Failure of Gradualism under Imperfect Monitoring*

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Abstract

I consider a two-player Prisoner's Dilemma type game with continuous actions, where players choose how much to contribute to a public project. This game is played infinitely many times and actions are irreversible: players cannot decrease their actions over time. While it is strictly dominant for players not to contribute in the stage game, some strictly positive level of contribution is Pareto optimal. It is known that when players perfectly observe each other's actions, cooperation can be achieved through gradual increases in contribution levels. I show that introducing an arbitrarily small amount of smooth noise in the monitoring makes cooperation impossible and players play the static Nash equilibrium of the stage-game forever.

Keywords: dynamic games, monotone games, public goods, voluntary contributions, gradualism, irreversibility, imperfect monitoring

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1 Introduction

In many economic settings, while cooperative outcomes are generally Pareto optimal, players often have incentives to free ride and benefit from the contributions of others without having to incur a private cost. This problem can usually be overcome through repeated interactions, which allow players to reward and punish each other over time. For example, in an infinitely repeated prisoner's dilemma, cooperation can be achieved with a simple grim-trigger strategy, which prescribes a return to the non-cooperative outcome if one player deviates from a given cooperative path.

While the ability that players have to punish each other over time is central to reaching cooperation, in various economic situations this ability is limited due to irreversibility or commitment constraints: after some degree of cooperation, threats to return to a non-cooperative outcome can no longer be made. For example, in a dynamic public good provision game, irreversibility arises when past contributions are not refundable: players cannot threaten to reduce their overall level of contributions to the public good as they cannot claim back past contributions. In this paper, I show that when player's cannot perfectly monitor each other's actions and there is irreversibility, cooperation can be impossible to achieve.

The main insight from the literature on irreversibility is that cooperation has to take the form of gradual increases in contribution levels over time. Marx and Matthews (2000) study a game of dynamic voluntary contribution to a public project where past contributions are not refundable and payoffs are linear in cumulative contributions, with a possible extra benefit when cumulative contributions are above a given threshold (the "completion point"). They construct an approximatively efficient subgame-perfect equilibrium when there is little discounting. Lockwood and Thomas (2002) study a similar setting, with no extra benefit, and characterize the efficient equilibria for any discount factor by the means of a second-order difference equation. Gale (2001) introduces the notion of monotone games with positive spillovers, a more general setting, and looks at the case without discounting. All those papers show that there can be cooperation when actions are irreversible, but that it has to take the form of gradual increases in contribution levels: because of irreversibility, the only threats that can be made are the reductions of future increases in the levels of contributions. This implies that cooperation has to be gradual. In a bargaining setting Compte and Jehiel (2004) show that, when players' outside options are history dependent and players have the option to terminate the game at any stage, equilibrium concessions will exhibit gradualism. The option that players have of terminating the game has the same role as the threat of discontinuing contributions to the public project.

Admati and Perry (1991) also study voluntary contributions to a public project and show that when past contributions are sunk players contribute gradually. The setting however is different as players move sequentially and do not enjoy intermediate flow payoffs. Instead, the benefit from the project is perceived only once the project is completed. Compte and Jehiel (2003) however show that when players value the project differently it can then be completed in only two stages.

In the papers reviewed, players always perfectly observe either individual contributions or total contributions. In particular players can condition their actions on the level of total contributions and detect any deviation from a given contribution path, possibly triggering a punishment phase. However, it is often the case that players cannot perfectly monitor each other's actions. Continuing with the public project example, players may only be able to observe the stage of development of the public project, which can be a noisy signal of total contributions. When players only observe a noisy signal instead of the actual actions of other players, cooperation might be more difficult to achieve.

In a repeated game with finite actions and signals, when signals are publicly observed and sufficiently informative, Fudenberg et al. (1994) show that cooperation can still be achieved in equilibrium, provided players are sufficiently patient, and establish a folk theorem for games with imperfect public monitoring. Cooperation is again possible with a continuum of actions and signals. Green and Porter (1984) and Porter (1983) study collusion in Cournot games with imperfect public monitoring where the action set (the quantity to produce) and the set of signals (the market price) are a continuum. They show that collusion is possible and Porter (1983) characterize the optimal collusive trigger strategy. Abreu et al. (1986) study optimal strategies in the Green and Porter (1984) model, without restricting attention to trigger strategies. However they depart from the Green and Porter (1984) model by restricting attention to a finite set of actions.

With a particular form of private monitoring ("network monitoring"), Wolitzky (2011) studies the level of cooperation that can be achieved in a repeated public good game where players perfectly observe the actions of their neighbors in a network but cannot observe the other players' level of cooperation.

The main question I address in this paper is whether cooperation can still be achieved when there is imperfect monitoring and the environment is non-stationary because actions are irreversible. I show that, under certain regularity assumptions about the payoff function and the monitoring technology, cooperation can no longer be achieved and players must play the unique stage-game Nash equilibrium for ever. This result is striking as it shows a stark discontinuity with the perfect monitoring case: the introduction of a little noise in the monitoring technology can render cooperation impossible.[1]

I consider a model in which a two-player Prisoner's Dilemma with continuous actions is played infinitely many times. In each period players choose

[1] In a recent paper Bonatti and Horner (2011) study inefficiencies that arise in teams in a dynamic moral hazard setting with incomplete information about the quality of the project. Players do not observe the actions of others in their team, but only observe whether and when the project succeeds. Under this particular monitoring structure they show that agents will under invest in effort.

a level of contribution (a number in $\mathbb{R}_+$). While it is strictly dominant not to contribute in the stage game, it is mutually beneficial to do so. Actions are irreversible, so contribution levels cannot decrease over time.[2] Crucially, players do not perfectly observe each other's actions. Instead in each period they receive a noisy signal of the action profile played. The signal is publicly observed by all players and drawn from a compact subset of $\mathbb{R}^k$ according to a known probability distribution. When the payoff function is continuously differentiable in actions, the monitoring technology has full support and is continuous in actions, I show that with irreversibility there can no longer be cooperation.[3]

Under perfect monitoring, with irreversibility, cooperation takes place in the form of gradual increases in contribution levels. At any point in time, the threat of maintaining contribution levels constant forever provides the necessary incentives to players to contribute today. That is, the losses from the withdrawal of future increases in contribution levels offset the instantaneous gain from a deviation.

However under imperfect monitoring this is no longer the case. If there are strictly positive contributions in equilibrium, it can first be shown that for a set of histories of positive measure, contributions will be arbitrarily close to an upper bound.[4] Close to this upper bound, a player will have an incentive to deviate by slightly reducing his contribution today and then resuming to the prescribed strategy tomorrow. The gain from this deviation is instantaneous and of a similar order of magnitude than the deviation.

[2]One interpretation of irreversibility is that the payoff-relevant variable is the stock of total contributions, which is irreversible as players can only contribute non-negative amounts in each period. See Lockwood and Thomas (2002, Section 4) for a discussion.

[3]I also show that the result holds for the "linear kinked" case, when the benefit from another player's contribution becomes null beyond a certain level of total contributions.

[4]This is because in order to provide incentives for players to increase their level of contributions today, contributions have to increase with strictly positive probability in the future. As contributions will be bounded from above in equilibrium, they must converge to a finite limit. Moreover this limit cannot be reached, as players would have an incentive to deviate just before reaching that limit.

The cost is two-folds: first, the deviation affects the distribution of signals. This effect is also of similar order than the deviation.[5] Secondly, given that signals are affected, the deviating player receives a lower continuation value. This loss is however arbitrarily small as contributions are arbitrarily close to an upper bound. The cost of a deviation no longer offsets the gain and cooperation in the form of gradual increases in contributions can no longer be sustained.

We can think of a number of examples of strategic situations in which cooperation is mutually desirable but myopic incentives are to defect and actions are irreversible or very costly to reverse. For example in an industry with a declining demand, competing firms might have a mutual interest in reducing their capacity, which can be considered irreversible. However in a one-shot game it is strictly dominant for firms not to reduce their capacity. Similarly, parties over-exploiting a common resource might mutually benefit from a destruction of their capital in order to reduce over-exploitation, even though in a one-shot interaction it is dominant for each party not to destroy capital. Other examples include environmental cooperation, where the installation of costly abatement technology is irreversible, and disarmament between warring parties.[6] In all these examples, it is possible that players might not perfectly observe each other's actions but only noisy signals of those actions.

The rest of the paper is organized as follows: in Section 2 I describe the model under the imperfect public monitoring framework; in Section 3 I characterize the unique *perfect public equilibrium* (PPE), in which players (almost) never contribute; Section 4 presents a counterexample where cooperation is possible, but where the monitoring technology is not continuous with respect to the players' actions; and Section 5 concludes.

[5]When the monitoring technology is smooth with respect to actions.

[6]Those examples are taken from Marx and Matthews (2000) and Lockwood and Thomas (2002).

2 The model under imperfect public monitoring

In this section I present the main features of the model under the assumption of imperfect public monitoring. I follow the model of Lockwood and Thomas (2002) and add the assumption that actions are not perfectly monitored. I start with the properties of the stage game, which has the structure of a Prisoner's Dilemma with continuous actions: Players choose a level of contribution (for example to a public project) in $\mathbb{R}_+$. While it is strictly dominant for players not to contribute in the stage game, players can both benefit from strictly positive levels of contributions. I then present the dynamic version of the model characterized by two main assumptions: actions are irreversible, so that the level of contributions has to be non-decreasing; and players do not perfectly observe each other's actions. Instead they observe a public noisy signal drawn from a know probability distribution on a compact subset of $\mathbb{R}^k$. I then describe the histories upon which players condition their actions and how they evaluate future streams of random payoffs.

2.1 The stage game

There are two players $i = 1, 2$.[7] Each player i chooses an action $c_i \in \mathbb{R}_+$, interpreted as his level of contribution to a public project. Both players simultaneously choose an action and the payoff to player i from the action profile $(c_1, c_2) \in \mathbb{R}_+^2$ is $\pi(c_i, c_j)$. It is assumed that π is continuously differentiable, decreasing in its first argument and increasing in its second argument. There exist contribution levels $c_1 > 0$ and $c_2 > 0$ such that it is mutually desirable for both players to reach those levels, providing the game with a Prisoner's Dilemma structure. Furthermore, the function $\pi(c_1, c_2) + \pi(c_2, c_1)$ is assumed to have a unique global maximizer on $\mathbb{R}_+^2$. Finally the marginal

[7] The main result of this paper can be generalized to the case of n players in a straightforward way.

cost of contributing is restricted to be bounded away from zero:

Assumption 1 (Smoothness). *The function* $(c_1, c_2) \mapsto \pi(c_1, c_2)$ *is continuously differentiable.*[8]

Assumption 2 (Prisoner's dilemma structure). *The function* π *is decreasing in its first argument and increasing in its second argument:* $\pi_1 \leq 0$ *and* $\pi_2 \geq 0$. *Moreover there exist* $c_1 > 0$ *and* $c_2 > 0$ *such that* $\pi(c_1, c_2) > \pi(0, 0)$ *and* $\pi(c_2, c_1) > \pi(0, 0)$.

Assumption 3 (Global maximizer). *The function* $(c_1, c_2) \mapsto \pi(c_1, c_2) + \pi(c_2, c_1)$ *has a unique global maximizer* (c_1^*, c_2^*) *on* $\mathbb{R}_+^2$, *such that* $\pi(c_1, c_2) + \pi(c_2, c_1)$ *is decreasing in* $c_1 + c_2$ *for* $c_1 + c_2 \geq c_1^* + c_2^*$.

Remark 1. Note that because $(c_1, c_2) \mapsto \pi(c_1, c_2) + \pi(c_2, c_1)$ is symmetric, if it has a unique maximizer then this maximizer is such that $c_1^* = c_2^* = c^*$.

Remark 2. The second part of Assumption 3 implies that the function $(c_1, c_2) \mapsto \pi(c_1, c_2) + \pi(c_2, c_1)$ is quasiconcave on the set $\{(c_1, c_2) \in \mathbb{R}_+^2 : c_1 + c_2 \geq c_1^* + c_2^*\}$. This assumption rules out the possibility that beyond the optimum, more contributions could be beneficial. In the example of environmental cooperation, it could be the case that even though players have over-invested, further contributions will produce a technological breakthrough which incremental benefit will outweigh the incremental cost.

This assumption can be dispensed with when focusing on symmetric equilibrium.

Assumption 4 (Strictly positive marginal cost). $\pi_1(x, y) > 0$ *for* $x > 0$.

2.2 The dynamic game and monitoring structure

The stage game is played infinitely many times. In each period $t = 0, 1, \ldots$ both players simultaneously choose an action $c_i^t \in \mathbb{R}_+$ that cannot be lower

[8] Note that this assumption excludes the linear kinked case considered by Lockwood and Thomas (2002). The result still holds in that case, and this is discussed in Remark 5.

than the action chosen in the previous period. One interpretation is that the payoff-relevant variable is the sum of all contributions and that increments in contributions have to be non-negative, so that the total level of contribution of each player will be non-decreasing:

Assumption 5 (Irreversibility)**.** $c_i^t \geq c_i^{t-1}, \quad i = 1, 2, \quad t \geq 1.$

There is a common discount factor $\delta \in (0, 1)$ and the players evaluate payoff streams using discounting. The sequence of action profiles $(c_i^t, c_j^t)_{t=0}^{\infty}$ generates a payoff for player i of:

$$(1 - \delta) \sum_{t=0}^{\infty} \delta^t \pi(c_i^t, c_j^t).$$

I consider a game with imperfect public monitoring: at the end of each period players only observe a public signal y drawn from a compact set $Y \subset \mathbb{R}^k$, $k \geq 1$. If $k = 1$ for example, the signal observed could be the sum of the players' contributions plus some random noise. If $k = 2$, the public signal could have a component for each of the players' actions. I assume that given a stage-game action profile (c_1, c_2), the distribution of signals has full support and is absolutely continuous with respect to the Lebesgue measure and has probability density function $f\,(y \mid c_1, c_2)$, where $\int_Y f\,(y \mid c_1, c_2)\, dy = 1$ for any $c_1,\, c_2 \geq 0$:

Assumption 6 (Monitoring technology)**.** *The distribution of signals conditional on action profiles is absolutely continuous with respect to the Lebesgue measure and has a probability density function f such that $f\,(y \mid c_1, c_2) > 0$, $\forall y \in Y$, $\forall c_1,\, c_2 \geq 0$.*

The monitoring technology is assumed to be Lipschitz continuous with respect to stage-game actions. This ensures that small changes in actions cannot lead to important changes in the distribution of public signals.

Assumption 7 (Lipschitz continuity)**.** *There exists a constant* K *such that* $|f(y \mid c_1 + \Delta, c_2) - f(y \mid c_1, c_2)| \leq K\Delta$ *and* $|f(y \mid c_1, c_2) - f(y \mid c_1, c_2 + \Delta)| \leq K\Delta$ *for any* $y \in Y$.

The public signal y is the only information each player has about his opponent's play. Therefore if player i receives payoffs at the end of each period they cannot convey additional information about the other player's action. Hence for a given ex ante payoff function (interpreted as an expected payoff) π and a given monitoring structure, the ex post or realized payoff function $\pi^* (c_i, y)$ is defined to satisfy the following identity:

$$\pi (c_i, c_j) = \int_Y \pi^* (c_i, y) f (y \mid c_i, c_j) \, dy, \quad \forall c_i, c_j.$$

Remark 3. Even though I treat actions in each period as being the level of total contributions, players in fact choose a non-negative increment to their level of contribution. In this setting, it may be natural to consider a public signal that does not depend on the total stock of contributions but on the increments in contributions.

However the context of the public project, the signal might reflect the stage of development of the project, which depends on total contributions. Similarly, the ex post payoff function, which depends on one's own action and the public signal, should reflect the benefit from the current state of the project, which again depends on total contributions.

Nonetheless the model could be restated in order to have a monitoring technology which depends on the size of the increments, and the result of this paper would still hold. The reason is that the order of magnitude of (6) would not change, provided that we make again the assumption of Lipschitz continuity of the monitoring technology, only this time with respect to the size of the increments.

2.3 Private and public histories

In each period, players only observe the public signal and the action they have played. They do not observe their payoffs. A private history for player i is a sequence of actions and signals $h_i^t = (c_i^0, y^0; c_i^1, y^1; \ldots; c_i^{t-1}, y^{t-1})$, and the set of all private histories for player i is $\mathcal{H}_i := \cup_{t\geq 0}(\mathbb{R}_+ \times Y)^t$, where $(\mathbb{R}_+ \times Y)^0 = \emptyset$. A public history h^t is a sequence of t public signals: $h^t = (y^0, y^1, \ldots, y^{t-1}) \in Y^t$. The set of all public histories is $\mathcal{H} := \cup_{t\geq 0} Y^t$. I will also use the notation $\mathcal{H}^t$ to denote the set of public histories of length t.

A pure public strategy σ_i for player i is a measurable function that specifies a level of contribution $\sigma_i(h^t)$ after any public history $h^t \in \mathcal{H}$ and that satisfies irreversibility:

$$\sigma_i : \begin{cases} \mathcal{H} & \longrightarrow \mathbb{R}_+ \\ h^t & \longmapsto \sigma_i(h^t) \end{cases},$$

such that for any $h^t \in \mathcal{H}$ and any $y \in Y$ we have $\sigma_i(h^t y) \geq \sigma_i(h^t)$, where $h^t y$ denotes the concatenation of histories h^t and y. For any strategy σ_i, player i's continuation strategy induced by $h^t \in \mathcal{H}^t$ is denoted by $\sigma_i|_{h^t}$ and $\sigma_i|_{h^t}(h) := \sigma_i(h^t h)$, $\forall h \in \mathcal{H}$.

Remark 4 (Pure vs. mixed strategies)**.** When focusing on pure strategies, there is no loss of generality in restricting attention to public strategies and looking at perfect public equilibrium as every pure strategy in a public monitoring game is realization equivalent to a public pure strategy.[9] However the set of equilibrium outcomes might be different when considering public mixed strategies and private mixed strategies. In this paper I focus on public pure strategies, although the result still holds when considering public mixed strategies. The proof however cannot be extended to the case of private mixed strategies and sequential equilibrium.[10]

[9] See for example Mailath and Samuelson (2006, Lemma 7.1.2).

[10] Note that the definition of sequential equilibrium is not clear when there is a continuum of actions.

2.4 Stochastic process of public signals

Let $\Omega := Y^{\mathbb{N}}$ be the space of infinite sequences of public signals. Along with a monitoring technology, a strategy profile $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ determines recursively a stochastic process of public signals, which induces a probability distribution on Ω that I denote by $\mathbb{P}_{\boldsymbol{\sigma}}$. Expectations with respect to that probability distribution will be denoted by $\mathbb{E}_{\boldsymbol{\sigma}}$.

An element $\omega \in \Omega$ is an infinite sequence of public signals and I denote by $h^t(\omega)$ the first t elements of ω. Let $V(\sigma_i, \sigma_j)$ be the expected payoff of player i from the strategy profile $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ and $V(\sigma_i, \sigma_j \mid h^\tau)$ be the continuation payoff from $\boldsymbol{\sigma}$ after the public history h^τ:[11]

$$V(\sigma_i, \sigma_j) := (1-\delta)\mathbb{E}_{\boldsymbol{\sigma}}\Big[\sum_{t=0}^{\infty} \delta^t \pi(\sigma_i(h^t(\omega)), \sigma_j(h^t(\omega)))\Big],$$

$$V(\sigma_i, \sigma_j \mid h^\tau) := (1-\delta)\mathbb{E}_{\boldsymbol{\sigma}}\Big[\sum_{t=0}^{\infty} \delta^t \pi(\sigma_i(h^{t+\tau}(\omega)), \sigma_i(h^{t+\tau}(\omega))) \mid h^\tau\Big].$$

A profile of public strategies (σ_1, σ_2) is a *perfect public equilibrium* (PPE) if for any $i \in \{1,2\}$, any strategy σ' and almost every public history $h^t \in \mathcal{H}$ we have that $V(\sigma_i, \sigma_j \mid h^t) \geq V(\sigma', \sigma_j \mid h^t)$.

2.5 Notation

For ease of readability I will use the following notations in the rest of the paper:

$$\pi(\sigma_i, \sigma_j \mid h^t) := \pi(\sigma_i(h^t), \sigma_j(h^t)),$$

and

$$f(y \mid \sigma_i, \sigma_j, h^t) := f(y \mid \sigma_i(h^t), \sigma_j(h^t)).$$

I also denote by $h^t|_\tau$, $1 \leq \tau \leq t-1$, the τ-truncation of a public history

[11] Even though any finite history occurs with probability zero because there is a continuum of signals, the probability conditional on a history h^τ is well defined as it is the probability measure induced by the continuation strategy profile $\boldsymbol{\sigma} \mid_{h^\tau}$.

h^t: If $h^t = (y^0, y^1, \ldots, y^{t-1})$ then $h^t|_\tau = (y^0, y^1, \ldots, y^{\tau-1})$.

3 Impossibility of Cooperation in Equilibrium

In this section I show that the only PPE of the dynamic game presented in Section 2 is to play (0,0) (no contribution) after almost every history: cooperation cannot be achieved with positive probability. In Section 4, Assumptions 6 and 7 are relaxed and it is shown that cooperation is again possible in equilibrium.

Theorem 1. *Let* $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ *be a* Perfect Public Equilibrium *of the dynamic contribution game with imperfect public monitoring. Then* $\sigma_i(h^t) = 0$*,* $i = 1, 2$*, for almost every public history* $h^t \in \mathcal{H}$*, under Assumptions 1 to 7.*

In Section 3.1 I provide a brief discussion of the proof. Section 3.2 introduces some preliminary results, while the formal proof is presented in Section 3.3.

3.1 Outline of the Proof

The main idea behind the proof of Theorem 1 is to consider, in each period, the essential supremum of a player's level of contribution. Contributions can only take values above the essential supremum on set of measure zero and there always exists a set of histories of positive measure for which contributions are arbitrarily close to the essential supremum.

It is first shown that at any point in time, there is a set of histories of positive measure for which contribution levels are close to their essential supremum (for one of the players) and for which contributions are expected to increase in the future. Because of irreversibility, the only way to provide incentives for players to contribute today is through the promise of future increases in contributions. No longer increasing contributions after histories for which contributions are close to the essential supremum would mean that along such histories there will be a last time where contributions increase, giving players an opportunity to profitably deviate. This result is analogous to Lockwood and Thomas (2002, Lemma 2.1 (ii)), who show that in an equilibrium with positive contributions, contributions cannot become constant after a certain time.

As the essential supremum of contributions converges (it is a sequence of increasing numbers bounded from above in equilibrium), we can consider a time after which it is arbitrarily close to its limit. The previous intuition now tells us that along histories for which contributions are close to the limit of the essential supremum, players are expected to increase contributions in the future. However as contributions are close to their upper bound, they cannot be expected to increase by a significative amount. To prove Theorem 1 I, then, show that along such histories a player can profitably deviate by increasing his contribution levels by less than what is prescribed in a putative equilibrium with positive contributions.

The gain from such a deviation consists of the instantaneous gain from reducing contribution levels. The cost is two-folds: first, a deviation will affect the distribution of signals; secondly, given that signals are affected, the deviating player will receive different (lower) continuation values.[12] However, when the monitoring technology is continuous with respect to players' actions, a small deviation will have an impact on the distribution of public signals of a similar order to the gain. Furthermore, as contributions of the other player are close to an upper bound, losses from lower future continuation values will be arbitrarily small. Hence the cost of deviating consists of one element that is of similar order to the gain and another element that is arbitrarily small, making the deviation profitable.

[12]Note that the use of the word "lower" might suggest that some assumption has been made about the informativeness of the signals (such as a monotone likelihood ratio assumption). The proof does not rely on any such assumption. However if the player expects higher continuation values after deviating then the proof becomes trivial.

3.2 Preliminary Results

In this section I present some general properties of the possible equilibria of the game described in Section 2.

First I show that after almost every history where contributions have increased, contributions are again expected to increase in the future. That is, the only force that provides incentives for players to increase their contributions today is the expectation of future increases in contributions from the other player.

Lemma 1. *In equilibrium, for almost every history* $h^t \in \mathcal{H}^t$ *such that* $\sigma_i(h^t) > \sigma_i(h^t|_{t-1})$, $i \in \{1,2\}$, *both players are expected to increase their levels of contributions in the future:*[13]

$$\mathbb{P}_{\boldsymbol{\sigma}}\left(\{h \in \mathcal{H} : \sigma_j(h^t h) > \sigma_j(h^t)\} \mid h^t\right) > 0, \quad j \in \{1,2\}.$$

Proof. Assume first that there is a set of histories of positive measure and length t such that $\sigma_i(h^t) > \sigma_i(h^t|_{t-1})$ but such that $\mathbb{P}_{\boldsymbol{\sigma}}(\{h \in \mathcal{H} : \sigma_j(h^t h) > \sigma_j(h^t)\} \mid h^t) = 0$.[14] Even though player i has increased his contribution, he does not expect player j to do so in the future. As contributing is strictly dominated, player i could profitably deviate by not increasing his contribution.

If player i increases his contribution, he then expects player j to also do so in the future. But if player j increases his contributions it is because he similarly expects player i to increase his contributions in the future. Hence we also have $\mathbb{P}_{\boldsymbol{\sigma}}(\{h \in \mathcal{H} : \sigma_i(h^t h) > \sigma_i(h^t)\} \mid h^t) > 0$. □

[13] Recall from footnote 11 that we can condition on finite histories even though any finite history has measure zero, as a continuation strategy induces a probability distribution on the set of continuation histories.

[14] Note that if a certain property holds for a set of histories of positive measure, then from the σ-additivity of $\mathbb{P}_{\boldsymbol{\sigma}}$ and because time is countable, there will be a certain t and a set of histories of positive measure and length t such that this property holds for those histories.

The next lemma shows that in equilibrium the value function of players is bounded from below by their current flow payoff. If this was not the case a player could always choose to maintain his contribution level constant. As payoffs are non-decreasing in the other player's contribution, which cannot decrease due to irreversibility, this would guarantee a future payoff of at least the current flow payoff.

Lemma 2. *If (σ_1, σ_2) is an equilibrium, then for almost every $y \in Y$ and almost every $h^t \in \mathcal{H}$ we have that $V(\sigma_i, \sigma_j \mid h^t y) \geq \pi(\sigma_i, \sigma_j \mid h^t)$. Furthermore $V(\sigma_i, \sigma_j \mid h^t) \geq \pi(\sigma_i, \sigma_j \mid h^t)$.*

Proof. Assume that there is an equilibrium where $V(\sigma_i, \sigma_j \mid h^t y) < \pi(\sigma_i, \sigma_j \mid h^t)$ for a set of histories of positive measure and a given t. Consider the strategy σ' for player i that coincides with σ_i except after a history h^t where $V(\sigma_i, \sigma_j \mid h^t y) < \pi(\sigma_i, \sigma_j \mid h^t)$, in which case player i stops contributing forever: $\sigma'(h^t h) = \sigma_i(h^t)$. Then for any $h \in \mathcal{H}$, $\pi(\sigma', \sigma_j \mid h^t h) \geq \pi(\sigma_i, \sigma_j \mid h^t)$. Hence $V(\sigma_i, \sigma_j \mid h^t y) \geq \pi(\sigma_i, \sigma_j \mid h^t)$, a contradiction.

Moreover,

$$\begin{aligned} V(\sigma_i, \sigma_j \mid h^t) &= (1-\delta)\pi(\sigma_i, \sigma_j \mid h^t) + \delta \mathbb{E}_{\boldsymbol{\sigma}}\Big[V(\sigma_i, \sigma_j \mid h^t y) \mid h^t\Big] \\ &\geq (1-\delta)\pi(\sigma_i, \sigma_j \mid h^t) + \delta \pi(\sigma_i, \sigma_j \mid h^t) \\ &= \pi(\sigma_i, \sigma_j \mid h^t), \end{aligned}$$

which completes the proof. □

In the next lemma I show that in equilibrium players' contributions are bounded from above.

Lemma 3. *In equilibrium, for almost every history $h \in \mathcal{H}$ we have $\sigma_1(h) + \sigma_2(h) < 2c^*$.*

Proof. Assume that this is not the case and there is a set of histories of positive measure and length t such that $\sigma_1(h^t) + \sigma_2(h^t) \geq 2c^*$. The sum of the players' value functions, given h^t, is a discounted sum of terms $\pi(\sigma_1, \sigma_2 \mid h^t h) + \pi(\sigma_2, \sigma_1 \mid h^t h)$, $h \in \mathcal{H}$. I consider in turn two cases.

First assume that players increase their contributions with positive probability after such histories. As $\pi(c_1, c_2)+\pi(c_2, c_1)$ is above its global maximum (Assumption 3) it is decreasing in c_1+c_2. Therefore $\pi(\sigma_1, \sigma_2 \mid h^t h)+\pi(\sigma_2, \sigma_1 \mid h^t h) \leq \pi(\sigma_1, \sigma_2 \mid h^t) + \pi(\sigma_2, \sigma_1 \mid h^t)$ $\forall h \in \mathcal{H}$, where the inequality is strict on a set of histories of positive measure. This implies that $V(\sigma_i, \sigma_j \mid h^t) < \pi(\sigma_i, \sigma_j \mid h^t)$ for at least one $i \in \{1, 2\}$, a contradiction with Lemma 2.

Secondly if instead contributions remain constant then for each h^t such that $\sigma_1(h^t) + \sigma_2(h^t) \geq 2c^*$ there is a time s, $0 \leq s < t$, at which one of the players makes a last increase, at which point he has an incentive to deviate and not perform that last increase. □

Lemma 3 implies that in equilibrium each player will not contribute above $2c^*$. I define this upper bound as $\bar{c}$: $\bar{c} := 2c^*$.

3.3 Proof of Theorem 1

Assume that there is a pure-strategy equilibrium $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ where players make strictly positive contributions. Without loss of generality, in equilibrium, at least one of the players contribute in period 0.[15] Suppose $\sigma_1(\emptyset) > 0$.

In what follows I consider the *essential supremum* of contribution levels, which is the supremum except on a set of measure zero and is defined as follows:[16]

$$\operatorname{ess\,sup}_{h^t \in \mathcal{H}^t} \sigma_i(h^t) := \inf \Big\{ a \in \mathbb{R} : \mathbb{P}_{\boldsymbol{\sigma}}(\{h^t \in \mathcal{H}^t : \sigma_i(h^t) > a\}) = 0 \Big\}.$$

Let $(\widehat{\sigma}_{it})_{t=0}^{\infty}$, $i = 1, 2$, be the deterministic sequence of a player's essential

[15]Consider a history h^t at which the first increase in contribution levels occurs. Then $(\sigma_1|_{h^t}, \sigma_2|_{h^t})$ is also an equilibrium of the original dynamic game.

[16]For a brief discussion of the essential supremum, see for example Doob (1994, Part V.17).

supremum contribution:

$$\widehat{\sigma}_{it} = \operatorname{ess\,sup}_{h^t \in \mathcal{H}^t} \sigma_i(h^t), \quad \forall t \geq 0,\, i = 1, 2. \tag{1}$$

By definition there always exists a set of histories of positive measure for which contributions are close to the essential supremum. I denote by $\mathcal{H}^t_{i\epsilon}$ the set of histories of length t for which contributions of player i are within ϵ of the essential supremum:

$$\mathcal{H}^t_{i\epsilon} := \left\{ h^t \in \mathcal{H}^t : |\hat{\sigma}_{it} - \sigma_i(h^t)| \leq \epsilon \right\} \quad \forall t,\, i = 1, 2. \tag{2}$$

Note that $\mathbb{P}_{\boldsymbol{\sigma}}(\mathcal{H}^t_{i\epsilon}) > 0$ and that the set of histories for which contributions are strictly higher than the essential supremum is of measure zero.

Lemma 4. *In equilibrium, the deterministic sequence $(\hat{\sigma}_{it})_t$ converges to a limit $\hat{\sigma}_{i\infty} < \infty$, $i = 1, 2$.*

Proof. The sequence $(\hat{\sigma}_{it})_t$ is weakly increasing, because of irreversibility, and bounded from above by $\bar{c}$. It therefore converges. □

In the next lemma I show that for any time t there is a set of histories of positive measure for which contributions are close to the essential supremum *and* will continue to increase in the future.

Lemma 5. *In an equilibrium where players contribute with positive probability, for any $i \in \{1, 2\}$, $\epsilon > 0$ and t, there is a set of histories $\widetilde{\mathcal{H}}^t_{i\epsilon}$ of length t and of positive measure such that:*

(i) Contributions of player i are close to their essential supremum: $|\sigma_i(h^t) - \widehat{\sigma}_{it}| \leq \epsilon$, $\forall h^t \in \widetilde{\mathcal{H}}^t_{i\epsilon}$; and

(ii) Player i increases his contribution with strictly positive probability in the future:

$$\mathbb{P}_{\boldsymbol{\sigma}} \left(\{ h \in \mathcal{H} : \sigma_i(h^t h) > \sigma_i(h^t) \} \mid h^t \right) > 0, \quad \forall h^t \in \widetilde{\mathcal{H}}^t_{i\epsilon},$$

where $\widehat{\sigma}_{it}$ is the essential supremum of player i's contribution in period t, as defined in (1).

Proof. Assume that the result does not hold and consider the set $\mathcal{H}^t_{i\epsilon}$ of histories of length t for which player i's contribution is within ϵ of his essential supremum contribution, as defined in (2): $|\sigma_i(h^t)-\widehat{\sigma}_{it}| \leq \epsilon$, $\forall h^t \in \mathcal{H}^t_{i\epsilon}$. Recall that $\mathbb{P}_{\boldsymbol{\sigma}}(\mathcal{H}^t_{i\epsilon}) > 0$. If Lemma 5 does not hold then there exists a time t and an $\epsilon > 0$ such that $\mathbb{P}_{\boldsymbol{\sigma}}\left(\{h \in \mathcal{H} : \sigma_i(h^t h) > \sigma_i(h^t)\} \mid h^t\right) = 0$, for almost every history in $\mathcal{H}^t_{i\epsilon}$. Without loss of generality I assume that $\widehat{\sigma}_{it} > 0$ as players make strictly positive contributions in equilibrium. I now consider in turn two cases.

First assume that for almost every $h^t \in \mathcal{H}^t_{i\epsilon}$ we have $\sigma_i(h^t) = \sigma_i(\emptyset)$. From irreversibility, for every $h^t \in \mathcal{H}^t$, $\sigma(h^t) \geq \sigma(\emptyset)$. Therefore $\forall h^t \in \mathcal{H}^t$ we have $|\sigma_i(h^t) - \widehat{\sigma}_{it}| \leq \epsilon$, so that $\mathcal{H}^t \subseteq \mathcal{H}^t_{i\epsilon}$. As $\mathcal{H}^t_{i\epsilon} \subseteq \mathcal{H}^t$ we have $\mathcal{H}^t_{i\epsilon} = \mathcal{H}^t$: Player i keeps his contribution level constant at $\sigma_i(\emptyset)$ with probability one. As player i never increases his contribution, player j will best respond by also maintaining his contribution constant throughout the game. But then player i has an incentive to deviate and keep his contribution constant at zero in the first period, a contradiction.

Assume now that there is a positive measure subset of $\mathcal{H}^t_{i\epsilon}$ such that for histories in that subset $\sigma_i(h^t) > \sigma_i(\emptyset)$. Then for each of such history there is a time $s < t$ such that $\sigma_i(h^t|_s) < \sigma_i(h^t|_{s+1}) = \ldots = \sigma_i(h^t|_{t-1}) = \sigma_i(h^t)$. Player i makes his last increase along history h^t at time s, but would then have an incentive to deviate and not perform that last increase, again a contradiction. □

The following corollary ensues:

Corollary 1. *In any equilibrium where players contribute with positive probability, for any $i \in \{1, 2\}$ and any $\epsilon > 0$, there exists a finite t and a set of histories $\mathcal{H}^{t\star}_{i\epsilon}$ of positive measure such that:*

(i) $|\sigma_i(h^t) - \widehat{\sigma}_{i\infty}| \leq \epsilon$, $\forall h^t \in \mathcal{H}^{t\star}_{i\epsilon}$ *; and*

(ii) *Player $j \neq i$ increases his contribution at h^t: $\sigma_j(h^t) > \sigma_j(h^t|_{t-1})$, $\forall h^t \in \mathcal{H}^{t\star}_{i\epsilon}$.*

Proof. Consider a time t' such that $|\widehat{\sigma}_{it'} - \widehat{\sigma}_{i\infty}| \leq \epsilon/2$ and the set $\widetilde{\mathcal{H}}^{t'}_{i\epsilon/2}$ introduced in Lemma 5. We know that this set has positive measure, and that for any history in that set $|\sigma_i(h^{t'}) - \widehat{\sigma}_{it'}| \leq \epsilon/2$. As t' is such that $|\widehat{\sigma}_{it'} - \widehat{\sigma}_{i\infty}| \leq \epsilon/2$, by the triangle inequality we have that $|\sigma_i(h^{t'}) - \widehat{\sigma}_{i\infty}| \leq \epsilon$, $\forall h^{t'} \in \widetilde{\mathcal{H}}^{t'}_{i\epsilon/2}$. Lemma 5 tells us that player i increases his contribution with positive probability after histories in $\widetilde{\mathcal{H}}^{t'}_{i\epsilon/2}$. By Lemma 1 this also implies that player j will increase his contribution with positive probability at a time $t \geq t'$. □

I now complete the proof of Theorem 1 by showing that it is profitable to deviate for (say) player 1 after a history in $\mathcal{H}^{t\star}_{2\epsilon}$, where player 1's strategy specify an increase in contribution levels while player 2's contribution level is ϵ-close to its upper bound:

$$\mathcal{H}^{t\star}_{2\epsilon} := \Big\{h^t \in \mathcal{H}^t : |\sigma_2(h^t) - \widehat{\sigma}_{2\infty}| \leq \epsilon \text{ and } \sigma_1(h^t) > \sigma_1(h^t|_{t-1})\Big\}.$$

To do so I consider a deviation σ' from σ_1 which prescribes lower increases in contribution levels after histories in $\mathcal{H}^{t\star}_{2\epsilon}$ but agrees with σ_1 otherwise:[17]

$$\sigma'(h^t) = \begin{cases} \sigma_1(h^t) - \nu(h^t) & \text{if } h^t \in \mathcal{H}^{t\star}_{2\epsilon}, \\ \sigma_1(h^t) & \text{otherwise.} \end{cases}$$

To show that $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ cannot be an equilibrium, I will show that player 1 can profitably deviate to σ' after any history in $\mathcal{H}^{t\star}_{2\epsilon}$:

$$V(\sigma', \sigma_2 \mid h^t) - V(\sigma_1, \sigma_2 \mid h^t) > 0, \quad \forall h^t \in \mathcal{H}^{t\star}_{2\epsilon}, \tag{3}$$

[17]For example we could take $\nu(h^t) = \frac{\sigma_1(h^t) - \sigma_1(h^t|_{t-1})}{2} > 0$.

where

$$V(\sigma, \sigma_2 \mid h^t) = (1-\delta)\pi(\sigma, \sigma_2 \mid h^t) + \delta \int_Y V(\sigma_1, \sigma_2 \mid h^t y) f(y \mid \sigma, \sigma_2, h^t) dy,$$

$\sigma \in \{\sigma_1, \sigma'\}$.

Note that for any $y_0 \in Y$ we have that

$$\int_Y \Big[f(y \mid \sigma_1, \sigma_2, h^t) - f(y \mid \sigma', \sigma_2, h^t) \Big] V(\sigma_1, \sigma_2 \mid h^t y) dy = \\ \int_Y \Big[f(y \mid \sigma_1, \sigma_2, h^t) - f(y \mid \sigma', \sigma_2, h^t) \Big] \Big[V(\sigma_1, \sigma_2 \mid h^t y) - V(\sigma_1, \sigma_2 \mid h^t y_0) \Big] dy,$$

as

$$\int_Y \Big[f(y \mid \sigma_1, \sigma_2, h^t) - f(y \mid \sigma', \sigma_2, h^t) \Big] V(\sigma_1, \sigma_2 \mid h^t y_0) dy = 0.$$

A deviation to σ' is profitable for player 1 if (3) holds, which can be rewritten as:

$$\int_Y \Big[f(y \mid \sigma_1, \sigma_2, h^t) - f(y \mid \sigma', \sigma_2, h^t) \Big] \Big[V(\sigma_1, \sigma_2 \mid h^t y) - V(\sigma_1, \sigma_2 \mid h^t y_0) \Big] dy < \\ \frac{1-\delta}{\delta} \Big[\pi(\sigma', \sigma_2 \mid h^t) - \pi(\sigma_1, \sigma_2 \mid h^t) \Big], \quad y_0 \in Y. \quad (4)$$

I now look for an upper bound of the left-hand side of (4). From Lemma 2 we know that the next period's value function is bounded from below by current flow payoffs. Hence we have the following inequality:

$$V(\sigma_1, \sigma_2 \mid h^t y) - V(\sigma_1, \sigma_2 \mid h^t y_0) \leq \pi(\sigma_1(h^t), \widehat{\sigma}_{2\infty}) - \pi(\sigma_1(h^t), \sigma_2(h^t)), \quad \forall y \in Y. \quad (5)$$

Using the Mean Value Theorem (recall from Assumption 1 that π is continuous and differentiable), there exists a $\widetilde{\sigma}_2 \in (\sigma_2(h^t), \widehat{\sigma}_{2\infty}) \subseteq \mathbb{R}_+$ such that $\pi(\sigma_1(h^t), \widehat{\sigma}_{2\infty}) - \pi(\sigma_1(h^t), \sigma_2(h^t)) = (\widehat{\sigma}_{2\infty} - \sigma_2(h^t))\pi_2(\sigma_1(h^t), \widetilde{\sigma}_2)$. As $h^t \in \mathcal{H}^{t\star}_{2\epsilon}$, we have that $\widehat{\sigma}_{2\infty} - \sigma_2(h^t) \leq \epsilon$. Moreover there is an upper bound λ on $\pi_2(\sigma_1(h^t), \widetilde{\sigma}_2)$ that is independent of h^t, so that $\pi(\sigma_1(h^t), \widehat{\sigma}_{2\infty}) - \pi(\sigma_1(h^t), \sigma_2(h^t)) \leq \lambda\epsilon$. This is because π_2 is a continuous function (Assumption 1) that, in equilibrium, takes values on the compact set $[0, \bar{c}] \times [0, \bar{c}]$ (Lemma 3). Combining this with (5), we have the following upper bound for left-hand side of (4):

$$\lambda\epsilon \int_Y \Big| f(y \mid \sigma_1, \sigma_2, h^t) - f(y \mid \sigma', \sigma_2, h^t) \Big| dy. \tag{6}$$

From Lipschitz continuity (Assumption 7), the integral in (6) is bounded from above by $\lambda(Y)K(\sigma_1(h^t) - \sigma'(h^t))$, where $\lambda(Y)$ is the Lebesgue measure of the set Y. Let $\mu := \lambda(Y)K$. We can now bound the left-hand side of (4) from above (in absolute value) by $\mu\lambda\epsilon(\sigma_1(h^t) - \sigma'(h^t))$. Hence, (4) holds if

$$\mu\lambda\epsilon(\sigma_1(h^t) - \sigma'(h^t)) < \frac{1-\delta}{\delta}\Big[\pi(\sigma', \sigma_2 \mid h^t) - \pi(\sigma_1, \sigma_2 \mid h^t)\Big].$$

Again by the Mean Value Theorem there exists a $\widetilde{\sigma}$ in $(\sigma'(h^t), \sigma_1(h^t)) \subseteq \mathbb{R}_+$ such that

$$\pi(\sigma', \sigma_2 \mid h^t) - \pi(\sigma_1, \sigma_2 \mid h^t) = -\pi_1(\widetilde{\sigma}, \sigma_2(h^t))(\sigma_1(h^t) - \sigma'(h^t)),$$

so that (4) holds if

$$\mu\lambda\epsilon < -\frac{1-\delta}{\delta}\pi_1(\widetilde{\sigma}, \sigma_2(h^t)). \tag{7}$$

The right-hand side of (7) is positive as π is decreasing in its first argument. Moreover it is bounded away from zero, independently of h^t. This is because π_1 is continuous and $(\widetilde{\sigma}, \sigma_2(h^t))$ take values on the compact set $[\sigma_1(\emptyset), \bar{c}] \times [0, \bar{c}]$, on which it is strictly positve, as $\sigma_1(\emptyset) > 0$ (Assumption 4).

For ϵ small enough (7) therefore holds, which ensures that (4) holds and that $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ cannot be an equilibrium, as player 1 has a profitable deviation on the set of histories in $\mathcal{H}^{t\star}_{2\epsilon}$. This concludes the proof of Theorem 1, and the only PPE is when players do not cooperate and contribution levels remain constant at zero.

Remark 5 (The linear kinked case)**.** The smoothness assumption (Assumption 1) excludes the "linear kinked" case discussed in Lockwood and Thomas (2002), where payoffs are as follows:

$$\pi(c_1, c_2) = \begin{cases} \pi_1 c_1 + \pi_2 c_2 & \text{if } c_1 + c_2 \leq 2c^*, \\ \pi_1 c_1 + \pi_2(2c^* - c_1) & \text{if } c_1 + c_2 > 2c^*, \end{cases}$$

where $\pi_1 < 0$, $\pi_2 > 0$ and $\pi_1 + \pi_2 > 0$. Under the linear kinked case most of the proof still holds, with the following simplifications: the right-hand side of (4) becomes $\frac{1-\delta}{\delta}\Big[\pi(\sigma', \sigma_2 \mid h^t) - \pi(\sigma_1, \sigma_2 \mid h^t)\Big] = \frac{1-\delta}{\delta}\pi_1\Big[\sigma'(h^t) - \sigma_1(h^t)\Big]$; and the right-hand side of (5) becomes $\pi(\sigma_1(h^t), \widehat{\sigma}_{2\infty}) - \pi(\sigma_1(h^t), \sigma_2(h^t)) = \pi_2(\widehat{\sigma}_{2\infty} - \sigma_2(h^t))$.[18] Therefore (4) holds if

$$\mu \pi_2 \epsilon < -\frac{1-\delta}{\delta}\pi_1,$$

which is satisfied for ϵ sufficiently small, as μ, π_1, π_2 and δ are constants.

4 A Counterexample: All-or-Nothing Monitoring

In this section I consider an example the monitoring technology is no longer absolutely continuous with respect to the Lebesgue measure and no longer smooth in the actions of players: in each period, both players observe each other's actions with probability $1 - \epsilon$, or observe the outcome of a random variable uniformly distributed on $[0, K]$, $K > 0$, with probability ϵ. Assumptions 6 and 7 are therefore violated, although the monitoring technology still has full support. I refer to this setting as the ϵ-*almost perfect monitoring game* and will show that cooperation can again be achieved.[19]

[18]Note that in equilibirum Lemma 3 tells us that $\sigma_1(h^t) + \sigma_2(h^t) < 2c^*$, so that $\pi(\sigma_1(h^t), \sigma_2(h^t)) = \pi_1\sigma_1(h^t) + \pi_2\sigma_2(h^t)$.

[19]This monitoring technology is not exactly "all-or-nothing" as with probability ϵ an uninformative signal is observed. The result of this section still holds if with probability

Let us consider a possible contribution path $\boldsymbol{c} := (c_0, c_1, \ldots)$ and the following strategy profile $\boldsymbol{\sigma_c}$: first, both players play c_0 in period 0. Then in period t both players play c_t if c_{t-1} was observed in period $t-1$. Otherwise players do not increase their levels of contributions from the previous period.

The strategy profile $\boldsymbol{\sigma_c}$ prescribes that players keep their levels of contributions constant forever after observing a deviation from the prescribed contribution path $\boldsymbol{c}$. Given this strategy profile, we can interpret ϵ as being the probability of a breakdown in cooperation. Hence in this particular setting, the effect of imperfect monitoring is to render players less patient than in the perfect monitoring case.

Let V_t denote the value from the strategy $\boldsymbol{\sigma_c}$ at time t when players have observed the sequence of action profiles $((c_0, c_0), \ldots, (c_{t-1}, c_{t-1}))$ up to time t.[20] After observing $((c_0, c_0), \ldots, (c_{t-1}, c_{t-1}))$, players should play the action profile (c_t, c_t) in period t, yielding a current flow payoff of $\pi(c_t, c_t)$. With probability $(1-\epsilon)$, players then observe that the action profile (c_t, c_t) was played, leading to a continuation payoff of V_{t+1}. With the complementary probability, players do not observe the action profile (c_t, c_t) and then play (c_t, c_t) forever. Hence we can express V_t as a function of V_{t+1} as follows:

$$V_t = (1-\delta)\pi(c_t, c_t) + \delta\{(1-\epsilon)V_{t+1} + \epsilon\pi(c_t, c_t)\}.$$

The strategy profile $\boldsymbol{\sigma_c}$ is an equilibrium of the ϵ-almost perfect monitoring game if there are no profitable one-shot deviations. Here I only consider deviations after histories of the type $((c_0, c_0), \ldots, (c_{t-1}, c_{t-1}))$, as after other histories players maintain their contributions constant, which is an equilibrium. The best one-shot deviation possible after such a history is not to increase at all the contribution level between time t and $t+1$, as after a de-

ϵ no signal was observed, but this would violate full support. This monitoring technology is similar to the "network monitoring" of Wolitzky (2011), although in this paper the monitoring technology remains public, whereas the network monitoring considered by Wolitzky (2011) is a special form of private monitoring.

[20]This history occurs with probability $(1-\epsilon)^t$.

viation players continue to cooperate with probability zero. Such a one-shot deviation is not profitable after history $((c_0, c_0), \ldots, (c_{t-1}, c_{t-1}))$ if:

$$(1-\delta)\pi(c_{t-1}, c_t) + \delta\pi(c_{t-1}, c_t) \leq (1-\delta)\pi(c_t, c_t) + \delta\{(1-\epsilon)V_{t+1} + \epsilon\pi(c_t, c_t)\}. \quad (8)$$

The left-hand side of (8) is the payoff from deviating. The current payoff is $\pi(c_{t-1}, c_t)$, and since there is a zero probability of observing the profile (c_t, c_t) the contribution levels become constant forever, yielding a continuation value of $\pi(c_{t-1}, c_t)$. The right-hand side of (8) is the payoff from following the prescribed strategies, which is V_t.

Proposition 1. *There exists a $\delta > 0$, an $\epsilon \in (0, 1)$ and a sequence $\boldsymbol{c} := (c_0, c_1, \ldots)$ such that $\boldsymbol{\sigma_c}$ is an equilibrium of the ϵ-almost perfect monitoring game such that there is a strictly positive probability of players contributing. The sequence $(c_t)_t$ satisfies the following difference equation:*

$$\pi(c_t, c_{t+1}) = \frac{1}{\delta(1-\epsilon)}[\pi(c_{t-1}, c_t) - \pi(c_t, c_t)] + \pi(c_t, c_t), \quad t > 0, \qquad (9)$$

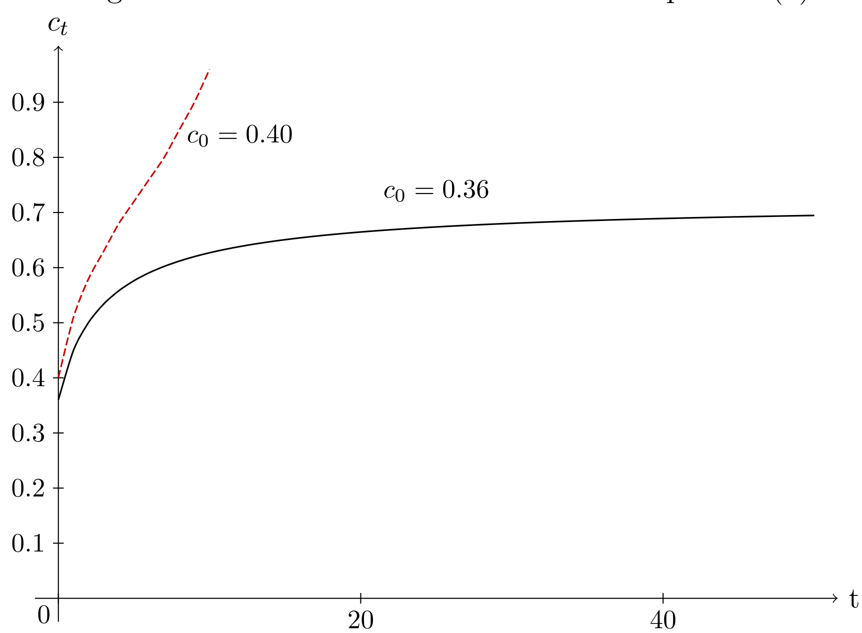
with initial conditions $\bar{c}_{-1} = 0$ and $\bar{c}_0 = c_0$.

The proof of Proposition 1 is presented in Appendix A. Note that (9) is similar to the difference equation (2.4) of Lockwood and Thomas (2002) but with a modified discount factor $\widetilde{\delta} = \delta(1-\epsilon)$. Indeed, as was argued previously, the effect of ϵ in this setting is to render players less patient than in the perfect monitoring setting.

Figure 1 shows two different paths that solve the difference equation (9) when $\pi(x, y) = -x^2/2 + y$, $\delta = 0.8$ and $\epsilon = 0.10$, with initial values for c_0 being 0.36 and 0.4.[21] The function π and the value of δ have been chosen as in the example of Lockwood and Thomas (2002, Figure 1). Note that the solution to (9) does not converge when the initial value is 0.4 and that the

[21]Note that the function $\pi(x, y) = -x^2/2 + y$ satisfies Assumptions 1 to 3.

Figure 1: Two solutions to the difference equation (9)



limit of the sequence when the initial value is 0.36 (the highest initial value consistent with convergence) is 0.72, that is, $\delta(1-\epsilon)$.

Remark 6 (Benefit jump and project completion)**.** A key ingredient in the proof of Theorem 1 is that in a putative equilibrium with positive contributions, increments in contribution levels decrease to zero while remaining strictly positive on a set of histories of positive measure. When the monitoring technology is smooth, small deviations will be hard to detect when increments are close to zero, while consequences will be limited, giving players incentives to deviate and reduce their level of contributions. In this section I presented a counterexample where the monitoring distribution was not smooth and the result of Theorem 1 did not hold.

Another way to depart from the assumptions of Section 2 would be to consider a jump in the payoff function when the project is completed, as in Marx and Matthews (2000). In their paper the benefit of contributions to player i is $\lambda_i X$, where X is the sum of contributions, if X is below a certain threshold X^* (the "completion point") and $V_i \geq \lambda_i C^*$ when X is above the completion point. If $V_i - \lambda_i C^* > 0$ then it is possible that in a putative equilibrium with positive contributions, increments are bounded away from zero (before the project is completed). The type of deviation considered in the proof of Theorem 1 might then no longer be profitable.

5 Conclusion

In a dynamic game where players can contribute to a public project and contributions are irreversible, I have showed that under imperfect public monitoring, when the monitoring technology is smooth and has full support, cooperation cannot be achieved and players do not contribute in equilibrium. This finding is in stark contrast with the perfect monitoring case, in which there exist equilibria with strictly positive contribution levels. However when the monitoring distribution is no longer smooth, small changes in contribution levels can be detected and cooperation is again possible.

The result relies on one player's knowledge that contributions of the other player are close to an upper bound. Such an argument cannot be used in the case of private monitoring, where each player assigns positive probability to any private history of the other player (assuming full support). A possible approach in that case would be to consider conditions under which player's contributions converge uniformly. There would then be a certain time after which player's cannot expect the other player's contribution to increase by much, even though they are unsure about the actual levels of contributions.

A Proof of Proposition 1

Recall that the strategy $\boldsymbol{\sigma_c}$ is an equilibrium of the ϵ-almost perfect monitoring if there are no one-shot deviations, that is if (8) holds for $t \geq 0$ (where $c_{-1} = 0$). First let us rewrite (8) both for t and $t+1$, assuming that the inequalities hold with equality:

$$\pi(c_{t-1}, c_t) = (1-\delta)\pi(c_t, c_t) + \delta\{(1-\epsilon)V_{t+1} + \epsilon\pi(c_t, c_t)\}, \tag{10}$$

and

$$\pi(c_t, c_{t+1}) = V_{t+1}. \tag{11}$$

Multiplying (11) by $-\delta(1-\epsilon)$ and adding it to (10), we obtain:

$$\pi(c_{t-1}, c_t) - \delta(1-\epsilon)\pi(c_t, c_{t+1}) = (1-\delta)\pi(c_t, c_t) + \delta\epsilon\pi(c_t, c_t).$$

Rearranging the terms then leads to (9).

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