Systemic Risk and Financial Development in a Monetary Model

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ABSTRACT

In a stochastic pure endowment economy with money but no financial markets, two types of agents trade one non-durable good using two alternative types of cash constraints. Simulations of the corresponding variants are compared to Arrow-Debreu and Autarky equilibriums. First, this illustrates how financial innovation or financial regression, including systemic risk, may arise in a neo-classical model with rational expectations and may or may not be countered. Second, the price and money partition dynamics generated by the two variants absent any macroeconomic shock, exhibit jumps as well as fat-tails and vary depending on the discount rate, underlining the potential usefulness of cash constraints for macro-modelling.

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INTRODUCTION

The model presented in this paper is very simple. It includes two infinitely-lived agents with rational expectations living in a pure endowment economy, i.e. in an economy with neither production nor investment processes. Each period, agents receive their individual endowment of one non-durable good and then trade it competitively in a centralised market. This economy has neither financial markets, nor financial intermediaries, nor even an active central bank. However, there is a fixed quantity of money in the economy, which agents, owing to cash constraints, use both to carry out transactions and to save. As the endowments of the two agents differ and are determined by a Markov process, the model is dynamic and stochastic. Finally, the model, the main characteristic of which is the interaction between heterogeneity and cash constraints, has two variants: in Variant 1, the constraint is a classical cash-in-advance one. In Variant 2, the constraint allows agents to settle their transactions only once overall trading has ended[1].

Despite or because of this extreme simplicity, *the model exhibits properties that illustrate in a novel way the links between financial innovation and/or regression and systemic risk in the context of a monetary model.* A systemic risk is the risk that an economy in apparent equilibrium suddenly reaches a collectively less efficient equilibrium as a result of an endogenous phenomenon possibly complementing or responding to an exogenous one. This definition is close to those of Aglietta and Moutot (1993), of the Committee on the Global Financial System (2010), and of many intermediary sources. Indeed, Variants 1 and 2 can be compared not only to each other, but also to two other closely similar models. In the Autarky model, each of our two agents lives in "autarky" i.e. transact in a separate centralised market with agents of his/her kind, instead of sharing a centralised good market with the other type of agents. The shift from one of the variants to this Autarky constitutes a *financial regression and a systemic risk event*. In the Arrow-Debreu model, by contrast, markets are complete and centralised; i.e. agents can trade their endowments before the start of transactions. Moving from one of the variants to the Arrow-Debreu model certainly constitutes a *financial innovation.*

[1] Except for the possibility to interpret these variants as turnpike models, the model does not explain further how money arises in this economy and does not include mechanisms to explain the genesis of the corresponding legal and/or technical apparatus.

The challenge when dealing with systemic risk in a set of recursive models is, however, to identify and simulate non-linear solutions with sufficient precision to be able to draw safe conclusions from them. Fortunately, as shown in Moutot (1991), non-linear solutions can be found in almost all cases for Variant 1, and the existence of solutions can even be formally proven in specific cases. Variant 2, the cash-at-the-end-of-the-day model, also has proven solutions in specific cases but does not have any solution for low discount rates. Nevertheless, for realistic values of the discount rate of utility, it is possible to numerically find and simulate equilibriums for Variant 2. Numerical simulations allow precise calculation of price and wealth distribution dynamics as well as agents' welfares[2], while checking that cash-constraints are never continuously binding or non-binding. In turn, this last feature makes the micro-economic foundations of cash constraints easier. Indeed, continuously binding cash constraints would be contradicted by available evidence on the flexibility of money velocity. The combination of these various points thus makes it logical to consider *how financial innovation and/or regression may or may not arise in the context of a neo-classical model with rational expectations* and, in particular, *how the risk of a systemic event may arise and be countered*[3].

The following results can be reported. First, *sufficient incentives and the corresponding legal apparatus are necessary to make the Variant 1 rational expectations solution a sustainable equilibrium*. Moreover, *whenever an unexpected change in the endowment's Markov process makes the above-mentioned legal apparatus of Variant 1 obsolete but does not prompt its adjustment in due time, a switch to the Autarky model may occur for a set of money partitions.* By contrast, *this never happens in Variant 2,*. Second, the partition of money across agents plays an essential role in Pareto-comparisons of the equilibriums generated by our models. In particular, *financial innovation under Variant 1 may occur only when the distribution of money across agents is not too unequal while, under Variant 2, all agents always support financial innovation*. *Moreover, transitions from Variant 1 to Variant 2 and*

[2] In the example chosen, agents share a common logarithmic utility function and face an endowment process with micro- but no macro-economic uncertainty.

[3] It should be recognised, however, that agents are assumed to care only for their own consumption and do not share collective ideals. Political or technical considerations and costs are not taken into account and neither are the computational difficulties that may be encountered in reaching a particular equilibrium. For instance, it may be argued that the costs associated with the identification of a fixed-point solution to Variant 2 should be contrasted with those of Variant 1 and those of the Autarky and the Arrow-Debreu model, which are negligible.

vice versa can happen only for one precise partition of money across agents.

Nevertheless – and it is another contribution of this paper – both Variants generate *volatile dynamics of prices and of money/wealth partitions across agents.* In particular, these dynamics *may exhibit jumps and fat-tails, depending on the level of the discount rate(s) of utility.* Such dynamics have often been associated with disequilibrium models advocated by so called "econo-physicists" such as Beinhocker (2006) and Bouchaud (2009). Consequently, and as hoped by Farmer and Geanakoplos in their (2008) paper on "The virtues and vices of equilibrium and the future of financial economics", it might be possible for such models, if complemented with further markets, agents and/or institutions, to generate some of the price dynamics that were initially pointed out by Cutler, Poterba and Summers (1987) or Schiller (1991).

This specific nature of prices and money/wealth dynamics has consequences for the assessment of financial development, systemic risk, and their trade-offs. *For example, the one and only partition of money across agents, at which transitions from Variant 1 to Variant 2 and vice versa could in principle happen, materializes rarely or never.* Furthermore, when the *systemic risk occurs in Variant 1, the authorities' best response is not necessarily the adaptation of the legal apparatus that would restore the full sustainability of Variant 1.* On the contrary, it may be *astute for authorities wishing to foster financial innovation and overall welfare to promote a shift to Variant 2, in view of facilitating a subsequent shift to the Arrow-Debreu model.* Consequently, the framework used for the assessment of systemic risk in this suite of simple models is useful in assessing trade-offs between the prevention of systemic risks and financial development[4].

Overall, describing and explaining these dynamics also allows us to confirm or infirm a set of points and/or assumptions, previously made in the literature on money demand, which remain relevant to current macro-models. For example, contrary to the views of Feenstra (1986) and Guidotti (1991), for realistic values of the discount

4 A caveat is useful however. While allowing the generation of a systemic events in connection with the partition of money/wealth, and while providing an estimate of its long term cost for each agent, this framework does not allow a description of all aspects of such a systemic event. For instance, it does not tell us how long this event would last nor the behaviour of prices and consumption during this interim period. This is because it does not make hypotheses concerning the functioning of the economy while moving from one type of equilibrium to another, i.e. it does not model disequilibrium. This is at odds with the more concrete approaches of systemic risk followed by authors like Acharya V. (2009) when developing a theory of systemic risk.

factor, cash-in-advance constraints 'à la Clower' are not equivalent to the inclusion of money into the utility function of sufficiently heterogeneous agents. For money may be held even when cash constraints are not binding and therefore serves as an asset or insurance on top of its role in transactions[5].

This point is particularly relevant for macro-modelling. Even when introducing state-of-the-art investment functions, and although constraints may occasionally be non-binding, velocity remains insufficiently flexible in representative agent models as demonstrated in Hodrick, Kocherlakota, and Lucas (1991). Therefore some (e.g. Woodford 2006) reject this inclusion, arguing that money plays no active role in the determination of the economic dynamics. By contrast, the most frequently used DSGE models with an active role for money (for instance, Christiano, Motto and Rostagno 2007) not only include money in the utility function of consumers but complement it with the inclusion of banks and investors for which continuously binding collateral constraints are imposed. In the case of Variant 1, however, the absence of complete markets and the frictions generated by the cash-in-advance constraint make money a natural saving and insurance instrument and make its velocity very flexible. This is in line with views first put forward by Bewley (1980) and Lucas (1980) and the subsequent literature on heterogeneous agents. Indeed, our approach rather supports similar views by Hansen and Imrohoroglu (1990) and Fuerst (1991), although the latter also mentions a preference for avoiding such a role for money. More recent approaches by Algan and Ragot (2010) and by Wen (2010) also give a role of insurance to money.

However, why do all those papers not mention results similar to ours in terms of price dynamics or systemic risk? As hinted at earlier, this is mostly because our modelling strategy and our numerical techniques differ from their approaches in several respects. First, we consider two distinct agents instead of using the methodology originally suggested by Lucas (1980), which brings the various agents together in one family, but as a consequence cannot consider the impact of wealth on agents' behaviour. Moreover, although we introduce cash-in-advance constraints and heterogeneity like Wen (2010), we do not assume that the dynamics of money or wealth has to stabilize

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As apparent in the comparison of Variants 1 and 2 was pointed out by many of the authors above, starting with Lucas (1980). For instance, we find a shift from Variant 1 to Variant 2 clearly diminishes the short term variability of prices while increasing their average level and thereby decreasing velocity.

after some finite time and do not concentrate on specific starting points, which in conjunction with value function techniques would stabilize such partition. This assumption, which is advised by many authors such as Burkhard and Mauβner (2005) or Liungqvist and Sargent (2004), is understandable with an infinity of agents and the possibility for monetary and/or other macro-economic policies to intervene in order to influence or stabilize such money/wealth distribution. It is natural and fair with one representative agent, as argued by Lucas (1980), or with complete and general equilibrium models, and is proven in those cases for fairly general statistical settings by Micio (2004). But it is a much less justified starting point in the context of financial development or systemic risk, which by nature cannot be consistent with a steady state or a constant distribution of money or wealth. Moreover, any approximation and linearisation is bound to impact precisely the occurrence, if any, of systemic risk, in such models. This is why the approach followed in this paper systematically avoids traditional techniques and assumptions for models with heterogeneous agents and, except for the use of grids, never makes approximations.

In the following, Section 2 presents the model and its two variants, explains why we chose to calculate the solution as the fixed point of a functional operator, and assesses whether and when cash constraints are binding or not. Section 3 illustrates the overall model dynamics by concentrating on a specific example and examines the link between the level of the money/wealth partition, cash-in-advance constraints and welfare. It shows the need for a legal framework in order to make the rational expectations equilibrium sustainable, and illustrates the existence of systemic risk and financial development. Section 4 focuses on the price and money partition dynamics in order to show that prices in Variants 1 and 2 may exhibit fat tails and jumps. It also describes the consequences of such dynamics for financial development and systemic risk. Section 5 concludes.

Section 2

A MODEL WITH TWO AGENTS UNDER CASH CONSTRAINTS,
ONE GOOD, ONE CURRENCY AND NO BOND MARKET

In this section, I describe the two variants of the model, define its equilibrium, outline the corresponding first order conditions and transform them into a functional operator. I also discuss the solutions of this operator and the numerical techniques used to simulate the model. Finally, I assess whether a cash-in-advance constraint is equivalent to the inclusion of money into the utility function.

Description of the model

The model is in its first variant a generalization of two well-known models. Although it is similar to the one-agent monetary model with cash-in-advance constraint developed by Lucas and Stokey (1987) or Coleman (1986), it has two agents and can be interpreted as a turnpike model along the lines of one first developed by Townsend (1980). However, instead of being a perfect foresight model like Townsend's and Manuelli and Sargent's (1988), it incorporates uncertainty using a stochastic framework borrowed from Lucas and Stokey (1987). Like Hansen and Imrohoroglu (1992) and the subsequent literature, it includes heterogeneous agents and cash constraints. However, as explained by Burkhard and Mauβner (2005) or by the first Chapter of Liungqvist and Sargent (2004), most of this heterogeneous agents' literature makes the assumption that the distribution of money across agents is constant as soon as the economy has converged towards a stationary equilibrium. This is not assumed by this paper.

The model is formulated in discrete time with an infinite horizon. The two agents are respectively named a and b. At the beginning of each period, t, each of them receives an endowment of a unique non-storable good. These endowments called respectively ξ_t^a and ξ_t^b are outcomes of a stochastic process (to be defined later) such that ξ_t^a and ξ_t^b are bounded away from zero. There is no private or asymmetric information. At the beginning of each period, each consumer observes his own endowment as well as the other consumer's endowment. Hence the information set I_t of the two consumers is identical and contains data on past and present endowments and prices. Each agent has preferences over his/her infinite lifetime consumption sequence $\{c_t^i\}_{t=0}^{\infty}$ as described by its time-separable utility function,

$$E\,[\sum_{t=0}^{\infty} \quad \beta^t\, U_i\left(c_t^i\right) \mid I_t] \quad i \in \{a,b\}$$

where $0 < \beta \leq 1$ is identical for Agents a and b, but where $U_a(.)$ and $U_b(.)$ can differ. Both $U_a(.)$ and $U_b(.)$ are assumed to be continuously differentiable, strictly increasing and strictly concave.

The only asset in this economy is money. Despite the availability of information, credit is unavailable, possibly due to the lack of an adequate judicial enforcement process combined with an absence of state records for the identification of individuals. The total amount of money in the economy is fixed to 1 and, at any given time t, this amount is divided between the two agents. Agent a possesses m_t^a units and Agent b possesses m_t^b units such that:

$$m_t^a + m_t^b = 1$$

From one period to the next, changes in the money holdings of the two agents are described by their budget constraints.

$$m_{t+1}^i = m_t^i + p_t\,(\xi_t^a - c_t^i) \quad i \in \{a,b\}$$

Indeed, after receiving their endowments, the two agents go to a market. In the **first variant of the model**, they sell their entire endowment and buy the amounts of the good (respectively, c_t^a and c_t^b) that they want to consume during period t independently and at a competitively determined price p_t. Both need to own enough cash at the beginning of the day to finance their consumption independently of the prospective receipts of their endowment's sale. Therefore the possession of money at the beginning of period t is essential in this variant, both in a common sense and in the Kocherlakota (1998) sense. Concerning the former, this is representative of a centralised but extended market with no clearing authority and no possibility to divide and sequence purchase and sale orders, where trust is limited and legal guarantees on the payment of intra-day debts are non-existent. Concerning the latter, and as will become apparent in Section 3, the trading mechanism, despite the availability of information on prices and quantities, is not a no-commitment mechanism[6]. In turn, this offers justifications for the use of fiat money and cash constraints. The use of

[6]On the contrary, as explained in Section 3, its actual functioning depends on the existence of a legal framework preventing its shift to autarky by imposing a fine, which is in this paper called a systemic event. This contravenes the use of the "money is memory" logic of Kocherlakota (1998).

Clower-type (1967) cash-in-advance constraints can therefore be justified by the "worker-shopper" idea of Lucas (1980), the idea of "portability-recognisability" put forward by Kocherlakota (1998) and others, and their association with a commitment mechanism. Moreover, agents have to put their endowment on sale before they consume[7].

$$p_t \;\; c_t^i \leq m_t^i \quad i \in \{a,b\}$$

In the **second variant of the model**, they can simultaneously sell their endowment and buy at a competitively determined price p_t their consumption during period t and can use the proceeds of their sale to guarantee their purchases. The possession of money is therefore constraining only at the end of period t trading when they need to settle all their transactions, which implies

$$m_{tt+1}^i \geq 0 \quad i \in \{a,b\}$$

or equivalently

$$p_t \;\; c_t^i \leq m_t^i + p_t \xi_t^i \quad i \in \{a,b\}$$

This, by contrast, implies that a clearing system and the legal and computational framework necessary to ensure its good functioning are available, although the reasons for its creation and the costs it generates are not accounted for by the model. Although this implies more sophistication than in the first variant, the two variants are not strongly inconsistent. The management of intra-day debt is relatively natural to central banks as they usually quickly obtain the authority to validate end-of-day positions in their accounts, even though their regulatory powers may otherwise be limited. In this case however, money is not essential to this economy in the Kocherlakota (1998) sense[8]. The equations corresponding to this variant will be numbered with a (..)' sign whenever different from those of the first variant. The obligation to put the endowment for sale is however justified in the same way as in

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[7] Imagine, for instance that raw endowments cannot be consumed, at least before the market authority has treated it with a substance of which it has the monopoly and that makes endowments proper for consumption.

[8] However, it may acquire this sense if one considers two elements: i) net clearing systems often need to reorder and sometimes delay transactions and ii) often demand guarantees from clearers, which would indicate some commitment technique, but this cannot be taken into account in the context of this model. Moreover, the only practical way to keep memory of such operations is to define them in monetary terms using the accounts of a bank. Clearing systems often ask for bank status.

Variant 1.

Overall, the two Variants illustrate the role of money in different ways. Together, they complement the duality between centralised moneyless macro-model approaches and decentralised search-based views of money, although the two sets of views have tended to be aggregated since Kiyotaki and Wright (1989), now followed by Aruoba, Waller, and Wright (2011) and Aruoba and Shorfheide (2011).

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Finally, the equilibrium of the good market requires that, at each period

$$c_t^a + c_t^b = \xi_t^a + \xi_t^b = \xi_t$$

Uncertainty is introduced through the definition of endowments. These endowments are time invariant functions of shocks generated by a first-order Markov process with a stationary transition function of density $\pi(.,.)$ such that

$$\xi_t^i = \xi^i(s_t) \qquad i \in \{a,b\} \quad \text{and}$$

$$P(s_{t+\mu+1} \in B \,|\, s_{t+\mu} = s) = P(s_{t+1} \in B \,|\, s_t = s) = \int_B \pi(s, ds')$$

whenever B belongs to the family of Borel Sets of S.

<u>Definition of an equilibrium</u>

An equilibrium in this economy is a set of processes $\{c_t^a\}$, $\{c_t^b\}$, $\{m_t^a\}$, $\{m_t^b\}$ and $\{p_t\}$ such that at any time t, c_t^a, c_t^b, m_{t+1}^a, m_{t+1}^b and p_t be the solutions of the two following maximization problems supplemented by two equilibrium conditions:

<u>Problem of Agent</u> $\quad i \quad i \in \{a,b\}$

$$\underset{c_t^i,\, m_{t+1}^i}{Max} \quad E\,[\sum_{l=0}^{l=\infty} \beta^l \, U_i(c_{t+l}) | \; I_t] \qquad i \in \{a,b\}$$

subject to:

$$I_t = \{ (p_{t-u}, \xi_{t-u}^a, \xi_{t-u}^b) | u \in (0,1,2,...,t)\}$$

$$p_t\ c_t^i \le m_t^i \qquad i \in \{a,b\} \tag{1.1}$$

$$p_t\ \ c_t^i \le m_t^i + p_t\,\xi_t^i \quad i \in \{a,b\} \tag{1.1'}$$

$$m_{t+1}^i = m_t^i + p_t\ (\xi_t^i - c_t^i) \qquad i \in \{a,b\} \tag{1.2}$$

General Equilibrium conditions

$$c_t^a + c_t^b = \xi \tag{1.3}$$

$$1 = m_t^a + m_t^b \tag{1.4}$$

Equilibria studied here are such that processes generated are time-homogeneous functions of shocks $\{s_t\}$ and of $\{m_t^a\}$ and do not depend on sunspot variables. Moreover, the equilibrium is not assumed to be necessarily sustainable if at some point in time, one agent prefers another model and no commitment mechanism prevents it, as we want to identify the need for such commitment, if any. This will be dealt with in Sections 3 and 4.

First order conditions,

If we call γ_t^i and θ_t^i the four Lagrange multipliers corresponding to (1.1) or (1.1') and (1.2), we have the following first order conditions:

$$U_i'(c_t^i) - (\gamma_t^i + \theta_t^i)\,p_t = 0 \qquad i \in \{a,b\} \tag{1.5}$$

$$\theta_t^i - \beta\ E_t(\theta_{t+1}^i + \gamma_{t+1}^i) = 0 \qquad i \in \{a,b\} \tag{1.6}$$

Second order conditions

$$\frac{d\theta^i(s, m_t^i)}{dm_t^i} \le 0$$

Transforming the first order conditions into a functional operator

This transformation is presented in Annex 1. However, it is useful to discuss the choice of such a technique rather than the more standard use of a value function, for instance assuming an "adequate" starting point and a constant distribution of wealth or consumption in order to find linear or close-to-linear solutions. Often, this is combined with a continuous set of individuals distributed along some probability distribution. Alternatively, these are sets of individuals belonging to the same family so as to go back to the representative agent framework. Indeed, the solutions put forward by Hansen and Imrohoroglu (1992) or Burkhard and Mauβner (2004) or even Ljungqvist and Sargent (2004) are all designed so as to generate stationary times series. This literature generates solutions that, on the one hand, do not make the probability distributions of consumption, of income, of wealth (or money in my model) "degenerate", but on the other hand, always assume some degree of approximation to facilitate the calculation of solutions.

This paper takes a different approach. The consideration of systemic risk, which was not the aim of the papers mentioned above, does not authorise the same kind of assumptions. One cannot just limit oneself to the "one path" that does not create problems of sustainability. On the contrary, one needs to take into account all possible starting values to identify those that may lead to systemic risk. One also needs to avoid assuming constant probability distributions of consumption or wealth, as such an assumption makes it difficult to explain that, at some point in time, one or several specific agents may want to default, i.e. move to autarky. Finally, the existence of a steady-state solution cannot be assumed, given that this would make the very notion of systemic event inconsistent. In fact, one has to accept the problem as highly non-linear, and not assume the existence of appropriately close-to-linear solutions.

The successive endogenous states cannot be assumed to stay in the neighbourhood of their former value, and two close endogenous states do not have to remain close across successive periods[9]. Hence, the first order conditions or the Bellman equation have to be satisfied assuming that s_t and m_t may take any value on $S\ X[0,1]$, given that all the paths and the infinity of possible starting points have to be considered in a systematic and comparable way. The most practical and safe approach is to consider all those cases simultaneously. Thus we replace s_t and m_t by two generic variables respectively called s and m. A solution now has to be the fixed point of an operator of functions rather than an operator of variables. Its dimension is therefore infinite and not bounded as in "path" solutions of the Bellman equation.

Then, $\theta_t^i = \theta^i(s,m)$ for $i \in \{a,b\}$ Moreover, let us also define $\theta(s,m)$ as:

$\theta(s,m) = \begin{bmatrix} \theta^a(s,m) \\ \theta^b(s,m) \end{bmatrix}$ and assume $\theta(s,m)$ belongs to C^2 x C^2. As shown in Annex 1, the first order conditions can then be written as:

$$\theta(s,m) = \Phi(\theta)(s,m) \qquad (1.13)$$

where for Variant 1,

$$\Phi(\theta)(s.m) = \begin{bmatrix} \beta \int_S Max\{\theta^a(s',m'), \frac{1}{h'}U_a'(\frac{m'}{h'})\} \quad \pi(s,ds') \\ \beta \int_S Max\{\theta^b(s',m'), \frac{1}{h'}U_b'(\frac{1-m'}{h'})\} \quad \pi(s,ds') \end{bmatrix}$$

and for Variant 2,

$$\Phi(\theta)(s.m) = \begin{bmatrix} \beta \int_S Max\{\theta^a(s',m'), \frac{1}{h'}U_a'(\frac{m'}{h'} + \xi^a(s'))\} \quad \pi(s,ds') \\ \beta \int_S Max\{\theta^b(s',m'), \frac{1}{h'}U_b'(\frac{1-m'}{h'} + \xi^b(s'))\} \quad \pi(s,ds') \end{bmatrix}$$

with $m' = M(m, \dot{\theta}(s,m), \dot{\xi}(s))$ and $h' = h(m', \dot{\theta}(s',m'), \dot{\xi}(s'))$ respectively defining the law of motion of the partition of money and the dynamics of prices.

Solutions can therefore be considered as the fixed points of a multi-dimensional functional operator Φ defined by (1.13). The knowledge of this fixed point combined

9 Finally, one cannot envisage the existence of a planner with the right to actively redistribute wealth on his own in order to eliminate a Pareto preference for autarky as in Kocherlakota (1996) and in Thomas and Warall (1988), given that the purpose is not to design a Pareto-superior equilibrium, but rather to discuss how systemic risk may arise in a competitive and rational expectations model.

with the knowledge of the stochastic process $\{s_t\}$ and of the initial partition of money m_0 determines all the other variables.

Existence and numerical calculation of solutions

Do functional operators like Φ have solutions and can we calculate them?

The answer to the first part of this question is still incomplete. Indeed, the Schauder Theorem, which is the standard tool used by mathematicians to prove the existence of fixed points, cannot be directly applied here given that Φ is not a compact operator. However, as shown in Moutot (1991), in a number of Variant 1 cases, it is possible to prove that its solutions are also solutions of more convoluted but nevertheless compact operators derived from Φ and to which the Schauder Theorem can be applied. Hence, solutions to (1.13) can be proved to exist in specific cases, including when endowments are constant and when β is small enough. However, such convoluted operators have not yet been derived for the most general cases, implying that full certainty about the existence of solutions to (1.13) cannot yet be reached.

However, in most Variant 1 and Variant 2 cases, solutions can be found by numerical techniques. Let us represent the range [0,1] over which m may vary over a grid with a finite number of points (usually 256 in the forthcoming sections, but occasionally 1024) and the functions $\theta(s,m)$ as matrices of dimension (2,S,256) where S is the finite number of possible shocks. As also shown by Moutot (1991), it is possible to search for solutions by iterations of Φ for fine enough grids. Solutions can be found across a number of choices for ß, for $U_a(.)$ and $U_b(.)$, and for various Markov processes. This answers positively the second part of the question above.

When is money in the utility function equivalent to a cash constraint?

One of the original reasons for developing the first Variant of our model is to answer the above question. Following Feenstra (1986) or Guidotti (1991), a cash-in-advance constraint is equivalent to the inclusion of money into the utility function. Indeed, if it is continuously binding, it is equivalent to the maximisation of the utility function under such constraint or to the addition to the utility function of a separable part including money. At the same time, continuously binding cash-in-advance constraints imply that the velocity of money in the economy remains constant over time, which is not realistic. However, as demonstrated by Moutot (1991), whenever the discount rate of utility is higher than a certain threshold value, the two cash constraints of Variant 1 cannot be simultaneously and continuously binding.

Theorem 1

Let $U_a(.)$ and $U_b(.)$ be two continuously differentiable, strictly increasing and strictly concave utility functions. Assume that the model considered is specified as a Variant 1 model and that β is strictly superior to 0. If a “stationary expectations" equilibrium exists and if, in this context, after a finite number of periods both agents are always under cash constraints,

$$\text{then: } \beta \leq \beta^1_{Max} = \underset{\substack{s\subset S \\ s''\subset S \\ i\in\{a,b\}}}{Min} \frac{\xi(s)U_i^{'}\frac{\xi^i(s'')}{\xi(s'')}\xi(s)}{\int_S \xi(s')U_i^{'}(\xi^i(s)\frac{\xi(s')}{\xi(s)})\pi(s,ds')} \quad \textbf{(2.1)}$$

Proof: see Annex 2 for an excerpt of the full theorem proved by Moutot (1991), showing the existence of solutions in such a case.

In the case where both utility functions are logarithmic, i.e. $U_a(.)$ and $U_b(.)$ are equal to log(.). , (2.1) is transformed into:

$$\beta \leq \beta^1_{Max} = \underset{\substack{s\subset S \\ s''\subset S \\ i\in\{a,b\}}}{Min} \frac{\xi^i(s'')}{\xi(s'')} \Big/ \frac{\xi^i(s)}{\xi(s)} \quad \textbf{(2.2)}$$

Hence, in Variant 1 and for logarithmic utility, the equivalence between money in the utility function and cash-in-advance constraints is only valid if the ratio of individual shares in the total endowment does not vary too strongly across time. If one assumes, as found by most studies, that the discount rate of utility is around 0.95 for quarterly models of the economy, this implies that the ratio of such shares across time should remain higher that 0.95. Equivalently, this share should never vary more than 5.2% across time. Obviously, this implies that the above-mentioned equivalence is probably not valid continuously, and is not likely to apply in financial crises where the partition of endowments may strongly vary.

In the context of Variant 2, cash constraints can, by definition, never be binding simultaneously. However, under Variant 1, cash constraints are binding whenever m is low or high enough, whatever the state, because without money, consumption is impossible. Is there, by analogy, a level of money partition under Variant 2 under which each agent becomes systematically constrained? The answer is no. Under Variant 2, an agent who owns no money is not necessarily cash-constrained, as shown by Theorem 2 in Annex 2.

It is thus interesting to ask whether, for some parameters, there is one cash-constrained agent at each point in time. This would determine "a contrario", expressing the values of β at which none of the agents is cash-constrained. Although this domain can be generally determined, (see Proposition A.3 in Annex 2), it cannot be captured in a simple and general formula like (2.1) or even (2.2). That is why I chose to present it for the specific type of Markov process and type of endowment variability for which a simple formula is available. Theorem 3 below also offers an actual proof of existence of solutions to (1.13) for cases in which one agent at a time is always under a cash constraint.

<u>Theorem 3</u>

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$. Suppose also that $\pi(s,s')$ is a two-state probability matrix equal to $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$. Suppose finally that the endowment matrix $\xi(.)$ is such that $\frac{\xi^a(1)}{\xi^b(1)} = \frac{\xi^b(2)}{\xi^a(2)}$=z. Then a Variant 2 solution to (1.13) exists if

$$\frac{2z}{1+z} \leq \beta < \beta^2_{Max} = \frac{-2z + \sqrt{(20z^2 + 16z)}}{2(1+z)}$$

and is such that cash constraints are always binding for one of the two agents.

<u>Proof</u> : See Annex 2.

Overall, limiting ourselves to the specific case of logarithmic utility, of two states with the above probability matrix and of endowments such that $\frac{\xi^a(1)}{\xi^b(1)} = \frac{\xi^b(2)}{\xi^a(2)}$=z, our findings on the binding character of cash constraints may be summarised in a simple chart such as Chart 1 below.

In the *area below the diagonal*, Variant 1 has solutions with binding cash constraints and defined prices. Indeed, (2.2) can now be written as: $\beta \leq \beta^1_{Max} = z$. By contrast, Variant 2 has no solution with determined prices. This does not mean that non-monetary equilibriums do not exist. They do, but do not give a well-defined value to goods and hence to money. Of course, the assumptions made by Feenstra (1986) and Guidotti (1989) cannot be relevant in such a case. *Above the diagonal*, Variant 1 has solutions, but they all include occasions when none of the cash constraints are binding. By contrast, above the diagonal, Variant 2 solutions with well-defined prices exist only above the black curve. Up to the red curve, they always have one agent under a cash constraint while the other is not constrained. Beyond the red curve, they also include episodes where none of the agents is cash constrained.

For instance, assuming β=0.95 again, z has to be lower than $\beta/2-\beta$ or 0.90 and higher than 0.76 in order to allow for the existence of a monetary equilibrium with one binding constraint all the time. Consequently, for a given agent, his/her share of the total endowment $\frac{\xi^a(s)}{\xi(s)}$ varies between $\frac{\xi^a(1)}{\xi(1)}=\frac{z}{1+z}$ and $\frac{\xi^a(2)}{\xi(2)}=\frac{1}{1+z}$ .Hence, an agent's share of the total endowment may vary up to 31% from one state to another and still be consistent with the existence of one binding cash constraint, but should vary more than 10% in order to be consistent with the existence of a monetary equilibrium. Hence, for realistic values of the discount factor and for logarithmic utility functions, *the equivalence between cash-in-advance models and models that include money into the utility function becomes uncertain* because, contrary to assumptions made by Feenstra (1986) and Guidotti (1989), cash constraints cannot remain continuously binding when the discount rate of utility is too high relative to economic uncertainty.

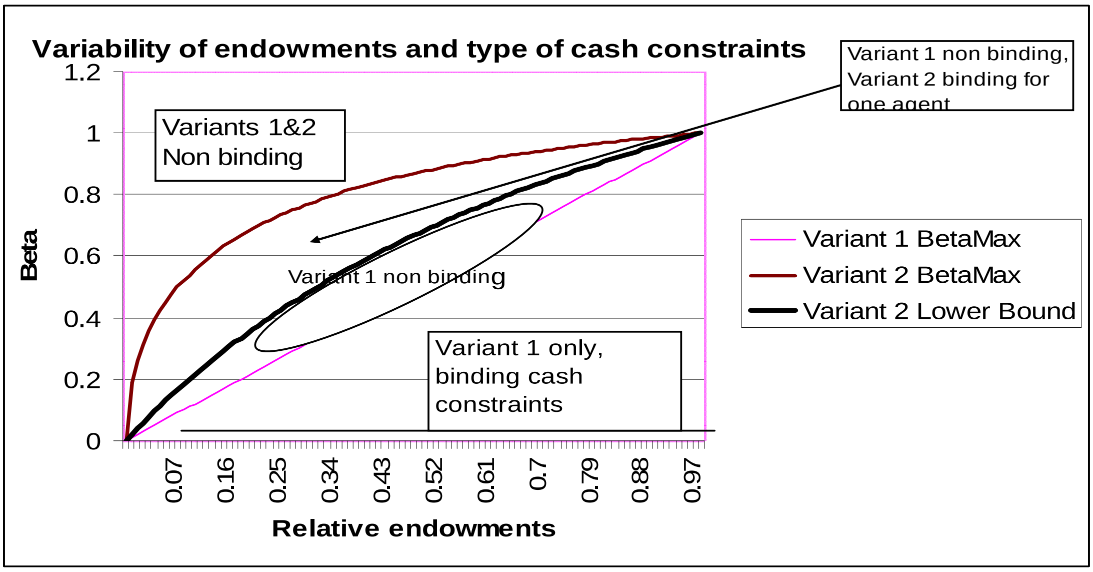

Chart 1

Section 3

THE LEVEL OF MONEY/WEALTH PARTITION AND THE CHOICE BETWEEN FINANCIAL DEVELOPMENT OR REGRESSION

Our model and its equilibriums are associated with an implicit assumption: agents have no opportunity to choose between Variants, could not refuse to trade together and cannot improve any of the two Variants by creating new markets, new financial products, or new institutions like banks. However, to the extent that agents in this model are fully rational and able to make the same calculations as presented in this article, it cannot be excluded that they compare their current welfare with the level of welfare that they would expect to attain in another model or in another variant.

For instance, at any point in time, they may decide to restart history and live in autarky forever. This would constitute *financial regression*. Such a decision may be made at the very start of the day, when agents do not yet know s_t, or later when s_t is known but purchases have not yet been carried out. Alternatively, if they are living in the Variant 1 model, they may consider whether they would like to jointly create a net clearing system so as to restart history in Variant 2, assuming this change in market rules would be preferable. Finally, they may compare their situation with the one they would have in an Arrow-Debreu model, in which agents are allowed to trade their endowments as securities before the start of history. To the extent that the equilibrium generated by this model is simple enough to be reproduced within our model with the creation of one insurance market, this last model is representative of most potential financial developments in the highly simplified economy that underlies Variants 1 or 2.

In order to examine such issues, it is essential to describe and differentiate the dynamics of the two Variants in terms of money/wealth partition. I will first simulate them and then calculate the levels of welfare reached in each model or Variant and, with the help of charts, examine whether the agents in these models prefer to stay in these Variants or move to another of the suite of models considered.

The simulation

Suppose that:

-both agents have a logarithmic utility function,

-both discount rates of utility are equal to 0.9,[10]

-the stochastic environment is described by the probability matrix $\begin{bmatrix} 0.50 & 0.50 \\ 0.50 & 0.50 \end{bmatrix}$,

-the endowment is equal to: $\begin{bmatrix} 5-xs & 5+xs \\ 5+xs & 5-xs \end{bmatrix}$ with xs=2.

Then the total goods endowment in the economy is constant over time and fixed at 10 units per period. However, agents' individual endowments vary: with xs=2, each agent has one chance out of two to receive 3 units of good while the other receives 7, and one chance out of two to receive 7 units while the other receives 3 only. ß is superior to β^1_{Max}, as: $0.9 = \beta \succ \beta^1_{Max} = \underset{\substack{s \subset S \\ s'' \subset S \\ i \in \{a,b\}}}{Min} \dfrac{\xi^i(s'')}{\xi(s'')} \Big/ \dfrac{\xi^i(s)}{\xi(s)} = 3/7$. Moreover, such a value of β is also in the range of values for which equilibriums with well-defined prices in Variant 2 have to include situations where none of the agents is cash constrained as shown by Theorem 3:

$$\beta^2_{Max} = \frac{-2z + \sqrt{(20z^2 + 16z)}}{2(1+z)} \text{ .=0.835782} < 0.9$$

Having identified the value of the fixed points of

$$\theta(s,m) = \Phi(\theta)(s,m) \qquad (1.13) \text{ and } (1.13')$$

we use them to calculate the values of $M(m, \dot{\theta}(s,m), \dot{\xi}(s))$ and $h(m, \dot{\theta}(s,m), \dot{\xi}(s))$ for all possible couples (s,m) in S X [0,1]. We then calculate the time series by giving an initial value to m and iterating up to 100, 1000 and occasionally 2000 periods.

$$m_{t+1} = M(m, \dot{\theta}(s_t, m_t), \dot{\xi}(s_t)) = F(m_t, s_t)$$

This very simple set-up immediately shows how these variants differ from usual macro-models. If the total goods endowment of an economy and its total money supply are constant, traditional macro-models without investment processes can only

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[10] I also tried with 0.95 and obtained very similar results.

generate constant prices and constant money velocity. The case of a single representative agent model such as that developed in Lucas and Stokey (1987) is similar. However, such is not the case in our two variants if the partition of the total endowment between agents is not constant over time and if cash constraints are not always binding (see Figures 1 and 2).

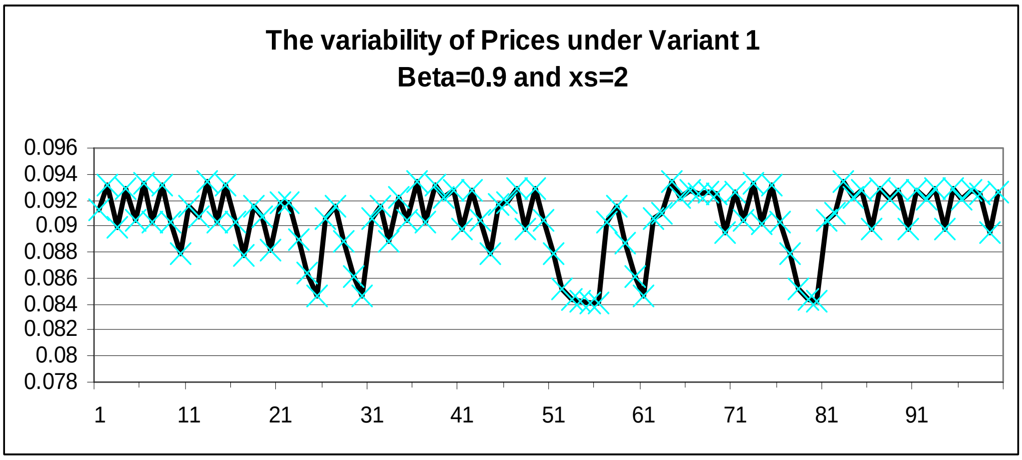


Figure 1

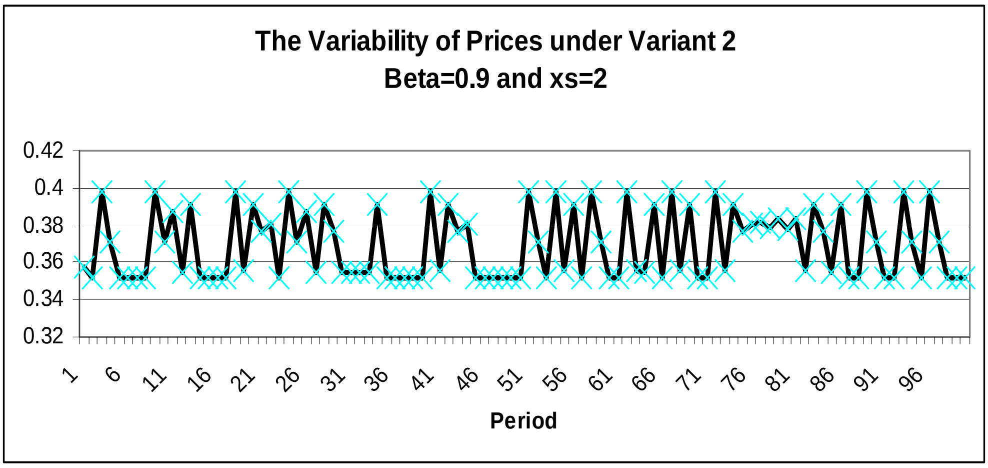


Figure 2

Indeed, the money partition, the price and therefore money velocity are highly variable in both Variants, although the level of prices is very different. As apparent in Table 1, it is 4.7 times higher in Variant 2, which is logical as the price is connected to the utility of net transactions rather than to their bulk. Also, the average deviation, the standard deviation and, more importantly the variance of prices is much lower in Variant 1[11].

11 Prices are more (and negatively) auto-correlated at the first lag in Variant 1. This negative

Table 1

Prices	Variant 1	Variant 2	Ratio
Average	0.085653397	0.403005117	4.705069
Average deviation	0.005251789	0.030751014	5.85534
Standard deviation	0.006110022	0.033598821	5.498969
Variance	3.73E-05	1.13E-03	30.23866
Autocorrelation			
lag 1	-0.378095537	0.072960957	
lag 2	-0.070243947	-0.134459375	
lag 3	-0.090740589	-0.03038133	

Explaining the dynamics of money partition and of prices in Variants 1 and 2

The dynamics of money partition can be quite different across variants and evolve with the value of β. This is easy to grasp on Charts 1.1 and 1.2, where the money partition of period $t+1$ is shown as a function of the money partition and of the state of the world at time t. In Variant 1, the money partition always converges into an interval formed by the intersection of the diagonal with the two curves describing the evolution of money partition from one period to the next, depending on the state of the world. Afterwards, it remains within this interval, i.e. between m_{Min} and m_{Max}, and may a priori reach any point of it, depending on the succession of exogenous states s_t. Somehow, money partition jumps according to the state inside a box delineated by the interval $[m_{Min}, m_{Max}]$ and two almost parallel lines.

---

correlation is logical: it reflects the limitations of a model which, despite heterogeneity and financial frictions, cannot adequately mimic a widely-discussed feature of asset markets (see Cutler, Poterba, and Summers (1989) or Schiller (1991)), namely their positive autocorrelation in the short term. Indeed, it includes neither investment nor loans and does not assume a positive correlation of its shocks. It therefore cannot offer its agents the possibility to envisage a continuation of recent trends in prices and therefore a positive short-term correlation. However, as we will see later, the model has some potential for exhibiting some other features of market data.

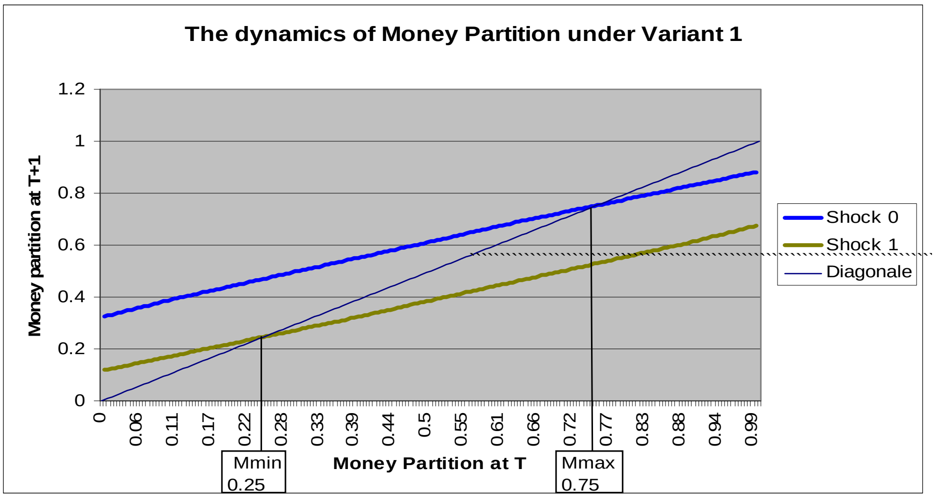


Chart 1.1

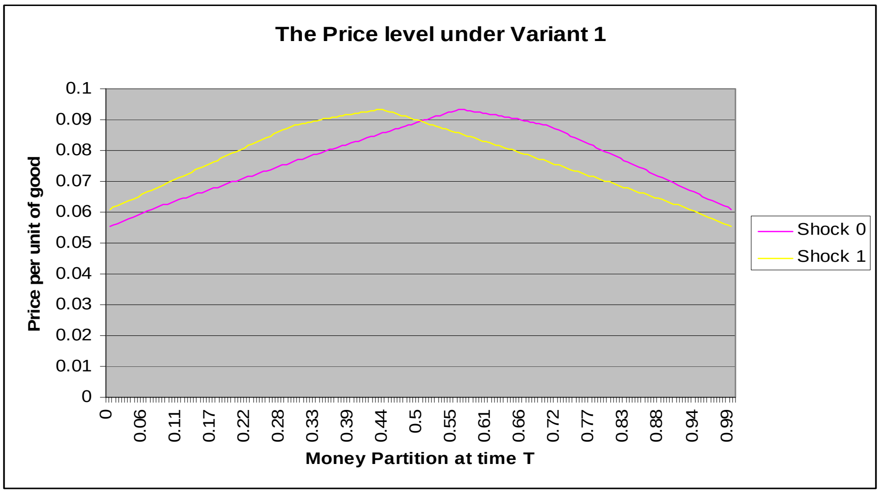


Chart1.2

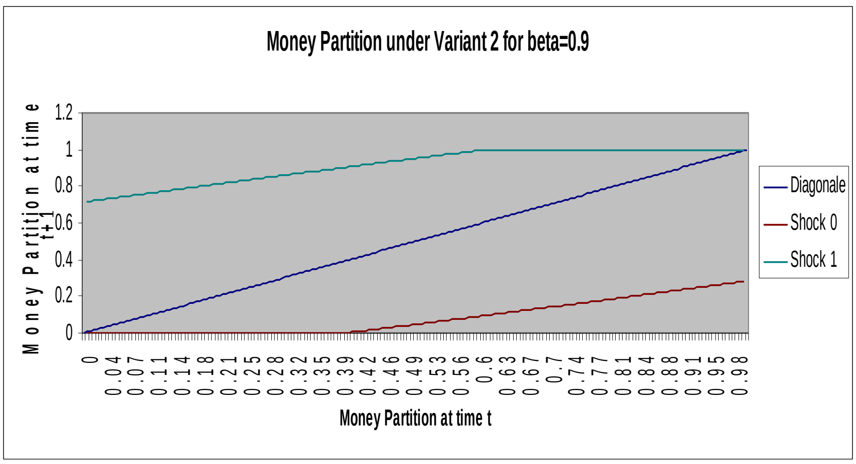


Chart 2.1

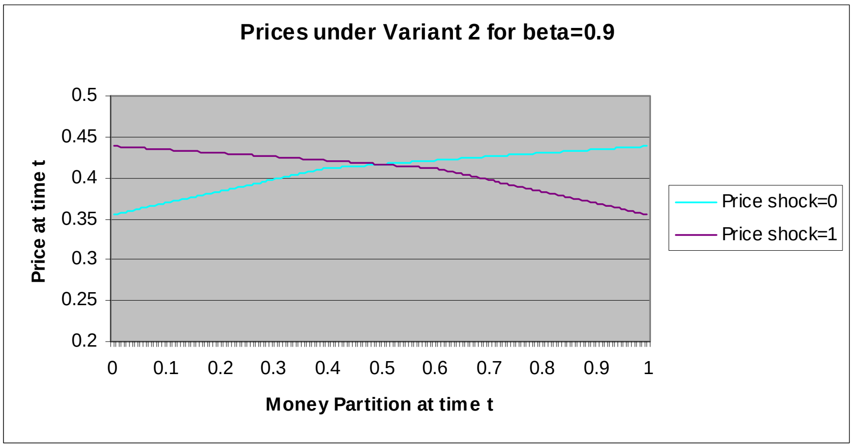


Chart 2.2

This is quite different under Variant 2 (see Charts 2.1 and 2.2). Whatever β, the partition of money may always become so unequal that it concentrates into the hands of one agent and reaches 0 or 1. Moreover, the distribution of money partition across time may show accumulation at 0 and 1: not only does it stay there if the state does not change (see Chart 1.2), but it is also attracted to 0 or 1 whenever the money partition gets into the neighbourhood of these extremes

As evidenced by Charts 1.2 and 2.2, the link between prices or velocity and the distribution of money across agents is two-dimensional, whereas such dimension is nil in usual macro-models and money-demand equations. Moreover, this link is non-linear and non-monotonous in Variant 1. By contrast, in Variant 2, it increases or decreases according to the state and is not far from linear.[12]

Comparing the levels of welfare generated by our suite of models

In order to calculate the level of welfare in the economy represented by our model, we estimated the expected utility of each agent at period t before he/she learns about the state of the world s_t and dependant on wealth as represented by the agent's money

[12] Finally, take $\beta = 0.999$. As apparent in Graphs 1.3 and 2.3, the dynamics of money partition become increasingly similar across the two variants but keep their characteristics. The range $[m_{Min}, m_{Max}]$ gets larger but remains within the segment [0, 1] for Variant 1 while the money/wealth partition reaches 0 or 1 under Variant 2. See graphs 1.3 and 2.3 in Annex 4.

holdings:

$$W_i^0(m_t) = E[\sum_{l=0}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_t] \qquad (4.1)$$

We also calculated such expected utility once the agent knows about the state of the world and can anticipate his/her consumption:

$$W_i^1(m_t, s_t) = U_i(c_t^i) + E[\sum_{l=1}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_{t+1}] \qquad (4.2)$$

More precisely, for each of these partitions, we randomly drew 100 processes of 100 random shocks each. Then we calculated the discounted utility generated by each of these processes for each of the two agents evaluated at time t. Then, following a Monte-Carlo approach, we calculated the mean of these 100 utilities in order to obtain an estimate of the expected utility of each agent for each original money partition. Finally, we calculated the level of welfare expected in the economy for each of these partitions by simply adding up the expected utilities of the two agents, thereby giving equal weights to each agent. This is done under Variant 1 and Variant 2. We also calculated the expected utilities generated by autarky and our specific Arrow-Debreu economy. Given the simplicity of the stochastic environment chosen, calculations of $W_i^{Au}(m)$ and $W_i^{AD}(m)$ were easy. In the autarky case, each agent may be considered as living in a monetary model similar to ours, but with one representative agent only instead of two. Moreover, each of these representative agents is forbidden to trade with the other representative agent. Hence, in the case of autarky,

$$W_i^{0Au}(m_t) = E[\sum_{l=0}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_t] = \frac{1}{2(1-\beta)}(\log(XSI[i,s_1] + \log XSI[i,s_2]$$

$$W_i^{1Au}(m_t, s_t) = U_i(c_t^i) + E[\sum_{l=1}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_{t+1}]$$

$$= \log XSI[i,s_t] + \frac{\beta}{2(1-\beta)}(\log(XSI[i,s_1] + \log XSI[i,s_2])$$

In the Arrow-Debreu case, all markets exist, and agents can therefore trade their future endowments. As securities on the endowments defined by our choice of Π and XSI can only have equal value at time t before the state in t and in following periods are known, the wealth of our two agents are identical. The equilibrium will therefore generate equal and constant consumption for both agents. Consequently,

$$W_i^{0AD}(m_t) = E[\sum_{l=0}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_t] = \frac{1}{(1-\beta)}(\log((XSI[i,s_1] + XSI[i,s_2])/2))$$

If however, the agents consider moving to an Arrow-Debreu model only after they know their current state and their current consumption, they would use up their consumption and

$$W_i^{1AD}(m_t, s_t) = U_i(c_t^i) + E[\sum_{l=1}^{l=\infty} \beta^l U_i(c_{t+l}^i | I_{t+1}]$$
$$= \log XSI[i, s_t] + \frac{\beta}{(1-\beta)}(\log((XSI[i, s_1] + XSI[i, s_2])/2)$$

The results

General welfare in the four types of economies is presented in Chart 3. Here, the general welfare is calculated by simply adding up the expected utility of the two agents. As expected, the Arrow-Debreu model is the best performer. Variant 1 dominates Variant 2 in all cases where money partition has reached its long term range in Variant 1. But Variant 2 dominates when agents start with extreme money partitions. However, it is important to note that the interval on which Variant 2 is collectively preferable to Variant 1 does not intersect with the interval $[m_{Min}, m_{Max}]$. Finally the autarky model may be collectively preferable to Variant 1 only for very extreme partitions of money.[13]

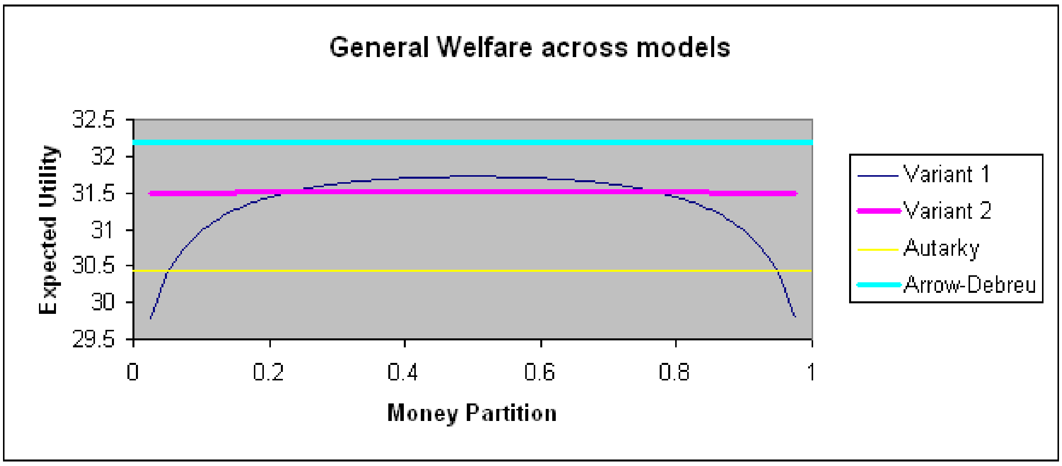


Chart 3

[13] The relative level of the general welfare created by Variants 1 and 2 was checked in order to ascertain that differences between them did not result from the randomness associated with the Monte Carlo method. The relative levels were found to have been estimated in a robust way.

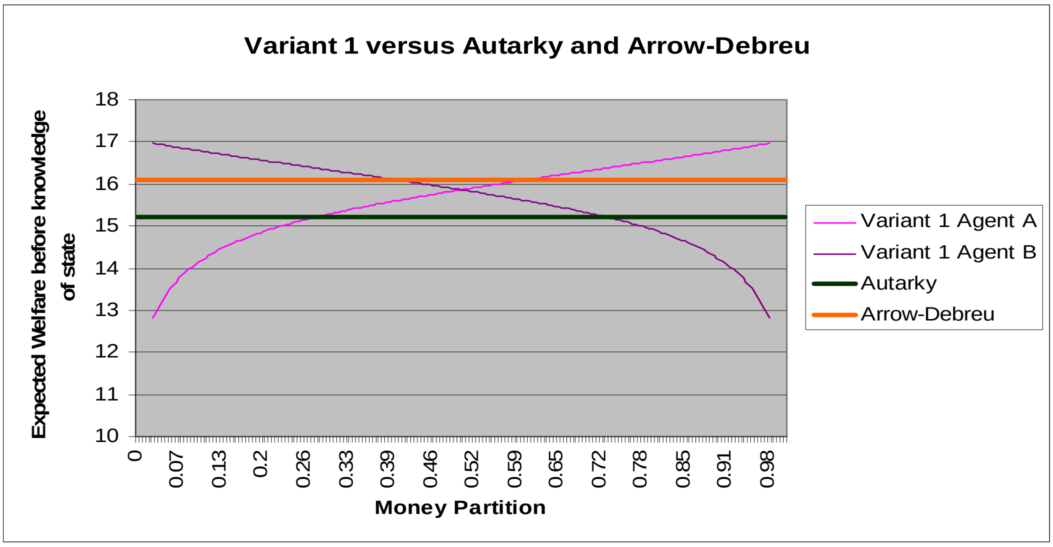


Chart 3.1

However, considering welfare individually allows for a more precise analysis and leads to different results. In Chart 3.1, two elements stand out. First, in Variant 1, at the beginning of each period before the current shock is known, autarky is preferable to one of the agents whenever the partition of money does not belong to the range [0.28, 0.72]. This range is narrower than the range $[m_{Min}, m_{Max}]$ which is equal to [0.25, 0.75]. Hence, even after the partition has converged to its long term range in Variant 1, it may *well reach the range [0.25, 0.28[, where agent A prefers autarky, or the range]0.72,0.75], where Agent B may also prefer autarky.*

Second, the move to an Arrow-Debreu model, i.e. *the creation of a more advanced financial system, will be seen as favourable by both agents only when money partition belongs to [0.4, 0.6]*. These elements have consequences from both legal and financial development viewpoints as will be seen later. By contrast, these difficulties do not arise with Variant 2, which is always Pareto superior to Autarky and Pareto inferior to Arrow-Debreu (see Chart 3.2). Hence, finding support for financial development in the context of Variant 2 should not be a problem.

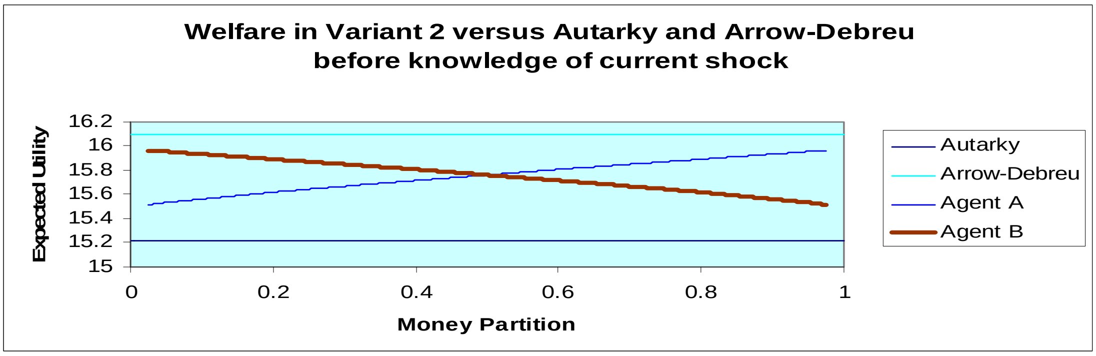


Chart 3.2

However, shifting from Variant 1 to Variant 2 or vice versa does not seem so natural. Whatever the partition of money, one of the agents does not have interest in abandoning the cash-in-advance model, except perhaps in the case of a partition of 0.5.

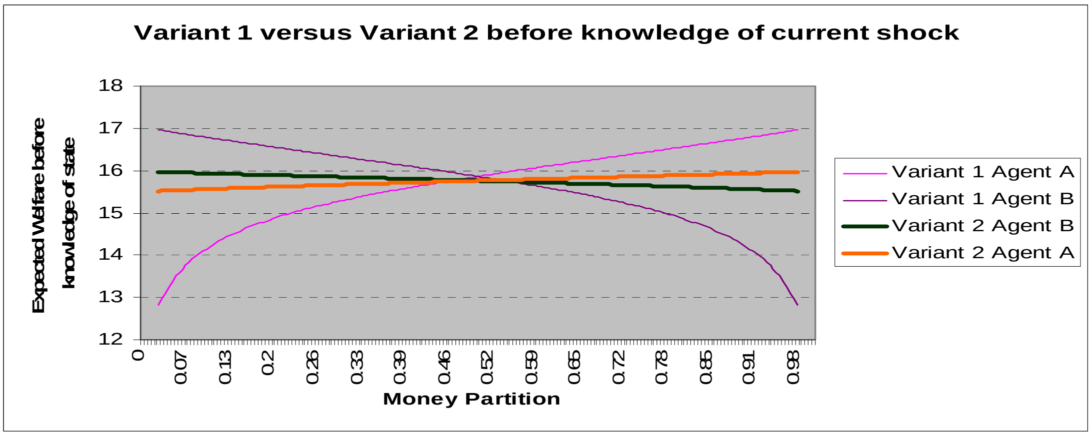


Chart 3.3

The legal framework of Variants 1 and 2: is a legal tender status needed?

As apparent in Chart 3.1, in the case of money partitions between 0.25 and 0.28 or between 0.72 and 0.75 and under Variant 1, one of the two agents may prefer autarky to the repayment of the collective debt represented by money. This agent knows that, in a situation where the amount of money he/she owns is particularly low, his/her endowment might be higher than his/her cash constraint will let him/her afford. As a result, the price of the unique good will fall to a low level at which the agent will have to sell anyway. Moreover, this interest in autarky may increase when the agent learns about the state s_t of the day. As apparent in Chart 4, when agent b learns that his/her endowment in state 0 is high and the amount of money he/she holds is limited, his/her interest is clear in the absence of further incentive: the agent prefers autarky given that the level of income insurance he/she obtains from the possession of money is too limited.

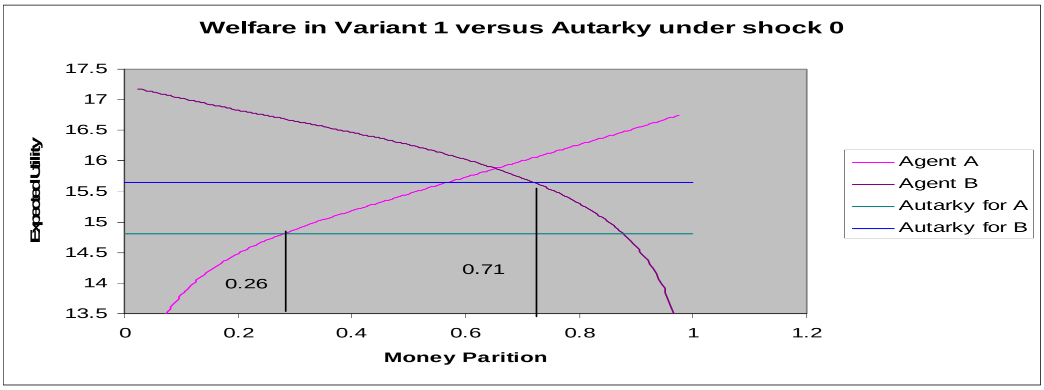


Chart 4

Therefore, the effect of introducing money into the economy is not as straightforward as suggested by models like Townsend (1980) or Manuelli and Sargent (1988) who limited themselves to specific money partitions. This phenomenon may explain the non-monetization or demonetization of some economies, either in developing countries or in mediaeval Europe, where money partition often became very uneven across agents or regions due to historical developments, subsequently leading to the disappearance of former currencies.

Making the rational equilibrium of Variant 1 sustainable may therefore imply a law on the legal tender status of money or on the functioning of markets[14]. This law needs to affect the incentives of agents to choose between autarky and Variant 1 when the partition of money becomes too unequal. In other words, in order to keep the market in Variant 1 operational, agents must be fined or penalised if they refuse to continue

[14] The term "legal tender" may be viewed by some as inappropriate as it usually characterises the payment of debts across agents which in this model do not exist formally.

participating in the centralised market of Variant 1 or if they shift to autarky. The fine or penalty must be sufficiently severe in order to have them prefer Variant 1 to autarky without affecting their incentives in terms of trading. The simplest solution is *to fine any agent moving to autarky by an amount at least equal, in terms of expected utility*, to the difference:

$$\underset{m\subset[0,25,0.28]}{Max}\{W_i^{0Au}(m_t)-W_i^{0}(m_t),\, W_i^{1Au}(m_t,s_t)-W_i^{1}(m_t,s_t)\,\}=$$

$$\text{Max}\{\frac{1}{2(1-\beta)}(\log(XSI[i,s_1]+\log XSI[i,s_2]-E[\sum_{l=0}^{l=\infty}\beta^l U_i(c_{t+l}^i|I_t],$$

$$\log XSI[i,s_t]+\frac{\beta}{(1-\beta)}(\log((XSI[i,s_1]+XSI[i,s_2])/2)-U_i(c_t^i)-E[\sum_{l=1}^{l=\infty}\beta^l U_i(c_{t+l}^i|I_{t+1}]\}$$

= Max{(15.22261-15.05499); 14.79896-15.47471} = 0.16762 Utils.

Such a fine, applied only to agents moving from Variant 1 to autarky, makes Variant 1 sustainable and corresponds to about 1/100 of the overall wealth of the agent. It entails very limited administrative capability. To the extent that it is convincing and timely, it does not impact trading itself.

By contrast, the need for a law on the legal status of money does not manifest itself under Variant 2.

Systemic risk and money in the absence of banks and other financial markets

Systemic risk exists when the occurrence of an exogenous event has the consequence that the functioning of the economy is durably altered and overall welfare is durably reduced (see Aglietta and Moutot (1993) and CGFS(2010). This happens in our suite of models if one of the agents prefers autarky and cannot be discouraged from doing so by an appropriate legal constraint.

Let us therefore assume that the economy is well described by Variant 1 as defined above, and that a legal tender status envisages fines up to 0.16762 units of utility, making this cash-in-advance economy perfectly stable and rational under its usual endowments.

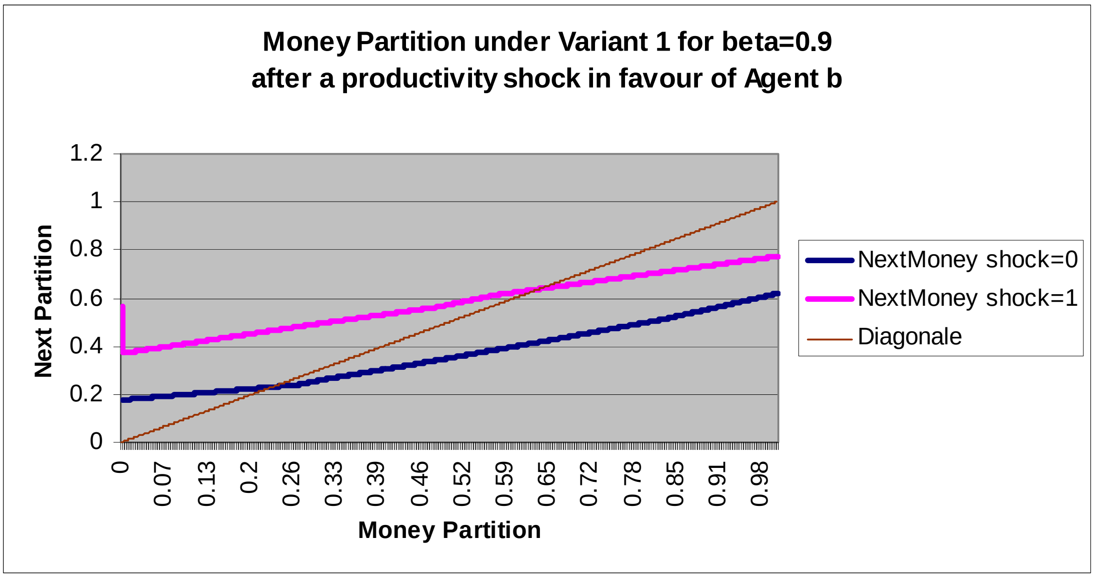


Chart 5.1

But let us imagine also that at a given point in time, while the partition of money is between 0.4 and 0.6, the endowments of one of the two agents, Agent b, unexpectedly shifts from alternating between 3 and 7 units of goods to alternating between 5 and 9. As shown in Chart 5.1, this leads to a new equilibrium in which money partition starts oscillating between two new values of m_{Min} and m_{Max} now forming the range [0.23, 0.64] which is narrower than the preceding range [0.25, 0.75].

Let us assume that the authorities do not immediately adjust the fines associated with the legal tender status and agents are not aware of this delay. If money partition reaches 0.64, the utility of agent b will switch from 18.7449 to 19.03331 if he/she shifts to autarky, implying that a higher fine, precisely a fine of at least 0.288408 Utils, is needed to protect the economy against systemic risk. If, however, the fine remains limited to 0.16762 Utils, the temptation to switch to autarky will exist for Agent b for all money partitions in the range [0.515, 0.64] as apparent from Chart 5.2. A systemic event may therefore occur about 14% of the time, leading to an overall average loss of general welfare of 0.938788 Utils.

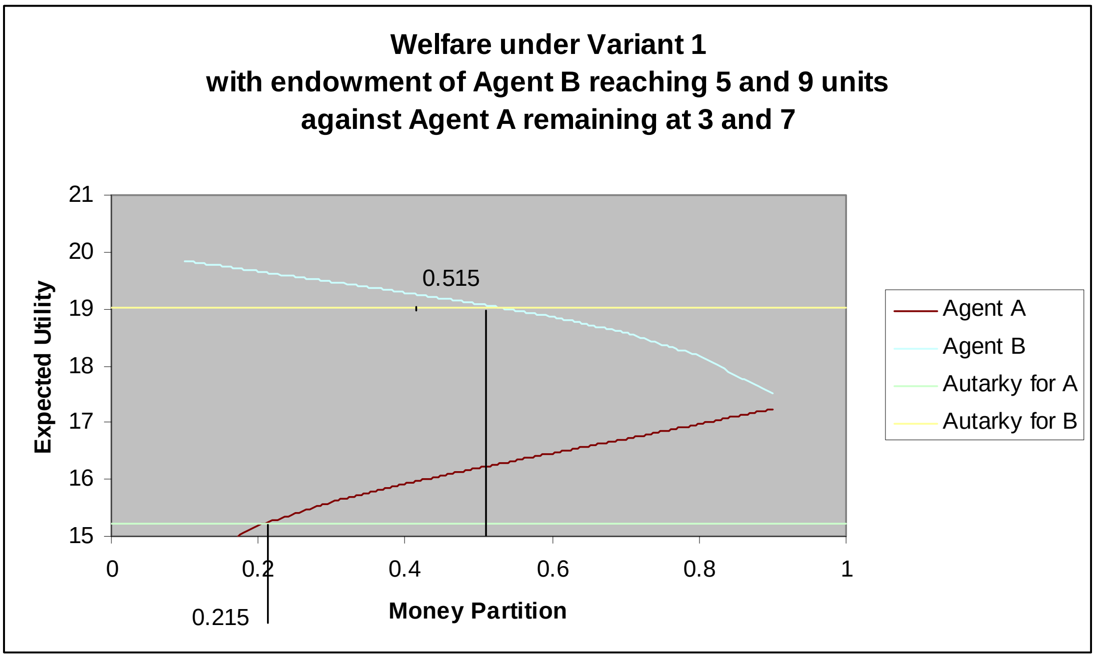


Chart 5.2

Hence, a systemic risk may exist and manifest itself in a model with centralised markets and rational expectations whenever financial frictions create the need for a legal environment in order to ensure financial stability, and this legal environment does not evolve in due time, i.e. does not adjust quickly enough to unexpected changes in the endowment process.

Let us note, moreover, that in this example, the inequality of endowments across agents increases from 7/3 to 9/3 and that such inequality is directly related to the need to strengthen regulation. This example may thus provide a practical and relatively simple illustration of the link between inequality and systemic risk made by Rajan (2010) in his book entitled “Fault Lines”.

Section 4

THE DYNAMICS OF MONEY PARTITION AND PRICES AND THE PRACTICALITY OF FINANCIAL DEVELOPMENT OR REGRESSION

However, the level of money partition in Variant 1 and 2 is not the only element to consider when discussing the choice between financial development and regression in Variants 1 and 2. Decisions take time and are easier, especially when made collectively, if the elements motivating them are more lasting. As a consequence, it is important to not only be aware of the level, but also of the dynamics of money partition and prices in models with cash constraints.

The dynamics of money partition across time in Variants 1 and 2

Chart 6.1 and 6.2 show the distribution of money partitions under Variants 1 and 2 when simulated over 1000 periods starting with an equal money partition, i.e. $m_0 = 0.5$ at time 0. In both cases, there is some accumulation at the edges of the respective intervals, i.e. $[m_{Min}, m_{Max}]$ and [0,1].

However, Charts 6.1 and 6.2, also show that the probability of a perfectly equal money partition is very low or nil, even though the starting partition was $m_0 = 0.5$. This implies that there is at least one point in the interval $[m_{Min}, m_{Max}]$ which is unlikely to be rapidly revisited, making the possibility of actual shifts between Variants 1 and 2 even more limited than envisaged in Section 3.

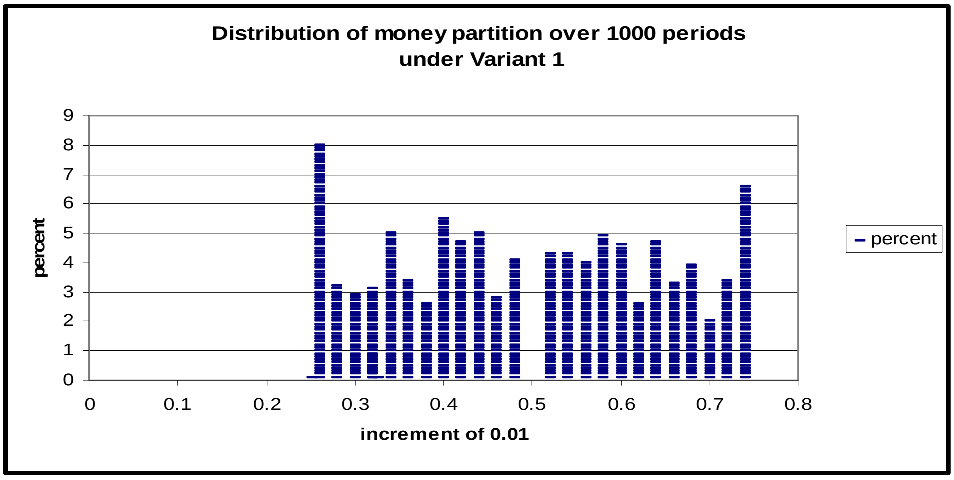


Chart 6.1

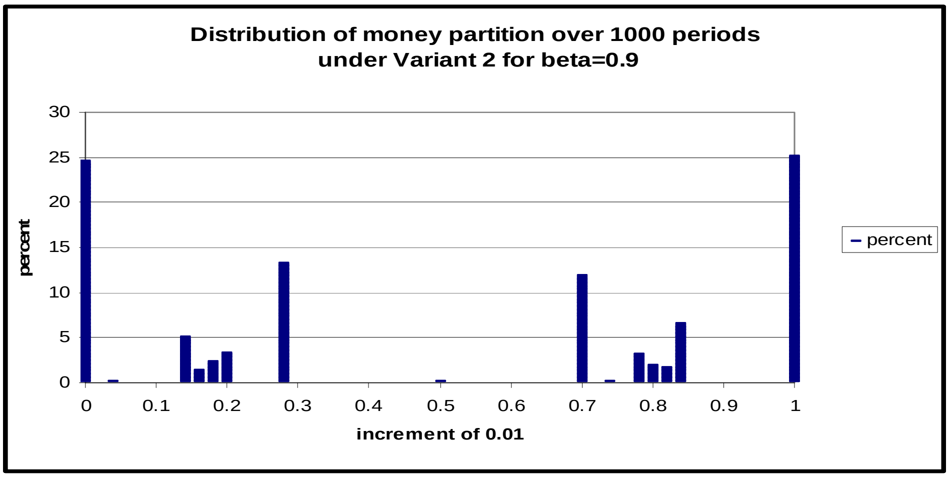


Chart 6.2

Reasoning by analogy, one may identify Variant 1 with gross payment systems and Variant 2 with net payment systems, leading to the conclusion that it may be difficult, as experienced historically, to replace net payments systems by gross payment systems without official interventions. Alternatively, one may identify Variant 1 with an over-the-counter market and Variant 2 with an integrated centralised market. One may conclude again that it may also be difficult to integrate over-the-counter operations into one centralised market without official interventions. In practice, some types of cash constraints may outlast their technical justifications due to the difficulty of collectively deciding to change them.

Chart 6.2 shows another important potential of models with cash constraints. It remains quite stable across runs, showing that money partition varies across a limited set of numbers across time in the example chosen and makes discrete jumps across more values, but not so many more than the number of microeconomic shocks. Hence, even without considering investment or stochastic decisions to trade, a rational expectations model with heterogeneous agents can create more discrete price and money partition jumps than the number of macro news. Moreover, it can exhibit more types of jumps than the number of micro news which create its dynamics. Hence, disequilibrium approaches are not the only possible approaches to a phenomenon that seems to characterize a number of markets, as argued by a number of authors (see for instance Bouchaud (2009) for an interesting and visual summary of this literature).

Moreover, the nature of the distributions of money partitions generated may vary strongly depending on β. For instance, when β is low (Chart 6.3) and makes cash-in-advance constraints always binding, the partition of money, although variable, always jumps after a few iterations between two values, thereby converging toward an invariant probability distribution, as standard for most general equilibrium models with one agent (Lucas (1980), Lucas and Stokey (1989), Medio (2004)).

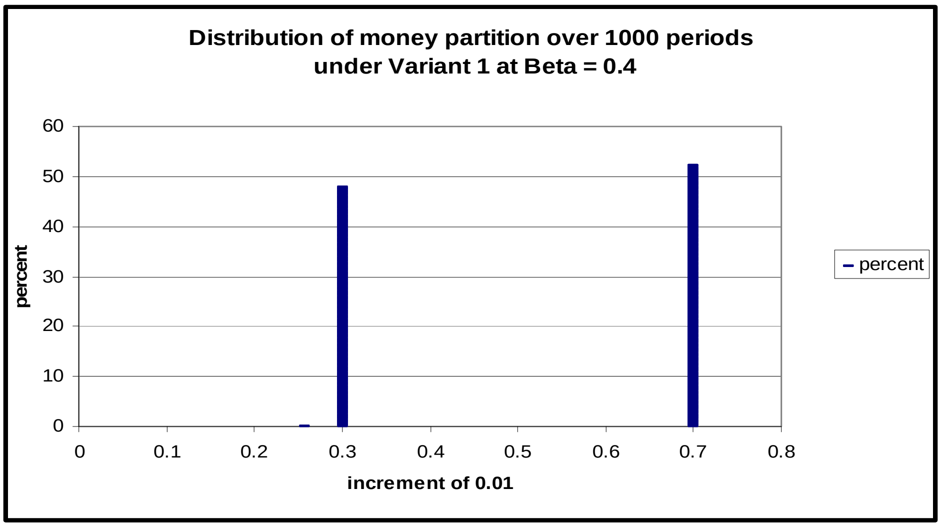


Chart 6.3

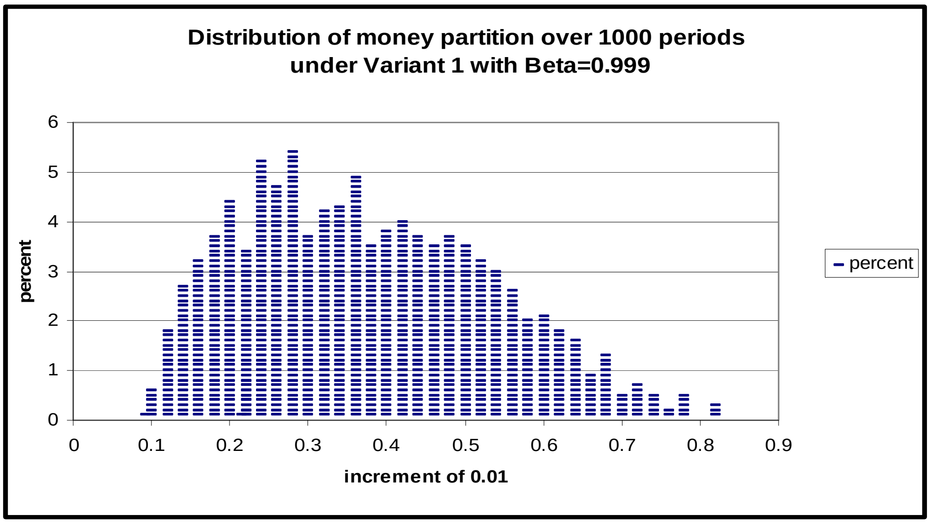


Chart 6.4

However, when β increases and becomes close to one, the distribution of the sequence of money partitions becomes more continuous and, under Variant 1, looks increasingly like a normal distribution (see Chart 6.4). But this is not the case under Variant 2 (see Chart 6.5), although almost all parts of [0,1] seem to be gradually reached.

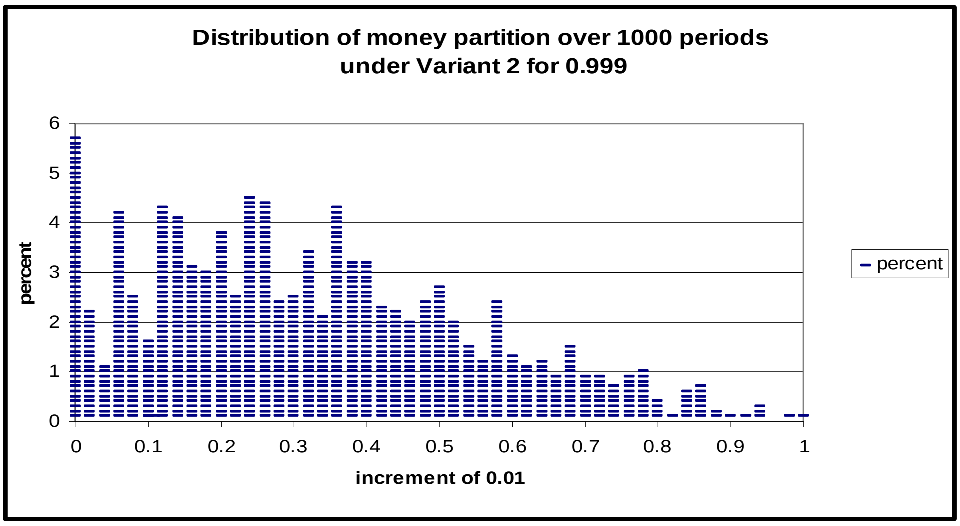


Chart 6.5

Overall, varying β within the context of the equilibrium models with rational expectations and heterogeneous agents constituted by Variants 1 and 2 allows the creation of a wealth of dynamics for the partition of money/wealth. As indicated by Farmer and Geneakoplos (2008), some authors would see them as characteristic of disequilibrium models.

<u>The dynamics of prices across time in Variants 1 and 2</u>

The dynamics of prices is also quite different from the dynamics that could be generated by more traditional models. To some extent, this just reflects the dynamics of money/wealth partition. In particular, price jumps are again common.[15]. However, it is also useful to look at the distribution of price changes. Again, under both Variant 1 and 2, fat-tails appear again for high values of β (see Charts 7.1 and 7.2).

[15] The corresponding graphs can be found in Annex 3.

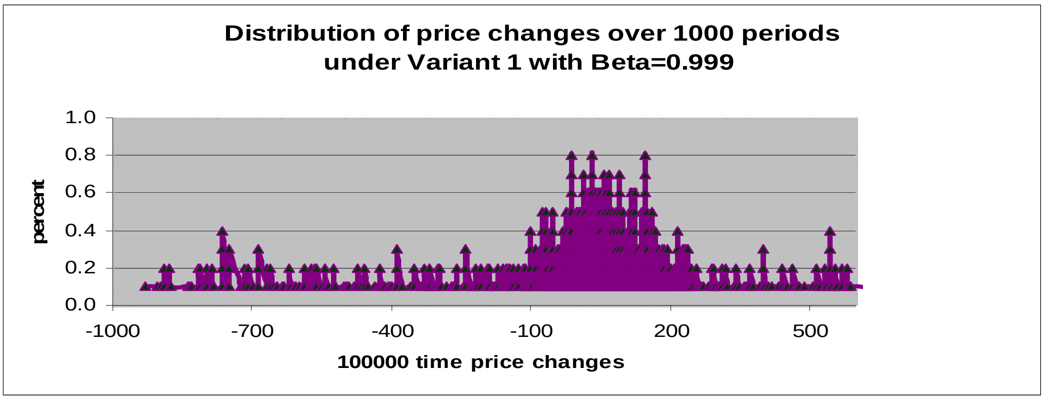


Chart 7.1.

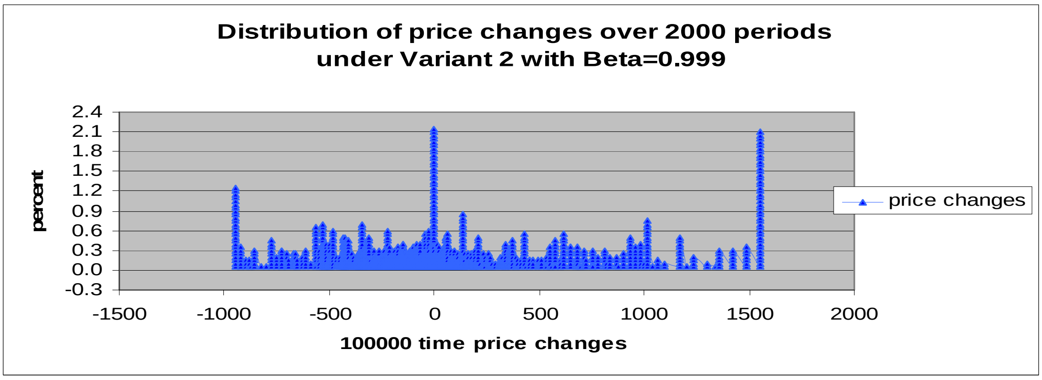


Chart 7.2

Financial development and money

As explained earlier, financial development is also dependant on the distribution of money/wealth and the degree of equality among agents. Under Variant 1, shifting to the Arrow-Debreu environment is Pareto superior only when the money partition is between 0.4 and 0.6. However, if making such a reform takes time, the dynamics of money partition influences the likelihood of a continuous agreement to carry out financial development.

In Table 2 below, one restarts history with $m_0 = 0.5$. Then one calculates how often financial development may happen under continuous Pareto-superiority, i.e. with m staying within the interval [0.4,0.6] from 1 to 6 periods. Money partition is between

0.4 and 0.6. about 40 percent of the time. However, if reforms need 6 periods to be carried out, the likelihood of financial development occurring once over 1000 periods falls to less than ½ percent. Under Variant 2 by contrast, this is always possible.

Table 2

How likely is it that money partition stays N periods in a row between 0.4 and 0.6 under Variant 1?					
N=1	N=2	N=3	N=4	N=5	N=6
39.70%	8.30%	3.60%	1.70%	0.90%	0.40%

However, shifting from Variant 1 to Variant 2 or vice versa is not easy either, as noticed before. Therefore, in a heterogeneous world with monetary frictions, excessive inequalities and variability in the partition of money may delay financial development. In view of the limited range of money partitions where our specific Arrow-Debreu equilibrium is Pareto-superior to the cash-in-advance equilibrium, barring any redistribution of ownership rights, it may often be difficult for authorities of a given country to garner the political support necessary to open new financial markets. This feature is certainly directly related to our care in avoiding any assumption on the existence of steady-state solutions.

Systemic risk as an occasion to foster financial development

Finally, it may be interesting to clarify that a systemic event may be an occasion for fostering financial development, if used adequately. Indeed, suppose that the economy is under Variant 1, and as described by Table 2, financial development is very unlikely. It may be rational for a government to refuse repairing the legal system as suggested above.

Indeed, if the authorities only offer to their two types of agents the possibility of a step to Variant 2 instead of a repair of the legal framework of Variant 1, the level of expected welfare of the economy will in aggregate be immediately reduced as apparent on Chart 3. But the chances of improving this less efficient economy in the time needed to shift from Variant 2 to the Arrow-Debreu model will increase to 100%. The overall loss of welfare generated by a shift from Variant 1 to Variant 2 is much smaller than the benefit of shifting in a few periods to the Arrow-Debreu. Hence, such a fall is, in the end, preferable to a return to the low likelihood of financial development reflected by Table 2. One could even think that manipulative or Machiavellian authorities would be glad to let systemic risk occur whenever they are sure of their ability to move in a short period of time from Variant 1 to Arrow-Debreu through Variant 2. This is in line with a Schumpeterian view of the world: as crises are necessary to development, no crisis should be wasted.

Conclusion

Two variants of a small model with two agents, money, rational expectations and non-continuously binding cash constraints have been presented and simulated. Together they show that the introduction of heterogeneous agents and cash constraints in macro-models may play a more important role than usually expected. First, it may help better reflect how financial frictions affect the link between money and prices in centralised markets. Second, it also enhances our understanding of the complex link between financial frictions and financial development, including the fact that some set-ups may be difficult to modify. In particular, they make it possible to consider and integrate systemic risk and financial development/regression. Third, cash constraints also allow the generation of price distributions that many authors, up to recently, related uniquely to disequilibrium models.

Much needs to be done, however, before macro-economists may use such models for practical purposes. Indeed, cash constraints will have to be complemented by debt and/or credit constraints if banks and investment have to be introduced. Obviously, the mathematical and computational difficulties associated with such models are important and will largely condition their development and use. But the benefits to be expected from models with heterogeneous agents and financial frictions are also particularly significant as they may help integrate systemic risk, financial development, dynamic macroeconomics, monetary policy and financial stability issues into the same framework.

Annex 1

Transforming the set of first-order conditions into a functional operator.

The determination of the current price p_t

If our model possesses an equilibrium, then p_t can be shown to be a function h(.) of m_t, the current partition of money between agents, of θ_t^a and θ_t^b, the Lagrange multipliers for the two budget constraints, and of $\dot{\xi}_t$, the vector of goods endowments.

To see this, let us first write (1.1) and (1.1') as respectively:

$$c_t^i \leq m_t^i / p_t \qquad i \in \{a,b\} \text{ with equality when } \gamma_t^i > 0$$

$$c_t^i \leq m_t^i / p_t + \xi_t^i \qquad i \in \{a,b\} \text{ with equality when } \gamma_t^i > 0$$

Since $U_i(.)$ is strictly increasing and strictly concave, its derivative $U'_i(.)$ has an inverse $U'^{-1}_i(.)$. It follows from (1.5) that:

$$c_t^i = U'^{-1}_i(p_t(\theta_t^i + \gamma_t^i)) \qquad i \in \{a,b\}$$

Like $U'_i(.)$, U'^{-1}_i is strictly decreasing. Also, the Lagrange multiplier γ_t^i can only be positive or equal to zero.

Therefore,

$$c_t^i \leq U'^{-1}_i(p_t\ \theta_t^i) \qquad i \in \{a,b\} \text{ with equality when } \gamma_t^i = 0$$

Consequently,

$$c_t^i = Min[m_t^i / p_t, U_i'^{-1}(\theta_t^i\ p_t)] \quad i \in \{a,b\} \qquad (1.7)$$

$$c_t^i = Min[m_t^i / p_t + \xi_t^i, U_i'^{-1}(\theta_t^i\ p_t)] \quad i \in \{a,b\} \qquad (1.7')$$

Finally, substituting (1.7) or respectively (1.7') for $i = a$ and $i = b$ into (1.3), writing m_t^a as m_t and using (1.4) to replace m_b^t by $1 - m_t$, and taking into account that in the second variant both cash constraints cannot be simultaneously binding, we get:

$$\xi_t = Min\{\frac{1}{p_t}, \frac{m_t}{p_t} + U'_b{}^{-1}(\theta_t^b p_t), \frac{1-m_t}{p_t} + U'_a{}^{-1}(\theta_t^a p_t), U'_a{}^{-1}(\theta_t^a p_t) + U'_b{}^{-1}(\theta_t^b p_t)\}$$

(1.8)

$$\xi_t = Min\{\frac{m_t}{p_t} + \xi_t^a + U'_b{}^{-1}(\theta_t^b p_t), \frac{1-m_t}{p_t} + \xi_t^b + U'_a{}^{-1}(\theta_t^a p_t), U'_a{}^{-1}(\theta_t^a p_t) + U'_b{}^{-1}(\theta_t^b p_t)\}$$

(1.8')

ξ_t is therefore equal to the minimum of four different functions of p_t in the first variant and only three in the case of the second variant, each of them decreasing and invertible. Therefore there exists for both variants, a function h such that:

$$p_t = h(m_t, \dot{\theta}_t, \dot{\xi}) \qquad (1.9)$$

where $\dot{\xi}_t$ is the vector (ξ_t^a, ξ_t^b) and $\dot{\theta}_t$ is the vector (θ_t^a, θ_t^b).

The nature of this function $h(.)$ is best understood by inverting each of the four or three functions of p_t on the right-hand side of (1.8) or (1.8') independently and writing $h(.)$, as the minimum of these four, respectively 3, inverted functions.

$$h(m_t, \dot{\theta}_t, \dot{\xi}_t) = Min\{h_1(\xi_t), h_2(m_t, \theta_t^b, \xi_t), h_3(m_t, \theta_t^a, \xi_t), h_4(\theta_t^a, \theta_t^b, \xi_t)\} \qquad (1.10)$$

$$h(m_t, \dot{\theta}_t, \dot{\xi}_t) = Min\{h_2(m_t, \theta_t^b, \xi_t^b), h_3(m_t, \theta_t^a, \xi_t^a), h_4(\theta_t^a, \theta_t^b, \xi_t)\} \qquad (1.10')$$

This shows that the price p_t shifts between 4, respectively 3, different regimes of determination, depending on which agent is or is not cash-constrained. $h_1(\xi_t)$ is the price which prevails when both consumers are under a binding cash constraint in the first variant and it is equal to $1/\xi_t$. $h_2(m_t^i, \theta_t^b, \xi_t)$, and respectively $h_2(m_t^i, \theta_t^b, \xi_t^b)$, defines p_t when Consumer a's cash constraint is binding, whereas Consumer b's cash constraint is not. $h_3(m_t, \theta_t^a, \xi_t)$, respectively $h_3(m_t, \theta_t^a, \xi_t^a)$, determines p_t when Agent b is under a binding cash constraint while Agent a is not. Finally, m_t does not enter the function $h_4(\theta_t^a, \theta_t^b, \xi_t)$ because $h_4(.)$ defines the price p_t in the case where neither a nor b are bound by cash constraints.

The partition m_{t+1} of the money supply

Using (1.7), respectively (1.7') and (1.2), one can write:

$$m_{t+1}^i = m_t^i + p_t \xi_t^i - p_t \, Min\{\frac{m_t^i}{P}, U'_i{}^{-1}(\theta_t^i p_t)\} = Max\{p_t \xi_t^i, m_t^i + p_t(\xi_t^i - U'_i{}^{-1}(\theta_t^i p_t))\} \quad i \in \{a,b\}$$

(1.11)

$$m_{t+1}^i = m_t^i + p_t \xi_t^i - p_t \, Min\{\frac{m_t^i}{P} + \xi_t^i, U'_i{}^{-1}(\theta_t^i p_t)\} = Max\{0, m_t^i + p_t(\xi_t^i - U'_i{}^{-1}(\theta_t^i p_t))\} \quad i \in \{a,b\}$$

(1.11')

Substituting (1.9) into (1.11), respectively (1.11') shows that m_{t+1} is determined by the same variables as p_t and can consequently be written:

$$m_{t+1} = M(m_t, \dot{\theta}_t, \dot{\xi}_t) \qquad (1.12)$$

Functions $h(.)$ and $M(.)$ specify p_t and m_{t+1} as functions of $m_t, \theta_t^a, \theta_t^b, \xi_t^a$ and ξ_t^b. To characterize the equilibrium law of motion m_{t+1} and hence p_{t+1}, we must solve for the Lagrange multipliers θ_t^a and θ_t^b.

Constructing the functional operator

We now make use of equations (1.6) and (1.11) for $i = a$ and $i = b$. Since both equations link variables at time t to expectations of variables at $t+1$, they will become the core of a functional operator :

For this, we will simultaneously take into account the set of all possible draws of the stochastic process $\{s_t\}$ and of all possible partitions of money, therefore replacing s_t and m_t by two generic variables respectively called s and m. Then,

if $s_t = s_{t+u} = s$ and $m_t = m_{t+u} = m$,

$\theta_t^i = \theta_{t+u}^i = \theta^i(s,m)$ for $t \geq 0$, $u \geq 0$, and $i \in \{a,b\}$

Let us also define $\theta(s,m)$ as:

$$\theta(s,m) = \begin{bmatrix} \theta^a(s,m) \\ \theta^b(s,m) \end{bmatrix}$$

and assume $\theta(s,m)$ belongs to C^2 x C^2 where C^2 is the set of continuous and bounded functions on S x [0,1].

From (1.5) and (1.7), respectively (1.7'):

$$\gamma_t^i + \theta_t^i = Max\{\theta_t^i, \frac{1}{p_t} U_i^{;} (\frac{m_t^i}{p_t})\}. \qquad i \in \{a,b\}$$

$$\gamma_t^i + \theta_t^i = Max\{\theta_t^i, \frac{1}{p_t} U_i^{;} (\frac{m_t^i}{p_t} + \xi_t^i)\}. \qquad i \in \{a,b\}$$

respectively

It follows from (1.6) and (1.12) that:

$$\theta(s,m) = \Phi(\theta)(s,m) \qquad (1.13)$$

where:

$$\Phi(\theta)(s.m) = \begin{bmatrix} \beta \int_S Max\{\theta^a(s',m'), \frac{1}{h'} U_a'(\frac{m'}{h'})\} \quad \pi(s,ds') \\ \beta \int_S Max\{\theta^b(s',m'), \frac{1}{h'} U_b'(\frac{1-m'}{h'})\} \quad \pi(s,ds') \end{bmatrix}$$

$$\Phi(\theta)(s.m) = \begin{bmatrix} \beta \int_S Max\{\theta^a(s',m'), \frac{1}{h'} U_a'(\frac{m'}{h'} + \xi^a(s'))\} \quad \pi(s,ds') \\ \beta \int_S Max\{\theta^b(s',m'), \frac{1}{h'} U_b'(\frac{1-m'}{h'} + \xi^b(s'))\} \quad \pi(s,ds') \end{bmatrix}$$

respectively,

with: $m' = M(m, \dot{\theta}(s,m), \dot{\xi}(s))$ and: $h' = h(m', \dot{\theta}(s',m'), \dot{\xi}(s'))$

Annex 2

Price regimes in the cash-in-advance and cash-at-the end-of-the-day variants of the model

Variant 1: some characteristics of solutions

The following theorem and lemmas were proved in Moutot (1991) in Appendix C.. By contrast, proofs concerning Variant 2 are new.

Lemma 1

Let h $= h(m, \theta^{\grave{a}}, \theta^{`b}, \xi)$. Then the following propositions are equivalent.

$\theta^a \leq \frac{1}{h} U_a'(\frac{m}{h})$ and $\theta^b \leq \frac{1}{h} U'_b(\frac{1-m}{h}) \Leftrightarrow h$ under h_1

$\theta^a \geq \frac{1}{h} U_a'(\frac{m}{h})$ and $\theta^b \geq \frac{1}{h} U'_b(\frac{1-m}{h}) \Leftrightarrow h$ under h_4

$\theta^a \geq \frac{1}{h} U_a'(\frac{m}{h})$ and $\theta^b \leq \frac{1}{h} U'_b(\frac{1-m}{h}) \Leftrightarrow h$ under h_3

$\theta^a \leq \frac{1}{h} U_a'(\frac{m}{h})$ and $\theta^b \geq \frac{1}{h} U'_b(\frac{1-m}{h}) \Leftrightarrow h$ under h_2

<u>Proof</u>

Suppose $\theta^a \leq \frac{1}{h} U_a'(\frac{m}{h})$. Then, as $U_a'(.)$ is decreasing, it is equivalent to

$$\frac{m}{h} \leq U_a^{'-1}(\theta^a h)$$

and therefore

$$\frac{1}{h} \leq \frac{1-m}{h} + U_a^{'-1}(\theta^a h)$$

$$\frac{m}{h} + U_b^{'-1}(\theta^b h) \leq U_a^{'-1}(\theta^a h) + U_b^{'-1}(\theta^b h)$$

As a consequence of (1.8) and (1.10), these two inequalities imply that:

$$\theta^a \le \frac{1}{h} U_a{}'(\frac{m}{h}) \Leftrightarrow h = h_1(\xi) \text{ or } h = h_2(m, \theta^b, \xi)$$

On the reverse, suppose $\theta^a \ge \frac{1}{h} U_a{}'(\frac{m}{h})$. By the same token,

$$\theta^a \ge \frac{1}{h} U_a{}'(\frac{m}{h}) \Leftrightarrow h = h_3(m, \theta^a, \xi) \text{ or } h = h_4(\theta^a, \theta^b, \xi)$$

By symmetry, the same manipulation of inequalities show that

$$\theta^b \le \frac{1}{h} U_b{}'(\frac{1-m}{h}) \Leftrightarrow h = h_1(\xi) \text{ or } h = h_3(m, \theta^a, \xi)$$

$$\theta^b \ge \frac{1}{h} U_b{}'(\frac{1-m}{h}) \Leftrightarrow h = h_2(m, \theta^b, \xi) \text{ or } h = h_4(\theta^a, \theta^b, \xi)$$

The two by two combinations of these four inequalities prove lemma 1.

Lemma 2

Let θ^a, θ^b, and ξ be three strictly positive real numbers. Let m be a real number in [0,1]. Then the two following propositions are equivalent:

$$\theta^a \le \xi U_a{}'(m\xi) \text{ and } \theta^b \le \xi U_b{}'((1-m)\xi) \Leftrightarrow h(m, \theta^a, \theta^b, \xi) = h_1(\xi) = \frac{1}{\xi}$$

Proof

Suppose first that: $\theta^a \le \xi U_a{}'(m\xi)$ and $\theta^b \le \xi U_b{}'((1-m)\xi)$.

By definition of h,

$$h(m, \theta^a, \theta^b, \xi) \le h_1(\xi) = \frac{1}{\xi} \qquad \text{for all m in [0,1]}$$

Given that $U_i^{'}(.)$ is strictly decreasing,

$$U_a^{'-1}(\theta^a h_1) \ge m\xi \text{ and } U_b^{'-1}(\theta^b h_1) \ge (1-m)\xi$$

Consequently, $\frac{m}{h_1} + U_b^{'-1}(\theta^b h_1) \ge \xi$

$$\frac{1-m}{h_1} + U_a^{'-1}(\theta^a h_1) \ge \xi$$

and $\quad U_a^{'-1}(\theta^a h_1) + U_b^{'-1}(\theta^b h_1) \ge \xi$

Hence, by definition of h in (1.8), $h(m, \theta^a, \theta^b, \xi) = h_1(\xi) = \frac{1}{\xi}$ for all m in [0,1].

Reciprocally, suppose that $h(m,\theta^a,\theta^b,\xi) = h_1(\xi) = \frac{1}{\xi}$.

By definition of h, $\frac{m}{h_1} + U_b^{'-1}(\theta^b h_1) \geq \xi$ and $\frac{1-m}{h_1} + U_a^{'-1}(\theta^a h_1) \geq \xi$.

These two inequalities again imply that:

$$\theta^a \leq \xi U_a'(m\xi) \text{ and } \theta^b \leq \xi U_b'((1-m)\xi). \text{ Q.E.D.}$$

Theorem 1

Let U_a (.) and U_b(.) be two continuously differentiable, strictly increasing and strictly concave utility functions. Be β strictly superior to 0.

If "a stationary expectations equilibrium" exists and if it is such that, after a finite number of periods, both agents are always under cash constraint, then:

$$\beta \leq \beta_{\max} = \underset{\substack{s\in S \\ s''\in S \\ i\in\{a,b\}}}{Min} \frac{\xi(s)U_i'(\frac{\xi^i(s'')}{\xi(s'')}\xi_i(s))}{\int_S \xi(s')U_i'(\xi^i(s)\frac{\xi(s')}{\xi(s)})\pi(s,ds')}, (3.1)$$

Proof

Suppose both agents are under cash constraint. Then

m'= $M(m,\theta^a(s,m),\theta^b(s,m),\xi(s)) = \frac{\xi^a(s)}{\xi(s)}$

which implies that, save for time 0 possibly, m always belongs to δ, the set of values $\{\frac{\xi^a(s'')}{\xi(s'')}\}$ for $s''\subset S$

Lemma 1 implies that

$$Max\{\theta^a(s',m'),\frac{1}{h'}U_a'(\frac{m'}{h'})\} = \xi(s')U_a'(\xi(s')\frac{\xi^a(s)}{\xi(s)})$$

$$Max\{\theta^b(s',m'),\frac{1}{h'}U_b'(\frac{1-m'}{h'})\} = \xi(s')U_b'(\xi(s')\frac{\xi^b(s)}{\xi(s)})$$

However, $\theta(s,m)$ being a solution to (1.13), this implies that:

$$\theta^i(s,,m) = \beta\int \xi(s')U_a'(\xi(s')\frac{\xi^a(s)}{\xi(s)})\pi(s,ds') \quad \forall(s,m)\subset SX\delta, \quad \forall i\in\{a,b\}.$$

However, Lemma 2 implies that if cash constraints are to be binding for all m in (0,1),

$\theta^i(s,m) \le \xi(s)U_i\,'(\frac{\xi^i(s''}{\xi(s'')}\xi(s)) \quad \forall i \in \{a,b\}, \forall s, \subset S,$ and for all $s'' \subset S$.

Therefore,

$$\beta \le \beta_{\max} = \underset{\substack{s\in S\\ s''\in S\\ i\in\{a,b\}}}{Min} \frac{\xi(s)U_i'(\frac{\xi^i(s'')}{\xi(s'')}\xi(s))}{\int_S \xi(s')U_i'(\xi^i(s)\frac{\xi(s')}{\xi(s)})\pi(s,ds')}$$

Variant 2 solutions: some characteristics

Lemma 1'

$\theta^a \ge \frac{1}{h}U_a\,'(\frac{m}{h}+\xi^a)$ and $\theta^b \ge \frac{1}{h}U'_b\,(\frac{1-m}{h}+\xi_b) \Leftrightarrow h$ under h_4

$\theta^a \ge \frac{1}{h}U_a\,'(\frac{m}{h}+\xi^a)$ and $\theta^b \le \frac{1}{h}U'_b\,(\frac{1-m}{h}+\xi_b) \Leftrightarrow h$ under h_3

$\theta^a \le \frac{1}{h}U_a\,'(\frac{m}{h}+\xi^a)$ and $\theta^b \ge \frac{1}{h}U'_b\,(\frac{1-m}{h}+\xi_b) \Leftrightarrow h$ under h_2

Proof

As $U_i^{'}(.)$ is by assumption decreasing,

$\theta^a \ge \frac{1}{h}U_a\,'(\frac{m}{h}+\xi^a)$ and $\theta^b \ge \frac{1}{h}U'_b\,(\frac{1-m}{h}+\xi_b) \Leftrightarrow$

$U_a^{'-1}(\theta_a h) \le \frac{m}{h}+\xi^a$ and $U_b^{'-1}(\theta_b h) \le \frac{m}{h}+\xi^b \Leftrightarrow$

$U_b^{'-1}(\theta_b h) + \frac{m}{h}+\xi^a \ge U_a^{'-1}(\theta_a h) + U_b^{'-1}(\theta_b h) \le \frac{m}{h}+\xi^b + U_a^{'-1}(\theta_a h)$

$\Leftrightarrow h$ under h_4 as a result of (1.8'). Similarly,

$\theta^a \ge \frac{1}{h}U_a\,'(\frac{m}{h}+\xi^a)$ and $\theta^b \le \frac{1}{h}U'_b\,(\frac{1-m}{h}+\xi_b) \Leftrightarrow$

$U_a^{'-1}(\theta_a h) \le \frac{m}{h}+\xi^a$ and $U_b^{'-1}(\theta_b h) \ge \frac{m}{h}+\xi^b \Leftrightarrow$

$$U_b^{'-1}(\theta_b h) + \frac{m}{h} + \xi^a \geq \frac{m}{h} + \xi^b + U_a^{'-1}(\theta_a h) \leq U_a^{'-1}(\theta_a h) + U_b^{'-1}(\theta_b h)$$

$\Leftrightarrow h$ under h_3 as a result of (1.8'). Finally,

$$\theta^a \leq \frac{1}{h} U_a'(\frac{m}{h} + \xi^a) \text{ and } \theta^b \geq \frac{1}{h} U'_b(\frac{1-m}{h} + \xi_b) \Leftrightarrow$$

$$U_a^{'-1}(\theta_a h) \geq \frac{m}{h} + \xi^a \text{ and } U_b^{'-1}(\theta_b h) \leq \frac{m}{h} + \xi^b \Leftrightarrow$$

$$\frac{m}{h} + \xi^b + U_a^{'-1}(\theta_a h) \geq \frac{m}{h} + \xi^a + U_b^{'-1}(\theta_b h) \leq U_a^{'-1}(\theta_a h) + U_b^{'-1}(\theta_b h) \backslash$$

h under h_2 as a result of (1.8').Q.E.D.

Proposition A.1

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$.

If $\text{h} = h_2(m, \theta^b, \xi^{b)} = \frac{1}{\xi^b}(m + \frac{1}{\theta^b})$, then $m \leq \frac{\xi^a}{\xi\theta^b} - \frac{\xi^b}{\xi\theta^a}$ and $M(m,\theta,\xi) = 0$.

If h= $h_3(m, \theta^a, \xi^{a)} = \frac{1}{\xi^a}(1 - m + \frac{1}{\theta^a})$, then $m \geq 1 - \frac{\xi^a}{\xi\theta^b} + \frac{\xi^b}{\xi\theta^a}$ and $M(m,\theta,\xi) = 1$.

If h= $h_4(m, \theta^a, \xi^{a)} = \frac{1}{\xi}(\frac{1}{\theta^a} + \frac{1}{\theta^b})$, then

$$\frac{\xi^a}{\xi\theta^b} - \frac{\xi^b}{\xi\theta^a} \leq m \leq 1 - \frac{\xi^a}{\xi\theta^b} + \frac{\xi^b}{\xi\theta^a} \text{ and } M(m,\theta,\xi) = m + \frac{\xi^a}{\xi\theta^b} - \frac{\xi^b}{\xi\theta^a}.$$

Proof

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$

(1.11') defines the three price regimes implied by the model.

1) If the price regime is h_2, then from (1.8')

$$\text{h} \leq h_4(m, \theta^a, \xi^{a)} = \frac{1}{\xi}(\frac{1}{\theta^a} + \frac{1}{\theta^b})$$

Therefore $\frac{\xi}{\xi^b} m + \frac{\xi}{\xi^b\theta^b} - \frac{1}{\theta^a} \leq 0$ and, $m\frac{\xi}{\xi^b} + \frac{\xi^a\xi}{\xi\xi^b}\frac{1}{\theta^b} - \frac{1}{\theta^a} \leq 0$

as $\frac{\xi^a}{\xi} \prec 1$ and θ^b is by definition positive.

Therefore $m \le \frac{\xi^b}{\xi\theta^a} - \frac{\xi^a}{\xi\theta^b}$. Moreover, using (1.11'),

$$M(m,\theta,\xi) = Max\{0, m + h(\xi^a - \frac{1}{\theta^a h})\} =$$

$$Max\{0, m + \frac{\xi^a}{\xi^b}(m + \frac{1}{\theta^b}) - \frac{1}{\theta^a}\} = Max\{0, m\frac{\xi}{\xi^b} + \frac{\xi^a \xi}{\xi\xi^b}\frac{1}{\theta^b} - \frac{1}{\theta^a}\} = 0$$

2) If the price regime is h_3, the proof is identical and the results can be deducted by changing a into b and 1 into 1-m.
3) If the price regime is h_4,

$$h_4(m,\theta^a,\xi^{a)} = \frac{1}{\xi}(\frac{1}{\theta^a} + \frac{1}{\theta^b}) \le h_2(m,\theta^b,\xi^{b)} = \frac{1}{\xi^b}(m + \frac{1}{\theta^b}) \text{ and}$$

$$h_4(m,\theta^a,\xi^{a)} = \frac{1}{\xi}(\frac{1}{\theta^a} + \frac{1}{\theta^b}) \le h_3(m,\theta^a,\xi^{a)} = \frac{1}{\xi^a}(1 - m + \frac{1}{\theta^a})$$

Therefore $\frac{\xi^a}{\xi\theta^b} - \frac{\xi^b}{\xi\theta^a} \le m \le 1 - \frac{\xi^a}{\xi\theta^b} + \frac{\xi^b}{\xi\theta^a}$ and

$$M(m,\theta,\xi) = Max\{0, m + h(\xi^a - \frac{1}{\theta^a h})\} = M(m,\theta,\xi) = Max\{0, m + h\xi^a - \frac{1}{\theta^a})\} =$$

$$Max\{0, m + \frac{\xi^a}{\xi}(\frac{1}{\theta^a} + \frac{1}{\theta^b}) - \frac{1}{\theta^a})\} = m - \frac{\xi^b}{\xi}\frac{1}{\theta^a} + \frac{\xi^a}{\xi\theta^b} \text{ Q.E.D}$$

Theorem 2

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$ Any solution to (1.13') for which prices are well defined is such that:

-at least for one s in S, prices are determined by h_4 or h_3 for m=0.

-at least for one s in S, prices are determined by h_4 or h_2 for m=1..

<u>Proof</u>

Suppose prices are, for $m = 0$, defined under h_2 for all s in S.

Then by Proposition A.1, $m' = M(0, \theta(\dot{s}, 0), \xi(\dot{s}))$ is equal to 0 for all s in S.

$$h_2(0,\theta^b(s,0),\xi^b(s)) = \frac{1}{\xi^b}(m+\frac{1}{\theta^b}) = \frac{1}{\xi^b(s)}.\frac{1}{\theta^b(s,0)} = h_2(m',\theta^b(s,m'),\xi^b(s))$$

Hence, applying Lemma 1', $\theta^a \leq \frac{1}{h}U_a'(\frac{m'}{h}+\xi^a)$ and $\theta^b \geq \frac{1}{h}U'_b(\frac{1-m'}{h}+\xi_b)$ .Hence

$$\Phi(\theta)(s.0) = \begin{bmatrix} \beta \int_S \frac{1}{h'} U_a'(\xi^a(s')) \quad \pi(s,ds') \\ \beta \int_S \theta^b(s',0') \quad \pi(s,ds') \end{bmatrix} = \begin{bmatrix} \beta \int_S \frac{\xi^b(s')\theta^b(s',0)}{\xi^a(s')} \quad \pi(s,ds') \\ \beta \int_S \theta^b(s',0') \quad \pi(s,ds') \end{bmatrix}$$

As Lagrangians are always positive or nil, $\theta^b(s,0)$ as well as $\theta^a(s,0)$ can only be nil for all s in S. This makes prices at m=0 undefined. Consequently, there must be some s in S such that h is determined under h_4 or h_3.

Symmetrically, if h is under h3 and m' is equal to 1,

$$h_3(m,\theta^a,\xi^a) = \frac{1}{\xi^a}(1-m+\frac{1}{\theta^a}) = \frac{1}{\xi^a}.\frac{1}{\theta^a}$$

Hence,

$$\Phi(\theta)(s.1) = \begin{bmatrix} \beta \int \theta^a(s',1) \quad \pi(s,ds') \\ \beta \int_S \xi^a(s')\theta^a(s',1) \, {}^{1}\!/_{\xi^b(s')} \pi(s,ds') \end{bmatrix}$$

and conclusions are identical.

If m' is equal to 0 (respectively 1) while h under h4, according to Proposition A.1 $m - \frac{\xi^b}{\xi}\frac{1}{\theta^a} + \frac{\xi^a}{\xi\theta^b}$ =0 which implies that h is also under h2 (respectively h3) and hence prices are again undefined. . Consequently, there must be some s in S such that h is determined under h_4 or h_2. Q.E.D

Proposition A.2

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$. Suppose that $\theta(s,m)$ is a Variant 2 solution to (1.13) such that cash constraints are always binding for one of the agents at least.

Then, for any given s in S,

either $h(m,\theta(s,m),\xi(s)) = h_2(m,\theta^b(s,m),\xi^b(s))$ for all m in [0,1],

or $h(m,\theta(s,m),\xi(s)) = h_3(m,\theta^a(s,m),\xi^a(s)$ for all m in [0,1].

Proof

Suppose that cash constraints are always binding for one of the agents at least. Then h is always under h2 or h3 and not under h4. Then, suppose that, for a given s in S, at least one point m_0 in]0,1[exists such that in the left (respectively right) neighbourhood of m_0:

$$h(m,\theta(s,m),\xi(s)) = h_2(m,\theta^b(s,m),\xi^b(s))$$

while in the right (respectively left) neighbourhood of a range m_0,

$$h(m,\theta(s,m),\xi(s)) = h_3(m,\theta^a(s,m),\xi^a(s)\,.$$

Then, by Proposition A.1, and in view of the continuity in m of $\theta(s,m)$ and the fact that the functions $h(.,.,.)$ can only take one value at a time,

$$h(m,\theta(s,m),\xi(s)) = \frac{1}{\xi^b(s)}(m+\frac{1}{\theta^b(s,m)}) = \frac{1}{\xi^a(s)}(1-m+\frac{1}{\theta^a(s,m)}) \text{ for } m_0 \text{ in } [0,1].$$

However, applying Proposition A.1 again, $M(m_0,\theta(s,m_0),\xi(s)) = 0$ and 1 which is a contradiction.

Hence, for a given s in S,

Either $h(m,\theta(s,m),\xi(s)) = h_2(m,\theta^b(s,m),\xi^b(s))$ for all m in [0,1],

Or $h(m,\theta(s,m),\xi(s)) = h_3(m,\theta^a(s,m),\xi^a(s)$ for all m in [0,1]. Q.E.D.

Proposition A.3

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$. Suppose also that $\pi(s,s')$ is a two-state probability matrix equal to $\begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$. Also, let us define $z_1 = \frac{\xi^a(1)}{\xi^b(1)}$ as well as $z_2 = \frac{\xi^b(2)}{\xi^a(2)}$ .and $k = \frac{0.5\beta}{1-0.5\beta}$.

If a Variant 2 solution to (1.13) exists and is such that cash constraints are always binding for one of the two agents, such a solution takes the following values:

$$\theta^b(1,m) = \frac{k^2 - z_1 z_2}{k(1+z_2)+(1+z_1)z_2} \, . \; \theta^a(2,m)) = \frac{k^2 - z_1 z_2}{k(1+z_1)+(1+z_2)z_1}$$

$$\theta^a(1,m) = 0.5\beta(k^2 - z_1 z_2)(\frac{1}{k(1+z_1)+(1+z_2)z_1} + \frac{1}{k(1+z_2)z_1+(1+z_1)z_2 z_1}) \text{ and}$$

$$\theta^b(2,m) = 0.5\beta(k^2 - z_1 z_2)(\frac{1}{k(1+z_2)+(1+z_1)z_2} + \frac{1}{k(1+z_1)z_2+(1+z_2)z_1 z_2})$$

Moreover, the two conditions below are simultaneously satisfied:

$$\frac{1}{\theta^a(1)}\frac{\xi^b(1)}{\xi(1)} - \frac{\xi^a(1)}{\xi(1)}\frac{1}{\theta^b(1)} \geq 1 \text{ and } \frac{\xi^a(2)}{\xi(2)}\frac{1}{\theta^b(2)} - \frac{1}{\theta^a(2)}\frac{\xi^b(2)}{\xi(2)} \geq 1$$

Proof

Let us apply Theorem 2 to the case of only two states in S. By assumption h is under h_4. Hence, there must be s_1 in S such that h is under h_2 for m=1 and s_2 in S such that h is under h_3 for m=0. Moreover, applying Proposition A.2, s_1 cannot be equal to s_2. Hence:

s_1 must be in S such that $h(m,\theta(s_1,m),\xi(s_1)) = h_2(m,\theta^b(s_1,m),\xi^b(s_1))$ for all m in [0,1], and

s_2 must be in S such that $h(m,\theta(s_2,m),\xi(s_2)) = h_3(m,\theta^a(s_2,m),\xi^a(s_2)$ for all m in [0,1].

Moreover, $M(m,\theta(s_1,m),\xi(s_1)) = 0$ and $M(m,\theta(s_2,m),\xi(s_2)) = 1$.

Let us then calculate $\frac{\xi^a(1))}{\xi^b(1)} / \frac{\xi^a(2)}{\xi^b(2)}$.

If $\frac{\xi^a(1))}{\xi^b(1)} / \frac{\xi^a(2)}{\xi^b(2)} \leq 1$, let us identify s_1 with 1 and s_2 with 2. If $\frac{\xi^a(1))}{\xi^b(1)} / \frac{\xi^a(2)}{\xi^b(2)} \succ 1$, let us call s_1 2 and s_2 1 so that in the end, $\frac{\xi^a(s_1)}{\xi^b(s_1)} / \frac{\xi^a(s_2)}{\xi^b(s_2)} \leq 1$,

Let us now calculate $\theta(s,m)$, the Variant 2 solution to (1.13) as

$$\Phi(\theta)(s.m) = \begin{bmatrix} \beta \int_S Max\{\theta^a(s',m'), \frac{1}{h'}U_a'(\frac{m'}{h'})\} \quad \pi(s,ds') \\ \beta \int_S Max\{\theta^b(s',m'), \frac{1}{h'}U_b'(\frac{1-m'}{h'})\} \quad \pi(s,ds') \end{bmatrix}$$

Applying Lemma 1', for s'=1, $h(m,\theta(s_1,m),\xi(s_1)) = h_2(m,\theta^b(s_1,m),\xi^b(s_1))$ implies

$\theta^a(1,m) \le \frac{1}{h}U_a'(\frac{m}{h}+\xi^a(1))$ and $\theta^b(1,m) \ge \frac{1}{h}U'_b(\frac{1-m}{h}+\xi_b(1))$ for all m in [0,1]

For s'=2, $h(m,\theta(s_2,m),\xi(s_2)) = h_3(m,\theta^a(s_2,m),\xi^a(s_2)$ implies

$\theta^a(2,m) \ge \frac{1}{h}U_a'(\frac{m}{h}+\xi^a(2))$ and $\theta^b(2,m) \le \frac{1}{h}U'_b(\frac{1-m}{h}+\xi_b(2))$ for all m in [0,1].

Moreover, $M(m,\theta(1,m),\xi(1)) = 0$ and $M(m,\theta(2,m),\xi(2)) = 1$. Hence,

$$\Phi(\theta)(1,m) = \begin{bmatrix} 0.5\beta\{\theta^a(2,0) + \dfrac{1}{h_2(0,\theta^b(1,0),\xi^b(1))}U_a'(\xi^a(1))) \\ 0.5\beta(\theta^b(1,0) + \dfrac{1}{h_3(0,\theta^a(2,0)\xi^a(2)}U_b'(\dfrac{1}{h_3(0,\theta^a(2,0)\xi^a(2))} + \xi^b(2))) \end{bmatrix}$$

$$\Phi(\theta)(2,m) = \begin{bmatrix} 0.5\beta\{\theta^a(2,1) + \dfrac{1}{h_2(1,\theta^b(1,,1\},\xi(1)}U_a'(\dfrac{1}{h_2(1,\theta^b(1,1),\xi(1))} + \xi^a(1))) \\ 0.5\beta(\theta^b(1,1) + \dfrac{1}{h_3(1,\theta^a(2,1),\xi^a(2))}U_b'(\xi^b(2))) \end{bmatrix}$$

<u>with</u>

$$h_2(0,\theta^b(1,0),\xi^b(1) = \frac{1}{\xi^b(1)}(\frac{1}{\theta^b(1,0)})$$

$$h_3(0,\theta^a(2,0),\xi^a(2) = \frac{1}{\xi^a(2)}(1+\frac{1}{\theta^a(2,0)})$$

$$h_3(1,\theta^a(2,1),\xi^a(2) = \frac{1}{\theta^a(2,1)\xi^a(2)} \qquad h_2(1,\theta^b(1,1),\xi^b(1) = \frac{1}{\xi^b(1)}(1+\frac{1}{\theta^b(1,1)})$$

Hence, replacing and simplifying,

$$\theta(1.m) = \begin{bmatrix} 0.5\beta(\theta^a(2,0) + \dfrac{\xi^b(1)}{\xi^a(1)}\theta^b(1,0)) \\ 0.5\beta(\theta^b(1,0) + \dfrac{1}{1+\dfrac{\xi^b(2)}{\xi^a(2)}\dfrac{\theta^a(2,1)+1}{\theta^a(2,1)}}) \end{bmatrix}$$

$$\theta(2.m)=\begin{bmatrix} 0.5\beta(\theta^a(2,1)+\dfrac{1}{1+\dfrac{\xi^a(1)(1+\theta^b(1,1))}{\xi^b(1)\theta^b(1,1)}}) \\ 0.5\beta(\theta^b(1,1)+\dfrac{\theta^a(2,1)\xi^a(2)}{\xi^b(2)}) \end{bmatrix}$$

Consequently, $\theta(s,m)=\theta(s)$ as θ does not depend on m. Hence,

$\theta^a(1)=0.5\beta(\theta^a(2)+\dfrac{\xi^b(1)}{\xi^a(1)}\theta^b(1))$ while $\theta^b(2)=0.5\beta(\theta^b(1)+\dfrac{\theta^a(2)\xi^a(2)}{\xi^b(2)})$.

Moreover,

$\theta^b(1)=0.5\beta(\theta^b(1)+\dfrac{1}{1+\dfrac{\xi^b(2)}{\xi^a(2)}\dfrac{\theta^a(2)+1}{\theta^a(2)}})$ and $\theta^a(2)=0.5\beta(\theta^a(2)+\dfrac{1}{1+\dfrac{\xi^a(1)(1+\theta^b(1)}{\xi^b(1)\theta^b(1)}})$

Hence, defining k as $k=\dfrac{0.5\beta}{1-0.5\beta}$ and $z_1=\dfrac{\xi^a(1)}{\xi^b(1)}$ as well as $z_2=\dfrac{\xi^b(2)}{\xi^a(2)}$.

$\theta^a(2)=\dfrac{k}{1+z_1+z_1\dfrac{1}{\theta^b(1)}}$ and $\theta^b(1)=\dfrac{k}{1+z_2+z_2\dfrac{1}{\theta^a(2)}}$ which implies that

$\theta^a(2)=\dfrac{k}{1+z_1+\dfrac{(1+z_2)z_1}{k}+\dfrac{z_1z_2}{k}\dfrac{1}{\theta^a(2)}})$ and $\theta^a(2)=\dfrac{k^2-z_1z_2}{k(1+z_1)+(1+z_2)z_1}$

Similarly, $\theta^b(1)=\dfrac{k^2-z_1z_2}{k(1+z_2)+(1+z_1)z_2}$. Consequently

$$\theta^a(1)=0.5\beta(k^2-z_1z_2)(\frac{1}{k(1+z_1)+(1+z_2)z_1}+\frac{1}{k(1+z_2)z_1+(1+z_1)z_2z_1})\text{ and}$$

$$\theta^b(2)=0.5\beta(k^2-z_1z_2)(\frac{1}{k(1+z_2)+(1+z_1)z_2}+\frac{1}{k(1+z_1)z_2+(1+z_2)z_1z_2}).$$

For this to be a solution, it is first necessary for θ to be positive or nil, i.e.

that $k^2\geq z_1z_2$ which implies: $\dfrac{0.5\beta}{1-0.5\beta}=k\ \geq\sqrt{z_1z_2}$ or $\beta\geq\dfrac{2\sqrt{\dfrac{\xi^a(1)}{\xi^b(1)}/\dfrac{\xi^a(2)}{\xi^b(2)}}}{1+\sqrt{\dfrac{\xi^a(1)}{\xi^b(1)}/\dfrac{\xi^a(2)}{\xi^b(2)}}})$.

Moreover, it has to be ensured that prices are never defined under h_4, i.e. that:

-for s= 1 and all m in [0,1], $\dfrac{1}{\xi(1)}(\dfrac{1}{\theta^a(1)}+\dfrac{1}{\theta^b(1)})\geq\dfrac{1}{\xi^b(1)}(m+\dfrac{1}{\theta^b(1)})$

-for s=2 and all m in [0,1], $\frac{1}{\xi(2)}(\frac{1}{\theta^a(2)}+\frac{1}{\theta^b(2)}) \geq \frac{1}{\xi^a(2)}(1-m+\frac{1}{\theta^a(2)})$

which implies:

1) $\frac{\xi^b(1)\theta^b(1)}{\xi^a(1)+\xi(1)\theta^b(1)} \geq \theta^a(1)$ and 2) $\frac{\theta^a(2)}{\theta^b(2)}) - (1+\frac{\xi^b(2)}{\xi^a(2)})\theta^a(2) \geq \frac{\xi^b(2)}{\xi^a(2)}$ .Q.E.D.

Theorem 3

Suppose that $U^i(.) = \log(.)$ for all $i \subset \{a,b\}$. Suppose also that $\pi(s,s')$ is a two-state probability matrix equal to $\begin{bmatrix} 0.5\ 0.5 \\ 0.5\ 0.5 \end{bmatrix}$. Suppose finally that the endowment matrix $\xi(.)$ is such that $\frac{\xi^a(1)}{\xi^b(1)} = \frac{\xi^b(2)}{\xi^a(2)}$=z.

Then a Variant 2 solution to (1.13) exists if

$\frac{2z}{1+z} \leq \beta < \beta_{Max} = \frac{-2z+\sqrt{(20z^2+16z)}}{2(1+z)}$ and is such that cash constraints are always binding for one of the two agents.

<u>Proof</u>

Applying Proposition A.3, $\theta^b(1,m) = \frac{k-z}{(1+z)}$. $= \theta^a(2,m))$

$$\theta^a(1,m) = 0.5\beta(k^2-z^2)(\frac{1}{(1+z)(k+z)}+\frac{1}{(1+z)(k+z)z})$$

$$= 0.5\beta(k^2-z^2)(\frac{1+z}{(1+z)(k+z)z}) = \frac{k(k-z)}{(k+1)z} = \theta^b(2,m).$$

Let us now check the two conditions put forward in Proposition A.3, $\frac{\xi^b(1)\theta^b(1)}{\xi^a(1)+\xi(1)\theta^b(1)} \geq \theta^a(1)$ can be written $\frac{z}{(1+z)} \geq k^2/(k+1) = \frac{\beta^2}{4-2\beta}$. This implies $(1+z)\beta^2 + 2\beta z - 4z \leq 0$ Roots of this polynomial are both positive and negative.

$$\text{Then } \frac{2z}{1+z} \le \beta < \beta_{Max} = \frac{-2z+\sqrt{20z^2}+16z)}{2(1+z)}$$

$\frac{\theta^a(2)}{\theta^b(2)}) - (1+\frac{\xi^b(2)}{\xi^a(2)})\theta^a(2) \ge \frac{\xi^b(2)}{\xi^a(2)}$ can be written $\frac{\theta^a(2)}{\theta^b(2)}) - (1+z)\theta^a(2) \ge z$, leading to the same condition. Q.E.D.

Annex 3

Various charts of interest to complement those in the paper and answer possible questions

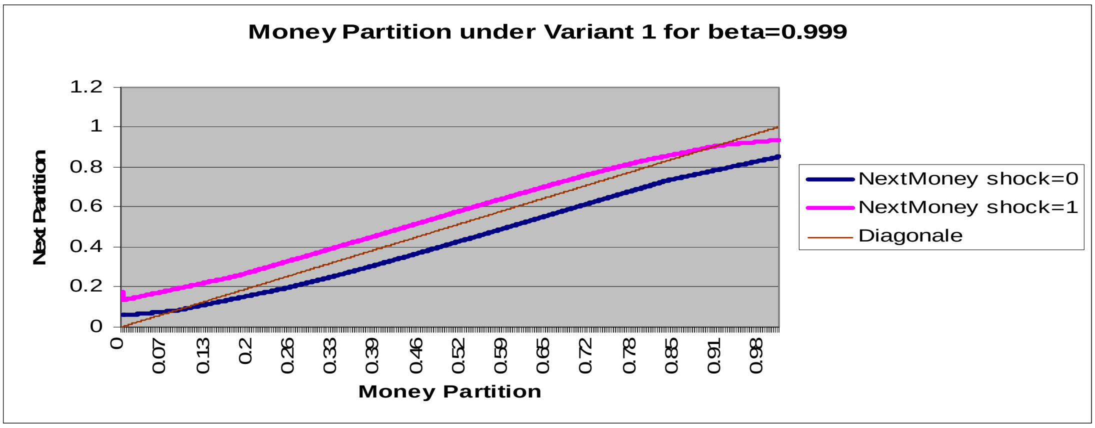


Chart A.1

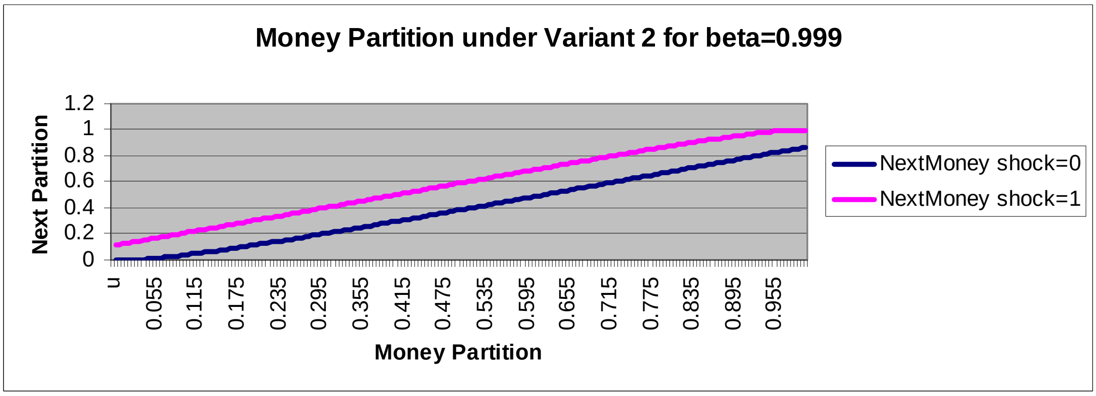


Chart A.2

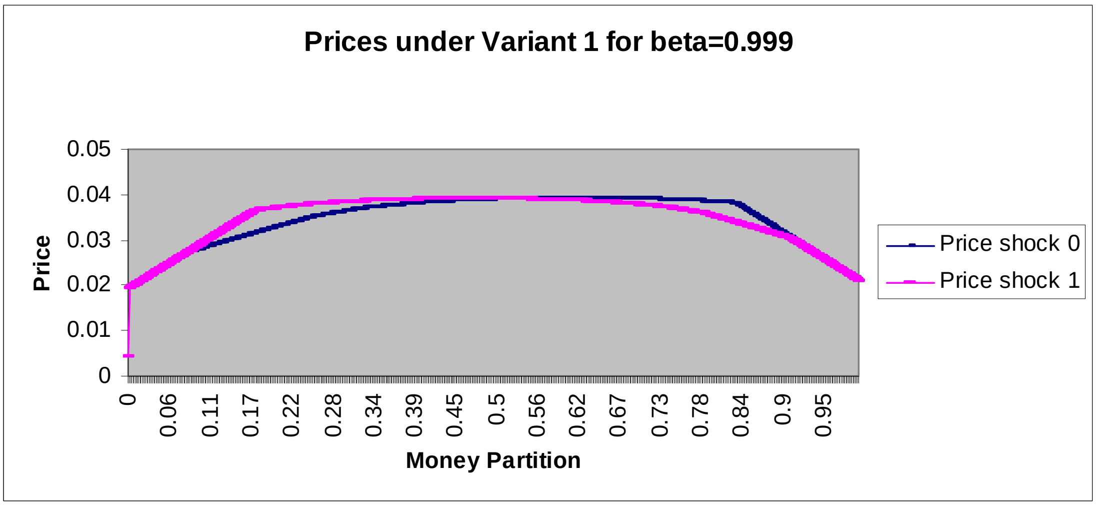


Chart A.3

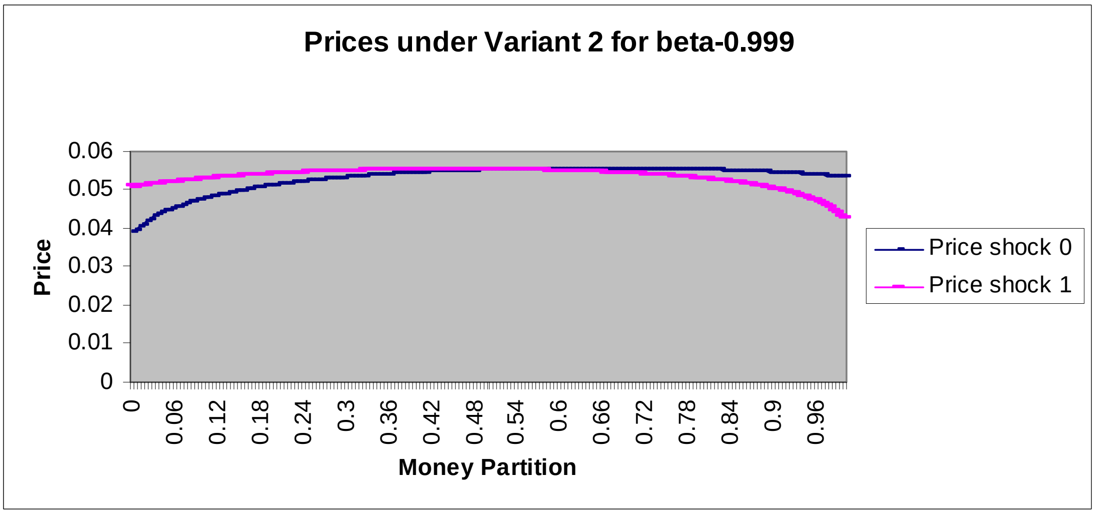


Chart A.4

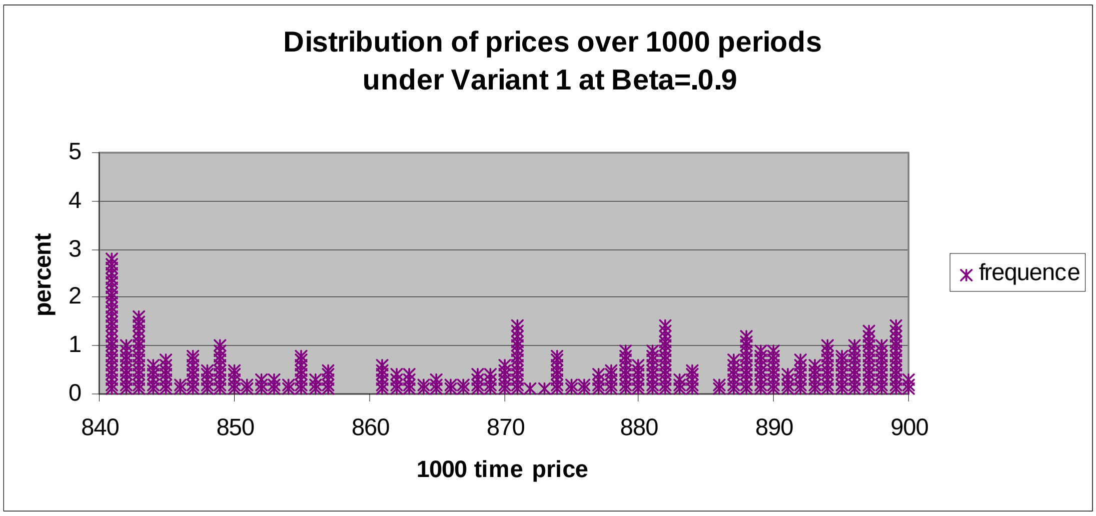


Chart A.5

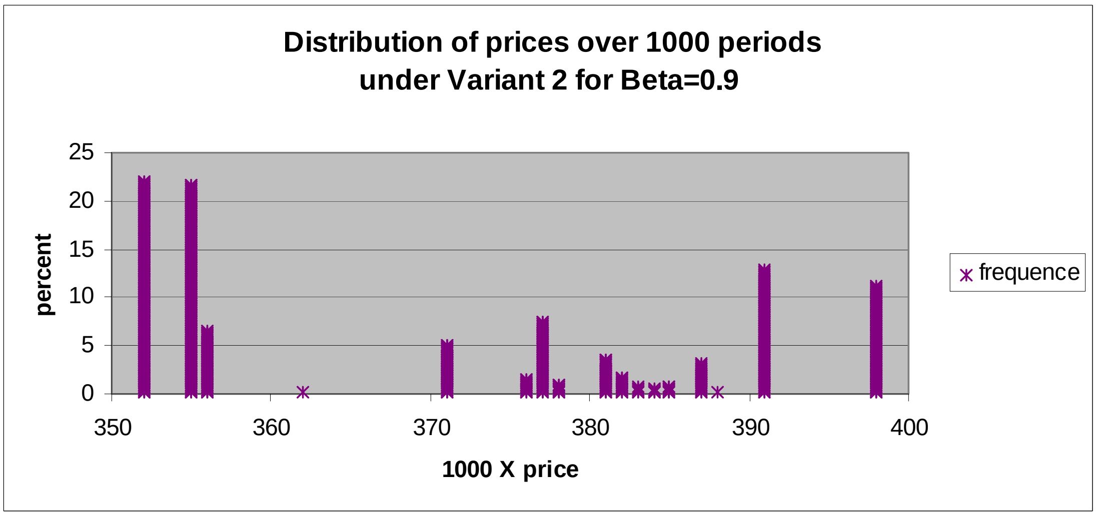


Chart A.6

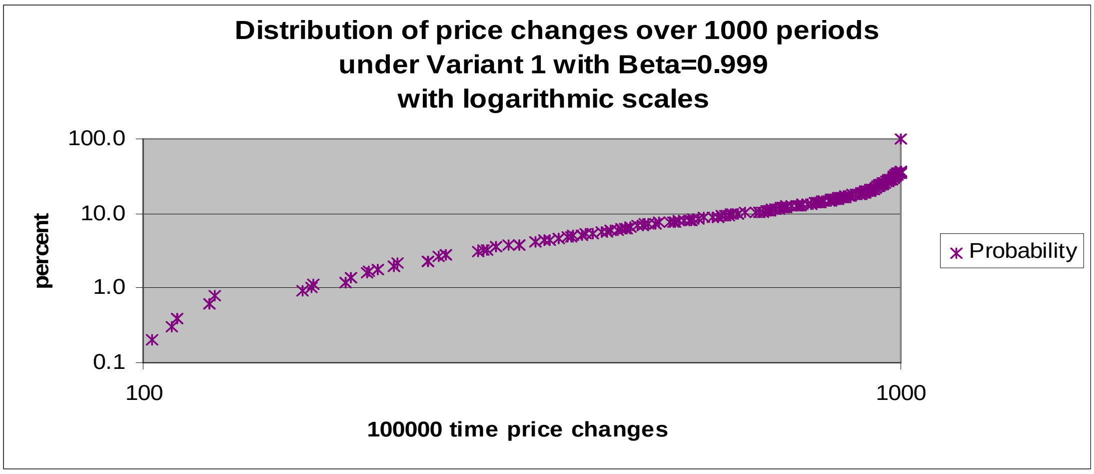


Chart A.7

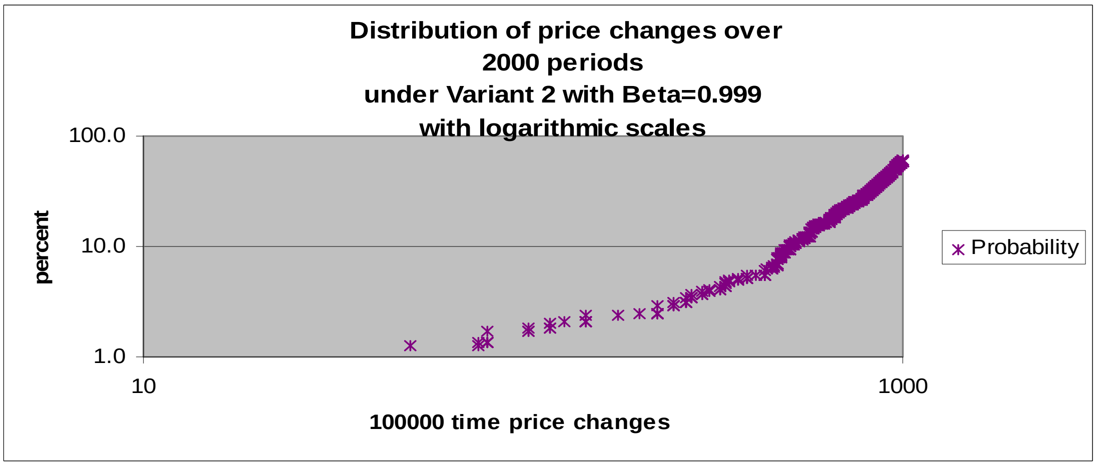


Chart A.8

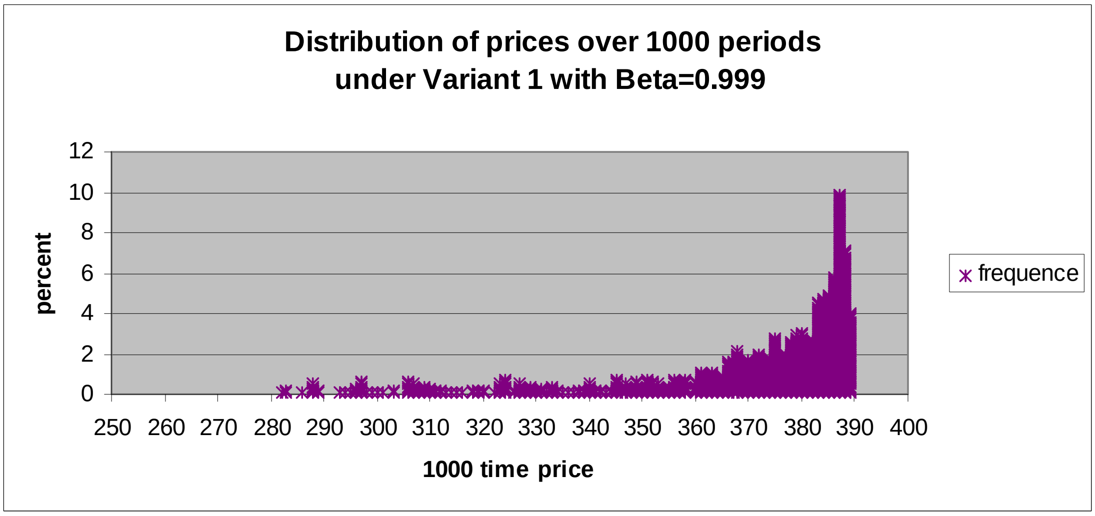


Chart A.9

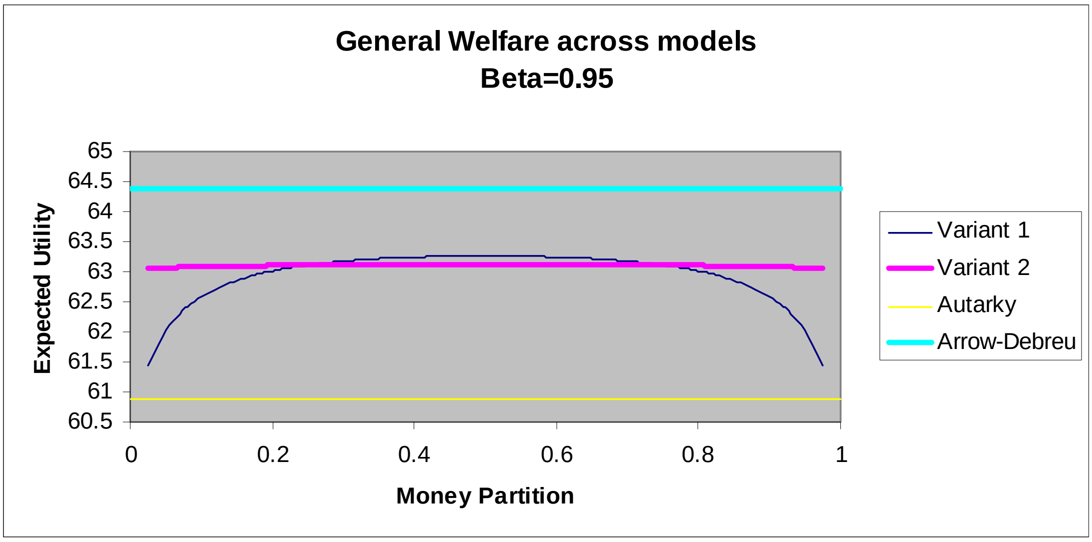


Chart A.10

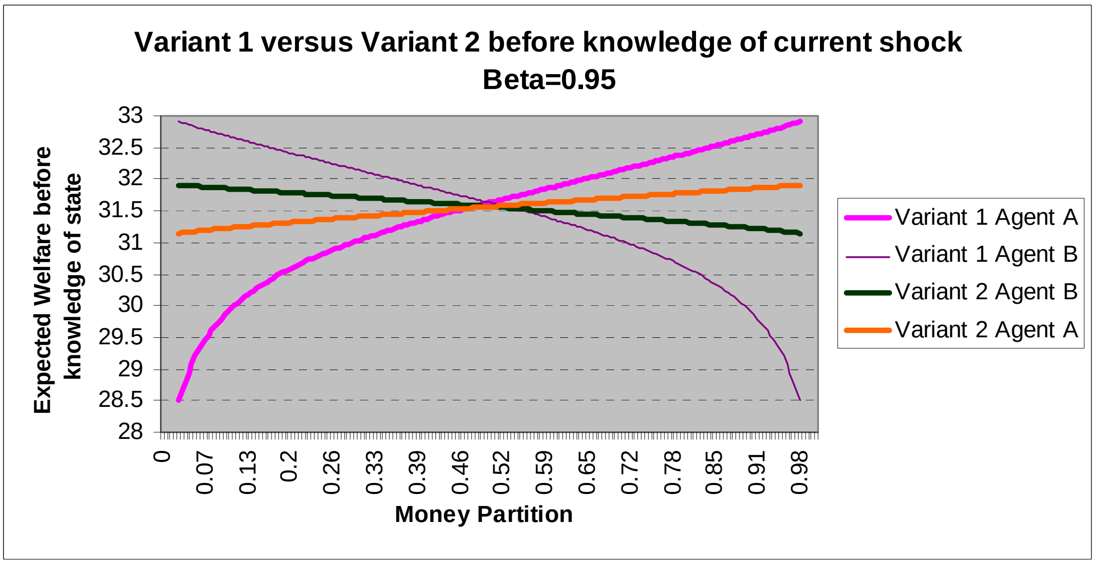
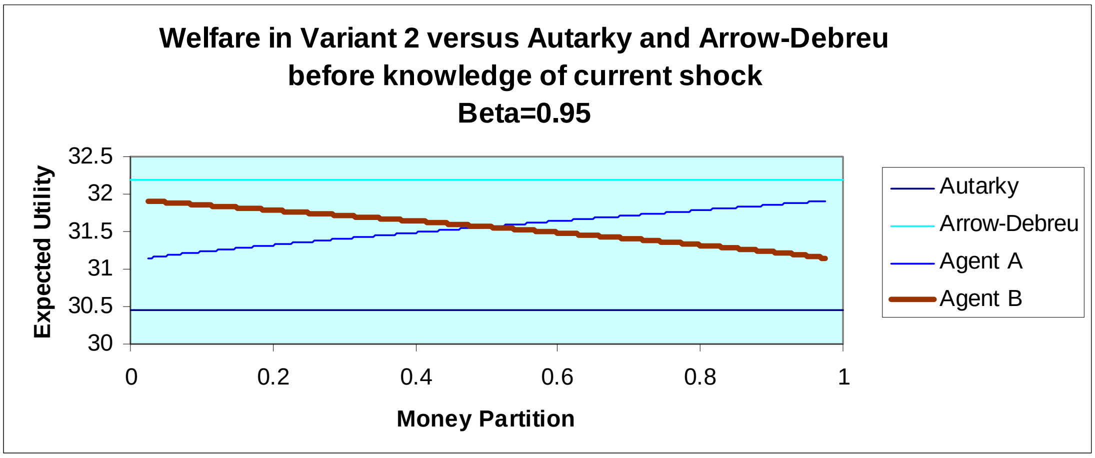


Chart A.11



Welfare in Variant 2 versus Autarky and Arrow-Debreu before knowledge of current shock
Beta=0.95
Expected Utility
32.5
32
31.5
31
30.5
30
0
0.2
0.4
0.6
0.8
1
Money Partition
Autarky
Arrow-Debreu
Agent A
Agent B

Chart A.12

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