analysis is done by varying one parameter (LGD, equity ratio or ratio of interbank assets to total assets in the banking system), as well as connectivity, and fixing all other parameters at their benchmark value set in Section 5.1.

5.2.2 Varying loss given default and connectivity

Figure 8 shows the not very surprising result that, for a given connectivity, the average percentage of assets affected by bank failure (and thus also the average number of bank failures) increases with an increasing loss given default. An interesting observation is that for a high LGD (= 100%) the effect of a disconnected system (equivalent to Figure 3 in Allen and Gale) is already visible between $p = 50\%$ and $p = 30\%$. Starting from $p = 50\%$ the average number of bank defaults decreases with decreasing connectivity. Hence, the impact of the disconnection of the financial system becomes more important for high rates of LGD where the average number of bank defaults tends to be very high.

Figure 9 shows that a LGD of 100% leads to a positive correlation between assets affected by bank failure and entropy, even for a high connectivity of the network. A LGD of 30%

[14] In the sections below, it is specified in more detail for which parameter values these results hold.

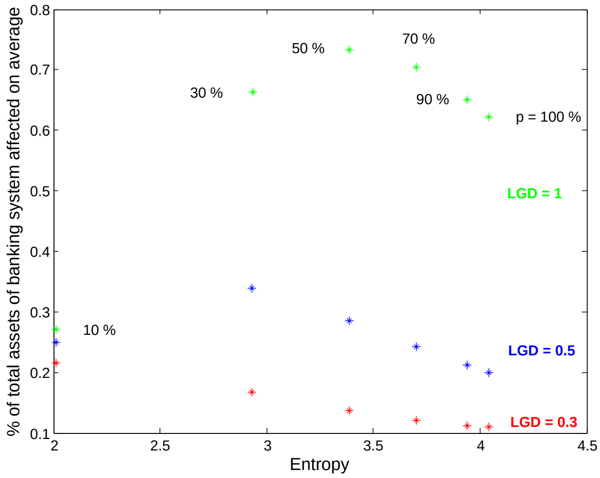

Figure 8: % of total assets of banking system affected on average dependent on entropy and connectivity

leads to a negative correlation between assets affected by bank failure and entropy, even for a low connectivity (up to $p = 30\%$) of the network. An explanation for this observation is that a high LGD makes a system vulnerable to interbank contagion as a high share of claims to the failing banks defaults. Thus, it can be assumed that contagion occurs with certainty. In this case, it is better to have a relatively unequal distribution of interbank exposures so that only few banks are hit by a second-round effect of contagion. For a low LGD, the system is rather resilient to interbank contagion. In this case, second-round effects only occur if interbank exposures are distributed very unequally, i.e. interbank claims are not well diversified among counterparties. This result is exactly in line with the findings of Mistrulli (2007), who shows that for low and medium LGDs, the maximum entropy assumption tends to underestimate the severity of contagion. For high LGDs, however, using the maximum entropy assumption leads to an overestimation of the severity of contagion.

5.2.3 Varying equity ratio and connectivity

Figure 10 shows, not surprisingly, that a lower equity ratio leads, for a given Erdös-Rényi probability, to a higher average number of bank assets affected by failure. Again, the impact of the disconnection of the financial system becomes more important for parameter constellations where the average number of bank defaults tends to be very high, i.e. for low equity ratios.

Looking at the effect of the distribution of claims for given connectivity in Figure 11, it can be

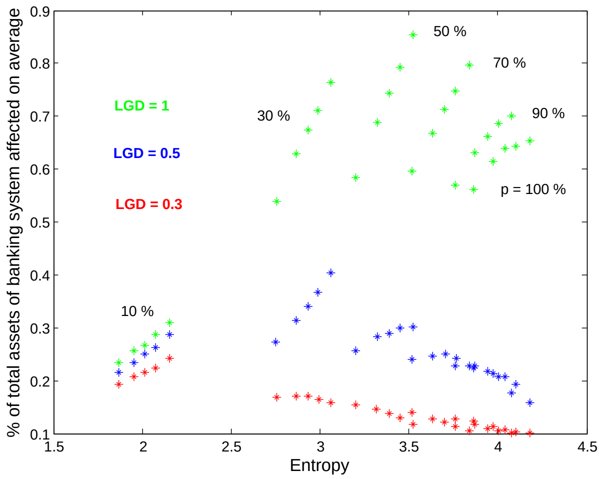

Figure 9: % of total assets of banking system affected on average dependent on entropy and connectivity

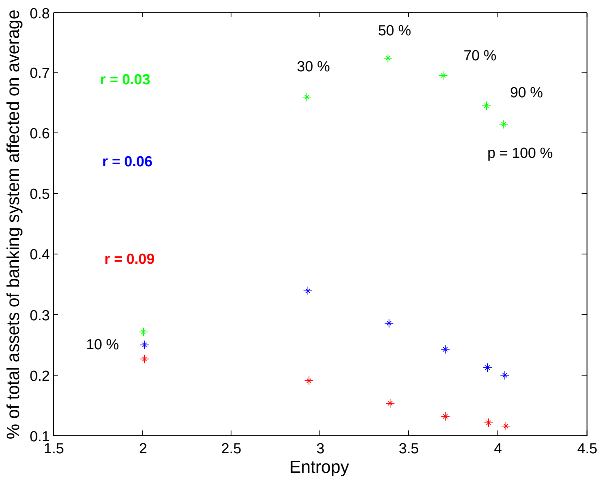

Figure 10: % of total assets of banking system affected on average dependent on entropy and connectivity

seen that for an equity ratio of 9% the correlation of entropy and percentage of assets affected by bank failure is negative for all values of p (except $p = 10\%$). For an equity ratio of 3% the correlation is always positive. Similar to the variation of LGD, systems with high equity ratios are more stable with an equal distribution of claims because second-round effects only occur when interbank claims are not well diversified. Systems with low equity ratios are, for a given Erdös-Rényi probability, more stable with an unequal distribution of claims so that second-round effects (that occur almost with certainty) only hit few counterparties in the financial system.

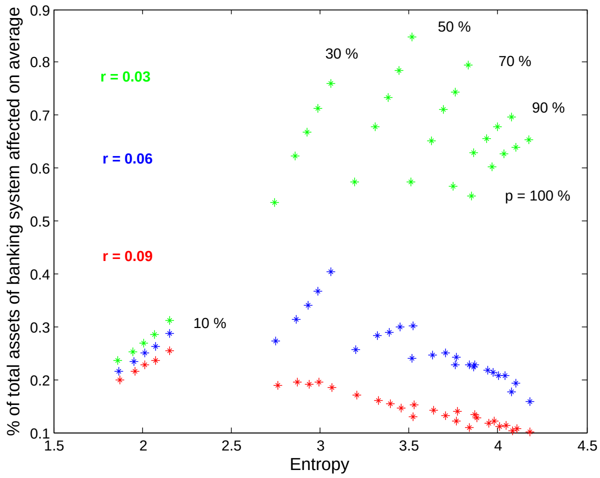

Figure 11: % of total assets of banking system affected on average dependent on entropy and connectivity

5.2.4 Varying ratio of interbank assets to total assets and connectivity

Additionally, the ratio of interbank assets to total assets ϕ can be varied. Figure 12 shows that a higher ratio of interbank assets to total assets leads, for a given Erdös-Rényi probability, to a higher average percentage of assets affected by bank failure. This is not surprising because, for a given equity ratio and LGD, banks become more vulnerable to the default of a neighboring bank. The reason is that, on average, the bilateral exposure per counterparty increases with an increasing ratio of interbank assets to total assets. Furthermore, for high numbers of bank failures (i.e. high numbers of ϕ) the effect of a disconnected system is again already visible for an Erdös-Rényi probability below 50%.

Figure 13 shows that the correlation between entropy and average percentage of assets affected by failure changes from positive to negative at an Erdös-Rényi probability of around 70%, inde-

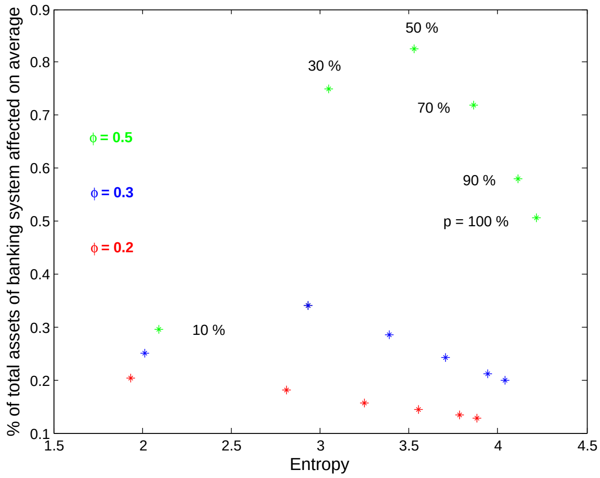

Figure 12: % of total assets of banking system affected on average dependent on entropy and connectivity

pendent of the value of ϕ. Thus, contrary to the variation of LGD and r, there is no influence of ϕ on the correlation between entropy and average number of banks defaults.[15] However, one has to be careful when comparing samples of matrices with different ϕ. Looking at the range of entropy values created for a given Erdös-Rényi probability in Figure 13 it can be seen that, for higher values of ϕ, the random generator on average creates matrices with a higher entropy. Thus, not the same entropy intervals can be compared when investigating the average percentage of bank assets affected by failure for different values of ϕ.

Paragraphs 5.2.2 to 5.2.4 can be summarized by the following result:

Result 4: For a given total amount of interbank assets, the sign of the correlation between the equality of the distribution of claims (entropy) and average percentage of bank assets affected by failure (or average number of banks defaulting, respectively) tends to be positive for parameters that make a system vulnerable to interbank contagion (i.e. a high LGD and low equity ratios). On the other hand, the sign of the correlation tends to be negative for parameters

[15] In additional simulations the LGD and the equity ratio were varied for different values of ϕ. The result is that high values of LGD or low values of r lead to a positive correlation between entropy and the average percentage of assets affected, and low values of LGD or high values of r lead to a negative correlation, independent of ϕ. Thus, a variation of LGD and r changes the correlation between entropy and average percentage of assets affected by failure; a variation of ϕ, however, does not.

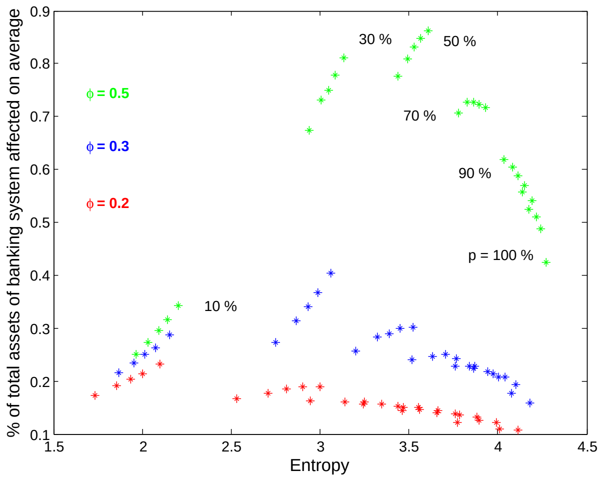

Figure 13: % of total assets of banking system affected on average dependent on entropy and connectivity

that make the system resilient to interbank contagion (i.e. a low LGD and high equity ratios). For intermediate parameter values, the sign of the correlation between equality of the distribution of claims and average number of banks defaulting changes from negative to positive with decreasing connectivity (see results in Section 5.1 and 5.2.1).

In further simulations, additional parameters are varied. At first, the total number of assets was changed. This, however, does not alter the results as relative numbers (for example equity ratio or ratio of interbank assets to total assets) do not change. Increasing the number of banks in the system makes it (all other variables kept equal) less vulnerable to the failure of one bank as the average amount of bilateral exposures per counterparty decreases with an increasing number of counterparties. Furthermore, these simulations were run for "extreme" parameter values, i.e. parameters that all make a financial system very unstable (high LGD, low equity ratio and high share of interbank assets to total assets) or stable, respectively. The results confirmed the main findings summarized in result 4.

5.3 Money center models

The analysis of the impact of the distribution of claims on financial stability in random graphs leads to some interesting results. It is, however, questionable, whether a random graph is a good description of real world financial networks. In the literature, it is sometimes stated that a more adequate model of a financial system is a scale-free network (see, for example, Boss et al. (2004) and Soramäki et al. (2006)). Furthermore, some recent studies prefer to model financial systems as *(multiple) money center systems* (see for example Craig and von Peter (2010)), where few large core banks are strongly interconnected (i.e. they form a complete network) and a larger number of small banks in the periphery are only connected to one core bank but not to other banks in the periphery. Figure 14 shows an example of a money center model (without the external sector).

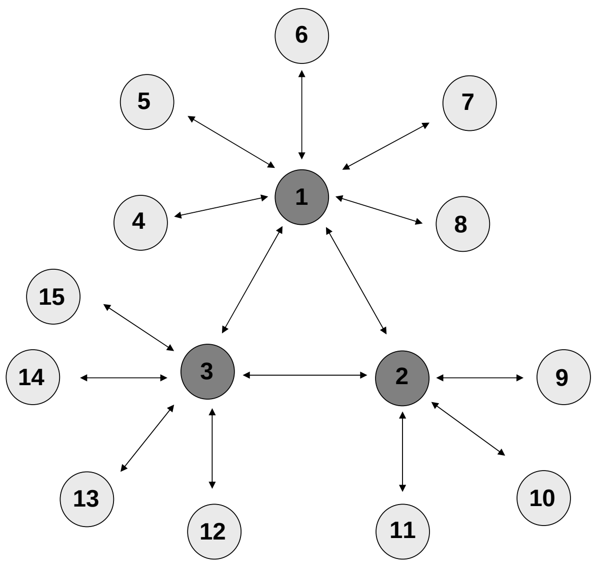

Figure 14: Example of a money center model with 15 banks (3 core banks and 12 banks in the periphery)

The financial network of Figure 14, including the external sector, can be described by an adjacency matrix:

$$
\mathbf{L} = \begin{array}{c} \\ L_{1,core} \\ L_{2,core} \\ L_{3,core} \\ \\ L_{4,per} \\ L_{5,per} \\ \vdots \\ L_{15,per} \\ L_N \end{array}
\begin{array}{c} A_{1,core} \quad A_{2,core} \quad A_{3,core} \qquad\quad A_{4,per} \quad A_{5,per} \quad \ldots \quad A_{15,per} \quad A_N \\
\left(\begin{array}{cccccccccc}
0 & 1 & 1 & | & 1 & 1 & \ldots & 0 & 1 \\
1 & 0 & 1 & | & 0 & 0 & \ldots & 0 & 1 \\
1 & 1 & 0 & | & 0 & 0 & \ldots & 1 & 1 \\
- & - & - & - & - & - & - & - & \\
1 & 0 & 0 & | & 0 & 0 & \ldots & 0 & 1 \\
1 & 0 & 0 & | & 0 & 0 & \ldots & 0 & 1 \\
\vdots & \vdots & \vdots & | & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 1 & | & 0 & 0 & \ldots & 0 & 1 \\
1 & 1 & 1 & & 1 & 1 & \ldots & 1 & 0
\end{array}\right)
\end{array}
$$

The interbank part of this adjacency matrix, i.e. the $(N-1) \times (N-1)$ matrix that is obtained when deleting the last row and the last column of $\mathbf{L}$, can be written in block matrix form:[16]

$$
\mathbf{L^{IB}} = \begin{pmatrix} \Lambda_1 & | & \Lambda_2 \\ - & - & - \\ \Lambda_3 & | & \Lambda_4 \end{pmatrix}
$$

By assumption, the money center model has to follow certain patterns. First, all core banks are strongly connected to each other. Thus, all off-diagonal elements of the top left corner of the adjacency matrix $\mathbf{L}^{IB}$ (Λ_1) have to be equal to one. Second, each bank in the periphery is linked to exactly one money center bank in both directions. Thus, the top right corner (Λ_2) has exactly one non-zero element per column and the bottom left corner of the adjacency matrix (Λ_3) is the exactly transposed version of the top right corner. Third, banks in the periphery are not linked to each other. Hence, the bottom right corner (Λ_4) contains only zeros. Thus, by construction, the strongly connected component always includes all financial institutions in the system.

[16]The way this money center system is constructed is very similar to the "block-model approach" of Craig and von Peter (2010). The conditions a money center system has to fulfill in this paper are, however, slightly different.

After constructing the adjacency matrix, the edges of the graph obtain weights that are, similar to the case of the random graph, created by a random generator. For given row and column sums, elements are again adjusted by using the RAS algorithm.

5.3.1 Varying the number of core banks and concentration ratio

At first, the stability of a money center model is investigated by varying two main parameters that characterize its pattern: the number of core banks n_{core} and the concentration ratio CR. To create a more reasonable ratio of core banks to periphery banks, the number of banks in the financial system is increased to 15, i.e. $N = 16$. Remaining parameters are set at their benchmark values, i.e. $LGD = 0.5$, $r = 0.06$ and $\phi = 0.3$. Total assets in the banking system are again set at $A^{banks} = 1,000$ and each simulation is run for a sample of $50,000$ randomly generated matrices. While for the random graph it does not matter which bank fails first, for the money center model it is assumed that a core bank fails.[17] Figures 15 to 18 show the average percentage of total assets that are affected by bank failure for $n_{core} = 5$ to $n_{core} = 2$. For each number of core banks the concentration ratio is varied from $CR = 0.99$ (core banks hold almost all assets in the banking system) to $CR = 0.4$ (core banks hold 40% of total assets in the banking system). The lower bound of the concentration ratio is set in such a way that, for all values of n_{core} investigated, it cannot happen that core banks have smaller balance sheet totals than periphery banks.

Not surprisingly, entropy decreases with an increasing concentration ratio as a higher concentration of assets among few core banks implies a more unequal distribution of claims within the financial system. Furthermore, for a given number of core banks, the average percentage of assets affected by bank failure increases with the concentration ratio. One reason is that the total amount of assets of the bank that fails first, and therefore the initial percentage of assets affected by bank failure, is higher. But this is not the only effect. With increasing size of core banks compared to banks in the periphery (i.e. a higher CR), and all other variables kept equal, the average weights of the links between core banks increase in the liability matrices generated. Thus the amount of interbank assets between core banks becomes larger on average, which makes core banks more vulnerable to interbank contagion. And the more core banks fail, the more banks in the periphery are on average affected by domino effects.

Comparing the stability of the financial system by varying the number of core banks n_{core} it can

[17] Additionally, simulations with a periphery bank failing were run. Not surprisingly, in this case, the financial system is more stable than in the case of a core bank failing.

be seen that, all other parameters kept equal, the stability of the financial system increases with an increasing number of core banks. This effect is again due to the size of the core banks. For a given concentration ratio, the size of the core banks decreases with an increasing number of core banks. Thus, the average amount of interbank assets between core banks also becomes smaller and the probability that domino effects between core banks occur, is reduced.

Hence, the main result of this paragraph is:

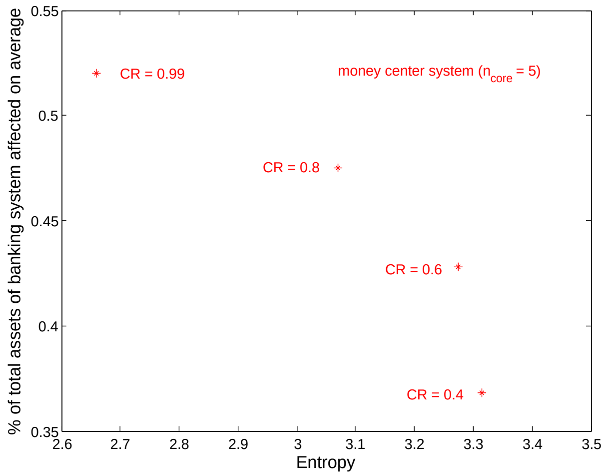

Figure 15: % of total assets of banking system affected on average dependent on entropy and concentration ratio

Result 5: Increasing asset concentration (a higher concentration ratio for a given number of core banks or a lower number of core banks for a given concentration ratio) within a money center system makes it more unstable on average.

5.3.2 Comparison to random graphs

Additionally, the stability of a money center system is compared to the stability of a random graph. As a benchmark, the investigation of the stability of random graphs with the same system size (in terms of total assets) and the same number of banks as in the money center system is included in each subsequent figure. When modelling the random graph, it is assumed that all

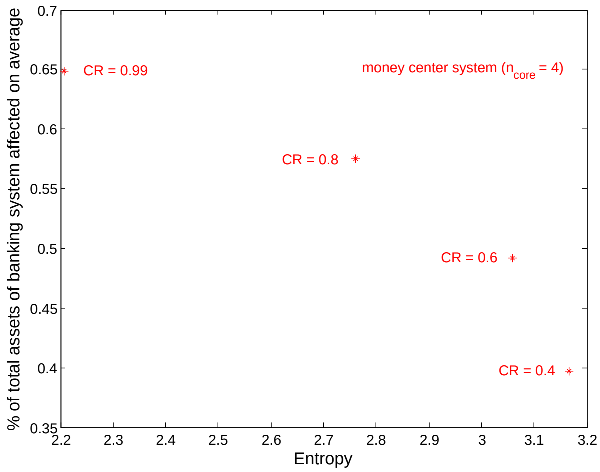

Figure 16: % of total assets of banking system affected on average dependent on entropy and concentration ratio

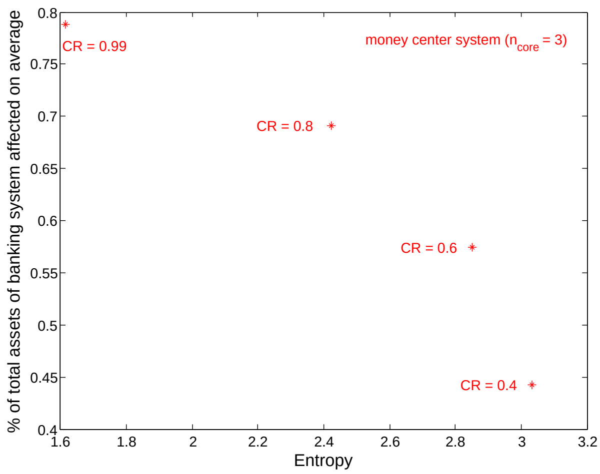

Figure 17: % of total assets of banking system affected on average dependent on entropy and concentration ratio

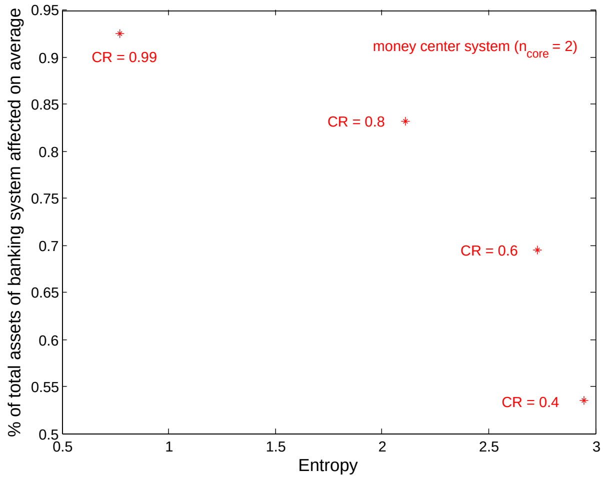

Figure 18: % of total assets of banking system affected on average dependent on entropy and concentration ratio

banks have homogeneous balance sheet totals and are linked randomly with a certain probability. The money center system is interlinked according to the description in Section 5.3.
However, one has to be careful when setting the parameter values for the money center system. Within a random graph, the average amount of bilateral interbank exposures (over the whole sample of generated matrices) is the same between all banks. In a money center model, though, if the concentration ratio is chosen too low (or the number of core banks is chosen too high), i.e. core banks do not have a balance sheet total that is large enough compared to periphery banks, the assumed network topology of the money center model leads to interbank connections where core banks have less exposures to each other than to banks in the periphery. The reason is that periphery banks have all their interbank exposures to one core bank by assumption. To obtain a valid liability matrix for given balance sheet totals, the result is a very low weight on interbank exposures among core banks. This does not fit to realistic banking systems. Thus, in the following simulations, it is assumed that the average amount of bilateral interbank assets each core bank holds against another core bank is at least as high as the average amount of bilateral interbank assets a core bank holds against a periphery bank. This amounts to a minimum concentration ratio of $CR = 0.25$ for $n_{core} = 2$, $CR = 0.35$ for $n_{core} = 3$, $CR = 0.4$ for $n_{core} = 4$ and $CR = 0.45$ for $n_{core} = 5$.[18]

[18]These results are obtained by calculating the average amount of bilateral interbank assets between two core banks and between a core and a periphery bank by simulation for different values of CR (in steps of 0.05). The

Figures 19 to 22 show that (for the same values of N, A^{banks}, ϕ, LGD and r) the random graph is always more stable on average than the money center system with the minimum concentration ratio derived above.[19]

Several reasons for this result can be mentioned. First, in the money center model, the initial percentage of assets affected by bank failure is larger than in the random graph. The reason is that, for the given minimum concentration ratio, core banks in the money center model are always larger than banks in a random graph. Second, a large balance sheet total of the failing core bank implies on average a high amount of interbank claims defaulting in the first round. Thus, the initial shock is larger compared to a random graph. Third, among core banks there are only limited possibilities of risk diversification, as they are only linked to the other core banks and to few periphery banks. Additionally, there is no sufficient risk diversification possible for periphery banks that hold all their claims against one core bank. Fourth, money center models are always strongly connected by assumption, i.e. they have only one strongly connected component. Hence, domino effects can never be curtailed by disconnection of the financial system.

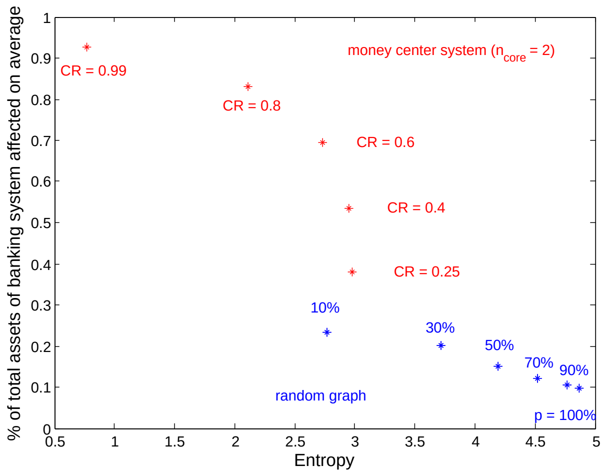

Figure 19: % of total assets of banking system affected on average dependent on entropy and concentration ratio (money center system) or connectivity (random graph)

value of CR, where the discrepancy between the average amount of interbank assets each core bank holds against another core bank and the average amount of interbank assets a core bank holds against a periphery bank is minimal, is then chosen as the minimum concentration ratio for further simulations.

[19]In additional simulations the same result was obtained for extreme values of LGD and r.

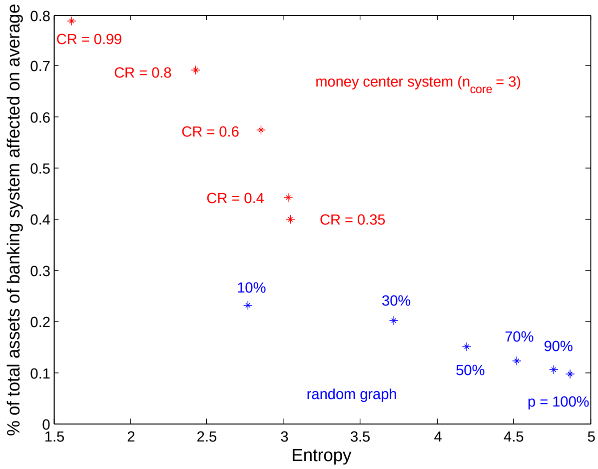

Figure 20: % of total assets of banking system affected on average dependent on entropy and concentration ratio (money center system) or connectivity (random graph)

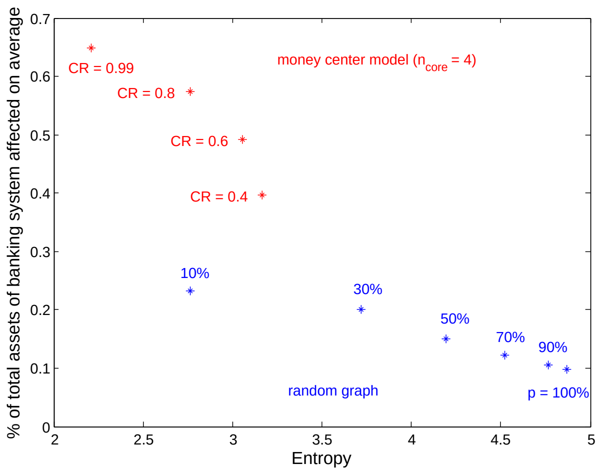

Figure 21: % of total assets of banking system affected on average dependent on entropy and concentration ratio (money center system) or connectivity (random graph)

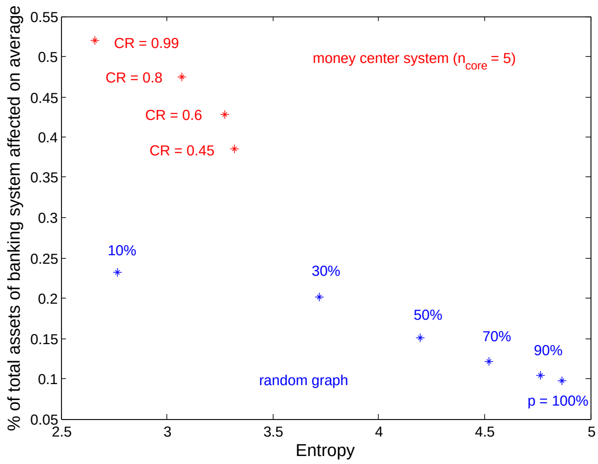

Figure 22: % of total assets of banking system affected on average dependent on entropy and concentration ratio (money center system) or connectivity (random graph)

Thus, to summarize the main findings of this paragraph:

Result 6: Under the assumption that the average amount of bilateral interbank assets between two core banks is at least as high as between a core and a periphery bank, a money center system with asset concentration among core banks is, in all previously conducted simulations, less stable on average than a system of banks with homogeneous size that follows a random graph.

6 Conclusion

In this paper, the impact of the structure of the interbank liabilities matrix on financial stability is analyzed. After characterizing the financial system according to the number of banks, total assets in the banking system, equity ratio, the ratio of interbank to total assets and loss given default, a large number of valid interbank liability matrices is created by random generator for given row and column sums. Thus, for the first time, interbank contagion is investigated for a large sample of interbank matrices. These matrices are then characterized by entropy and relative entropy to the maximum entropy solution, which constitute measures of the equality of the distribution of interbank exposures. As a next step, domino effects resulting from the default of one bank are modeled. As long as banks are assumed to be of equal size (to be able to only focus on the effects of the structure of the liability matrix), it does not matter which bank fails first. Additionally, as a large number of valid matrices is generated, results do not depend on the sample of matrices.

The first simulations are conducted for complete networks and "intermediate" parameter values. The main result is that a more equal distribution of interbank claims leads to a more stable financial system. These results, however, change if an incomplete network is considered. Starting with a random graph and, again, "intermediate" parameter values, it can be seen that the sign of the correlation between equality of distribution of claims and percentage of assets affected by bank failures changes with decreasing connectivity. Furthermore, a crucial result of these simulations is that not only completeness and interconnectedness of a financial network, as investigated theoretically in Allen and Gale (2000), matters, but also the *distribution of claims* within the financial network. In this paper, cases can be shown where, contrary to the findings of Allen and Gale, a complete network (with an unequal distribution of claims) is less stable than an incomplete but perfectly interconnected network (with an equal distribution of claims).

As a next step, further sensitivity analysis is implemented by varying loss given default, banks' equity ratio and the ratio of interbank assets to total assets. The main result in this context is that the sign of the correlation between entropy and the average percentage of assets affected by bank failure depends on connectivity, loss given default and equity ratio. For high values of LGD and low values of r (i.e. parameters that make a financial system vulnerable to interbank contagion) the sign of the correlation between entropy and the average percentage of assets affected tends to be positive, while for low values of LGD and high values of r (i.e. parameters that make a financial system resilient to interbank contagion) the sign of the correlation tends to be negative. For "intermediate" parameter values the sign of the correlation changes from negative to positive with decreasing connectivity.

A second, probably more realistic, approach to modeling incomplete networks is to consider money center systems. The main idea of money center models is to distinguish between large core banks that are strongly connected to each other and small banks in the periphery that are only linked to one core bank. Not surprisingly, the more concentrated assets are within a money center system, the less stable it is. Additionally, using reasonable parameters for the number of core banks and the concentration of assets among core banks, it turns out that, for all simulations run, the money center system is less stable than a random graph with homogeneous bank size.

As a conclusion, this paper extends the existing literature on interbank contagion within a financial network by explicitly considering the distribution of claims within the financial system, and therefore gives a variety of insights into the determinants of financial stability. This approach can be widened to aspects of interbank contagion extending domino effects (for example contagion due to liquidity problems). Therefore, this approach leaves a lot of new topics for future research.

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7 Appendix

7.1 Appendix 1: Generation of the maximum entropy solution of an interbank liability matrix

The starting point is a matrix X with given row sums L_i, $i \in \{1, ..., N\}$ and column sums A_j, $j \in \{1, ..., N\}$.[20]

$$\mathbf{X} = \begin{array}{c} \\ L_1 \\ L_2 \\ \vdots \\ L_N \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_N \end{array} \\ \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \end{array}$$

with $\sum_{i=1}^{N} L_i = L$, $\sum_{j=1}^{N} A_j = A$ and $A = L$.
As entropy methods must be applied on probability fields, some normalization of row and column sums is necessary:

$$\mathbf{X^p} = \begin{array}{c} \\ l_1^p \\ l_2^p \\ \vdots \\ l_N^p \end{array} \begin{array}{c} \begin{array}{cccc} a_1^p & a_2^p & \cdots & a_N^p \end{array} \\ \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \end{array}$$

with $l_i^p = \frac{L_i}{L}$ and $a_j^p = \frac{A_j}{A}$.
Furthermore $\sum_{i=1}^{N} l_i^p = \sum_{j=1}^{N} a_j^p = 1$.

The a^p's and l^p's are interpreted as realizations of the marginal distributions $f(a)$ and $f(l)$, the elements of the liability matrix $X^p \left(= x_{ij}^p\right)$ as realizations of their joint distribution $f(a,l)$. If $f(a)$ and $f(l)$ are independent, the elements x_{ij}^p of the normalized matrix are given by $x_{ij}^p := l_i^p \; a_j^p$.

[20] See Upper and Worms (2004)

This results in maximizing the entropy of X^p.

$$\mathbf{X^p} = \begin{array}{c} \\ l_1^p \\ l_2^p \\ \vdots \\ l_N^p \end{array} \begin{array}{c} \begin{array}{cccc} a_1^p & a_2^p & \cdots & a_N^p \end{array} \\ \begin{pmatrix} x_{11}^p & x_{12}^p & \cdots & x_{1N}^p \\ x_{21}^p & x_{22}^p & \cdots & x_{2N}^p \\ \vdots & \vdots & \vdots & \vdots \\ x_{N1}^p & x_{N2}^p & \cdots & x_{NN}^p \end{pmatrix} \end{array}$$

with $\sum_{j=1}^{N} x_{ij}^p = l_i^p$ and $\sum_{i=1}^{N} x_{ij}^p = a_j^p$.

The problem is that the matrix X^p has non-zero elements on the main diagonal which means that banks lend to themselves. To avoid this phenomenon, a new matrix X_0^p with zero elements on the diagonal has to be created, i.e. x_{ij}^p is set equal to zero for $i = j$.

$$\mathbf{X_0^p} = \begin{array}{c} \\ l_1^p \\ l_2^p \\ \vdots \\ l_N^p \end{array} \begin{array}{c} \begin{array}{cccc} a_1^p & a_2^p & \cdots & a_N^p \end{array} \\ \begin{pmatrix} 0 & x_{12}^p & \cdots & x_{1N}^p \\ x_{21}^p & 0 & \cdots & x_{2N}^p \\ \vdots & \vdots & \vdots & \vdots \\ x_{N1}^p & x_{N2}^p & \cdots & 0 \end{pmatrix} \end{array}$$

The new matrix should deviate from the maximum entropy solution as little as possible. Thus, out of all possible normalized matrices $\overline{X^p}$, a matrix X^* has to be created that minimizes the relative entropy with respect to the matrix X_0^p.

$$x^* = argmin \; \overline{x^{p'}} \cdot ln \frac{\overline{x^p}}{x_0^p} \tag{8}$$

s.t. $x^* \geq 0$ and $Ax^* = \left[a^{p'}, l^p\right]'$,

where x^* and x_0^p are $(N^2 - N) \times 1$ vectors containing the off-diagonal elements of X^* and X_0^p, a^p and l^p are the row and column sums of X^p ($\left[a^{p'}, l^p\right]'$ has the size $2N \times 1$), and A is a $2N \times \left(N^2 - N\right)$ matrix containing zeros and ones so that the restrictions concerning row and column sums are fulfilled.

This minimization problem can be either solved using the RAS algorithm (see Blien and Graef (1991)) or using the "fmincon-command" of MATLAB's optimization toolbox. Both approaches lead to the same results.

After solving the minimization problem, a matrix X^* is obtained that deviates from the assumption of independence as little as possible.

$$\mathbf{X}^* = \begin{array}{c} \\ l_1^p \\ l_2^p \\ \vdots \\ l_N^p \end{array} \begin{array}{c} \begin{array}{cccc} a_1^p & a_2^p & \cdots & a_N^p \end{array} \\ \begin{pmatrix} 0 & x_{12}^* & \cdots & x_{1N}^* \\ x_{21}^* & 0 & \cdots & x_{2N}^* \\ \vdots & \vdots & \vdots & \vdots \\ x_{N1}^* & x_{N2}^* & \cdots & 0 \end{pmatrix} \end{array}$$

As a last step, X^* can be transformed back into a "real" liability matrix X by multiplying each element x_{ij}^* as well as the row and column sums l_i^p and a_j^p with L or A.

7.2 Appendix 2: Specification of the linear relationship between entropy and relative entropy to the maximum entropy solution

Consider a network of N nodes ($N-1$ banks and one external sector) and, in particular, the $(N-1)\times(N-1)$ interbank liability matrix that is normalized by the total amount of interbank assets / liabilities. Normalization implies that the sum of row as well as the sum of column sums has to be equal to 1. The characteristic of the maximum entropy solution of the interbank matrix is that claims are distributed as equally as possible for given row and column sums. Thus, under the assumption that all banks are of equal size, the general result of the maximum entropy solution is a matrix with row and column sums of $\frac{1}{N-1}$ and off-diagonal elements of $\frac{1}{(N-1)(N-2)}$, respectively:

$$\mathbf{X}^* = \begin{array}{c} \\ \frac{1}{N-1} \\ \frac{1}{N-1} \\ \vdots \\ \frac{1}{N-1} \end{array} \begin{array}{c} \begin{array}{cccc} \frac{1}{N-1} & \frac{1}{N-1} & \cdots & \frac{1}{N-1} \end{array} \\ \begin{pmatrix} 0 & \frac{1}{(N-1)(N-2)} & \cdots & \frac{1}{(N-1)(N-2)} \\ \frac{1}{(N-1)(N-2)} & 0 & \cdots & \frac{1}{(N-1)(N-2)} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{(N-1)(N-2)} & \frac{1}{(N-1)(N-2)} & \cdots & 0 \end{pmatrix} \end{array}$$

The relative entropy is given by:
$RE = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} l_{ij}^p ln\left(\frac{l_{ij}^p}{x_{ij}^p}\right)$
with $0 \cdot ln(0) := 0$ and $0 \cdot ln(\frac{0}{0}) := 0$.

This equation can be rearranged:
$RE = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} l_{ij}^p ln\left(l_{ij}^p\right) - \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} l_{ij}^p ln\left(x_{ij}^p\right)$

Assuming that x_{ij}^p constitutes an element of the matrix of the maximum entropy solution, yields the following equation:
$RE = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} l_{ij}^p ln\left(l_{ij}^p\right) - \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} l_{ij}^p ln\left(\frac{1}{(N-1)(N-2)}\right)$
$= \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} l_{ij}^p ln\left(l_{ij}^p\right) - ln\left(\frac{1}{(N-1)(N-2)}\right)\left(\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} l_{ij}^p\right)$
$= \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} l_{ij}^p ln\left(l_{ij}^p\right) - ln\left(\frac{1}{(N-1)(N-2)}\right) \cdot 1$

Entropy is given by:
$ENT = -\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} l_{ij}^p \cdot ln\left(l_{ij}^p\right)$

with $0 \cdot ln(0) := 0$.

Inserting the equation for entropy yields the following result:

$RE = -ENT - ln\left(\frac{1}{(N-1)(N-2)}\right)$

or

$RE = ln\left((N-1)(N-2)\right) - ENT$

As an example, consider 11 nodes in the system (10 banks and one external sector). The relationship between entropy and relative entropy to the maximum entropy solution is thus given by:

$RE = 4.4998 - ENT$

This equation can be confirmed by simulation (see Figure 23).

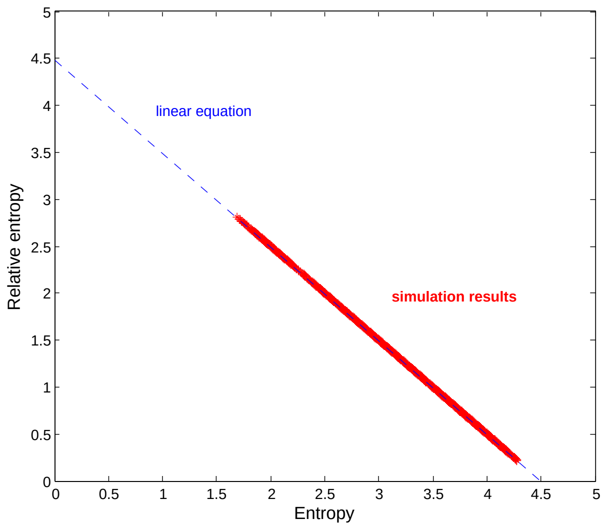

Figure 23: Negative linear relationship between entropy and relative entropy to the maximum entropy solution for $N = 11$