On the Structure of Cities: Emergence of Residential and Industrial Areas under Environmental Policy.

Efthymia Kyriakopoulou*

University of Gothenburg

Anastasios Xepapadeas†

Athens University of Economics and Business and Beijer Fellow

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Abstract

We study the internal structure of a city, in an urban model, where industry and housing compete for scarce land to locate. We analyze a spatial model of a city in which a single good is produced using land, labor and emissions of a pollutant, and in which people consume goods, residential land and dislike pollution. The agglomeration effects, caused by trade-offs between centripetal and centrifugal forces, in the form of knowledge spillovers, stringency of environmental policy and commuting cost, determine the emergence of industrial and residential clusters across space.

JEL classification: R3, Q5, H2.

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*Department of Economics, University of Gothenburg (e-mail: efi.kyriakopoulou@economics.gu.se).

†Department of International and European Economic Studies, Athens University of Economics and Business and Beijer Fellow (e-mail: xepapad@aueb.gr).

1 Introduction

Until the recent emergence of the literature of New Economic Geography, economic models had ignored space assuming that economic agents had no incentive to locate in any spatial point. These simplifying assumptions lead to a uniform distribution of population and activity, which is far from reality. Contrary to these results, firms choose their location basing their decisions on various factors such as the location decisions of their competitors, the access to the consumers, the transportation cost of inputs used in the production process, the regulations in different locations etc. The same is true for households which consider the quality characteristics of different neighborhoods, the distance to their workplaces, the different levels of pollution and other social factors such as crime or education and recreational opportunities. As a result, in the interior of a city we observe pure industrial areas - mostly in the centre of the city - pure residential areas - mostly in the suburbs - and some mixed areas where firms and households locate near each other.

The fact that areas of various sizes inside cities are used for different purposes (either industrial or residential) indicates the power of agglomeration forces and can also explain the renaissance of location theory after many decades. New Economic Geography studies the unequal, spatial distribution of agents resulting from different agglomeration forces: such as externalities in production, residential externalities, first nature advantages, transportation and commuting costs, regulations and others. As stated in the monograph by Combes et al.[3] and will become obvious from the results of our study, "the new element specific to economic geography is the fact that, in all places, what is near has more influence than what is far".

In this context, this paper examines the effect of environmental externalities on the location decisions of firms and households. We explicitly assume that pollution diffuses in space. More specifically, firms produce a single good using land, labor and emissions of a pollutant.[1] However, emissions generated at a spatial point affect the concentration of

[1] The use of emissions / pollution as an input in the production function was introduced by Brock [2]. This concept was later used by other authors, eg. Jouvet et al. [5], Kyriakopoulou and Xepapadeas [7], Rauscher [13], Stokey [14], Tahvonen and Kuuluvainen [15], Xepapadeas [17]. This idea captures the fact

pollution in nearby areas as well. On the other hand, households are negatively affected by pollution and prefer to locate in clean areas. Accordingly, the higher the pollution levels, the higher the disutility they receive. As a result, the production process pollutes the environment and consumers have an incentive to locate far from firms to avoid the higher concentration of pollution. In that areas, consumers are willing to pay higher rents in order to enjoy better environmental amenities.

Previous studies on the agglomeration effects of pollution, such as Lange and Quaas [8] and van Marrewijk [16], assume local pollution. The assumption of non-local pollution was used by Arnott et al. [1] who investigate the role of space in the control of pollution externalities. Kyriakopoulou and Xepapadeas [7], also, examined how the stringency of environmental policy affects the distribution of economic activity, by combining the assumption of pollution diffusing in space with other forces that affect the concentration of economic activity, such as externalities in production in the form of positive knowledge spillovers and a first nature advantage location.

The different patterns observed inside cities can be illustrated by many examples. Paris, one of the world's leading business centres, hosts 38 of the Fortune Global 500 companies in several business districts, notably La Défense, the largest purpose - built business district in Europe.[2] Also, according to 2007 estimates, the Paris urban agglomeration is Europe's second biggest city economy and the sixth largest in the world.[3] On the other hand, the population of Paris was estimated to be 2.14 million in 2004, much lower than its historical peak of 2.9 million in 1921.[4] The city's population loss mirrors the experience of most other core cities in the developed world. The main factors of this trend are a significant decline in household size and a dramatic migration of residents to the suburbs. This migration can be explained by several factors such as de-industrialisation, the transformation of living space into offices, the improvement and the decreasing cost of means of transport and the environmental amenities provided in the suburbs.

that techniques of production are less costly in terms of capital input if more emissions are generated.

[2] Fortune: "Global Fortune 500 by countries: France" (see http://money.cnn.com/magazines/fortune/global500/2009/countries/France.html.).

[3] World Urbanization Prospects: The 2007 Revision Population Database. The United Nations.

[4] Institut National de la Statistique et des Études Économiques. "Enquêtes annuelles de recensement 2004 et 2005".

The contribution of this paper is in studying how environmental externalities - which affect consumers negatively and imply the enforcement of environmental policy - combined with other agglomeration forces such as externalities in production and different commuting costs will finally determine the internal structure of a city. One of the main reasons for the clustering of economic activity is the presence of production externalities that affect aggregate production. That's the reason we decided to use as the basic agglomeration force of our model the existence of communications among firms. That type of communication permits the exchange of information and knowledge between firms. Thus, other things being equal, each firm has an incentive to locate closer to the other firms forming, in that way, business clusters. On the other hand, the formation of a pure business center increases the average commuting cost of workers and gives rise to higher wages and land rents in the area surrounding the cluster. This process acts as a centrifugal force which impedes further agglomeration of firms. The trade-off of production externalities and commuting costs has been explained extensively in a lot of studies, such as in Lucas and Rossi-Hansberg [9] and Fujita and Thisse [4] (Chapter 6). Except for adding environmental externalities in the above forces that have been extensively studied, there is also another differentiation, which is worth noticing, from the previous literature. More specifically, we use a novel approach in solving systems of integral equations with symmetric kernels. In that way, we explicitly take into account the effect of knowledge spillovers and aggregate pollution in the solution of the problem and thus we remove simplifying assumptions, that might make the solution less general.

In this context, we try to examine how pollution generated from production and the enforcement of environmental policy will affect the spatial structure of a city. Thus, firms will be obliged to pay a site-specific pollution tax, which depends on the marginal damage of pollution on the site they will decide to locate. However, the more industries locate in a spatial domain, the more polluted this domain will be. So, if firms decide to locate close to each other so as to benefit from positive knowledge spillovers, they will have to pay a higher pollution tax. In other words, the pollution tax works to discourage the agglomeration of economic activity. As for the consumers they are negatively affected by pollution and prefer to locate in clean areas. But, in that case, they move farther from the firms and have to incur higher commuting costs. The balance among these opposite forces, as well as the use of land for both production and residential purposes will finally define industrial and residential areas.

The results of the paper can be summarized in the following way. Residential land-rents decrease when pollution increases as agents prefer to move to cleaner areas. As for business land-rents, they decrease when the government imposes stricter environmental regulations, which decrease firms' profits. The comparison of these two kinds of land-rents defines the residential and industrial clusters analyzed below. Some first numerical experiments suggest that when knowledge spillovers are strong, environmental policy will lead to the formation of a polycentric city. This is expected because if a lot of firms locate around one spatial point, the cost of environmental taxation will be high.

In Sections 2 and 3, we present the model, its mathematical solution that provides the equilibrium distribution and the proof of existence and uniqueness of the equilibrium. In Section 4, we discuss the different equilibrium patterns of land use that can be formed given the agglomeration forces assumed in our model. In Section 5, we present some numerical experiments, giving different values to the parameters of the model that are consistent with the empirical literature. In the final section, we summarize the results of our paper and give some ideas for future research.

2 The model

We consider a one-dimensional space $S = [0, 2\pi]$ which is interpreted as a single city that constitutes a small part of a large economy. The spatial domain of the city is used for industrial and residential purposes. More specifically, there is a large number of industrial firms and households which can both be located anywhere inside the city. Firms produce a single good using land, labor and emissions. More specifically, the production process generates emissions which diffuse in space and increase the total concentration of pollution in the city. This fact is reinforced in areas with high concentration of economic activity, where a lot of firms operate and pollute the environment. The use of emissions in the production and the negative consequences that follow cause environmental regulation. The environmental policy is stricter in areas with high concentration of pollution and laxer in the rest of them. This means that if firms decide to locate close to other firms they will have to pay more money in the form of environmental taxation. As a result, environmental policy acts as a centrifugal force among firms. On the other hand, firms are positively affected when locating near other firms because of externalities in production in the form of knowledge spillovers. The trade-off between these two opposite forces defines the industrial areas in our circular economy.

In our model, consumers receive negative utility from pollution. Accordingly, they tend to locate far from firms to avoid polluted areas. So, aggregate pollution promotes the formation of pure residential and industrial clusters. This argument is stronger when aggregate pollution increases. On the other hand, consumers work in the business sector devoting one unit of time, a part of which is spent travelling to work. Thus, they have an incentive to locate close to their workplace so as not to spend much time / money commuting to it. As a result, commuting costs promote the formation of mixed areas where people will live next to their workplaces.

The objective of this paper is in examining the different equilibrium patterns of land use. The trade-off between the above forces will define residential, industrial or mixed areas inside the city.

2.1 Consumers' Problem

Consumers receive utility from the consumption of the good produced by the firms and the quantity of residential land, while they receive negative utility from pollution. Thus, a household located at the spatial point r receives utility $U(c(r), l(r), P(r))$,where c is the consumption of the produced good, l is residential land, P is aggregate pollution. Consumers at every location must receive the reservation utility level ($\bar{u}$):

$$U(c(r), l(r), P(r)) = \bar{u} \tag{1}$$

so as no household will have an incentive to move to another spatial point inside or outside the city.

Assuming that consumers live at r and work at s, the total income they earn is equal to $w(r) = w(s)e^{-\kappa|r-s|}$. This equation corresponds to a spatially discounted accessibility, which has been used extensively in spatial models of interaction. Now, if a consumer lives at s and works at r, the wage function becomes $w(r) = w(s)e^{\kappa|r-s|}$. If r is a mixed area, people who live there work there as well, and $w(r)$ denotes both a wage rate paid by firms and net wage earned by workers.

Consumers also receive a lump-sum subsidy that is provided by the government, which redistributes the revenues from the environmental taxation imposed on firms for the emissions they generate.[5] The lump sum subsidy per unit of land is $\frac{T}{S}$. The total revenues are spent on the land they rent at a price $q(r)$ per unit of land and on consumption, $c(r)$.

So, consumers minimize their expenditures:

$$w(r) + \frac{T}{S} = q(r)l(r) + c(r) = \min_{l,c}[q(r)l + c] \tag{2}$$

subject to

$$U(c, l, P) \geq \bar{u} \tag{3}$$

To solve for the equilibrium, we assume that a consumer living at site r considers the amount of aggregate pollution $P(r)$ at that spatial point as given. To derive a closed-form solution, we assume that the utility U is expressed as

$$U(r) = c(r)^a l(r)^{1-a} - P(r)^\phi \tag{4}$$

where $0 < a < 1$ and $\phi \geq 1$.

Taking the Lagrangian of the problem,

$$L = q(r)l(r) + c(r) + \lambda[\bar{u} - c^a l^{1-a} + P^\phi] \tag{5}$$

[5] For more details, see *Firms' Problem* below.

we obtain the following first order conditions (FOC):

$$q(r) = (1-a)\lambda l^{-a} c^{a} \tag{6}$$

$$1 = a\lambda c^{a-1} l^{1-a} \tag{7}$$

Dividing the FOC:

$$q(r) = \frac{1-a}{a}\frac{c}{l} \Rightarrow \tag{8}$$

$$c = \frac{a}{1-a} ql \tag{9}$$

Substituting 9 into 3, we get:

$$\bar{u} = \left(\frac{a}{1-a}\right)^{a} l^{a} q^{a} l^{1-a} - P^{\phi} \Rightarrow \tag{10}$$

$$l^{*} = \frac{(\bar{u} + P^{\phi})}{q^{a}} \left(\frac{1-a}{a}\right)^{a} \tag{11}$$

and

$$c^{*} = \left(\frac{a}{1-a}\right)^{1-a} q^{1-a} (\bar{u} + P^{\phi}) \tag{12}$$

Now, substituting 11 and 12 into the budget constraint:

$$w(r) + \frac{T}{S} = q^{1-a} (\bar{u} + P^{\phi}) \left(\frac{1-a}{a}\right)^{a} \frac{1}{1-a}$$

$$q^{*}(r) = \left[\frac{w(r) + \frac{T}{S}}{(\bar{u} + P(r)^{\phi}) \left(\frac{1-a}{a}\right)^{a} \frac{1}{1-a}}\right]^{\frac{1}{1-a}} \tag{13}$$

So, $q^{*}(r)$ is the rent per unit of land that a worker bids at location r while working at s

and enjoying the utility level $\bar{u}$. We observe that $\frac{\vartheta q(r)}{\vartheta P(r)} < 0$. This means that residential land rents are lower in areas with high concentration of pollution. In other words, people are willing to spend more money on areas with better environmental amenities. This is reassured by the fact that the highest residential rents in the real world are observed in purely residential areas which are in the suburbs of the cities, far from the polluted business centers.

Finally, assuming that the land density is 1, we can define the equilibrium population density N at each spatial point

$$N\ l = 1 \Longrightarrow N = \frac{1}{l^*}$$

$$N^*(r) = \frac{\left(w(r) + \frac{T}{S}\right)^{\frac{a}{1-a}}}{(\bar{u} + P^{\phi})^{\frac{1}{1-a}} \left(\frac{1-a}{a}\right)^{\frac{a}{1-a}} \left(\frac{1}{1-a}\right)^{\frac{a}{1-a}}}$$

It is obvious that the population distribution moves upward when the net wage increases and when the concentration of pollution at the same spatial point decreases.

2.2 Firms' Problem

All firms produce a single good which is sold around the world at a competitive price, that is considered as given for our small economy. The production is characterized by a constant returns to scale function of land, labor $L(r)$ and emissions $E(r)$.There is also an externality in production which relates the production at each spatial point with the employment density at nearby areas. Production per unit of land at location r is given by:

$$f(r) = g(z(r))x(L(r), E(r)) \tag{14}$$

In the numerical simulations presented below, the function g and x are considered to be of the form:

$$
\begin{aligned}
g(z) &= e^{\gamma z} \\
x(L, E) &= AL^b E^c
\end{aligned}
$$

The production externality is assumed to be linear and to decay exponentially at a rate δ with the distance between (r, s):

$$z(r) = \delta \int_0^S e^{-\delta(r-s)^2} \theta(s) \ln L(s) ds$$

Note that $\theta(r)$ is the proportion of land occupied by firms at the spatial point r and $1-\theta(r)$ is the proportion of land occupied by households at r. The generation of emissions during the production of the output damages the environment. The damage function per unit of land is given by

$$D(r) = \; P(r)^{\phi} \tag{15}$$

where D is the damage per unit of land and $\phi \geq 1$, $D'(P) > 0$, $D''(P) \geq 0$.[6] Each firm has to pay a tax for each unit of emissions it generates which is site-specific and depends on the marginal damage of aggregate pollution:

$$\tau(r) = \xi \; MD(r) = \xi \; \phi \; P(r)^{\phi-1} = \psi \; P(r)^{\phi-1} \tag{16}$$

where $0 \leq \theta \leq 1$, $\psi = \theta\phi$ and $\xi = 1$ means that the full marginal damage at point r is charged as a tax. It is obvious that the per unit tax paid by each firm depends not only on its own emissions but also on the aggregate pollution at the point it chooses to locate.

[6] In the modelling of the damage function, we follow Koldstad [6], who defines damages at a specific location as a function of aggregate emissions of the location. He does not relate environmental damages on a spatial point to the number of people living in that location. In other spatial models, the pollution damage is taken into account in residential areas only, with the interpretation that people are those who are negatively affected by pollution. Thus, environmental damage is zero if there are no residents in the specific location, but damages are positive in a nearby location where residents exist. We prefer to consider pollution damages in the whole spatial domain as a function of the aggregate emissions of that point independent of the number of residents. In this way, we avoid the potential contradiction of assigning very low damages to a heavily polluted area which does not have high residential density. In any case, people work in business areas and if those areas are heavily polluted, workers are negatively affected as well.

In the solution of the problem, the logarithm of the tax function is used:

$$\ln \tau(r) = \ln \psi + (\phi - 1)\ \ln P(r)$$

where

$$\ln P(r) = \int_0^S e^{-\zeta(r-s)^2} \theta(s) \ln E(s) ds \tag{17}$$

From equation 17, we notice that aggregate pollution P at a spatial point r is given by a weighted average of emissions generated in nearby places. The kernel here, $k_2(r,s) = e^{-\zeta(r-s)^2}$, describes the idea that pollution diffuses in space. A higher ζ value means that emissions in r pollute only nearby places.

It is clear from the above analysis, that the stringency of environmental policy depends on the degree to which the taxation internalizes fully or partly the marginal damage. In case that $\xi = 1$, the marginal damage is fully internalized in the taxation and a strict environmental policy is enforced, while in case that $\xi < 1$, the marginal damage is partly internalized and the environmental policy is laxer.

To derive the equilibrium solution, we assume that a firm located at the spatial point r chooses labor and emissions to maximize its profits:

$$\hat{Q}(r) = g(z(r))x(L(r), E(r)) - w(r)L(r) - \tau(r)E(r) = \max_{L,E}\{g(z(r))x(L,E) - w(r)L - \tau(r)E\}$$

The solution will be a function of z, w, τ: $L = L^*(w, z, \tau)$ and $E = E^*(w, z, \tau)$. The maximized profits at each spatial $Q = Q^*(w, z, \tau)$ point can also be interpreted as the business land rent which is the land rent that a firm is willing to pay so as to operate in that spatial point.

The maximized profit per unit of land, $Q^*(r)$, at location r, for the specific production function we presented above, is given by:

$$Q^*(r) = \max_{L,K,E} A e^{\gamma z(r)} L(r)^b E(r)^c - w(r)L(r) - \tau(r)E(r) \tag{18}$$

where $w(r)$ is the wage rate at each spatial point and the price of output is normalized to 1.

A firm located at site r treats the concentration of pollution $P(r)$ and the effect of knowledge spillovers in the production process $z(r)$ as exogenous parameter P^e and z^e respectively. This assumption implies that the tax $\tau(r)$ is also treated as a parameter at each spatial point.

The first order necessary conditions (FONC) for profit maximization are:

$$bAe^{\gamma z}L(r)^{b-1}E(r)^{c} = w(r) \tag{19}$$

$$dAe^{\gamma z}L(r)^{b}E(r)^{c-1} = \tau(r) \tag{20}$$

So, we solve explicitly for:

$$L^*(z,w,\tau) = \left(\frac{c^c b^{1-c} Ae^{\gamma z}}{\tau^c w^{1-c}}\right)^{\frac{1}{1-b-c}} \tag{21}$$

$$E^*(z,w,\tau) = \left(\frac{c^{1-b} b^{b} Ae^{\gamma z}}{\tau^{1-b} w^{b}}\right)^{\frac{1}{1-b-c}} \tag{22}$$

Substituting 21 and 22 into the maximized profit function, we solve explicitly for the industrial land rents:

$$Q^*(z,w,\tau) = \left(\frac{e^{\gamma z} A b^b c^c}{\tau^c w^b}\right)^{\frac{1}{1-b-c}} (1-b-c) \tag{23}$$

which represents the highest price a firm is willing to pay for a unit piece of land at r so as to operate at that spatial point.

Note that the total revenues from taxation $\int_0^S \tau(s)E(s)ds$ should be equal to T which is the total amount of money redistributed to households by the government.

In the explicit solution for L, E and Q presented above, there are two integral equations, the one describing knowledge spillovers and the other describing the concentration of pollution at each spatial point.[7] Most authors who have studied knowledge spillovers

[7]There are kernels in the right-hand side of equations 21-23 (see the definition of $z(r)$ and $\tau(r)$ above).

of this form use simplifying assumptions about the values that the kernels take at each spatial point. However, in that way we force firms to locate around the spatial domain with the highest assumed arbitrary values of knowledge spillovers. In that way, we do not take into account that $L(s)$ and $E(s)$, where $s \in S$, exist in the right hand side of 21-22, and that these equations have to be solved as a system of simultaneous integral equations. Instead of following this approach, we choose to use a novel method of solving systems of integral equations. More specifically, if we take logs on both sides of equations 19-20 and do some transformations which are described in the Appendix, the FONC result in a system of second kind Fredholm integral equations with symmetric kernels:

$$\frac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2}y(s)ds + \frac{c(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2}\varepsilon(s)ds + g_1(r) = y(r) \qquad (24)$$

$$\frac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2}y(s)ds + \frac{(1-b)(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2}\varepsilon(s)ds + g_2(r) = \varepsilon(r) \qquad (25)$$

where $y(r) = \ln L(r)$, $\varepsilon(r) = \ln E(r)$ and $g_1(r), g_2(r)$ are some known functions..

Proposition 1 *Assume that: (i) the kernel $k(r,s)$ defined on $[0,2\pi] \times [0,2\pi]$ is an L_2-kernel which generates the compact operator W, defined as $(W\phi)(r) = \int_a^b k(r,s)\,\phi(s)\,ds$, $a \leq s \leq b$ (ii) $1-b-c$ is not an eigenvalue of W, and (iii) G is a square integrable function, then a unique solution determining the equilibrium distribution of inputs and output exists.*

The proof of existence and uniqueness of the equilibrium is presented in the following steps:[8]

- A function $k(r,s)$ defined on $[a,b] \times [a,b]$ is an L_2-kernel if it has the property that $\int_a^b \int_a^b |k(r,s)|^2\, drds < \infty$.

  The kernels of our model have the following formulation: $e^{-\xi\,(r-s)^2}$ with $\xi = \delta,\ \zeta$ (positive numbers) and are defined on $[0,2\pi] \times [0,2\pi]$.

[8]See Moiseiwitsch [11] for more detailed definitions.

We need to prove that $\int_0^{2\pi}\int_0^{2\pi}\left|e^{-\xi\ (r-s)^2}\right|^2 drds < \infty$.

Rewriting the left part of inequality, we get: $\int_0^{2\pi}\int_0^{2\pi}\left|\frac{1}{e^{\xi\ (r-s)^2}}\right|^2 drds$.

The term $\frac{1}{e^{\xi\ (r-s)^2}}$ takes its highest value when $e^{\xi\ (r-s)^2}$ is very small. But the lowest value of $e^{\xi\ (r-s)^2}$ is obtained when either $\xi = 0$ or $r = s$ and in that case $e^0 = 1$. So, $0 < \left|\frac{1}{e^{\xi\ (r-s)^2}}\right| < 1$. When $\left|\frac{1}{e^{\xi\ (r-s)^2}}\right| = 1$, then $\int_0^{2\pi}\int_0^{2\pi}\left|\frac{1}{e^{\xi\ (r-s)^2}}\right|^2 drds = 4\ \pi^2 < \infty$. Thus, the kernels of our system are L_2-kernels.

- If $k(r,s)$ is an L_2- kernel, the integral operator

$$(W\phi)(r) = \int_a^b k(r,s)\,\phi(s)\,ds\ , a \leq s \leq b$$

that it generates is bounded and

$$\|W\| \leq \left\{\int_a^b\int_a^b |k(r,s)|^2\,drds\right\}^{\frac{1}{2}}$$

So, in our model the upper bound of the norm of the operator generated by the L_2-kernel is $\|W\| \leq \left\{\int_a^b\int_a^b |k(r,s)|^2\,drds\right\}^{\frac{1}{2}} = \left\{\int_0^{2\pi}\int_0^{2\pi}\left|\frac{1}{e^{i\ (r-s)^2}}\right|^2 drds\right\}^{\frac{1}{2}} \leq 2\pi$.

- If $k(r,s)$ is an L_2- kernel and W is a bounded operator generated by k, then W is a compact operator.

- If $k(r,s)$ is an L_2-kernel and generates a compact operator W, then the integral equation :

$$Y - \left(\tfrac{1}{1-a-b-c}\right) W\ Y = G \tag{26}$$

has a *unique* solution for all square integrable functions G, if $(1-a-b-c)$ is not an eigenvalue of W (Moiseiwitsch [11]). If $(1-a-b-c)$ is not an eigenvalue of W, then $\left(I - \frac{1}{1-a-b-c}W\right)$ is invertible.

- As we show in Appendix B, the system (24-25) can be transformed into a second kind Fredholm Integral equation of the form 26. Thus a unique equilibrium distribution of inputs and output exists.□

To solve the system (24-25) numerically we use a modified Taylor-series expansion method (Maleknejad et al. [10]). More precisely, a Taylor-series expansion can be made for the solutions $y(s)$ and $\varepsilon(s)$ in the integrals of the system (24-25). We use the first two terms of the Taylor-series expansion (as an approximation for $y(s)$ and $\varepsilon(s)$) and substitute them into the integrals of (24-25). After some substitutions, we end up with a linear system of ordinary differential equations of the form:

$$\theta_{11}(r)\ y(r) + \theta_{12}(r)\ y'(r) + \theta_{13}\ y''(r) + \sigma_{11}\ \varepsilon(r) + \sigma_{12}\ \varepsilon'(r) + \sigma_{13}\ \varepsilon''(r) = g_1(r) \quad (27)$$

$$\theta_{21}(r)\ y(r) + \theta_{22}(r)\ y'(r) + \theta_{23}\ y''(r) + \sigma_{21}\ \varepsilon(r) + \sigma_{22}\ \varepsilon'(r) + \sigma_{23}\ \varepsilon''(r) = g_2(r) \quad (28)$$

In order to solve the linear system (27-28), we need an appropriate number of boundary conditions. We construct them and then we obtain a linear system of three algebraic equations that can be solved numerically.

The maximized value of the firm's profits $Q^*(r)$ is also the land-rent per unit of land that a firm would be willing to pay to operate with these cost and productivity parameters at location r. Since the decision problem at each location is completely determined by the technology level z, the wage rate w and the concentration of pollution P, the FONC of the maximization problem give us the equilibrium values of labor and emissions used at each location: $L = L^*(z, w, P)$ and $E = E^*(z, w, P)$.

3 Equilibrium Land Use

Having solved the consumers' and firms' problems, we are able to define the equilibrium land use. Our city is strictly defined in the spatial domain $[0, S]$ and firms and households can locate nowhere else. Thus, a spatial equilibrium is reached when all firms receive zero profits, all households receive the same utility level $\bar{u}$, land is allocated to its highest values and rents and wages clear the land and labor markets.

Consumers dislike pollution, which means that they have an incentive to locate far from business areas. On the other hand, consumers work at the firms and if they locate far from them, they will suffer a higher commuting cost. It is obvious that the high commuting costs promote the formation of mixed areas, so as to reduce the distance between workers and their workplace. The trade-off between these two forces will define the residential location decisions.

Firms have a strong incentive to locate close to each other so as to benefit from the positive knowledge spillovers. However, if all firms locate around a spatial point, that spatial interval will be very polluted and thus, the high concentration of pollution will increase the cost of environmental policy. In that way, firms will be obliged to pay a higher environmental tax. In other words, the environmental regulations impede the concentration of economic activity. The trade-off between these two forces will define the size of the industrial areas. Finally, the use of land for residential and industrial purposes deters both firms and consumers from locating at a unique spatial point.

The equilibrium conditions are described in the following steps:

1. land rents equilibrium: at each spatial point $r \in S$,

$$R(r) = \max\{Q^*(r), q^*(r), 0\} \tag{29}$$

$$Q^*(r) = R(r) \quad \text{if } \theta(r) > 0 \text{ and } Q^*(r) > q^*(r) \tag{30}$$

$$q^*(r) = R(r) \quad \text{if } \theta(r) < 1 \text{ and } q^*(r) > Q^*(r) \tag{31}$$

2. commuting equilibrium: at each spatial point $r \in S$,

$$w(r) = w(s)e^{-\kappa|r-s|} = \max_{s \in S}[w(s)e^{-\kappa|r-s|}] \tag{32}$$

As people choose s so as to maximize their net wage, this means that in equilibrium

$$w(s)e^{-\kappa|r-s|} \leq w(r) \leq w(s)e^{\kappa|r-s|} \tag{33}$$

This is the wage arbitrage condition which implies that no one can gain by changing his job location.

3. labor market equilibrium: at each spatial point $r \in S$,

$$\int_0^S (1-\theta(s))N^*(s)ds = \int_0^S \theta(s)L^*(s)ds \tag{34}$$

4. Industries' and households' population constraints:

$$\int_0^S (1-\theta(s))N^*(s)ds = \bar{N} \tag{35}$$

$$\int_0^S \theta(s)L^*(s)ds = \bar{L} \tag{36}$$

where $\bar{N}$ is the total number of residents and $\bar{L}$ the total number of workers.

5. land use equilibrium: at each spatial point $r \in S$,

$$\begin{aligned} 0 &\leq \theta(r) \leq 1 \\ \theta(r) &= 1 \text{ if } r \text{ is a pure industrial area} \\ \theta(r) &= 0 \text{ if } r \text{ is a pure residential area} \\ 0 &< \theta(r) < 1 \text{ if } r \text{ is a mixed area} \end{aligned} \tag{37}$$

Equations 29-31 mean that each location is occupied by the agents who offer the highest bid rent. Condition 32 implies that a worker living in r will choose his working location s so as to maximize her net wage. Condition 34 ensures the equality of labor supply and demand in the whole spatial domain. This condition will determine the equilibrium wage rate at each spatial point, $w^*(r)$. Finally, conditions 35-36 mean that the sum of residents in all residential areas is equal to the total number of residents in the city and aggregate labor in all industrial areas equals the total number of workers in the city.

4 Numerical Experiments

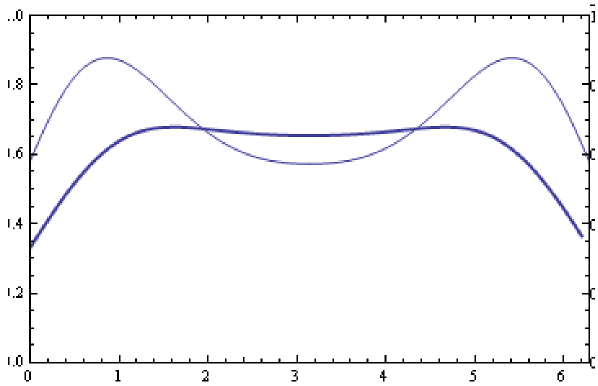
Giving different values to the parameters of the present model, consistent with the relevant literature, we can obtain different maps explaining the residential and the industrial clusters formed in our city. To put it differently, the equilibrium spatial distribution of residential and business land rents will determine the location of firms and households in our domain. The numerical method of Taylor-series expansion, described above, will give us the equilibrium values of land rents.

Let's describe the simulation values now. The share of consumption in the utility function is set to $a = 0.8$, which lets the corresponding share of residential land be 0.2. The constant utility is $\bar{u} = 0.6$. We set A, that describes the technology level equal to 1. The value that describes the effect of knowledge spillovers, γ, is set equal to 0.01. The share of labor is $b = 0.9$ and the share of emissions is $c = 0.05$. Given these values, we let the implied share of land be 0.05. The length of the city is considered to be $S = 2\pi$. We assume that the damage function is increasing and convex, which implies $\phi = 1.5$. Finally, the ζ parameter describing the diffusion of pollution at nearby places is set to 0.5.[9]

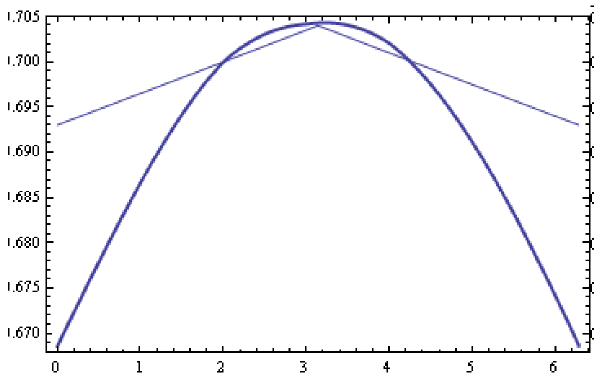
To study the different residential and industrial clusters formed under the agglomeration forces already analyzed, we hold the above parameters constant and let the parameters describing the commuting cost κ, the strength of knowledge spillovers δ and the strigency of environmental policy ψ vary. Some first results derived by giving different values to these parameters are presented and explained below.

In the first graph of Figure 1, we observe the residential (solid line) and industrial (thick line) land rents for the parameter values $\delta = 1$, $\kappa = 0.005$ and $\psi = 1.5$, which means full internalization of the marginal damage. It's obvious that in the center of the city, firms are willing to pay higher land rents as they try to locate close to each other so as to benefit from positive knowledge spillovers. The outcome of this concentration around the center is that this spatial domain is more polluted. Thus, workers who dislike pollution,

[9] This value of ζ was chosen so that the emissions generated at the city centre ($\bar{r} = \pi$) have a negligible effect on the aggregate level of pollution at the two boundary points ($r = 0,\ 2\pi$).



Residential (solid line) and Industrial (thick line) land rents.



Mixed wage curve (thick line) and wage curve (solid line)

Figure 1: The Monocentric City

are willing to spend more at the boundaries of the city, where aggregate pollution is lower. So, there is a central business district, defined by the interval $[2, 4.2]$, surrounded by two residential areas of equal size at the boundaries, $[0, 2]$ and $[4.2, 2\pi]$.

As stated above, in the present numerical experiment, firms are located at the centre of the city. For all workers to have the same net wage, the wage curve must take its highest value at the spatial point $r = \pi$ and decline exponentially according to the function $e^{-\kappa|r-\pi|}$. This is the solid line in the second graph. The thick line depicts the mixed wage curve, i.e. the curve that equates the residential and the industrial land rents. The construction of the equilibrium can be described as follows: we choose an initial wage w^* at the spatial point $r = 0$. If w^* follows a path like the one presented by the solid line in the second graph, it must make the supply and the demand of labor in the interval of our city equal. We try different initial wages until we find the one that satisfies the above equality. In our numerical experiment presented in Figure 1, $w^* = 0.693$. At the spatial point $r = 0$, w^* is higher than the value of the mixed wage curve, which means that location 0 is a pure residential location. People travel to the right to get to work. Also, as long as w^* remains higher compared to the mixed wage curve, $\theta(r) = 0$. When the associated path meets the mixed wage curve, we pass into a pure industrial area. So, now land use changes and in the interval $[2, 4.2]$, the value of $\theta(r)$ is equal to 1. In the same context, people who live at the right residential district travel to the left to get to work. In that way, the two graphs of Figure 1 reassure the fact that there is one industrial core and two residential districts.

In figure 2, we present only the graph with the land rents. We use the same parameter values, except for δ which is assumed to be equal to 3. In this case, knowledge spillovers are stronger which means that firms have more incentives to locate closer to other firms as benefits decline faster with distance. In Figure 2, the thick line presents again the industrial land rents and the solid line the residential ones. We observe that in equilibrium, there are two industrial areas in the intervals $[1.6, 2.6]$ and $[3.7, 4.7]$. These areas are surrounded by residential areas, $[0,\ 1.6]$, $[2.6,\ 3.7]$ and $[4.7,\ 2\pi]$. The concept is similar to the above analysis: people who live at the left residential boundary travel

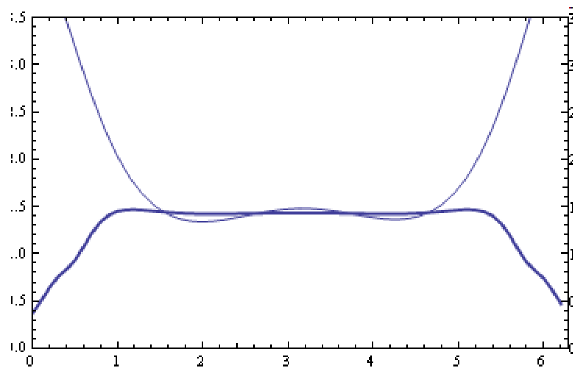


Figure 2: Residential (solid line) and Industrial (thick line) land rents.

to the right to go to work, while people who live at the right one, travel to the left to go to work. Now, people who live at the center are divided in two areas: those living in $[2.6, \pi]$ travel to the left to go to work and those living in $[\pi, 3.7]$ travel to the right to go to work. This restriction is imposed by the non cross-commuting assumption explained above.

As far as the two industrial clusters are concerned, the explanation is the following: in equilibrium firms want to locate close to each other, but if they do so, they will be obliged to pay more money in the form of the environmental taxation. As a result, instead of forming one industrial center, they are forming two industrial areas and the cost of taxation is lower.

5 Conclusion

The agglomeration forces - studied in this paper - affect the internal structure of our city in the following way. Pollution caused during the production process discourages agents from locating near their workplaces and as a result, it promotes the formation of residential districts. However, the commuting costs give an incentive to agents to locate closer to their workplaces. The trade-off among these forces determines consumers' locations decisions and the residential land-rents they are willing to pay at any spatial point.

As far as firms are concerned, their decisions are more easily predicted. On the one hand, they take into account the benefit of locating near each other so as to take advantage of positive knowledge spillovers. But, on the other hand, if there is a high concentration of economic activity around that spatial point, the environmental policy will be stricter. These two forces define the equilibrium distribution of business land-rents across space.

The contribution of this paper is to combine the assumption of pollution diffusing in space with other forces that have been proved to affect economic agglomerations in the literature of new economic geography, such as externalities in production and commuting costs. We also use a novel approach of solving systems of integral equations, which allows us to take explicitly into account the effect of labor and emissions used in nearby areas in equilibrium. Furthermore, environmental issues have not been studied extensively in models of spatial interactions, even though they have been proven to influence spatial location decisions. But, even in cases that they were taken into account, the assumption of diffusing pollution was rarely used in analytical models. To the best of our knowledge, this force has never been combined with externalities in production and commuting costs in the study of spatial location decisions.

The pattern of land use in the interior of our city is determined by the comparison of residential and industrial land-rents. Under the assumptions of our model, in equilibrium, firms always locate around one or more industrial clusters so as to benefit from positive knowledge spillovers. If knowledge spillovers are strong, firms have an incentive to be more and more concentrated. However, in that case, the cost of environmental policy will be very high, impeding the concentration around one point and leading to two industrial clusters. As for the consumers, they locate around the industrial area and form two residential clusters in the case of the monocentric city or more than two clusters in the case that more than one industrial areas are formed. In other words, under the assumption of pollution, firms and consumers do not locate close to each other. These results seem to explain the emergence of cities with multiple industrial and residential areas. We also plan to study how abatement policy, which leads to lower concentrations of pollution, but is an extra cost for the firms, will affect the spatial patterns derived. Another issue is to study the optimal patterns of land use and the differences they have with the equilibrium.

These thoughts are left for future research.

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Appendix A: Firms' Problem: Solving a system of second kind Fredholm integral equations, following the modified Taylor-series expansion method (Maleknejad et al., 2006).

Solving for the Rational Expectations Equilibrium: we take the logs of (19)-(20).

$$\ln b + \ln A + \gamma\delta \int_0^S e^{-\delta(r-s)^2} \ln L(s)ds + (b-1)\ln L(r) + c\ln E(r) = \ln w(r)$$

$$\begin{aligned} & \ln c + \ln A + \gamma\delta \int_0^S e^{-\delta(r-s)^2} \ln L(s)ds + b\ln L(r) + (c-1)\ln E(r) \\ = \; & \ln\psi + (\phi-1)\int_0^S e^{-\zeta(r-s)^2} \ln E(s)ds \end{aligned}$$

Note that the right-hand side of the last equation is equal to $\ln\tau(r)$.

Setting $\ln L = y$, $\ln E = \varepsilon$, we obtain the following system:

$$\gamma\delta\int_0^S e^{-\delta(r-s)^2}y(s)ds+(b-1)y(r)+c\varepsilon(r) = \ln w(r)-\ln b-\ln A$$
$$\gamma\delta\int_0^S e^{-\delta(r-s)^2}y(s)ds+by(r)+(c-1)\varepsilon(r)+(1-\phi)\int_0^S e^{-\zeta(r-s)^2}\varepsilon(s)ds = \ln\psi-\ln c-\ln A$$

We transform the system in order to obtain a system of second kind Fredholm integral equations with symmetric kernels:

$$\underbrace{\begin{pmatrix}\gamma\delta & 0\\ \gamma\delta & (1-\phi)\end{pmatrix}\begin{pmatrix}\int_0^S e^{-\delta(r-s)^2}y(s)ds\\ \int_0^S e^{-\zeta(r-s)^2}\varepsilon(s)ds\end{pmatrix}+\begin{pmatrix}\ln A+\ln b-\ln w(r)\\ \ln A+\ln c-\ln\psi\end{pmatrix}}_{\text{B}}$$
$$=\underbrace{\begin{pmatrix}1-b & -c\\ -b & 1-c\end{pmatrix}}_{\text{A}}\underbrace{\begin{pmatrix}y(r)\\ \varepsilon(r)\end{pmatrix}}_{\text{Z}}$$

$$B=AZ\Rightarrow A^{-1}B=Z \quad \text{where} \quad A^{-1}=\tfrac{1}{1-b-c}\begin{pmatrix}1-c & c\\ b & 1-b\end{pmatrix}$$

From $A^{-1}B=Z$, we derive the following system of second kind Fredholm integral equations:

$$\frac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2}y(s)ds+\tfrac{c(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2}\varepsilon(s)ds+g_1(r) = y(r) \quad (38)$$
$$\frac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2}y(s)ds+\tfrac{(1-b)(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2}\varepsilon(s)ds+g_2(r) = \varepsilon(r) \quad (39)$$

where:

$$g_1(r) = \tfrac{1}{1-b-c}\{(1-c)\ [\ln A+\ln b-\ln w(r)]+c\ [\ln A+\ln c-\ln\psi]\}$$
$$g_2(r) = \tfrac{1}{1-b-c}\{b\ [\ln A+\ln b-\ln w(r)]+(1-b)\ [\ln A+\ln c-\ln\psi]\}$$

A Taylor-series expansion can be made for the solutions $y(s)$ and $\varepsilon(s)$:

$$\begin{aligned}
y(s) &= y(r) + y'(r)(s-r) + \frac{1}{2}y''(r)(s-r)^2 \\
\varepsilon(s) &= \varepsilon(r) + \varepsilon'(r)(s-r) + \frac{1}{2}\varepsilon''(r)(s-r)^2
\end{aligned}$$

Substituting the expansions into the integrals of the system (38)-(39), we get:

$$\begin{aligned}
y(r) &= \frac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2}\{y(r) + y'(r)(s-r) + \frac{1}{2}y''(r)(s-r)^2\}\, ds + \\
&\quad \tfrac{c(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2}\{\varepsilon(r) + \varepsilon'(r)(s-r) + \frac{1}{2}\varepsilon''(r)(s-r)^2\}\, ds + g_1(r) \quad (40)
\end{aligned}$$

$$\begin{aligned}
\varepsilon(r) &= \frac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2}\{y(r) + y'(r)(s-r) + \frac{1}{2}y''(r)(s-r)^2\}\, ds + \\
&\quad \tfrac{(1-b)(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2}\{\varepsilon(r) + \varepsilon'(r)(s-r) + \frac{1}{2}\varepsilon''(r)(s-r)^2\}\, ds + g_2(r) \quad (41)
\end{aligned}$$

Rewriting the equations we have:

$$\begin{aligned}
g_1(r) &= \left[1 - \frac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2} ds\right] y(r) - \left[\tfrac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2}(s-r) ds\right] y'(r) \quad (42) \\
&- \left[\frac{1}{2}\,\tfrac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2}(s-r)^2 ds\right] y''(r) - \left[\tfrac{c(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2} ds\right] \varepsilon(r) \\
&- \left[\tfrac{c(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2}(s-r) ds\right] \varepsilon'(r) - \left[\frac{1}{2}\,\tfrac{c(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2}(s-r)^2 ds\right] \varepsilon''(r)
\end{aligned}$$

$$\begin{aligned}
g_2(r) &= \varepsilon(r) - \left[\frac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2} ds\right] y(r) - \left[\tfrac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2}(s-r) ds\right] y'(r) \quad (43) \\
&- \left[\frac{1}{2}\,\tfrac{\gamma\delta}{1-b-c}\int_0^S e^{-\delta(r-s)^2}(s-r)^2 ds\right] y''(r) - \left[\tfrac{(1-b)(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2} ds\right] \varepsilon(r) \\
&- \left[\tfrac{(1-b)(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2}(s-r) ds\right] \varepsilon'(r) - \left[\frac{1}{2}\,\tfrac{(1-b)(1-\phi)}{1-b-c}\int_0^S e^{-\zeta(r-s)^2}(s-r)^2 ds\right] \varepsilon''(r)
\end{aligned}$$

If the integrals in equations (42)-(43) can be solved analytically, then the bracketed quantities are functions of r alone. So (42)-(43) become a linear system of ordinary differential equations that can be solved, if we use an appropriate number of boundary conditions.

To construct boundary conditions we differentiate (38) and (39):

$$y'(r) = \tfrac{\gamma\delta}{1-b-c}\int_0^S -2\delta(r-s)e^{-\delta(r-s)^2}y(s)ds + \tfrac{c(1-\phi)}{1-b-c}\int_0^S -2\zeta(r-s)e^{-\zeta(r-s)^2}\varepsilon(s)ds + g_1'(r) \tag{44}$$

$$\begin{aligned} y''(r) &= \tfrac{\gamma\delta}{1-b-c}\int_0^S \left[-2\delta + 4\delta^2(r-s)^2\right] e^{-\delta(r-s)^2}y(s)ds + \\ & \tfrac{c(1-\phi)}{1-b-c}\int_0^S \left[-2\zeta + 4\zeta^2(r-s)^2\right] e^{-\zeta(r-s)^2}\varepsilon(s)ds + g_1''(r) \end{aligned} \tag{45}$$

$$\varepsilon'(r) = \tfrac{\gamma\delta}{1-b-c}\int_0^S -2\delta(r-s)e^{-\delta(r-s)^2}y(s)ds + \tfrac{(1-b)(1-\phi)}{1-b-c}\int_0^S -2\zeta\ (r-s)\ e^{-\zeta(r-s)^2}\ \varepsilon(s)\ ds + g_2'(r) \tag{46}$$

$$\begin{aligned} \varepsilon''(r) &= \tfrac{\gamma\delta}{1-b-c}\int_0^S \left[-2\delta + 4\delta^2(r-s)^2\right] e^{-\delta(r-s)^2}y(s)ds + \\ & \tfrac{(1-b)(1-\phi)}{1-b-c}\int_0^S \left[-2\zeta + 4\zeta^2\ (r-s)^2\right]\ e^{-\zeta(r-s)^2}\ \varepsilon(s)\ ds + g_2''(r) \end{aligned} \tag{47}$$

We substitute $y(r)$ and $\varepsilon(r)$ for $y(s)$ and $\varepsilon(s)$ in equations (44) - (47):

$$y'(r) = \left[\frac{\gamma\delta}{1-b-c}\int_0^S -2\delta(r-s)e^{-\delta(r-s)^2}ds\right]y(r) + \left[\frac{c(1-\phi)}{1-b-c}\int_0^S -2\zeta(r-s)e^{-\zeta(r-s)^2}ds\right]\varepsilon(r) + g_1'(r) \tag{48}$$

$$\begin{aligned} y''(r) &= \left[\frac{\gamma\delta}{1-b-c}\int_0^S \left[-2\delta + 4\delta^2(r-s)^2\right] e^{-\delta(r-s)^2}ds\right] y(r) + \\ & \left[\frac{c(1-\phi)}{1-b-c}\int_0^S \left[-2\zeta + 4\zeta^2(r-s)^2\right] e^{-\zeta(r-s)^2}ds\right]\varepsilon(r) + g_1''(r) \end{aligned} \tag{49}$$

$$\varepsilon'(r) = \left[\frac{\gamma\delta}{1-b-c}\int_0^S -2\delta(r-s)e^{-\delta(r-s)^2}ds\right]y(r) + \left[\tfrac{(1-b)(1-\phi)}{1-b-c}\int_0^S -2\zeta\ (r-s)\ e^{-\zeta(r-s)^2}\ ds\right]\varepsilon(r) + g_2'(r) \tag{50}$$

$$\begin{aligned} \varepsilon''(r) &= \left[\frac{\gamma\delta}{1-b-c}\int_0^S \left[-2\delta + 4\delta^2(r-s)^2\right] e^{-\delta(r-s)^2}ds\right] y(r) + \\ & \left[\tfrac{(1-b)(1-\phi)}{1-b-c}\int_0^S \left[-2\zeta + 4\zeta^2\ (r-s)^2\right]\ e^{-\zeta(r-s)^2}\ ds\right]\varepsilon(r) + g_2''(r) \end{aligned} \tag{51}$$

In equations (48)-(51), we observe that $y'(r),\ y''(r),\ \varepsilon'(r), \varepsilon''(r)$ are functions of $y(r),\ \varepsilon(r),$ $g_1'(r),\ g_1''(r),\ g_2'(r), g_2''(r)$. Substituting them into (42)-(43), we have a linear system of three algebraic equations that can be solved using Mathematica.

Appendix B: Transformation of the system of equations (24-25) to a single Fredholm equation of 2nd kind (Polyanin and Manzhirov [12]).

We define the functions $Y(r)$ and $G(r)$ on $[0, 2S]$, where $Y(r) = y_i(r-(i-1)S)$ and $G(r) = g_i(r-(i-1)S)$ for $(i-1)S \leq r \leq iS$.[10] Next, we define the kernel $C(r, \bar{r})$ on the square $[0, 2S] \times [0, 2S]$ as follows: $C(r, \bar{s}) = k_{ij}(r-(i-1)S,\ \bar{r}-(j-1)S)$ for $(i-1)S \leq r \leq iS$ and $(j-1)S \leq \bar{r} \leq jS$.

So, the system of equations (24-25) can be rewritten as the single Fredholm equation: $Y(r) - \frac{1}{1-b-c}\int_0^{2S} C(r,s)\ Y(s)\ ds = G(r),$ where $0 \leq r \leq 2S$.

If the kernel $k_{ij}(r,s)$ is square integrable on the square $[0,S] \times [0,S]$ and $g_i(r)$ are square integrable functions on $[0,S]$, then the kernel $C(r,s)$ is square integrable on the new square: $[0,2S] \times [0,2S]$ and $G(r)$ is square integrable on $[0,2S]$.

[10] We assume that $y_1 = y$ and $y_2 = \varepsilon$, so as to follow the notation of our model.