Expectations Traps and Coordination Failures: Selecting among Multiple Discretionary Equilibria*

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Abstract

Discretionary policymakers cannot manage private-sector expectations and cannot coordinate the actions of future policymakers. As a consequence, expectations traps and coordination failures can occur and multiple equilibria can arise. To utilize the explanatory power of models with multiple equilibria it is first necessary to understand how an economy arrives to a particular equilibrium. In this paper, we employ notions of learnability, self-enforceability, and properness to motivate and develop a suite of equilibrium selection criteria. Central among these criteria are whether the equilibrium is learnable by private agents and jointly learnable by private agents and the policymaker. We use two New Keynesian policy models to identify the strategic interactions that give rise to multiple equilibria and to illustrate our equilibrium selection methods. Importantly, unless the Pareto-preferred equilibrium is learnable by private agents, we find little reason to expect coordination on that equilibrium.

Key Words: Discretionary policymaking, multiple equilibria, coordination, equilibrium selection.

JEL References: E52, E61, C62, C73

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1 Introduction

Discretionary policymakers can fall foul of expectations traps and coordination failures. When private agents are forward-looking their expectations, shaped by anticipations about future policy, can influence importantly how policy today is conducted. The discretionary policymaker's Achilles heel is that when formulating policy it is unable to manage private sector expectations, and this inability, although essential for time-consistent policymaking, leaves ajar the door to multiple equilibria. When expectations cannot be managed, private agents can form expectations that, although unwelcome from the policymaker's perspective, lead private agents to react in a manner that traps the policymaker into implementing a policy that validates those expectations. The trap is closed when a policy that renders those unwelcome expectations without foundation is more costly and hence less attractive to the discretionary policymaker than a policy that accommodates them.

The fact that multiple equilibria produced by the policymaker's inability to manage private sector expectations can beset discretionary control problems is troublesome, yet hugely important. Troublesome, because efforts to solve or mitigate the time-consistency problem associated with optimal policymaking rely invariably on there being a unique discretionary equilibrium. A Rogoff-style (Rogoff (1985)) approach of delegating objectives to a discretionary policymaker (as per Jensen and McCallum (2002) and Walsh (2003), among others) is unlikely to be successful unless it also solves the coordination problem. Similarly, to the extent that an optimal contract (Walsh (1995)) can successfully overcome the time-consistency problem, it too should address the coordination problem. Important, because it means that discretionary policy behavior can be considerably richer and more varied than is commonly appreciated, with switches among equilibria becoming a potential source of economic volatility. Moreover, because the mechanisms that produce multiple equilibria involve the strategic interactions between agents over time, they are not precluded by linear constraints and quadratic objectives. As a consequence, much research analyzing discretionary policymaking since Kydland and Prescott (1977) may have inadvertently considered only one of several equilibria, potentially overlooking essential aspects of discretionary policy behavior.

It is not unusual for economies to transition between periods of high and low inflation, a phenomenon that expectations traps have the potential to explain (Albanesi et al. (2003)). Similarly, transitions from one equilibrium to another offers an explanation for policy regime changes, like those analyzed by Davig and Leeper (2006). Accordingly, an explanation for the change in U. S. inflation behavior between the 1970s and the 1980s could be that Volcker's appointment to Federal Reserve Chairman served to coordinate expectations and behavior, switching the economy from one discretionary equilibrium to another. However, in order to utilize the explanatory power of multiple equilibria it is necessary to first consider how an economy arrives at a particular equilibrium. In the words of Benabib and Farmer (1999), pp. 438, "in any model with multiple equilibria one must address the issue of how an equilibrium comes about".

In this paper, we study multiple equilibria in the class of infinite-horizon linear-quadratic discretionary control problems. We describe the control problem facing the discretionary policymaker and, drawing on Oudiz and Sachs (1985) and Currie and Levine (1985), Currie and Levine (1993), reinterpret the control problem as a dynamic game between policymakers at different points in time. An important aspect of this game is that within a period the policymaker is a (Stackelberg) leader with respect to private agents. Feedback equilibria to the discretionary control problem correspond to Markov-perfect Stackelberg-Nash equilibria to the dynamic game. We show how strategic interaction among current and future policymakers, operating through endogenous state variables and private sector expectations, leads to a form of strategic complementarity (Cooper and John (1988)) and makes expectations traps and coordination failures possible.

We approach the coordination problem inherent in equilibrium selection from two angles. First, we want to understand whether the economic agents can rationalize all discretionary equilibria, based on a simple and relevant behavioural model. We consider learning as a coordinating mechanism for equilibrium selection (Evans (1986)), drawing on the large literature that employs learning to analyze coordination in rational expectations models (Guesnerie and Woodford (1992), Evans and Guesnerie (1993), Evans and Guesnerie (2003), Evans and Guesnerie (2005), Evans and Honkapohja (2001)). With agents learning, and allowing private agents and/or the policymaker to be learning, we develop three expectational stability conditions whose satisfaction determines whether private agents and/or the policymaker might coordinate on a particular equilibrium. Among these three sets of stability conditions, we show that the key conditions are those indicating whether an equilibrium is learnable by private agents in isolation and by private agents and the policymaker jointly.

Second, we want to see if the policy maker can help to resolve the coordination problem. Although unilateral deviations from a Stackelberg-Nash equilibrium are not beneficial, several consequent policy makers can form a coalition. We consider whether the potential for such non-cooperative coalitions to form might effectively rule out some equilibria (Bernheim et al. (1987)). Pursuing this idea, we examine whether the Stackelberg-Nash equilibria we obtain are self-enforceable.

We illustrate these selection criteria by means of two examples of models with multiple discretionary equilibria. The first model is a version of the sticky price New Keynesian model with government debt adapted from Leeper (1991) by Blake and Kirsanova (2008). The second model is a sticky price New Keynesian model in the spirit of Woodford (2003) Ch.5, but with partial inflation indexation. In each model, the task confronting the policymaker is to stabilize inflation without impacting unduly the real economy. Inflation, in these models, is determined by the expected path of real marginal costs, so the policy challenge is to generate an appropriate path for real marginal costs. Since inflation depends on the entire expected path for real marginal costs while the discretionary policymaker can choose only today's policy, the policy chosen today depends necessarily on expected future policy. At the same time, the decisions that future policymakers make depend materially on the economic circumstances that they find themselves in, and hence on the choices previous policymakers have made. This interaction between policymakers over time produces coordination failure and leads to multiple equilibria.

The model with government debt is extremely simple and this allows us to explore the proposed equilibrium selection mechanisms and derive most of our results in analytical form. In contrast, the second model is a version of the standard dynamic stochastic general equilibrium model and is especially notable because it resides at the core of many New Keynesian models, such as those developed by Christiano et al. (2005) and Christiano et al. (2005) and Smets and Wouters (2007). As such, it is much more complex, can only be solved numerically, but it allows us to demonstrate how the selection mechanisms work in empirically relevant, but analytically less tractable, models.

The remainder of the paper is structured as follows. In Section 2 we present the approach to the equilibrium selection in the general LQ RE framework. In Section 3 we study the simple NK model with government debt and obtain all results in analytical form. In Section 4 we employ the selection criteria to the core DSGE model. Section 5 concludes.

2 Equilibrium Selection in LQ RE Models

2.1 The discretionary control problem

In this section, we outline the well known control problem facing a discretionary policymaker. We then reinterpret this control problem as a non-cooperative dynamic game and show that the standard optimal discretionary policy is a symmetric Markov-perfect Nash equilibrium of a dynamic game in which the policymaker is a Stackelberg leader and private agents are followers. To make explicit the game's leadership structure, we call this equilibrium a symmetric Markov-perfect Stackelberg-Nash equilibrium. Finally, we show that solving for a symmetric Markov-perfect Stackelberg-Nash equilibrium in this game requires solving a particular fix-point problem.

2.1.1 Constraints and objectives

The economic environment is one in which n_1 predetermined variables, $\mathbf{x}_t$, and n_2 nonpredetermined variables, $\mathbf{y}_t$, $t = 0, 1, ..., \infty$, evolve over time according to

$$\begin{aligned} \mathbf{x}_{t+1} &= \mathbf{A}_{11}\mathbf{x}_t + \mathbf{A}_{12}\mathbf{y}_t + \mathbf{B}_1\mathbf{u}_t + \mathbf{v}_{\mathbf{x}t+1}, & (1) \\ \mathrm{E}_t\mathbf{y}_{t+1} &= \mathbf{A}_{21}\mathbf{x}_t + \mathbf{A}_{22}\mathbf{y}_t + \mathbf{B}_2\mathbf{u}_t, & (2) \end{aligned}$$

where $\mathbf{u}_t$ is a $p \times 1$ vector of control variables, $\mathbf{v}_{\mathbf{x}t} \sim i.i.d.\,[\mathbf{0}, \mathbf{\Sigma}]$ is an $v \times 1$ $(1 \leq v \leq n_1)$ vector of white-noise innovations, and E_t is the mathematical expectations operator conditional upon period t information. Equations (1) and (2) capture aggregate constraints and technologies and the behavior (aggregate first-order conditions) of private agents. For their part, private agents are comprised of households and firms who are ex ante identical, respectively, infinitely lived, and atomistic. The matrices $\mathbf{A}_{11}$, $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, $\mathbf{A}_{22}$, $\mathbf{B}_1$, and $\mathbf{B}_2$ are conformable with $\mathbf{x}_t$, $\mathbf{y}_t$, and $\mathbf{u}_t$ as necessary and contain the parameters that govern preferences and technologies. Importantly, the matrix $\mathbf{A}_{22}$ is assumed to have full rank.

In addition to private agents, the economy is populated by a large player, a policymaker. For each period t, the period-t policymaker's objectives are described by the loss function

$$L_t = \mathrm{E}_t \sum_{k=t}^{\infty} \beta^{(k-t)} \left[\mathbf{z}_k'\mathbf{W}\mathbf{z}_k + 2\mathbf{z}_k'\mathbf{U}\mathbf{u}_k + \mathbf{u}_k'\mathbf{Q}\mathbf{u}_k\right], \quad (3)$$

where $\beta \in (0, 1)$ is the discount factor and $\mathbf{z}_k = \left[\begin{array}{cc} \mathbf{x}_k' & \mathbf{y}_k' \end{array}\right]'$. We assume that the weighting matrices $\mathbf{W}$ and $\mathbf{Q}$ are symmetric and, to ensure that the loss function is convex, that the matrix $\left[\begin{array}{cc} \mathbf{W} & \mathbf{U} \\ \mathbf{U}' & \mathbf{Q} \end{array}\right]$ is positive semi-definite.[1] We assume that the policymaker is a Stackelberg leader

[1] It is standard to assume that the weighting matrices, $\mathbf{W}$ and $\mathbf{Q}$, are symmetric positive semi-definite and

and that private agents are followers; we further assume that the policymaker does not have access to a commitment technology and that policy is conducted under discretion.[2] With policy conducted under discretion, the policymaker sets its control variables, $\mathbf{u}_t$, each period to minimize equation (3), taking the state, $\mathbf{x}_t$, and the decision rules of all future agents as given. Since the policymaker is a Stackelberg leader, the period-t policy decision is formulated taking equation (2) as well as equation (1) into account.

The control problem described above has many of the characteristics of an infinite horizon non-cooperative dynamic game, and is commonly viewed as such. Following Oudiz and Sachs (1985), Currie and Levine (1993), and Cohen and Michel (1988), the strategic players in the game are the (infinite) sequence of policymakers with private agents behaving competitively. Although individual private agents are not strategic players in aggregate they are not inconsequential. Private agents are important because private-sector expectations are the conduit through which strategic interaction between current and future policymakers occurs. In this decision problem, policy behavior is described by a policy strategy, private-agent behavior is described by a private sector strategy, the expectations operator (E_t) and policy loss (payoff) are induced by the policy and private sector strategies, and the equilibrium that we seek to analyze is a symmetric Markov-perfect Stackelberg-Nash equilibrium.

2.1.2 Some useful definitions and equilibrium concepts

In the previous section we emphasized that the discretionary control problem can be modeled as a non-cooperative dynamic game, with the decisions of the policymaker and of private agents taking the form of strategies. Further, we noted that because the policymaker is assumed to be a Stackelberg leader the discretionary equilibrium that we are interested in is a symmetric Markov-perfect Stackelberg-Nash equilibrium. We now make these terms precise.[3]

Definition 1 *A policy strategy* $\mathbf{S}$ *is a sequence of policy rules* $\{\mathbf{F}_t\}_0^\infty$*, where* $\mathbf{F}_t$ *is a function that maps* $\{\mathbf{x}_t\}_0^t$ *to* $\mathbf{u}_t$*. A policy strategy is said to be a Markov policy strategy if and only if each policy rule* $\mathbf{F}_t$ *is a function that maps* $\mathbf{x}_t$ *to* $\mathbf{u}_t$*. We denote by* $\mathbf{S}_{-t}$ *the sequence of policy rules* $\{\mathbf{F}_s\}_0^\infty$ *excluding* $\mathbf{F}_t$*.*

Definition 2 *A private sector strategy* $\mathbf{T}$ *is a sequence of decision rules* $\{\mathbf{H}_t\}_0^\infty$*, where* $\mathbf{H}_t$ *is a function that maps* $\{\mathbf{x}_t\}_0^t$ *to* $\mathbf{y}_t$*. A private sector strategy is said to be a Markov private sector strategy if and only if each decision rule* $\mathbf{H}_t$ *is a function that maps* $\mathbf{x}_t$ *to* $\mathbf{y}_t$*. We denote by* $\mathbf{T}_{-t}$ *the sequence of decision rules* $\{\mathbf{H}_s\}_0^\infty$ *excluding* $\mathbf{H}_t$*.*

symmetric positive definite, respectively (see Anderson, Hansen, McGrattan, and Sargent (1996), for example). However, since many economic applications involve a loss function that places no penalty on the control variables, we note that the requirement of $\mathbf{Q}$ being positive definite can be weakened to $\mathbf{Q}$ being positive semi-definite if additional assumptions about other system matrices are met (Clements and Wimmer, 2003).

[2] Events within a period occur as follows. After observing the state, $\mathbf{x}_t$, decisions are made first by the incumbent policymaker and subsequently by private agents. At the end of the period the shocks $\mathbf{v}_{\mathbf{x}t+1}$ are realized.

[3] Although the discretionary control problem described in section 2.1 is standard in the monetary policy literature (it is the formulation used by Clarida, Galí, and Gertler (1999), for example) there are other notions of discretion in the literature. These different notions of discretion are associated either with different dynamic games or with different equilibrium concepts. Cohen and Michel (1988), de Zeeuw and van der Ploeg (1991), and Chow (1997, chapter 6) provide useful discussions.

Definition 3 *A policy strategy* $\mathbf{S}$ *is a Stackelberg-Nash equilibrium if for every decision period* t*: i)* $\mathbf{F}_t$ *minimizes equation (3) subject to equations (1) and (2) and* $\mathbf{x}_t$ *known, taking* $\mathbf{S}_{-t}$ *and* $\mathbf{T}_{-t}$ *as given; and ii)* $\mathbf{H}_t$ *satisfies equations (1) and (2), taking* $\mathbf{S}$ *and* $\mathbf{T}_{-t}$*, as given.*

Definition 4 *A policy strategy* $\mathbf{S}$ *is a perfect Stackelberg-Nash equilibrium if for every decision period* t *and any history* $\{\mathbf{F}_s, \mathbf{H}_s\}_0^{t-1}$*: i)* $\mathbf{F}_t$ *minimizes equation (3) subject to equations (1) and (2) and* $\mathbf{x}_t$ *known, taking* $\mathbf{S}_{-t}$ *and* $\mathbf{T}_{-t}$ *as given; and ii)* $\mathbf{H}_t$ *satisfies equations (1) and (2), taking* $\mathbf{S}$ *and* $\mathbf{T}_{-t}$ *as given.*

A perfect Stackelberg-Nash equilibrium is time-consistent because it is subgame perfect. However, the strategies that characterize equilibrium are not necessarily Markov strategies and, as a consequence, trigger-strategy equilibria, and other equilibria supported by threats and punishments are not ruled out. The sustainable equilibria studied by Chari and Kehoe (1990), Ireland (1997), and Kurozumi (2008) as well as the "reputational" equilibria examined by Barro and Gordon (1983) are all examples of perfect Stackelberg-Nash equilibria.

Definition 5 *A policy strategy* $\mathbf{S}$ *is a Markov-perfect Stackelberg-Nash equilibrium if restricting* $\mathbf{S}$ *to be a Markov policy strategy and* $\mathbf{T}$ *to be a Markov private sector strategy, for every time period* t *and any history of Markov policy and decision rules* $\{\mathbf{F}_s, \mathbf{H}_s\}_0^{t-1}$*: i)* $\mathbf{F}_t$ *minimizes equation (3) subject to equations (1) and (2) and* $\mathbf{x}_t$ *known, taking* $\mathbf{S}_{-t}$ *and* $\mathbf{T}_{-t}$ *as given; and ii)* $\mathbf{H}_t$ *satisfies equations (1) and (2), taking* $\mathbf{S}$ *and* $\mathbf{T}_{-t}$ *as given.*

Definition 6 *A policy strategy* $\mathbf{S}$ *is a symmetric Markov-perfect Stackelberg-Nash equilibrium if and only if: i)* $\mathbf{S}$ *is a Markov-perfect Stackelberg-Nash equilibrium in which* $\mathbf{F}_t = \mathbf{F}$, $\forall\ t$*; and ii)* $\mathbf{T}$ *is a Markov private sector strategy in which* $\mathbf{H}_t = \mathbf{H}$, $\forall\ t$*.*

2.1.3 Characterizing equilibrium

For the decision problem summarized by equations (1)—(3), we now describe the equilibrium conditions that characterize a symmetric Markov-perfect Stackelberg-Nash equilibrium, focusing on equilibria for which the decision rules are linear in the state vector.

First, if a symmetric Markov-perfect Stackelberg-Nash equilibrium exists, then in this equilibrium the behavior of the policymaker and private agents in all states, $\mathbf{x}_t$, and in all decision periods, $t = 0, ..., \infty$, is described by the linear rules

$$\mathbf{u}_t = \mathbf{F}\mathbf{x}_t, \tag{4}$$
$$\mathbf{y}_t = \mathbf{H}\mathbf{x}_t, \tag{5}$$

respectively. In this equilibrium, the law-of-motion for the predetermined variables is given by

$$\mathbf{x}_{t+1} = \mathbf{M}\mathbf{x}_t + \mathbf{v}_{\mathbf{x}t+1},$$

where the spectral radius of $\mathbf{M}$ is less than $\beta^{-\frac{1}{2}}$. Further, since the loss function is quadratic and the constraints are linear, the payoff to the policymaker in period t that corresponds to these rules is summarized by the quadratic state-contingent value function

$$V(\mathbf{x}_t) = \mathbf{x}_t'\mathbf{V}\mathbf{x}_t + d,$$

where $\mathbf{V}$ is symmetric positive semi-definite. Importantly, because the policy rule, $\mathbf{F}$, and the decision rule, $\mathbf{H}$, in a symmetric Markov-perfect Stackelberg-Nash equilibrium apply in all states, the subgames one needs to consider when solving for a symmetric Markov-perfect Stackelberg-Nash equilibrium are those indexed only by time.

Second, if a symmetric Markov-perfect Stackelberg-Nash equilibrium exists for the subgame beginning in period $t+1$, then one can condition the subgame beginning in period t on the $\overline{\mathbf{H}}$, $\overline{\mathbf{F}}$, $\overline{\mathbf{M}}$, $\overline{\mathbf{V}}$, and $\overline{d}$ that characterize the equilibrium of the subgame beginning in period $t+1$. Thus, the decision problem facing the policymaker in the subgame beginning in period t is to choose a rule for setting $\mathbf{u}_t$ in order to minimize

$$\begin{aligned}\mathbf{x}_t'\mathbf{V}\mathbf{x}_t + d &= \mathbf{x}_t'\mathbf{W}_{11}\mathbf{x}_t + \mathbf{x}_t'\mathbf{W}_{12}\mathbf{y}_t + \mathbf{y}_t'\mathbf{W}_{21}\mathbf{x}_t + \mathbf{y}_t'\mathbf{W}_{22}\mathbf{y}_t + 2\mathbf{x}_t'\mathbf{U}_1\mathbf{u}_t + 2\mathbf{y}_t'\mathbf{U}_2\mathbf{u}_t + \mathbf{u}_t'\mathbf{Q}\mathbf{u}_t \\ &\quad +\beta \mathrm{E}_t\left(\mathbf{x}_{t+1}'\overline{\mathbf{V}}\mathbf{x}_{t+1} + \overline{d}\right), \qquad (6)\end{aligned}$$

subject to equations (1) and (2) and

$$\mathbf{u}_{t+1} = \overline{\mathbf{F}}\mathbf{x}_{t+1}, \qquad (7)$$

$$\mathbf{y}_{t+1} = \overline{\mathbf{H}}\mathbf{x}_{t+1}, \qquad (8)$$

and $\mathbf{x}_t$ known. Importantly, although $\overline{\mathbf{H}}$ and $\overline{\mathbf{V}}$ are functions of $\overline{\mathbf{F}}$, the problem's structure means that $\overline{\mathbf{F}}$ does not have a separate, explicit, effect on the current period payoff, $V(\mathbf{x}_t) = \mathbf{x}_t'\mathbf{V}\mathbf{x}_t + d$. Consequently, as this decision problem is formulated, equation (7) does not bind as a separate constraint.

Using equation (8) to form $\mathrm{E}_t\mathbf{y}_{t+1}$, substituting the resulting expression into equation (2), and exploiting equation (1), we obtain the aggregate private-sector reaction function

$$\mathbf{y}_t = \mathbf{J}\mathbf{x}_t + \mathbf{K}\mathbf{u}_t, \qquad (9)$$

where

$$\mathbf{J} = \left(\mathbf{A}_{22} - \overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{A}_{11} - \mathbf{A}_{21}\right), \qquad (10)$$

$$\mathbf{K} = \left(\mathbf{A}_{22} - \overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{B}_{1} - \mathbf{B}_{2}\right). \qquad (11)$$

Provided $rank\,(\mathbf{K}) \neq \mathbf{0}$, equation (9) implies that the period-t policymaker is a Stackelberg leader with respect to the period-t private sector. Then, substituting equation (9) into equations (6) and (1), the decision problem facing the policymaker in the subgame beginning in period t is to choose a rule for setting $\mathbf{u}_t$ in order to minimize

$$\mathbf{x}_t'\mathbf{V}\mathbf{x}_t + d = \mathbf{x}_t'\widehat{\mathbf{W}}\mathbf{x}_t + 2\mathbf{x}_t'\widehat{\mathbf{U}}\mathbf{u}_t + \mathbf{u}_t'\widehat{\mathbf{Q}}\mathbf{u}_t + \beta\mathrm{E}_t\left(\mathbf{x}_{t+1}'\overline{\mathbf{V}}\mathbf{x}_{t+1} + \overline{d}\right), \qquad (12)$$

subject to

$$\mathbf{x}_{t+1} = \widehat{\mathbf{A}}\mathbf{x}_t + \widehat{\mathbf{B}}\mathbf{u}_t + \mathbf{v}_{\mathbf{x}t+1}, \qquad (13)$$

where

$$\widehat{\mathbf{W}} = \mathbf{W}_{11} + \mathbf{W}_{12}\mathbf{J} + \mathbf{J}'\mathbf{W}_{21} + \mathbf{J}'\mathbf{W}_{22}\mathbf{J}, \qquad (14)$$

$$\widehat{\mathbf{U}} = \mathbf{W}_{12}\mathbf{K} + \mathbf{J}'\mathbf{W}_{22}\mathbf{K} + \mathbf{U}_1 + \mathbf{J}'\mathbf{U}_2, \qquad (15)$$

$$\widehat{\mathbf{Q}} = \mathbf{Q} + \mathbf{K}'\mathbf{W}_{22}\mathbf{K} + 2\mathbf{K}'\mathbf{U}_2, \qquad (16)$$

$$\widehat{\mathbf{A}} = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{J}, \qquad (17)$$

$$\widehat{\mathbf{B}} = \mathbf{B}_1 + \mathbf{A}_{12}\mathbf{K}. \qquad (18)$$

Conditional on $\overline{\mathbf{H}}$ and $\overline{\mathbf{V}}$ (and $\overline{\mathbf{F}}$), equations (12) and (13) describe a standard linear-quadratic dynamic programming problem. To guarantee existence of a solution, we need $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}\right)$ to be a controllable pair and $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{W}}\right)$ to be a detectable pair (Laub (1979), Anderson et al. (1996)). Suppose that, for a given $\mathbf{J}$ and $\mathbf{K}$, $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}\right)$ is a controllable pair and $\left(\widehat{\mathbf{A}}, \widehat{\mathbf{W}}\right)$ is a detectable pair, then the solution to the subgame beginning in period t has the form of rules (4) and (5), with

$$\mathbf{F} = -\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{B}}\right)^{-1}\left(\widehat{\mathbf{U}}' + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{A}}\right), \tag{19}$$

$$\mathbf{0} = \overline{\mathbf{H}}\mathbf{A}_{12}\mathbf{H} - \mathbf{A}_{22}\mathbf{H} + \overline{\mathbf{H}}\left(\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}\right) - \mathbf{A}_{21} - \mathbf{B}_2\mathbf{F}, \tag{20}$$

$$\mathbf{V} = \widehat{\mathbf{W}} + 2\widehat{\mathbf{U}}\mathbf{F} + \mathbf{F}'\widehat{\mathbf{Q}}\mathbf{F} + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)'\overline{\mathbf{V}}\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right), \tag{21}$$

$$d = \beta tr\left(\mathbf{V}\mathbf{\Sigma}\right) + \beta\overline{d}. \tag{22}$$

From $\mathbf{F}$ and $\mathbf{H}$, the matrix $\mathbf{M}$ in the law-of-motion for the predetermined variables is then given by

$$\mathbf{M} = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} + \mathbf{B}_1\mathbf{F}. \tag{23}$$

Because $\overline{\mathbf{H}}$, $\overline{\mathbf{F}}$, $\overline{\mathbf{M}}$, $\overline{\mathbf{V}}$, and $\overline{d}$ represent a symmetric Markov-perfect Stackelberg-Nash equilibrium for the subgame beginning in period $t+1$, any fix-point of equations (19)—(23) in which $\mathbf{H} = \overline{\mathbf{H}}$, $\mathbf{F} = \overline{\mathbf{F}}$, $\mathbf{M} = \overline{\mathbf{M}}$, $\mathbf{V} = \overline{\mathbf{V}}$, and $d = \overline{d}$, such that $\mathbf{V}$ is symmetric positive semi-definite and $\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)$ has full rank, is a symmetric Markov-perfect Stackelberg-Nash equilibrium for the subgame beginning in period t.

2.2 Equilibrium selection

In this section we discuss several mechanisms which have a potential to reduce the number of empirically relevant equilibria. Specifically, we focus on three coordination/selection mechanisms: expectational stability (Evans (1986)) and self-enforceability (Bernheim et al. (1987), Bernheim and Whinston (1987)).

2.2.1 Learning and expectational stability

In order to arrive to a rational expectations equilibrium the agents may need to to undertake a thought process to revise how they form expectations based on how these expectations affect the actual economy. In other words, the agents may need to rationalize, or equate, a perceived low-of-motion of the economy with the actual law-of-motion. If this 'natural revision rule' returns the system to an equilibrium, the equilibrium is said to be 'expectationally stable', see Evans (1986). The required revisions occur in meta-time, i.e. eductive in nature, and constitute the process of learning.

Evans (1986) motivates expectational stability as a selection criterion in rational expectations models with multiple equilibria. Loosely speaking, a rational expectations equilibrium is expectationally stable if, following small deviations to the expectation formation process, the system returns to that equilibrium under a 'natural revision rule'. There is a close connection between expectational stability and real-time least-squares learnability of a rational expectations equilibrium (Marcet and Sargent (1989), Evans and Honkapohja (2001)).

Like Evans (1986) and Evans and Guesnerie (2003), Evans and Guesnerie (2005) we view learning as a mechanism through which agents may coordinate on an equilibrium. We go further than the existing studies, as we explicitly treat the policy maker as a rational expectations agent who also is learning. Somewhat more restrictively and differently, the existing studies investigate the stability of rational expectation *private sector* equilibria, in which the private sector rationally responds to a policy *rule*. It is usually assumed that such rule is imposed by the policy maker in every period, and no future reoptimizations are expected by the private sector, even if the *exact form and parameters of this rule* is a result of discretionary optimization (see Evans and Honkapohja (2003)). As a consequence, we analyze learning problems where the learning is eductive in nature with agents revising their behavior in meta-time based on the outcomes of thought experiments. The notion of stability under learning that we consider is iterative expectational stability (IE-stability) and we derive all conditions in Section 2. Specifically, we consider private sector learning, and joint learning.[4]

Recall that a symmetric Markov-perfect Stackelberg-Nash equilibrium is characterized by $\{\mathbf{H}, \mathbf{F}, \mathbf{M}, \mathbf{V}, d\}$. Because $\mathbf{M}$ and d follow immediately and uniquely from $\mathbf{F}$, $\mathbf{H}$, and $\mathbf{V}$, we implement the partitioning $\{\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}, \{\mathbf{M}, d\}\}$ and focus on $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$ in what follows. Specifically, we consider:

1. Private sector learning, where we analyze whether private agents can learn $\mathbf{H}$, conditional on $\{\mathbf{F}, \mathbf{V}\}$.
2. Policymaker learning, where we analyze whether the policymaker can learn $\{\mathbf{F}, \mathbf{V}\}$, conditional on $\{\mathbf{H}\}$.
3. Joint learning, where we analyze whether private agents and the policymaker can learn $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$ jointly.

Preliminaries To place the three learning problems in a unified framework, let us denote by $\mathbf{\Phi}$ the object(s) to be learned. Thus, in the case where only private agents are learning $\mathbf{\Phi} = \{\mathbf{H}\}$. Then, to determine whether $\mathbf{\Phi}$ is learnable we construct and analyze the T-map that relates a perception of $\mathbf{\Phi}$, denoted $\overline{\mathbf{\Phi}}$, to an actual $\mathbf{\Phi}$, $\mathbf{\Phi} = T\left(\overline{\mathbf{\Phi}}\right)$.

Definition 7 *A fix-point, $\mathbf{\Phi}^*$, of the T-map, $\mathbf{\Phi} = T\left(\overline{\mathbf{\Phi}}\right)$, is said to be IE-stable if*

$$\lim_{k\uparrow\infty} T^k\left(\overline{\mathbf{\Phi}}\right) = \mathbf{\Phi}^*,$$

for all $\overline{\mathbf{\Phi}} \neq \mathbf{\Phi}^*$.

It follows that $\mathbf{\Phi}^*$ is IE-stable if and only if it is a stable fix-point of the difference equation

$$\mathbf{\Phi}_{k+1} = T\left(\mathbf{\Phi}_k\right), \tag{24}$$

[4] We prove in Appendix that any equilibrium is IE-stable under the policy maker's learning so we do not consider it here.

where index k denotes the step of the updating process. Similarly,

Definition 8 *A fix-point, $\mathbf{\Phi}^*$, of the T-map, $\mathbf{\Phi} = T\left(\overline{\mathbf{\Phi}}\right)$, is said to be locally IE-stable if*

$$\lim_{k\uparrow\infty} T^k\left(\overline{\mathbf{\Phi}}\right) = \mathbf{\Phi}^*,$$

for all $\overline{\mathbf{\Phi}}$ about a neighborhood of $\mathbf{\Phi}^*$.

Let the derivative of the T-map be denoted $DT\left(\mathbf{\Phi}^*\right)$, then it is straightforward to prove the following Lemma (see Appendix A).

Lemma 1 *Assume that the derivative map, $DT\left(\mathbf{\Phi}^*\right)$, has no eigenvalues with modulus equal to 1. A fix-point, $\mathbf{\Phi}^*$, of the T-map, $\mathbf{\Phi} = T\left(\overline{\mathbf{\Phi}}\right)$, is locally IE-stable if and only if all eigenvalues of the derivative map, $DT\left(\mathbf{\Phi}^*\right)$, have modulus less than 1.*

Eductive learning by private agents We begin with the case in which only private agents are learning and examine whether private agents can learn $\mathbf{H}$, given $\{\mathbf{F}, \mathbf{V}\}$. For a given policy rule, $\mathbf{u}_t = \mathbf{F}\mathbf{x}_t$, and a postulated private sector decision rule

$$\mathbf{y}_t = \overline{\mathbf{H}}\mathbf{x}_t,$$

the actual private sector decision rule takes the form

$$\mathbf{y}_t = \mathbf{H}\mathbf{x}_t,$$

where

$$\mathbf{H} = \left(\overline{\mathbf{H}}\mathbf{A}_{12} - \mathbf{A}_{22}\right)^{-1}\left[\mathbf{A}_{21} + \mathbf{B}_2\mathbf{F} - \overline{\mathbf{H}}\left(\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}\right)\right]. \tag{25}$$

Equation (25) describes the T-map, $T(\overline{\mathbf{H}})$, from $\overline{\mathbf{H}}$ to $\mathbf{H}$; it is, of course, equivalent to equation (20). Appendix B proves the following Lemma.

Lemma 2 *A symmetric Markov-perfect Stackelberg-Nash equilibrium is locally IE-stable under private sector learning if and only if all eigenvalues of*

$$-\left[\mathbf{I} \otimes \left(\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22}\right)\right]^{-1}\left[\left(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} + \mathbf{B}_1\mathbf{F}\right)' \otimes \mathbf{I}\right]$$

have modulus less than 1.

Because the eigenvalues of $\mathbf{M} = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H} + \mathbf{B}_1\mathbf{F}$ are all strictly less than $\beta^{-\frac{1}{2}}$, equilibria that are not locally IE-stable under private sector learning are those for which $(\mathbf{H}\mathbf{A}_{12} - \mathbf{A}_{22})$ is close to equaling the null matrix.

Eductive learning by the leader We now turn to the case where the policymaker is learning, but private agents are not. Here we examine whether the policymaker can learn $\{\mathbf{F}, \mathbf{V}\}$, given $\{\mathbf{H}\}$. We show that although learning by policymakers is interesting and important in many contexts, here this local IE-stability criterion cannot discriminate among equilibria.

For a given private sector decision rule, $\mathbf{y}_t = \mathbf{H}\mathbf{x}_t$, and a postulated policy rule

$$\mathbf{u}_t = \overline{\mathbf{F}}\mathbf{x}_t,$$

and a postulated value function matrix $\overline{\mathbf{V}}$, the T-map $T(\overline{\mathbf{F}}, \overline{\mathbf{V}})$, from $\{\overline{\mathbf{F}}, \overline{\mathbf{V}}\}$ to $\{\mathbf{F}, \mathbf{V}\}$ is described by the following updating relationships

$$\mathbf{F} = -\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{B}}\right)^{-1}\left(\widehat{\mathbf{U}}' + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{A}}\right), \tag{26}$$

$$\mathbf{V} = \widehat{\mathbf{W}} + 2\widehat{\mathbf{U}}\mathbf{F} + \mathbf{F}'\widehat{\mathbf{Q}}\mathbf{F} + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)'\overline{\mathbf{V}}\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right), \tag{27}$$

where $\widehat{\mathbf{W}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{Q}}$, $\widehat{\mathbf{A}}$, and $\widehat{\mathbf{B}}$ are defined by equations (14)—(18) and do not depend on $\mathbf{F}$ or $\mathbf{V}$ (or on $\overline{\mathbf{F}}$ or $\overline{\mathbf{V}}$). Notice, that $\mathbf{F}$, given $\mathbf{H}$, is uniquely determined by $\mathbf{V}$, so the key to learning $\mathbf{F}$ is to learn $\mathbf{V}$. As a consequence, without loss of generality we can substitute equation (26) into equation (27) and analyze the the learning problem using the concentrated T-map $T(\overline{\mathbf{V}}) = \mathbf{V}$. Appendix C

Lemma 3 *All symmetric Markov-perfect Stackelberg-Nash equilibria are locally IE-stable under policymaker learning.*

Joint eductive learning Finally, we analyze the case in which both private agents and the policymaker are learning. The postulated policy and decision rules are

$$\mathbf{y}_t = \overline{\mathbf{H}}\mathbf{x}_t,$$
$$\mathbf{u}_t = \overline{\mathbf{F}}\mathbf{x}_t,$$

and the postulated value function matrix is $\overline{\mathbf{V}}$. Then the actual policy and decision rules are given by

$$\mathbf{H} = \mathbf{J} + \mathbf{K}\mathbf{F}, \tag{28}$$

$$\mathbf{F} = -\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{B}}\right)^{-1}\left(\widehat{\mathbf{U}} + \beta\widehat{\mathbf{B}}'\overline{\mathbf{V}}\widehat{\mathbf{A}}\right), \tag{29}$$

$$\mathbf{V} = \widehat{\mathbf{W}} + 2\widehat{\mathbf{U}}\mathbf{F} + \mathbf{F}'\widehat{\mathbf{Q}}\mathbf{F} + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)'\overline{\mathbf{V}}\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right), \tag{30}$$

where

$$\mathbf{J} = \left(\mathbf{A}_{22} - \overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{A}_{11} - \mathbf{A}_{21}\right), \tag{31}$$

$$\mathbf{K} = \left(\mathbf{A}_{22} - \overline{\mathbf{H}}\mathbf{A}_{12}\right)^{-1}\left(\overline{\mathbf{H}}\mathbf{B}_{1} - \mathbf{B}_{2}\right), \tag{32}$$

and $\widehat{\mathbf{W}}$, $\widehat{\mathbf{U}}$, $\widehat{\mathbf{Q}}$, $\widehat{\mathbf{A}}$, and $\widehat{\mathbf{B}}$ are defined by equations (14)—(18) and are functions of $\mathbf{J}$ and $\mathbf{K}$.

Given equations (31) and (32), equations (28)—(30) describe the T-map, $T\left(\overline{\mathbf{H}}, \overline{\mathbf{F}}, \overline{\mathbf{V}}\right)$, from $\left\{\overline{\mathbf{H}}, \overline{\mathbf{F}}, \overline{\mathbf{V}}\right\}$, to $\{\mathbf{H}, \mathbf{F}, \mathbf{V}\}$. Appendix D proves the next Lemma.

Lemma 4 *A symmetric Markov-perfect Stackelberg-Nash equilibrium is locally IE-stable under joint learning if and only if all eigenvalues of the matrix* $\mathbf{P}^{-1}\mathbf{L}$ *in*

$$vec\left[d\left(\mathbf{G}\right)\right]=\mathbf{P}^{-1}\ \mathbf{L}vec\left[d\left(\overline{\mathbf{G}}\right)\right],$$

where $vec\left[d\left(\mathbf{G}\right)\right] = \left[\begin{array}{ccc} vec\left[d\left(\mathbf{H}\right)\right]' & vec\left[d\left(\mathbf{F}\right)\right]' & vec\left[d\left(\mathbf{V}\right)\right]' \end{array}\right]'$ *and* $\mathbf{P}$ *and* $\mathbf{L}$ *are characterized below, have modulus less than* 1.

Before leaving this section, we wish to emphasize that the IE-stability criteria associated with private sector learning and joint learning, although connected, are distinct. Joint learnability of an equilibrium neither implies nor is implied by private sector learnability of that equilibrium.

2.2.2 Self-enforceability

Multiple equilibria arise because of lack of coordination. The current policy maker chooses policy which is based on prediction that the atomistic private sector will coordinate on this equilibrium because of its members' beliefs about the next-period policy. Discretionary equilibria are symmetric Stackelberg-Nash equilibria and unilateral deviations are not beneficial. However, the ability of several consequent policy makers to form a coalition can help to coordinate on one of the equilibria. The obvious interpretation of such a coalition can be the policy maker's tenure spanning multiple decision periods. As a consequence, we model them in terms of sequential players.[5]

We approach the coordination problem by asking whether an equilibrium is self-enforceable (Bernheim et al. (1987), Bernheim and Whinston (1987)), robust to the potential formation of non-cooperative coalitions. Intuitively, policymakers can more easily coordinate on an equilibrium if that equilibrium is self-enforceable, and no group of policymakers finds it beneficial to form a coalition and deviate from equilibrium play.

Assume that the model has N symmetric Markov-perfect Stackelberg-Nash equilibria. Because the economic environment is one in which there is complete and perfect information, the existence and nature of all N equilibria is known to all agents. Moreover, the N equilibria can (invariably) be welfare ranked and, as a consequence, agents are not indifferent to which equilibrium prevails. Can the policy makers form a coalition to ensure selection of an equilibrium, and how many members should such coalition have?

Consider two equilibria, equilibria labelled $\mathfrak{G}$ for 'good' and $\mathfrak{B}$ for 'bad'. We treat policy rules associated with each equilibrium as a set of policy actions, which can be described by $\mathbf{F}^j$, $j \in \{\mathfrak{G}, \mathfrak{B}\}$.

No policy maker will want to deviate unilateraly from playing $\mathbf{F}^{\mathfrak{B}}$ once it is believed that the private sector expects that equilibrium $\mathfrak{B}$ will realize at $t = 1, 2...$ However, if the policy maker expects to stay in the office for K consequent periods, he may find it beneficial to deviate in each of K periods, even if it is believed that the private sector expects that equilibrium $\mathfrak{B}$ will

[5] One might view the group of deviating policymakers to be small if it numbers less than a policymaker's average tenure. In the U. S., Federal Reserve chairmen are appointed to a four year term, but the average tenure is somewhat longer. In the U. K., monetary policy committee members have three-year contracts that overlap to prevent members from retiring simultaneously.

realize in the post-coalition periods. The first policy maker, therefore, forms a coalition with future himself for K periods. Once such coalition is formed and $\mathbf{F}^{\mathfrak{G}}$ is known to be played for K periods, this forces the private sector to reply with $\mathbf{H}^{\mathfrak{G}}$ in each of these periods because unilateral deviations are not beneficial in a Nash equilibrium.

The minimal size of the coalition which ensures the switch to equilibrium $\mathfrak{G}$ can be determined by the following reasoning. Suppose it is believed that equilibrium $\mathfrak{B}$ will prevail in all future periods. The first policy maker will find it beneficial to form a coalition of size K and play $\mathbf{F}^{\mathfrak{G}}$ if his loss from such decision is less than the expected loss of policy $\mathbf{F}^{\mathfrak{B}}$ for K periods.

Policy maker at any intermediate time $s = K - n$, $n = 0, ..K - 2$ knows that even if the private sector in period $s = K - n$ believes that the remaining $K - n - 1$ policy makers will all play $\mathbf{F}^{\mathfrak{G}}$, all agents know that such policy is not optimal and the coalition is not sustainable, so the private sector has to react according to its reaction function with coefficients in each period

$$\mathbf{H}^{(s)} = \left(\mathbf{H}^{(s+1)}\mathbf{A}_{12} - \mathbf{A}_{22}\right)^{-1}\left[\mathbf{A}_{21} + \mathbf{B}_2\mathbf{F}^{\mathfrak{G}} - \mathbf{H}^{(s+1)}\left(\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}^{\mathfrak{G}}\right)\right]. \tag{33}$$

where $\mathbf{H}^{(s)} = \mathbf{H}^{\mathfrak{B}}$ for $s = K + 1, ...$ Reaction (64)-(65) will hold for $n = 0, .., K - 2$ periods, and the corresponding (evolution of the state

$$\mathbf{M}^{(s)} = \mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{H}^{(s)} + \mathbf{B}_1\mathbf{F}^{\mathfrak{G}}, \quad s = 1, .., K$$

should be used to to compute the welfare gain due to the coalition. Once the welfare for the first-period policy maker is higher than in equilibrium $\mathfrak{B}$, the first-period policy maker's best response becomes $\mathbf{F}^{\mathfrak{G}}$. The first-period private sector knows that it is welfare-improving for the first K policy makers to form a coalition, and so will find it optimal to react with $\mathbf{H}^{\mathfrak{G}}$ to policy $\mathbf{F}^{\mathfrak{G}}$ in all K periods.

We know that if $K = 1$, then the first-period policymaker's best response is to play $\mathbf{F}^{\mathfrak{B}}$. However, as K increases, the period-t policymaker's best response can switch from $\mathbf{F}^{\mathfrak{B}}$ to $\mathbf{F}^{\mathfrak{G}}$. For the equilibrium with $\mathbf{F}^{\mathfrak{B}}$, we calculate the number of periods of multilateral deviation K required to switch the first-period policymaker's best response from $\mathbf{F}^{\mathfrak{B}}$ to $\mathbf{F}^{\mathfrak{G}}$. Of course, although the first-period policymaker's best response may switch from $\mathbf{F}^{\mathfrak{B}}$ to $\mathbf{F}^{\mathfrak{G}}$ as K increases, it need not. In fact, whether the first-period policymaker's best response switches from $\mathbf{F}^{\mathfrak{B}}$ to $\mathbf{F}^{\mathfrak{G}}$ as K increases turns on whether equilibrium $\mathfrak{G}$ is Pareto-preferred to equilibrium $\mathfrak{B}$ and on whether equilibrium $\mathfrak{G}$ is locally IE-stable under private sector learning, see the proof in Appendix E.

Lemma 5 *The first-period policymakers best response will switch from $\mathbf{F}^{\mathfrak{B}}$ to $\mathbf{F}^{\mathfrak{G}}$ in the limit as $K \uparrow \infty$ if and only if equilibrium $\mathfrak{G}$ is Pareto-preferred to equilibrium $\mathfrak{B}$ and equilibrium $\mathfrak{G}$ is locally IE-stable under private sector learning.*

An additional issue that we consider is whether coalition forming can generate a switch from the prevailing equilibrium to the Pareto-preferred equilibrium and, if so, how large of a coalition is required to generate such a switch. It follows from Lemma 5 that the Pareto-preferred equilibrium must be locally IE-stable under private sector learning if such a switch is to occur.

3 A Simple New Keynesian Model with Government Debt

We adopt the model from Benigno and Woodford (2003).[6] We assume that all public debt consists of riskless one-period bonds. The nominal value $\mathcal{B}_t$ of end-of-period public debt then evolves according to a law of motion:

$$\mathcal{B}_t = (1 + i_{t-1})\,\mathcal{B}_{t-1} + P_t G_t - \tau_t P_t Y_t, \tag{34}$$

where τ_t is the share of national product Y_t that is collected by the government in period t, and government purchases G_t are treated as exogenously given and time-invariant. P_t is aggregate price level and i_t is interest rate on bonds. The national income identity yields

$$Y_t = C_t + G_t \tag{35}$$

where C_t is private consumption. For analytical convenience we introduce the real value of debt at maturity $B_t = (1+i_{t-1})\mathcal{B}_{t-1}/P_{t-1}$, observed at the beginning of period t, so that (34) becomes

$$B_{t+1} = (1 + i_t)\left(B_t \frac{P_{t-1}}{P_t} - \tau_t Y_t + G_t\right). \tag{36}$$

We assume that fiscal authorities operate with simple mechanistic feedback rule that relates the tax rate τ_t and B_t

$$\tau_t = \tau_o \left(\frac{B_t}{B_o}\right)^{\mu \frac{B_o}{Y_o}} \tag{37}$$

where τ_o and B_o are steady state values of tax rate and real debt correspondingly.

Log-linearizing (36) and (37) yields

$$b_{t+1} = \frac{B_o}{Y_o}\iota_t + \frac{1}{\beta}\left((1 - \mu\tau_o)\, b_t - \frac{C_o}{Y_o}\tau_o c_t - \frac{B_o}{Y_o}\pi_t\right) \tag{38}$$

where $b_t = \frac{B_o}{Y_o}\ln\left(\frac{B_t}{B_o}\right)$, $c_t = \ln\left(\frac{C_t}{C_o}\right)$, $\iota_t = \ln\frac{(1+i_t)}{(1+i_o)}$ and subscript o denotes steady state values of corresponding variables in zero inflation steady state. The private sector's discount factor $\beta = 1/(1 + i_o)$. To make the model particularly simple we assume $B_o = 0$, which eliminates the first-order effect of the interest rate and inflation on debt, and obtain the final version of linearized debt accumulation equation:

$$b_{t+1} = \rho b_t - \eta c_t \tag{39}$$

where the parameter $\rho = (1 - \mu\tau_o)/\beta$ is a function of the tax rate, implying that with stronger feedback μ the debt is stabilized faster; and where the parameter $\eta = C_o\tau_o/(\beta Y_o)$ describes the sensitivity of debt to the tax base.

[6] We delegate all technical details to the Online Appendix and only present a sketch of standard derivations. The Online Appendix and all necessary MATLAB programs are available from www.people.ex.ac.uk/tkirsano or upon request from the authors.

The derivation of the appropriate Phillips curve that describes Calvo-type price-setting decisions of monopolistically competitive firms is standard (Benigno and Woodford (2003), Sec. A.5) and marginal cost is a function of output and taxes. A log-linearization of the aggregate supply relationship around the zero-inflation steady state yields the following (deterministic) New Keynesian Phillips curve

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \left(\left(\frac{1}{\sigma} + \frac{\theta}{\psi} \right) c_t + \frac{\tau_o}{(1-\tau_o)} \tau_t \right) + \eta_t$$

where κ is the slope of Phillips curve, $\tau_t = \ln\left(\frac{\tau_t}{\tau_o}\right)$ and σ and ψ are parameters of the private sector utility function, and $\theta = C_o/Y_o$. η_t is an AR(1) cost push shock with parameter ρ_η. Substituting in the log-linearized (35) and (37) yields

$$\pi_t = \beta E_t \pi_{t+1} + \lambda c_t + \nu b_t + \eta_t \tag{40}$$

where $\nu = \mu\kappa\tau_o/(1-\tau_o)$ and $\lambda = \kappa(1/\sigma + \theta/\psi)$.

Finally, the model is described by two equations, the debt accumulation equation (39) and Phillips curve (40). As in the previous example, the aggregate agents' decision variable is inflation, π_t, and we assume that the policy maker chooses consumption c_t. Debt b_t is the aggregate predetermined state variable in period t. The economy evolves according to (39) and (40) and the initial state $\bar{b}$ is known to all agents.

The inter-temporal welfare criterion of the policy maker is defined by the quadratic loss function[7]

$$L_t = \frac{1}{2} E_t \sum_{s=t}^{\infty} \beta^{s-t} \left(\pi_s^2 + \alpha c_s^2 \right). \tag{41}$$

The policy maker knows the laws of motion (39)-(40) of the aggregate economy and takes them into account when formulating policy. The policy maker finds the best action every period, knows that future policy makers have freedom to change policy, and knows that future policy makers will apply the same decision process.

3.1 Discretionary Equilibria

Under discretion at every point in time t the decision rules of each agent are linear functions of the current state

$$c_t = c_\eta \eta_t + c_b b_t, \tag{42}$$

$$\pi_t = \pi_\eta \eta_t + \pi_b b_t. \tag{43}$$

From (40) it follows

$$\pi_\eta \eta_{t+1} + \pi_b b_{t+1} = \pi_t - \frac{\lambda}{\beta} c_t - \frac{\nu}{\beta} b_t - \frac{1}{\beta} \eta_t = \pi_\eta \rho_\eta \eta_t + \pi_b (\rho b_t - \eta c_t)$$

[7] The criterion is derived under the assumption of steady state labour subsidy. Here parameter α is a function of model parameters, $\alpha = \theta\lambda/\epsilon$, and ϵ is the elasticity of substitution between any pair of monopolistically produced goods.

and so the private sector reaction function is

$$\pi_t = \left(\beta\pi_\eta\rho_\eta + 1\right)\eta_t + \left(\beta\pi_b\rho + \nu\right)b_t + \left(\lambda - \beta\pi_b\eta\right)c_t. \tag{44}$$

The following Bellman equation characterizes discretionary policy:

$$\begin{aligned} & S_{\eta\eta}\eta_t^2 + 2S_{\eta b}\eta_t b_t + S_{bb}b_t^2 \\ = & \min_{c_t}\Big(\left(\left(\beta\pi_\eta\rho_\eta + 1\right)\eta_t + \left(\beta\pi_b\rho + \nu\right)b_t + \left(\lambda - \beta\pi_b\eta\right)c_t\right)^2 \\ & + \alpha c_t^2 + \beta\left(S_{\eta\eta}\rho_\eta^2\eta_t^2 + 2S_{\eta b}\rho_\eta\eta_t\left(\rho b_t - \eta c_t\right) + S_{bb}\left(\rho b_t - \eta c_t\right)^2\right)\Big). \end{aligned} \tag{45}$$

Here components of the value function is denoted by S. The optimal policy response can be written in form of (42) with

$$c_\eta = -\frac{\left(\left(\lambda - \beta\pi_b\eta\right)\left(\beta\pi_\eta\rho_\eta + 1\right) - \eta\beta S_{\eta b}\rho_\eta\right)}{\left(\beta\eta^2 S_{bb} + \left(\lambda - \beta\pi_b\eta\right)^2 + \alpha\right)} \tag{46}$$

$$c_b = -\frac{\left(\left(\lambda - \beta\pi_b\eta\right)\left(\beta\pi_b\rho + \nu\right) - \eta\beta S_{bb}\rho\right)}{\left(\beta\eta^2 S_{bb} + \left(\lambda - \beta\pi_b\eta\right)^2 + \alpha\right)} \tag{47}$$

and so the components of the value function satisfy the following equations

$$S_{\eta\eta} = \left(\left(\left(\beta\pi_\eta\rho_\eta + 1\right) + \left(\lambda - \beta\pi_b\eta\right)c_\eta\right)^2 + \alpha c_\eta^2 + \beta\left(\rho^2 S_{\eta\eta} - 2\rho\eta S_{\eta b}c_\eta + \eta^2 S_{bb}c_\eta^2\right)\right), \tag{48}$$

$$\begin{aligned} S_{\eta b} = & \left(\left(\beta\pi_\eta\rho_\eta + 1\right) + \left(\lambda - \beta\pi_b\eta\right)c_\eta\right)\left(\left(\beta\pi_b\rho + \nu\right) + \left(\lambda - \beta\pi_b\eta\right)c_b\right) + \alpha c_\eta c_b \\ & + \beta S_{\eta b}\rho_\eta\left(\rho - \eta c_b\right) - \beta S_{bb}\eta c_\eta\left(\rho - \eta c_b\right), \end{aligned} \tag{49}$$

$$S_{bb} = \left(\left(\beta\pi_b\rho + \nu\right) + \left(\lambda - \beta\pi_b\eta\right)c_b\right)^2 + \beta S_{bb}\left(\rho - \eta c_b\right)^2 + \alpha c_b^2. \tag{50}$$

Therefore, (44) can be written in the form of (43) with coefficients

$$\pi_\eta = \beta\pi_\eta\rho_\eta + 1 + \left(\lambda - \beta\pi_b\eta\right)c_\eta \tag{51}$$

$$\pi_b = \beta\pi_b\rho + \nu + \left(\lambda - \beta\pi_b\eta\right)c_b \tag{52}$$

Solution to (46)-(52) gives the discretionary equilibrium described by $\{c_\eta, c_b, \pi_\eta, \pi_b, S_{\eta\eta}, S_{bb}\}$.

Note that discretionary equilibrium is fully characterized by the deterministic component of the solution, by set $\{\pi_b, c_b, S\}$. Indeed, system (46)-(52) is recursive. We can solve (47), (50) and (52) for $\{c_b, \pi_b, S_{bb}\}$ and then solve the rest of the system for the stochastic component of the solution. We use this well known fact[8] to find all discretionary equilibria in the following simple and illustrative way.

Suppose the policy maker guesses the response of the private sector to the state, π_b. Then the optimal discretionary policy is given by the pair (47) and (50). We find c_b and then the optimal response π_b^* of the private sector is given by (52). Therefore, for every – not necessarily optimal – π_b we can compute a unique π_b^* and plot the dependence $\pi_b^*(\pi_b)$, see the first panel in Figure 1, Panel I. Clearly, if $\pi_b = \pi_b^*$ we have a solution to the discretionary problem.

[8] See Anderson et al. (1996) on certainly equivalence in this class of models or Blake and Kirsanova (2008) for explicit formulae for stochastic compnents as functions of deterministic components for discretionary models.

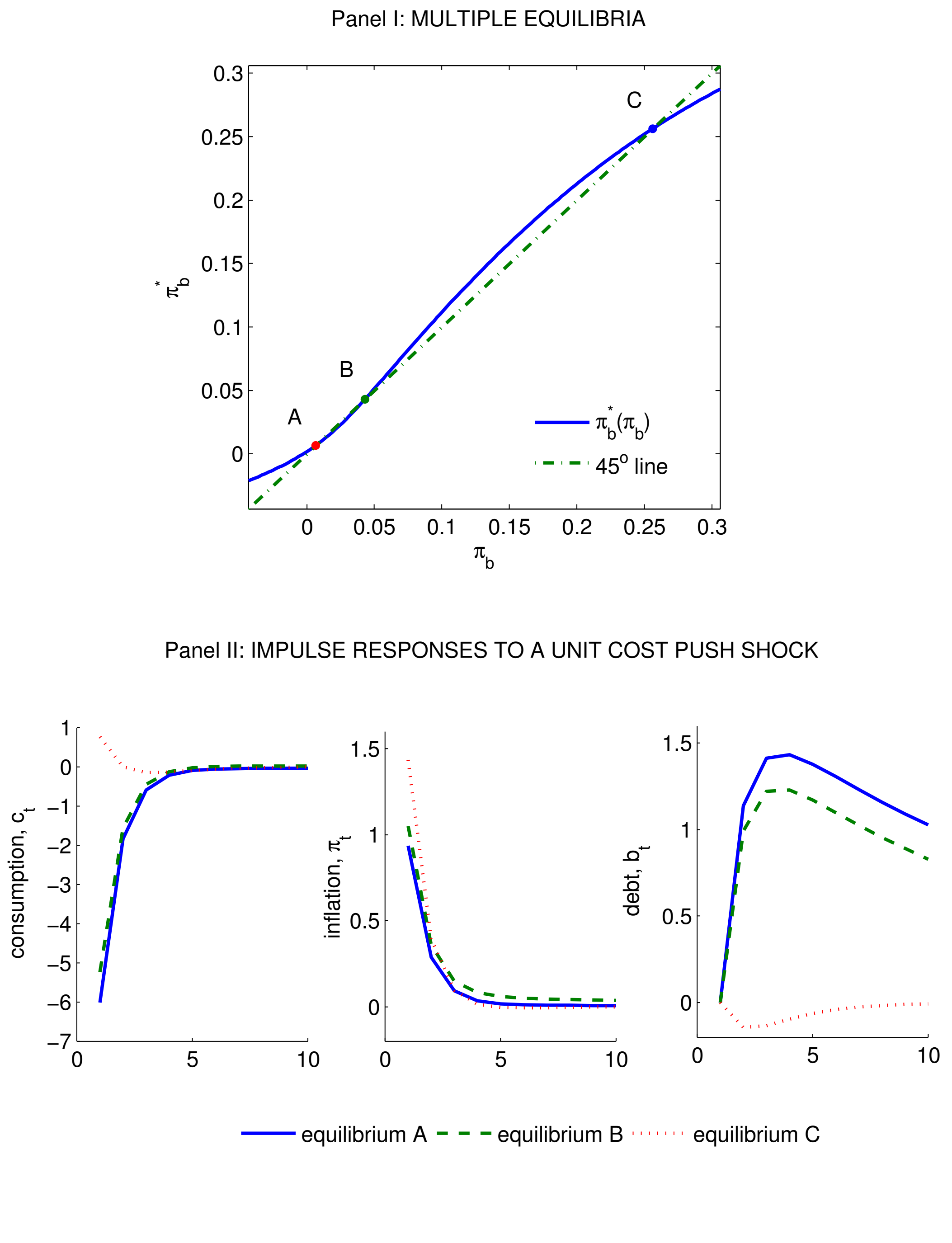

Figure 1: New Keynesian model with government debt.

Eqm	Policy Reaction	Private Sector Reaction	Loss Matrix	Speed of Adjustment	Average Loss
	$\begin{bmatrix} c_\eta & c_b \end{bmatrix}$	$\begin{bmatrix} \pi_\eta & \pi_b \end{bmatrix}$	$\begin{bmatrix} S_{\eta\eta} & S_{\eta b} \\ S_{\eta b} & S_{bb} \end{bmatrix}$	b_b	L
A	$\begin{bmatrix} -6.0135 & -0.0343 \end{bmatrix}$	$\begin{bmatrix} 0.9349 & 0.0066 \end{bmatrix}$	$\begin{bmatrix} 1.3156 & 0.0117 \\ 0.0117 & 0.0004 \end{bmatrix}$	0.9408	0.2695
B	$\begin{bmatrix} -5.2396 & 0.0155 \end{bmatrix}$	$\begin{bmatrix} 1.0485 & 0.0430 \end{bmatrix}$	$\begin{bmatrix} 1.5348 & 0.0780 \\ 0.0780 & 0.0131 \end{bmatrix}$	0.9314	0.3344
C	$\begin{bmatrix} 0.7611 & 1.6403 \end{bmatrix}$	$\begin{bmatrix} 1.4335 & 0.2561 \end{bmatrix}$	$\begin{bmatrix} 2.2227 & 0.4482 \\ 0.4482 & 0.1449 \end{bmatrix}$	0.6237	0.5098

Table 1: Policy and Private Sector Reactions in Equilibrium

For our base line calibration the graph of $\pi_b^* (\pi_b)$ intersects the 45^o degree line in three points labelled A, B and C, so we have three discretionary policy equilibria.[9]

In Table 1, we report the policy rule, $c = \{c_\eta, c_b\}$, and the private-sector decision rules, $\pi = \{\pi_\eta, \pi_b\}$, for all three equilibria.

The three equilibria, which characteristics are presented in Table 1 result in qualitatively and quantitatively different dynamics of the economy. Figure 1, Panel II, which shows the responses of key variables to a unit markup shock. Focusing first on equilibria A and B, inflation rises following the markup shock and the policy response is to defer consumption (by raising the nominal interest rate sufficiently high, this is implicit in our model). The decline in consumption lowers output and government tax revenues, which leads to a rise in government debt. In subsequent periods, although interest rates are lowered to stimulate the economy and bring it out of recession, government debt is brought back to baseline predominantly through (primary) fiscal surpluses, rather than through a decline in the cost of financing government debt.

In equilibrium C monetary policy responds to the markup shock by stimulating consumption and output, raises real marginal costs, and causes inflation to rise by more than it otherwise would. This monetary policy causes tax revenues to rise and leads to a decline in government debt. To stabilize government debt, future policymakers raise the cost of financing government debt, which causes consumption, output, and real marginal costs to decline and places downward pressure on inflation. Specifically, in the spirit of Leeper (1991) monetary policy can be thought of as being active in equilibria A and B and passive in equilibrium C. Table 1 reveals this trade-off between the response to government debt and the response to the markup shock: the more active

[9] The benchmark calibration follows Schaumburg and Tambalotti (2007) and Blake and Kirsanova (2008). The model's frequency is quarterly. The subjective discount rate β is set to 0.99, the government share of total output $1 - \rho$ is 0.25. The elasticity of intertemporal substitution σ is 1/2, the Frisch elasticity of labor supply $\varphi = 1/2$, an elasticity of demand ϵ of 7. The Calvo parameter $\gamma = 0.75$. Fiscal feedback μ is set to 0.075.

the policy the more aggressively interest rates are raised in response to the markup shock.

Equilibrium A is Pareto-preferred to the other equilibria, see Table 1, but should we expect equilibrium A to prevail over or realize more often than the others? In the next section we apply the methods discussed in Section 2.

3.2 Equilibrium Selection

3.2.1 Learning and expectational stability

Learning by private agents. We begin with the case, studied in Evans and Honkapohja (2003), in which the policy maker applies the same equilibrium policy every period and only private agents are learning their rational expectations response. Recall that a symmetric Markov-perfect Stackelberg-Nash equilibrium is fully characterized by the set $\{\pi_b, c_b, S\}$. We want to examine whether private agents can learn $\{\pi_b\}$, given $\{c_b, S\}$. Suppose it is anticipated that the policy maker implements (72) every period. Suppose the private sector starts with the following perceived low of motion

$$\pi_\tau = \bar{\pi}_\eta \eta_\tau + \bar{\pi}_b b_\tau. \tag{53}$$

Because the private sector is atomistic, there is no collective image of the future and the aggregated private sector can rationally start the eductive process with some guess. Suppose this guessed reacton is thought to be implemented in meta-time period $\tau + 1$:

$$\pi_{\tau+1} = \bar{\pi}_\eta \eta_{\tau+1} + \bar{\pi}_b b_{\tau+1}.$$

This perceived reaction of the private sector will be a RE equilibrium reaction if it is supported by the evolution of the economy; and (69)-(71) imply:

$$\begin{aligned} \beta\left(\bar{\pi}_\eta \eta_{\tau+1} + \bar{\pi}_b b_{\tau+1}\right) &= \pi_\eta \eta_\tau + \pi_b b_\tau - \lambda\left(c_\eta \eta_\tau + c_b b_\tau\right) - \nu b_\tau - \eta_\tau \\ &= \beta\left(\bar{\pi}_\eta \rho_\eta \eta_\tau + \bar{\pi}_b\left(\rho b_t - \eta\left(\left(c_\eta \eta_\tau + c_b b_\tau\right)\right)\right)\right) \end{aligned} \tag{54}$$

From where

$$\pi_\eta = \beta \bar{\pi}_\eta \rho_\eta - \beta \bar{\pi}_b \eta c_\eta + \lambda c_\eta + 1 \tag{55}$$
$$\pi_b = \beta \bar{\pi}_b\left(\rho - \eta c_b\right) + \lambda c_b + \nu \tag{56}$$

A fixed point of this natural revision mapping, from the initial guess of the reaction $\{\bar{\pi}_\eta, \bar{\pi}_b\}$ to the updated reaction $\{\pi_\eta, \pi_b\}$, results in the law of motion of the economy which is consistent with RE equilibrium. The fixed point the mapping needs to be locally Lyapunov-stable to allow the (aggregated) private sector to learn the RE equilibrium. Following Evans and Honkapohja (2001) we say that a symmetric Markov-perfect Stackelberg-Nash equilibrium is locally IE-stable under private sector learning if and only if all eigenvalues of the linearised mapping given by (85)-(87) are inside the unit circle.

The Jacobian of mapping (55)-(56) takes the form

$$J = \begin{bmatrix} \frac{\partial \pi_\eta}{\partial \bar{\pi}_\eta} & \frac{\partial \pi_\eta}{\partial \bar{\pi}_b} \\ \frac{\partial \pi_b}{\partial \bar{\pi}_\eta} & \frac{\partial \pi_b}{\partial \bar{\pi}_b} \end{bmatrix} = \begin{bmatrix} \beta \rho_\eta & -\beta \eta c_\eta \\ 0 & \beta\left(\rho - \eta c_b\right) \end{bmatrix}$$

	Equilibrium		
Characteristic	A	B	C
(1) Average loss	0.2695	0.3344	0.5259
(2) IE-stable (Private sector)	yes	yes	yes
(3) IE-stable (Joint)	yes	no	yes
(4) Switch to eq. A	—	35	18
(5) Self-enforceable	yes	no	no

Table 2: Equilibrium characteristics

and has two eigenvalues $z_1 = \beta\rho_\eta < 1$, and $z_2 = \beta(\rho - \eta c_b) = 1 - \tau_o\left(\mu + c_b \frac{C_o}{Y_o}\right)$. The size of z_2 is determined by the given policy c_b. For our model $|z_2| < 1$ in all cases.

Table 2 which summarizes all results on equilibrium selection. We rank the equilibria by the average loss, reported in line (1). We compute the numerical values of all eigenvalues and Line (2) confirms all three equilibria can be learned under the 'natural revision rule', so the private sector can discover them, provided the equilibrium policy response does not need to be learned.

Joint Learning. Under the joint learning the policy maker also seeks to rationalize its decisions when reoptimizing. The policy maker knows that the private sector will react to the policy, should the policy change, so (54) becomes:

$$\begin{aligned}\beta\left(\bar{\pi}_\eta \eta_{\tau+1} + \bar{\pi}_b b_{\tau+1}\right) &= \pi_\tau - \lambda c_\tau - \nu b_\tau - \eta_\tau \\ &= \beta\left(\bar{\pi}_\eta \rho_\eta \eta_\tau + \bar{\pi}_b\left(\rho b_\tau - \eta c_\tau\right)\right)\end{aligned}$$

From where

$$\pi_\tau = \left(\beta\bar{\pi}_\eta\rho_\eta + 1\right)\eta_\tau + \left(\beta\bar{\pi}_b\rho + \nu\right)b_\tau + \left(\lambda - \beta\bar{\pi}_b\eta\right)c_\tau$$

where the second equality is obtained when (88) is substituted.

The policy maker solves the following Bellman equation when rationalizes the choice of the policy reaction function:

$$\begin{aligned}& S_{\eta\eta}\eta_\tau^2 + 2S_{\eta b}\eta_\tau b_\tau + S_{bb}b_\tau^2 \qquad (57)\\ =\ & \min_{c_\tau}\Big(\left(\left(\beta\bar{\pi}_\eta\rho_\eta + 1\right)\eta_\tau + \left(\beta\bar{\pi}_b\rho + \nu\right)b_\tau + \left(\lambda - \beta\bar{\pi}_b\eta\right)c_\tau\right)^2 \\ & +\alpha c_\tau^2 + \beta\left(\bar{S}_{\eta\eta}\rho_\eta^2\eta_\tau^2 + 2\bar{S}_{\eta b}\rho_\eta\eta_\tau\left(\rho b_\tau - \eta c_\tau\right) + \bar{S}_{bb}\left(\rho b_\tau - \eta c_\tau\right)^2\right)\Big).\end{aligned}$$

Here components of the perceived value function is denoted by $\bar{S}$. The optimal policy response can be written as

$$\begin{aligned}c_\tau &= -\frac{\left(\left(\lambda - \beta\bar{\pi}_b\eta\right)\left(\beta\bar{\pi}_\eta\rho_\eta + 1\right) - \eta\beta\bar{S}_{\eta b}\rho_\eta\right)\eta_\tau + \left(\left(\lambda - \beta\bar{\pi}_b\eta\right)\left(\beta\bar{\pi}_b\rho + \nu\right) - \beta\eta\rho\bar{S}_{bb}\right)b_\tau}{\left(\beta\eta^2\bar{S}_{bb} + \left(\lambda - \beta\bar{\pi}_b\eta\right)^2 + \alpha\right)} \qquad (58)\\ &= c_\eta\left(\bar{S}, \bar{\pi}\right)\eta_\tau + c_b\left(\bar{S}, \bar{\pi}\right)b_\tau\end{aligned}$$

and so the revised value function becomes

$$\begin{aligned} S_{\eta\eta} &= \left(\left(\beta\bar{\pi}_\eta\rho_\eta+1\right)+\left(\lambda-\beta\bar{\pi}_b\eta\right)c_\eta\left(\bar{S},\bar{\pi}\right)\right)^2+\alpha c_\eta\left(\bar{S},\bar{\pi}\right)^2 && (59)\\ &\quad+\beta\left(\rho_\eta^2\bar{S}_{\eta\eta}-2\rho_\eta\eta\bar{S}_{\eta b}c_\eta\left(\bar{S},\bar{\pi}\right)+\eta^2\bar{S}_{bb}c_\eta\left(\bar{S},\bar{\pi}\right)^2\right) \\ S_{\eta b} &= \left(\left(\beta\bar{\pi}_\eta\rho_\eta+1\right)+\left(\lambda-\beta\bar{\pi}_b\eta\right)c_\eta\left(\bar{S},\bar{\pi}\right)\right)\left(\left(\beta\bar{\pi}_b\rho+\nu\right)+\left(\lambda-\beta\bar{\pi}_b\eta\right)c_b\left(\bar{S},\bar{\pi}\right)\right) && (60)\\ &\quad+\alpha c_\eta\left(\bar{S},\bar{\pi}\right)c_b\left(\bar{S},\bar{\pi}\right)+\beta\bar{S}_{\eta b}\rho_\eta\left(\rho-\eta c_b\left(\bar{S},\bar{\pi}\right)\right)-\beta\bar{S}_{bb}\eta c_\eta\left(\bar{S},\bar{\pi}\right)\left(\rho-\eta c_b\left(\bar{S},\bar{\pi}\right)\right) \\ S_{bb} &= \left(\left(\beta\bar{\pi}_b\rho+\nu\right)+\left(\lambda-\beta\bar{\pi}_b\eta\right)c_b\left(\bar{S},\bar{\pi}\right)\right)^2+\beta\bar{S}_{bb}\left(\rho-\eta c_b\left(\bar{S},\bar{\pi}\right)\right)^2+\alpha c_b\left(\bar{S},\bar{\pi}\right)^2 && (61) \end{aligned}$$

and, finally, the revision process of the private sector depends on revision of the policy maker:

$$\pi_\tau=\left(\beta\bar{\pi}_\eta\rho_\eta+1+\left(\lambda-\beta\bar{\pi}_b\eta\right)c_\eta\left(\bar{S},\bar{\pi}\right)\right)\eta_\tau+\left(\beta\rho\bar{\pi}_b+\nu+\left(\lambda-\beta\eta\bar{\pi}_b\right)c_b\left(\bar{S},\bar{\pi}\right)\right)b_\tau$$

so that

$$\pi_\eta=\left(\beta\bar{\pi}_\eta\rho_\eta+1+\left(\lambda-\beta\bar{\pi}_b\eta\right)c_\eta\left(\bar{S},\bar{\pi}\right)\right)=\pi_\eta\left(\bar{S},\bar{\pi}\right) \quad (62)$$

$$\pi_b=\left(\beta\bar{\pi}_b\rho+\nu+\left(\lambda-\beta\bar{\pi}_b\eta\right)c_b\left(\bar{S},\bar{\pi}\right)\right)=\pi_b\left(\bar{S},\bar{\pi}\right) \quad (63)$$

Equations (59)-(61) and (62)-(63) determine mapping from the perceived reaction $\left\{\bar{\pi},\bar{S}\right\}$ to the actual reaction $\{\pi,S\}$ and therefore policy reaction via (58).[10] The fixed point of this natural revision mapping results in the low of motion of the economy which is consistent with RE equilibrium. The fixed point the mapping needs to be locally Lyapunov-stable to allow all agents to learn the RE equilibrium. We say that a symmetric Markov-perfect Stackelberg-Nash equilibrium is locally IE-stable under joint learning if and only if all eigenvalues of the linearized mapping given by (59)-(61) and (62)-(63) are inside the unit circle.

The Jacobian of the mapping takes the form

$$\begin{bmatrix} \frac{dS_{\eta\eta}}{d\bar{S}_{\eta\eta}} & \frac{dS_{\eta\eta}}{d\bar{\pi}_\eta} & \frac{dS_{\eta\eta}}{d\bar{S}_{\eta b}} & \frac{dS_{\eta\eta}}{d\bar{\pi}_b} & \frac{dS_{\eta\eta}}{d\bar{S}_{bb}} \\ \frac{d\pi_\eta}{d\bar{S}_{\eta\eta}} & \frac{d\pi_\eta}{d\bar{\pi}_\eta} & \frac{d\pi_\eta}{d\bar{S}_{\eta b}} & \frac{d\pi_\eta}{d\bar{\pi}_b} & \frac{d\pi_\eta}{d\bar{S}_{bb}} \\ \frac{dS_{\eta b}}{d\bar{S}_{\eta\eta}} & \frac{dS_{\eta b}}{d\bar{\pi}_\eta} & \frac{dS_{\eta b}}{d\bar{S}_{\eta b}} & \frac{dS_{\eta b}}{d\bar{\pi}_b} & \frac{dS_{\eta b}}{d\bar{S}_{bb}} \\ \frac{d\pi_b}{d\bar{S}_{\eta\eta}} & \frac{d\pi_b}{d\bar{\pi}_\eta} & \frac{d\pi_b}{d\bar{S}_{\eta b}} & \frac{d\pi_b}{d\bar{\pi}_b} & \frac{d\pi_b}{d\bar{S}_{bb}} \\ \frac{dS_{bb}}{d\bar{S}_{\eta\eta}} & \frac{dS_{bb}}{d\bar{\pi}_\eta} & \frac{dS_{bb}}{d\bar{S}_{\eta b}} & \frac{dS_{bb}}{d\bar{\pi}_b} & \frac{dS_{bb}}{d\bar{S}_{bb}} \end{bmatrix} = \begin{bmatrix} \beta\rho^2 & \frac{dS_{\eta\eta}}{d\bar{\pi}_\eta} & \frac{dS_{\eta\eta}}{d\bar{S}_{\eta b}} & \frac{dS_{\eta\eta}}{d\bar{\pi}_b} & \frac{dS_{\eta\eta}}{d\bar{S}_{bb}} \\ 0 & \frac{d\pi_\eta}{d\bar{\pi}_\eta} & \frac{d\pi_\eta}{d\bar{S}_{\eta b}} & \frac{d\pi_\eta}{d\bar{\pi}_b} & \frac{d\pi_\eta}{d\bar{S}_{bb}} \\ 0 & \frac{dS_{\eta b}}{d\bar{\pi}_\eta} & \frac{dS_{\eta b}}{d\bar{S}_{\eta b}} & \frac{dS_{\eta b}}{d\bar{\pi}_b} & \frac{dS_{\eta b}}{d\bar{S}_{bb}} \\ 0 & 0 & 0 & \frac{d\pi_b}{d\bar{\pi}_b} & \frac{d\pi_b}{d\bar{S}_{bb}} \\ 0 & 0 & 0 & \frac{dS_{bb}}{d\bar{\pi}_b} & \frac{dS_{bb}}{d\bar{S}_{bb}} \end{bmatrix}$$

where we compute full derivatives taking into the account their dependence on policy. It is easy to demonstrate that the IE-stability properties of this system can be fully described by the properties of its deterministic component. Indeed, the Jacobian is bloc-diagonal and its eigenvalues are $z_0=\beta\rho^2<1$, z_1 and z_2 are eigenvalues of

$$J_s=\begin{bmatrix} \frac{d\pi_\eta}{d\bar{\pi}_\eta} & \frac{d\pi_\eta}{d\bar{S}_{\eta b}} \\ \frac{dS_{\eta b}}{d\bar{\pi}_\eta} & \frac{dS_{\eta b}}{d\bar{S}_{\eta b}} \end{bmatrix}$$

[10] We assume that policy c_η and c_b is substituted out.

and z_3 and z_4 are eigenvalues of

$$J_d = \begin{bmatrix} \frac{d\pi_b}{d\bar{\pi}_b} & \frac{d\pi_b}{d\bar{S}_{bb}} \\ \frac{dS_{bb}}{d\bar{\pi}_b} & \frac{dS_{bb}}{d\bar{S}_{bb}} \end{bmatrix}.$$

It is easy to obtain

$$z_{1,2} = \frac{1}{2}\beta\rho \frac{\left(1 \pm \sqrt{1 - 4\alpha\rho \frac{\left(\alpha + (\lambda - \beta\eta\pi_b)^2 + \beta\eta^2 S_{bb}\right)}{(\alpha + \alpha\rho + \beta\eta^2 S_{bb} + (\lambda\rho + \nu\eta)(\lambda - \beta\eta\pi_b))^2}}\right)}{\frac{\left(\alpha + (\lambda - \beta\eta\pi_b)^2 + \beta\eta^2 S_{bb}\right)}{(\alpha + \alpha\rho + \beta\eta^2 S_{bb} + (\lambda\rho + \nu\eta)(\lambda - \beta\eta\pi_b))}}$$

and they lie within the unit circle. These roots are zero for i.i.d. shock.

Roots z_3 and z_4 can be written in an analytical form, they will be functions of parameters of the model and the solution to the deterministic component of the model. Line (3) in Table 2 presents the result of the numerical examination of these eigenvalues. Only two equilibria, A and C, are jointly learnable so the agents can rationalize them. In this model the joint learning criterion is more restrictive than the private sector learning.

Before leaving this section, we note that the IE-stability criteria associated with private sector learning and joint learning, although connected, are distinct. Joint learnability of an equilibrium neither implies nor is implied by private sector learnability of that equilibrium. In this case both these criteria cannot discriminate between equilibria A and C.

3.2.2 Self-enforceability

The set of equilibrium policy actions can be described by $c^j = \left\{c^j_\eta, c^j_b\right\}$ and $\pi^j = \left\{\pi^j_\eta, \pi^j_b\right\}$, $j \in \{A, B, C\}$. Recall that we have a sequence of distinct policy makers and each policy maker reoptimizes within one period.

The minimal size K of the coalition which ensures the switch to the best equilibrium A can be determined as described above. The private sector has to react according to its reaction function with coefficients in each period:

$$\pi^{(s)}_\eta = \beta\rho_\eta\pi^{(s+1)}_\eta - \beta\eta\pi^{(s+1)}_b c^A_\eta + \lambda c^A_\eta + 1 \qquad (64)$$

$$\pi^{(s)}_b = \beta\left(\rho - \eta c^A_b\right)\pi^{(s+1)}_b + \lambda c^A_b + \nu \qquad (65)$$

where $\pi^{(s)}_\eta = \pi^m_\eta, \pi^{(s)}_b = \pi^m_b$ for $s = K+1, .., m \in \{B, C\}$. Reaction (64)-(65) will hold for $n = 0, .., K-2$ periods, and the corresponding evolution of the state

$$b_{s+1} = \rho b_s - \eta c^A_s, \quad s = 1, .., K$$

should be used to to compute the welfare gain due to the coalition.

Line (4) in Table 2 reports the minimum number of members in the coalition which will decide to leave bad equilibria B and C and switch to the good equilibrium A.

In particular, we obtain that if equilibrium C is expected to prevail in the future then it is optimal for the first 18 members to form a coalition and jointly precommit to play c^A. This represents four and half year, which is close to widely used average tenure terms for a (monetary) policy maker. It is more difficult to switch from equilibrium B, as 35 coalition members are required. Equilibrium A is self-enforceable as no coalition will find it optimal to deviate from it; this is reported in line (5) in Table 2.

The size of the minimal coalition is sensitive to the fiscal parameter: with stronger fiscal feedback it becomes easier to switch to the best equilibrium. The size of the minimum coalition required to switch from equilibria B and C to A is reduced to 7 and 6 quarters correspondingly once the fiscal feedback is sufficiently large.[11]

4 A DSGE Model with Capital Accumulation and Inflation Inertia

Following Woodford (2003) Ch.5, the economy is populated by households, intermediate-good producing firms, final-good producing firms, and a central bank. Households are identical and infinitely lived, choosing consumption, c_t, labor, l_t, and nominal holdings of next period bonds, b_{t+1}, to maximize expected discounted utility subject to a budget constraint. On the production side, a unit-continuum of monopolistically competitive intermediate-good producing firms, indexed by $\omega \in [0,1]$, produce by combining labor services hired in a perfectly competitive market with their firm-specific capital. These intermediate-good producing firms make labor and investment decisions, seeking to maximize their value subject to their production technology

$$Y_t(\omega) = e^{u_t} K_t(\omega)^{\alpha} L_t(\omega)^{(1-\alpha)},$$

their capital accumulation equation

$$I_t(\omega) = I\left(\frac{K_{t+1}(\omega)}{K_t(\omega)}\right) K_t(\omega),$$

where $I(1) = \delta$, $I'(1) = 1$, and $I''(1) = \eta$, and a Calvo (1983) price rigidity, where firms that cannot optimally set their price in a given period are assumed to index their price to lagged aggregate inflation (Smets and Wouters (2003)). Profits are aggregated and returned to households (shareholders) in the form of a lump-sum dividend. The final-good producing firms purchase intermediate goods, aggregate them into a final good according to a Dixit and Stiglitz (1977) production technology, and sell these final goods in a perfectly competitive market to households and firms to consume and invest, respectively.

After aggregating and log-linearizing about a zero-inflation nonstochastic steady state, the

[11] We compute these numbers for $\mu = 0.125$. This fiscal feedback is still sufficiently small to keep the economy in the area of multiplicity.

model's constraints and first-order conditions are

$$
\begin{aligned}
\pi_t &= \frac{\theta}{1+\theta\beta}\pi_{t-1} + \frac{\beta}{1+\theta\beta}\mathrm{E}_t\pi_{t+1} + \frac{(1-\xi)(1-\beta\xi)}{(1+\theta\beta)\,\xi}mc_t + v_t, \\
c_t &= \mathrm{E}_t c_{t+1} - \frac{1}{\sigma}\left(r_t - \mathrm{E}_t\pi_{t+1} - g_t + \mathrm{E}_t g_{t+1}\right), \\
k_{t+1} &= \frac{1}{1+\beta}k_t + \frac{\beta}{1+\beta}\mathrm{E}_t k_{t+2} + \frac{1-\beta(1-\delta)}{(1+\beta)\,\eta}\mathrm{E}_t ms_{t+1} - \frac{1}{(1+\beta)\,\eta}\left(r_t - \mathrm{E}_t\pi_{t+1}\right) \\
mc_t &= w_t - y_t + l_t, \\
w_t &= \chi l_t + \sigma c_t - g_t, \\
y_t &= (1-\gamma)\,c_t + \frac{\gamma}{\delta}\left[k_{t+1} - (1-\delta)\,k_t\right], \\
y_t &= u_t + \alpha k_t + (1-\alpha)\,l_t, \\
ms_t &= w_t - k_t + l_t
\end{aligned}
$$

where $\beta \in (0,1)$ is the discount factor, $\rho \equiv \frac{1-\beta}{\beta}$ is the discount rate, $\gamma \equiv \frac{\alpha\delta}{\rho+\delta}\frac{\varepsilon-1}{\varepsilon}$ is the steady-state share of investment in output, $\varepsilon > 1$ is the steady-state elasticity of substitution between intermediate goods, $\delta \in (0,1)$ is the depreciation rate, and $\eta > 0$ is the elasticity of the investment-to-capital ratio with respect to Tobin's q evaluated at steady state (Eichenbaum and Fisher (2007)).

Although the model allows for three stochastic elements: an aggregate consumption-preference shock, g_t; an aggregate markup shock, v_t; and an aggregate technology shock, u_t, we zero-out g_t and u_t in order to focus on the policy trade-offs associated with the markup shock, v_t, which is assumed to evolve over time according to

$$v_{t+1} = \rho_v v_t + \epsilon_{vt+1}, \tag{66}$$

where $\rho_v \in (-1,1)$ and $\epsilon_{v_{t+1}}$ is *i.i.d.* distributed with zero mean and finite variance.[12]

The central bank's loss function is assumed to have the form

$$L_t = (1-\beta)\,\mathrm{E}_t \sum_{k=t}^{\infty} \beta^{(k-t)} \left[\pi_k^2 + \frac{(1-\xi)(1-\beta\xi)}{(1+\theta\beta)\,\xi\varepsilon} y_k^2\right].$$

This objective is, in a certain sense, *ad hoc*, but we argue that the precise form of the objective function is not essential for the results we want to demonstrate. This is a reasonable objective which is likely to be used (at least as a first approximation) when a realistic central bank is faced with any moderately complex economic model.

4.1 Discretionary Equilibria

The monetary policy aims to stabilize inflation and does this via controlling the marginal cost. The existence of several channels of transmission leads to multiplicity of discretionary equilibria, as

[12] To parameterize the model, we set the discount factor, β, to 0.99, the Calvo price rigidity, ξ, to 0.75, the inflation indexation parameter, θ, to 0.60, the Cobb-Douglas production function parameter, α, to 0.36, the capital adjustment costs parameter to 6.0, the labor supply elasticity, χ, to 1, the elasticity of intertemporal substitution, σ, to 2, the depreciation rate, δ, to 0.025, the elasticity of subititution between goods, ε, to 11, and the shock persistece, ρ_v, to 0.3.

we now explain. Adapting a result from Dennis and Sóderstróm (1990), the forward representation of the inflation equation is given by

$$\pi_t = \theta\pi_{t-1} + \frac{(1-\xi)(1-\beta\xi)}{\xi}\mathrm{E}_t\sum_{k=t}^{\infty}\beta^{(k-t)}mc_k + \frac{1+\theta\beta}{1-\rho_v\beta}v_t. \tag{67}$$

Moreover, real marginal costs can be expressed as

$$mc_t = \left(\frac{\alpha+\chi}{1-\alpha} + \frac{\sigma}{1-\gamma}\right)y_t + \left[\frac{\sigma\gamma(1-\delta)}{(1-\gamma)\delta} - \frac{\alpha(\alpha+\chi)}{1-\alpha}\right]k_t - \frac{\sigma\gamma}{(1-\gamma)\delta}k_{t+1}. \tag{68}$$

It is apparent that movements in mc_t and mc_{t+1} are highly substitutable in terms of their effect on π_t and that, for any initial value of inflation, there are multiple paths for mc_t that will return inflation to target. These different paths for real marginal costs are associated with different monetary policies and with different performance in terms of loss. Equation (68) shows that monetary policy can affect mc_t through two distinct channels. To lower real marginal costs, the central bank can raise the real interest rate, weakening aggregate demand and thereby causing y_t to decline. It can also lower the real interest rate to stimulate investment and thereby boost the future capital stock. Notice that raising (lowering) the real interest rate causes both y_t and k_{t+1} to decline (rise) and that y_t and k_{t+1} have countervailing effects on mc_t. As a consequence, the desirability of each policy from the perspective of the period-t policymaker turns on how future policymakers are expected to respond to the capital stock.

Consider the case where future policymakers are expected to lower the interest rate in response to a rise in the capital stock. Following a positive markup shock, the policy of raising the real interest rate and causing y_t and k_{t+1} to decline will successfully deliver lower real marginal costs and inflation because the boost in future real marginal costs caused by the decline in the capital stock is offset by higher interest rates in the future. Under this approach, monetary policy responds to the positive markup shock by contracting demand, lowering real marginal costs and inflation, and by then lowering interest rates as inflation declines allowing the economy to recover, producing an equilibrium. Alternatively, if future policymakers are expected to raise the interest rate in response to a higher capital stock, then a policy that lowers the real interest rate and stimulates investment can bring about a decline in inflation, despite the boost to y_t and mc_t today, because future policymakers respond to the higher capital stock by tightening monetary policy, producing another equilibrium.

More formal investigation demonstrates that there are three discretionary equilibria, the described above equilibrium in which the real interest rate is raised and we label it A, the described above equilibrium in which the real interest rate is lowered and we label it C, and a 'middle' equiliblrium in which the interest rate is only weakly raised, we label it B.

As above, we can obtain these equilibria looking for a fixed point of private sector response to the stock variable, capital. Note that the evolution of the economy can be written in the reduced form:

$$\begin{aligned}
\pi_t &= \frac{\theta}{1+\theta\beta}E_t\pi_{t+1} + \frac{\beta}{1+\theta\beta}E_t\pi_{t+1} + \lambda_c c_t + \lambda_o k_{t+1} - \lambda_k k_t + v_t && (69)\\
k_{t+1} &= \nu_o E_t k_{t+2} + \nu_k k_t + \nu_c c_t - \nu_r(i_t - E_t\pi_{t+1}) && (70)\\
c_t &= E_t c_{t+1} - \sigma(i_t - E_t\pi_{t+1}) && (71)
\end{aligned}$$

with all coefficients given in Appendix X. There are three predetermined state variables, endogenous capital k_t and past inflation π_{t-1}, and exogenous cost push shock v_t. Because of this, equilibrium policy and private sector strategies can be written in the form of linear rules

$$i_t = \iota_v v_t + \iota_k k_t + \iota_\pi \pi_{t-1} \quad (72)$$
$$k_{t+1} = k_v v_t + k_k k_t + k_\pi \pi_{t-1} \quad (73)$$
$$\pi_t = \pi_v v_t + \pi_k k_t + \pi_\pi \pi_{t-1} \quad (74)$$
$$c_t = c_v v_t + c_k k_t + c_\pi \pi_{t-1}. \quad (75)$$

As in the previous model, the solution to the discretionary optimization problem can be obtained in two steps: we can find the deterministic component of the solution and they restore the stochastic component in the unique way (see e.g. Blake and Kirsanova (2008)). However, because we have two endogenous predetermined states, past inflation and capital, we will be looking for private sector's response to capital, conditional on the optimal response to inflation. Suppose the (arbitrary) reaction of the private sector to the change in capital stock is the set of nine feedback coefficients $\{k, \pi, c\}$ (we omit subscripts for compactness). We can compute the three feedback components of the optimal policy response and the value function, $\{\iota, S\}$. Next, we can compute a private sector response $\{k, \pi, c\}$. Of course this response will depend on policy, which depends on initial guess $\{k, \pi, c\}$. We now update all components except π_k and find the next approximation of optimal policy $\{\iota, S\}$ and the next update $\{k, \pi, c\}$. If this procedure converges, we obtain $\{k^*, \pi^* \setminus \pi_k, c^*\}$ where k^* and c^* are optimal consumption and investment decisions of the private sector *conditional on price setting component* π_k, and $\pi^* \setminus \pi_k$ include all components of the price-setting decisions except π_k. We can now compute $\{\iota^*, S^*\}$ as function of $\{k^*, \pi^* \setminus \pi_k, c^*\}$ and then update *once* $\{k^*, \pi^*, c^*\}$. This gives us the optimal price setting decision $\pi_k^* = \pi_k^*(\pi_k)$. If $\{\iota, k, \pi, c, S\}$ is a discretionary equilibrium then $\pi_k^* = \pi_k$. We can look at all points of intersection of $\pi_k^* = \pi_k^*(\pi_k)$ with 45^0 line and check that these points constitute a discretionary equilibrium. This line is plotted in Figure 2, Panel I. The line intersects the 45^0 line in three points which are the three discretionary equilibria described above. The policy rule and the private-sector decision rules for each equilibria are reported in Table 3.

The economy's behavior in the different equilibria are shown in 2, Panel II, which displays the responses of key variables to a unit markup shock. Focusing first on equilibria A and B, following the markup shock the interest rate is raised by more than the increase in inflation, causing the real interest rate to rise. The higher real interest rate generates a decline in consumption and investment, which lowers output and real marginal costs. Further, the fall in investment leads to a decline in the capital stock. In subsequent periods, the decline in real marginal costs causes inflation to moderate. With inflation declining back to baseline, monetary policy responds by lowering the interest rate and stimulating demand. In these two equilibria, monetary policy stabilizes the economy in the traditional way, contracting output and hence real marginal costs in order to keep inflationary pressures contained.

In contrast, in equilibrium C the interest rate is lowered in response to the positive markup shock, generating a big decline in the real interest rate. The lower real interest rate stimulates consumption and investment, which pushes up output and real marginal costs and further boosts inflation. However, the rise in investment causes the capital stock to increase and the capital build up eventually lowers real marginal costs while inducing tighter monetary policy. Although

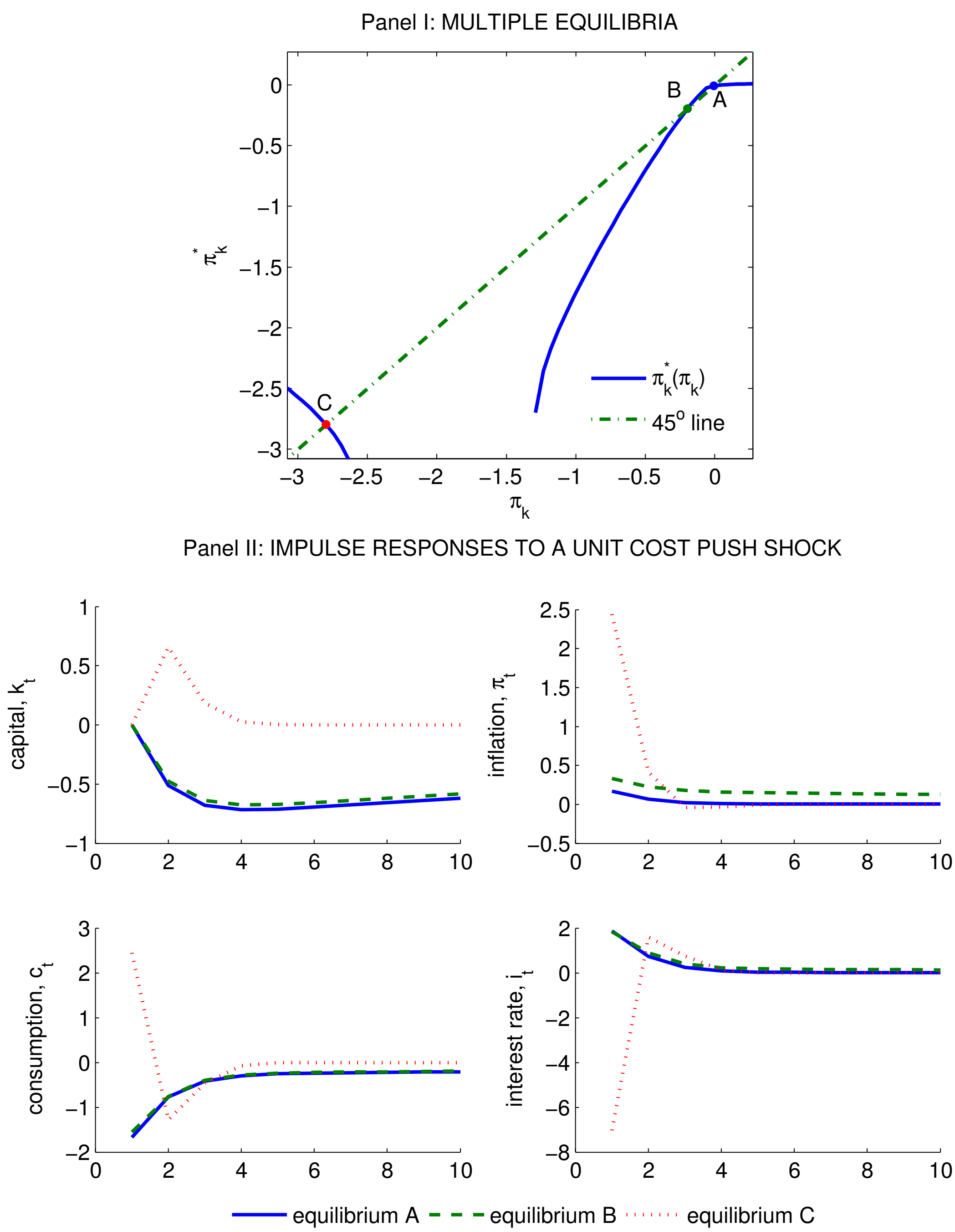

Figure 2: A DSGE model with inflation inertia

Eqm	Policy Reaction	Private Sector Reaction	Average Loss
	$\begin{bmatrix} \iota_v & \iota_k & \iota_\pi \end{bmatrix}$	$\begin{bmatrix} \pi_v & \pi_k & \pi_\pi \\ c_v & c_k & c_\pi \\ k_v & k_k & k_\pi \end{bmatrix}$	L
A	$\begin{bmatrix} 1.8651 & -0.0191 & 1.0340 \end{bmatrix}$	$\begin{bmatrix} 0.1683 & -0.0051 & 0.0673 \\ -1.6748 & 0.3302 & -0.6152 \\ -0.5102 & 0.9696 & -0.1802 \end{bmatrix}$	0.2436
B	$\begin{bmatrix} 1.8248 & -0.0024 & 1.0199 \end{bmatrix}$	$\begin{bmatrix} 0.3319 & -0.1944 & 0.1075 \\ -1.5541 & 0.2006 & -0.5847 \\ -0.4739 & 0.9302 & -0.1711 \end{bmatrix}$	0.7653
C	$\begin{bmatrix} 1.8248 & -0.0024 & 1.0199 \end{bmatrix}$	$\begin{bmatrix} 0.3319 & -0.1944 & 0.1075 \\ -1.5541 & 0.2006 & -0.5847 \\ -0.4739 & 0.9302 & -0.1711 \end{bmatrix}$	6.5921

Table 3: Policy and Private Sector Reactions in Equilibrium

the policy tightening is aimed primarily at lowering investment, it also serves to lower output, which causes a further decline in real marginal costs. In this equilibrium, monetary policy responds to the markup shock by stimulating the economy in order to boost capital spending. This policy succeeds in stabilizing the economy because the higher capital stock causes future real marginal costs to decline and future monetary policy to tighten.

As in the previous example, the economy behaves very differently in equilibria A and B than it does in equilibrium C. The conventional policy associated with equilibrium A is welfare superior to the unconventional policy associated with equilibrium C, but the model does not suggest which equilibrium is likely to realize. We apply the same selection criteria as in the previous section to understand if and how the best equilibrium can come about.

4.2 Equilibrium Selection

4.2.1 Learning and expectational stability

Learning by private agents A discretionary equilibrium in this model is characterized by the set of feedback coefficients $\{k, c, \pi, \iota, S\}$. Here we want to examine whether private agents can learn $\{k, c, \pi\}$, given $\{\iota, S\}$. Suppose it is anticipated that the policy maker implements (72) every period. Suppose the private sector starts with the following perceived low of motion

$$\begin{aligned} k_{\tau+1} &= \bar{k}_v v_\tau + \bar{k}_k k_\tau + \bar{k}_\pi \pi_{\tau-1} && (76)\\ \pi_\tau &= \bar{\pi}_v v_\tau + \bar{\pi}_k k_\tau + \bar{\pi}_\pi \pi_{\tau-1} && (77)\\ c_\tau &= \bar{c}_v v_\tau + \bar{c}_k k_\tau + \bar{c}_\pi \pi_{\tau-1} && (78)\end{aligned}$$

Because the private sector is atomistic, there is no collective image of the future and the aggregated private sector can rationally start the eductive process with some guess. Suppose this guessed reacton is thought to be implemented in meta-time period $\tau+1$:

$$\begin{aligned} k_{\tau+2} &= \bar{k}_v v_{\tau+1} + \bar{k}_k k_{\tau+1} + \bar{k}_\pi \pi_\tau && (79)\\ \pi_{\tau+1} &= \bar{\pi}_v v_{\tau+1} + \bar{\pi}_k k_{\tau+1} + \bar{\pi}_\pi \pi_\tau && (80)\\ c_{\tau+1} &= \bar{c}_v v_{\tau+1} + \bar{c}_k k_{\tau+1} + \bar{c}_\pi \pi_\tau && (81)\end{aligned}$$

This perceived reaction of the private sector will be a RE equilibrium reaction if it is supported by the evolution of the economy; and (69)-(71) imply:

$$\begin{aligned} & \bar{k}_v \rho_v v_\tau + \bar{k}_k k_{\tau+1} + \bar{k}_\pi \pi_\tau && (82)\\ = \; & \frac{\nu_r + \theta\beta\nu_r + \theta\nu_r\iota_v}{\theta\nu_o} v_\tau + \frac{\theta\nu_r\iota_\pi + \beta\nu_r}{\theta\nu_o}\pi_{\tau-1} + \frac{\theta\nu_r\iota_k - \theta\nu_k - \lambda_k\left(\nu_r + \theta\beta\nu_r\right)}{\theta\nu_o} k_\tau \\ & + \frac{\theta + \lambda_o\left(\nu_r + \theta\beta\nu_r\right)}{\theta\nu_o} k_{\tau+1} - \frac{\left(1+\theta\beta\right)\nu_r}{\theta\nu_o}\pi_\tau + \frac{\lambda_c\left(\nu_r + \theta\beta\nu_r\right) - \theta\nu_c}{\theta\nu_o} c_\tau \\ & \bar{\pi}_v \rho v_\tau + \bar{\pi}_k k_{\tau+1} + \bar{\pi}_\pi \pi_\tau && (83)\\ = \; & -\frac{\left(\theta\beta+1\right)}{\theta} v_\tau - \frac{\beta}{\theta}\pi_{\tau-1} + \frac{\lambda_k\left(\theta\beta+1\right)}{\theta} k_\tau \\ & -\frac{\lambda_o\left(\theta\beta+1\right)}{\theta} k_{\tau+1} + \frac{\left(\theta\beta+1\right)}{\theta}\pi_\tau - \frac{\lambda_c\left(\theta\beta+1\right)}{\theta} c_\tau \\ & \bar{c}_v \rho_v v_\tau + \bar{c}_k k_{\tau+1} + \bar{c}_\pi \pi_\tau && (84)\\ = \; & \sigma\left(\frac{1+\theta\beta+\theta\iota_v}{\theta}\right) v_\tau + \left(\frac{\theta\iota_\pi + \beta}{\theta}\right)\sigma\pi_{\tau-1} + \sigma\left(\frac{\theta\iota_k - \lambda_k\left(1+\theta\beta\right)}{\theta}\right) k_\tau \\ & + \frac{\sigma\lambda_o\left(1+\theta\beta\right)}{\theta} k_{\tau+1} - \frac{\sigma\left(1+\theta\beta\right)}{\theta}\pi_\tau + \left(\frac{\sigma\lambda_c\left(1+\theta\beta\right)}{\theta} + 1\right) c_\tau \end{aligned}$$

From where

$$\begin{aligned} k_{\tau+1} &= k_k\left(\bar{k},\bar{c},\bar{\pi}\right) k_\tau + k_\pi\left(\bar{k},\bar{c},\bar{\pi}\right)\pi_{\tau-1} + k_v\left(\bar{k},\bar{c},\bar{\pi}\right) v_\tau && (85)\\ c_\tau &= c_k\left(\bar{k},\bar{c},\bar{\pi}\right) k_\tau + c_\pi\left(\bar{k},\bar{c},\bar{\pi}\right)\pi_{\tau-1} + k_v\left(\bar{k},\bar{c},\bar{\pi}\right) v_\tau && (86)\\ \pi_\tau &= \pi_k\left(\bar{k},\bar{c},\bar{\pi}\right) k_\tau + \pi_\pi\left(\bar{k},\bar{c},\bar{\pi}\right)\pi_{\tau-1} + \pi_v\left(\bar{k},\bar{c},\bar{\pi}\right) v_\tau && (87)\end{aligned}$$

and all coefficients are given in Appendix.

A fixed point of this natural revision mapping, from the initial guess of the reaction $\left\{\bar{k},\bar{c},\bar{\pi}\right\}$ to the updated reaction $\{k,c,\pi\}$, results in the low of motion of the economy which is consistent with RE equilibrium. The fixed point the mapping needs to be locally Lyapunov-stable to allow the (aggregated) private sector to learn the RE equilibrium. A symmetric Markov-perfect

	Equilibrium		
Characteristic	A	B	C
(1) Average loss	0.2436	0.7653	6.5921
(2) IE-stable (Private sector)	yes	yes	yes
(3) IE-stable (Joint)	yes	no	yes
(4) Switch to eq.A (welfare)	—	3	69
(5) Self-enforceable	yes	no	no

Table 4: Equilibrium characteristics

Stackelberg-Nash equilibrium is locally IE-stable under private sector learning if and only if all eigenvalues of the linearised mapping given by (85)-(87) are inside the unit circle.

We present the result of the examination of the eigenvalues is given in Table 4.

The conventional policy is superior to the unconventional policy, see row (1). Rows (2) and (3) show that all equilibria and learnable by private agents.

Joint learning Similar to the above, the perceived law of motion of the economy is described by reactions (76)-(78) and

$$i_\tau = \bar{\iota}_v v_\tau + \bar{\iota}_k k_\tau + \bar{\iota}_\pi \pi_{\tau-1} \tag{88}$$

and the perceived value function with components $\bar{S}$.

The policy maker now also seeks to rationalize its decisions when reoptimizing. The policy maker knows that the private sector will react to the policy, should the policy change, so (82)-(84) become

$$\begin{aligned}
& \bar{k}_v \rho v_\tau + \bar{k}_k k_{\tau+1} + \bar{k}_\pi \pi_\tau \\
= \; & \frac{(\nu_r + \theta\beta\nu_r)}{\theta\nu_o} v_t + \frac{\beta(\nu_r + \theta\beta\nu_r)}{\theta\nu_o(\theta\beta+1)}\pi_{t-1} - \frac{\theta\nu_k + \lambda_k(\nu_r + \theta\beta\nu_r)}{\theta\nu_o} k_\tau \\
& + \frac{\theta + \lambda_o(\nu_r + \theta\beta\nu_r)}{\theta\nu_o} k_{t+1} - \frac{(1+\theta\beta)\nu_r}{\theta\nu_o}\pi_t + \frac{\lambda_c(\nu_r + \theta\beta\nu_r) - \theta\nu_c}{\theta\nu_o} c_t + \frac{\nu_r}{\nu_o} i_t \\
& \bar{\pi}_v \rho v_\tau + \bar{\pi}_k k_{\tau+1} + \bar{\pi}_\pi \pi_\tau \\
= \; & -\frac{(\theta\beta+1)}{\theta} v_t - \frac{\beta}{\theta}\pi_{t-1} + \frac{\lambda_k(\theta\beta+1)}{\theta} k_t - \frac{\lambda_o(\theta\beta+1)}{\theta} k_{t+1} \\
& + \frac{(\theta\beta+1)}{\theta}\pi_t - \frac{\lambda_c(\theta\beta+1)}{\theta} c_t \\
& \bar{c}_v \rho v_\tau + \bar{c}_k k_{\tau+1} + \bar{c}_\pi \pi_\tau \\
= \; & \frac{\sigma(1+\theta\beta)}{\theta} v_t + \frac{\beta\sigma(1+\theta\beta)}{\theta(\theta\beta+1)}\pi_{t-1} - \frac{\sigma\lambda_k(1+\theta\beta)}{\theta} k_t \\
& + \frac{\sigma\lambda_o(1+\theta\beta)}{\theta} k_{t+1} - \frac{\sigma(1+\theta\beta)}{\theta}\pi_t + \left(\frac{\sigma\lambda_c(1+\theta\beta)}{\theta} + 1\right) c_t + \sigma i_t
\end{aligned}$$

and so

$$
\begin{aligned}
k_{\tau+1} &= k_{\nu S}\left(\bar{k},\bar{c},\bar{\pi}\right)\nu_\tau + k_{kS}\left(\bar{k},\bar{c},\bar{\pi}\right)k_\tau + k_{\pi S}\left(\bar{k},\bar{c},\bar{\pi}\right)\pi_{\tau-1} + k_P\left(\bar{k},\bar{c},\bar{\pi}\right)i_\tau \quad (89)\\
&= k_\nu\left(\bar{k},\bar{c},\bar{\pi},\bar{\iota}\right)\nu_\tau + k_k\left(\bar{k},\bar{c},\bar{\pi},\bar{\iota}\right)k_\tau + k_\pi\left(\bar{k},\bar{c},\bar{\pi},\bar{\iota}\right)\pi_{\tau-1}\\
\pi_\tau &= \pi_{\nu S}\left(\bar{k},\bar{c},\bar{\pi}\right)\nu_\tau + \pi_{kS}\left(\bar{k},\bar{c},\bar{\pi}\right)k_\tau + \pi_{\pi S}\left(\bar{k},\bar{c},\bar{\pi}\right)\pi_{\tau-1} + \pi_P\left(\bar{k},\bar{c},\bar{\pi}\right)i_\tau \quad (90)\\
&= \pi_\nu\left(\bar{k},\bar{c},\bar{\pi},\bar{\iota}\right)\nu_\tau + \pi_k\left(\bar{k},\bar{c},\bar{\pi},\bar{\iota}\right)k_\tau + \pi_\pi\left(\bar{k},\bar{c},\bar{\pi},\bar{\iota}\right)\pi_{\tau-1}\\
c_\tau &= c_{\nu S}\left(\bar{k},\bar{c},\bar{\pi}\right)\nu_\tau + c_{kS}\left(\bar{k},\bar{c},\bar{\pi}\right)k_\tau + c_{\pi S}\left(\bar{k},\bar{c},\bar{\pi}\right)\pi_{\tau-1} + c\left(\bar{k},\bar{c},\bar{\pi}\right)i_\tau \quad (91)\\
&= c_\nu\left(\bar{k},\bar{c},\bar{\pi},\bar{\iota}\right)\nu_\tau + c_k\left(\bar{k},\bar{c},\bar{\pi},\bar{\iota}\right)k_\tau + c_\pi\left(\bar{k},\bar{c},\bar{\pi},\bar{\iota}\right)\pi_{\tau-1}
\end{aligned}
$$

where the second equality is obtained when (88) is substituted. (All coefficients are given in Appendix X.)

The policy maker solves the following Bellman equation when rationalizes the choice of the interest rate reaction function:

$$
\begin{aligned}
& S_{\nu\nu}\nu_\tau^2 + S_{kk}k_\tau^2 + S_{\pi\pi}\pi_{\tau-1}^2 + 2S_{\nu k}\nu_\tau k_\tau + 2S_{\nu\pi}\nu_\tau\pi_{\tau-1} + 2S_{\pi k}\pi_{\tau-1}k_\tau \quad (92)\\
= & \min_{i_\tau}\Bigg(\left(\pi_{\nu S}\nu_\tau + \pi_{kS}k_\tau + \pi_{\pi S}\pi_{\tau-1} + \pi_P i_\tau\right)^2 + \alpha\left(\left(1-\gamma\right)c_\tau + \frac{\gamma}{\delta}\left[k_{\tau+1} - \left(1-\delta\right)k_\tau\right]\right)^2\\
& +\beta\left(\bar{S}_{\nu\nu}\nu_{\tau+1}^2 + \bar{S}_{kk}k_{\tau+1}^2 + \bar{S}_{\pi\pi}\pi_\tau^2 + 2\bar{S}_{\nu k}\nu_{\tau+1}k_{\tau+1} + 2\bar{S}_{\nu\pi}\nu_{\tau+1}\pi_\tau + 2\bar{S}_{\pi k}\pi_\tau k_{\tau+1}\right)\Bigg).
\end{aligned}
$$

Here components of the perceived value function is denoted by $\bar{S}$. Substitute future states out using (66), (89)-(90) and differentiate with respect to i_τ to obtain the optimal policy response

$$
i_\tau = c_\eta\left(\bar{S},\bar{\pi}\right)\eta_\tau + c_b\left(\bar{S},\bar{\pi}\right)b_\tau i_\tau = \iota_v\left(\bar{S},\bar{k},\bar{c},\bar{\pi}\right)v_\tau + \iota_k\left(\bar{S},\bar{k},\bar{c},\bar{\pi}\right)k_\tau + \iota_\pi\left(\bar{S},\bar{k},\bar{c},\bar{\pi}\right)\pi_{\tau-1} \quad (93)
$$

Its substitution into the Bellman equation (92) yields the revision rule for components of the value function $\bar{S}$.

This yields a mapping from the perceived reaction $\left\{\bar{k},\bar{c},\bar{\pi},\bar{\iota},\bar{S}\right\}$ to the actual reaction $\left\{k,c,\pi,\iota,S\right\}$. The fixed point of this natural revision mapping results in the low of motion of the economy which is consistent with RE equilibrium. The fixed point the mapping needs to be locally Lyapunov-stable to allow all agents to learn the RE equilibrium. As before, we examine whether eigenvalues of the linearized mapping are inside the unit circle and report results in Table 4, line (3). The joint learning rules out the middle equilibrium B, but both equilibrium A and equilibrium C are IE-stable under the joint learning and can realize if both, the private sector and the policy maker learn jointly.

4.2.2 Self-enforceability

Similar to the previous example, when deciding on the size of the minimum coalition K which allows to switch away from equilibria B and C to equilibrium A, the policy maker takes into account the reaction of the private sector with feedback coefficients which can be computer recursively:

$$
\begin{aligned}
k^{(s)} &= k\left(k^{(s+1)},c^{(s+1)},\pi^{(s+1)},\iota^A\right) \quad (94)\\
c^{(s)} &= c\left(k^{(s+1)},c^{(s+1)},\pi^{(s+1)},\iota^A\right) \quad (95)\\
\pi^{(s)} &= \pi\left(k^{(s+1)},c^{(s+1)},\pi^{(s+1)},\iota^A\right). \quad (96)
\end{aligned}
$$

Here we omit state subscripts for simplicity. The complete formulae are given in Appendix X. The implied evolution of the state is

$$k_{s+1} = k_k^{(s)} k_s + k_\pi^{(s)} \pi_{s-1} + k_v^{(s)} v_s, \quad s = 1, .., K \tag{97}$$

$$\pi_s = \pi_k^{(s)} k_s + \pi_\pi^{(s)} \pi_{s-1} + \pi_v^{(s)} v_s, \quad s = 1, .., K \tag{98}$$

will be used to assess the gain due to the coalition.

Line (4) in Table 4 reports the minimum number of members in the coalition which will decide to leave equilibria B and C and switch to the Pareto-optimal equilibrium A. The switch to the best equilibrium presents much bigger challenge for the policy maker in this model than in the model with government debt as the size of the minimal coalition is large, the tenure in the office required to switch from equilibrium C is close to 20 years. As before, the best equilibrium A is self-enforceable as no coalition will find it optimal to deviate from it, this is reported in line (5) in Table 4.

5 Conclusion

Discretionary policymakers can manage neither the expectations of private agents nor the actions of future policymakers. As a consequence, discretionary policymakers are susceptible to expectations traps and coordination failures and discretionary control problems can have multiple equilibria. Recognizing this potential for multiple equilibria, this paper addresses the important issue of equilibrium selection, an issue related intrinsically to the capacity for agents to coordinate. One contribution of this paper is to cast the discretionary control problem as a dynamic game, allowing us to explain clearly the strategic interactions that give rise to multiple equilibria. However, the paper's main contribution is to develop several of equilibrium selection criteria, criteria motivated by expectational stability and self-enforceability.

We illustrate these equilibrium selection criteria by applying them to two New Keynesian models. In the first model, the Pareto-preferred equilibrium is one of two equilibria that is both jointly learnable and learnable by private agents. Since, the Pareto-preferred is the only equilibrium that is self-enforceable in this model, the equilibrium selection criteria indicate that agents might plausibly coordinate upon it. We demonstrate how a coalition can ensure a switch to the Pareto-optimal equilibrium. In the second model, the Pareto-preferred equilibrium is one of two equilibrium that is jointly learnable, and all equilibria are private sector learnable. Nevertheless, the Pareto-preferred equilibrium is selected as the equilibrium of interest because it is self-enforceable. However, our results demonstrate that in some cases the switch to the Pareto-preferred equilibrium might require a relatively long tenure.

Although these selection criteria happen to point to the Pareto-preferred equilibrium as the equilibrium of interest in these two models, this need not have been the case. Our experience is that the Pareto-preferred equilibrium is jointly learnable, but that it is not necessarily private sector learnable. It is entirely possible, therefore, that in other models these selection criteria could point toward equilibria (or an equilibrium) that is Pareto-dominated. Finally, while we have described and applied three selection criteria in this paper, there are, of course, other approaches to selecting among equilibria. One such approach might be to select an equilibrium using minimax-loss or minimax-regret; another might be to identify an equilibrium from the limiting behavior of quasi-commitment policies. We leave the study and application of these criteria, and an investigation into whether multiple discretionary equilibria is a general feature of New Keynesian monetary policy models, for future work.

A Proof of Lemma (1)

Proof. Following Evans (1985), to analyze the local stability of equation (24) we linearize the equation about $\mathbf{\Phi}^*$. Using matrix calculus results from Magnus and Neudecker (1999), Ch.9 we obtain

$$d\left(vec\left(\mathbf{\Phi}_{k+1}\right)\right) = DT\left(\mathbf{\Phi}^*\right) d\left(vec\left(\mathbf{\Phi}_k\right)\right)$$

where $DT\left(\mathbf{\Phi}^*\right) = \partial\left(vec\left(T\left(\mathbf{\Phi}^*\right)\right)\right)/\partial\left(vec\left(\mathbf{\Phi}\right)\right)'$. Applying standard results for linear difference equations, if all of the eigenvalues of $DT\left(\mathbf{\Phi}^*\right)$ have modulus less than one, then $\mathbf{\Phi}^*$ is locally stable. In contrast, if one or more of the eigenvalues of $DT\left(\mathbf{\Phi}^*\right)$ have modulus greater than one, then $\mathbf{\Phi}^*$ is not locally stable. ■

B Proof of Lemma (2)

Proof. Applying standard matrix calculus rules to equation (25), the total differential can be written as

$$\left(\mathbf{HA}_{12} - \mathbf{A}_{22}\right) d\left(\mathbf{H}\right) + d\left(\overline{\mathbf{H}}\right)\mathbf{A}_{12}\mathbf{H} + d\left(\overline{\mathbf{H}}\right)\left(\mathbf{A}_{11} + \mathbf{B}_1\mathbf{F}\right) = \mathbf{0},$$

which after vectorizing can be rearranged to give

$$vec\left[d\left(\mathbf{H}\right)\right] = -\left[\mathbf{I}\otimes\left(\mathbf{HA}_{12} - \mathbf{A}_{22}\right)\right]^{-1}\left[\left(\mathbf{A}_{11}+\mathbf{A}_{12}\mathbf{H} + \mathbf{B}_1\mathbf{F}\right)'\otimes\mathbf{I}\right] vec\left[d\left(\overline{\mathbf{H}}\right)\right].$$

We apply Lemma 1 to obtain the required result. Note that invertibility of $\left(\mathbf{HA}_{12} - \mathbf{A}_{22}\right)$ is virtually ensured by the assumption that $\mathbf{A}_{22}$ has full rank. ■

C Proof of Lemma (3)

Proof. Applying standard matrix calculus rules to equations (26) and (27), total differentials are given by

$$\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right) d\left(\mathbf{F}\right) + \beta\widehat{\mathbf{B}}' d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right) = \mathbf{0}, \tag{99}$$

$$2\left[\widehat{\mathbf{U}} + \mathbf{F}'\widehat{\mathbf{Q}} + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)'\mathbf{V}\widehat{\mathbf{B}}\right] d\left(\mathbf{F}\right) + \beta\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)' d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right) = \mathbf{I}d\left(\mathbf{V}\right), \tag{100}$$

Using equation (99) to solve for $d\left(\mathbf{F}\right)$ and substituting the resulting expression into equation (100) yields, upon rearranging,

$$\beta\left[-2\left(\widehat{\mathbf{U}} + \beta\widehat{\mathbf{A}}'\mathbf{V}\mathbf{B}\right)\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)^{-1}\widehat{\mathbf{B}}' - 2\mathbf{F}'\widehat{\mathbf{B}}' + \left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right)'\right] d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}} + \widehat{\mathbf{B}}\mathbf{F}\right) = \mathbf{I}d\left(\mathbf{V}\right),$$

which, given equation (26), collapses to

$$\beta\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)^{'}d\left(\overline{\mathbf{V}}\right)\left(\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}\right)=\mathbf{I}d\left(\mathbf{V}\right). \tag{101}$$

After vectorizing and recognizing that $\mathbf{M}=\widehat{\mathbf{A}}+\widehat{\mathbf{B}}\mathbf{F}$, equation (101) can be written as

$$vec\left[d\left(\mathbf{V}\right)\right]=\beta\left(\mathbf{M}^{'}\otimes\mathbf{M}^{'}\right)vec\left[d\left(\overline{\mathbf{V}}\right)\right].$$

The matrix $\beta\left(\mathbf{M}^{'}\otimes\mathbf{M}^{'}\right)$ defines the derivative map $DT\left(\mathbf{V}\right)$. Applying Lemma 1, a symmetric Markov-perfect Stackelberg-Nash equilibria $\{\mathbf{H},\mathbf{F},\mathbf{M},\mathbf{V},d\}$ is a local IE-stable policy equilibrium if and only if all of the eigenvalues of $DT\left(\mathbf{V}\right)$ have modulus less than 1. Because the eigenvalues of $\mathbf{M}$ all have modulus less than $\beta^{-\frac{1}{2}}$ in all symmetric Markov-perfect Stackelberg-Nash equilibria the result follows. ■

D Proof of Lemma (4)

Proof. Total differentials of equations (28)—(32) about the point $\{\mathbf{H},\mathbf{F},\mathbf{V},\mathbf{J},\mathbf{K}\}$ are given by

$$\mathbf{0} = d\left(\mathbf{J}\right)+d\left(\mathbf{K}\right)\mathbf{F}+\mathbf{K}d\left(\mathbf{F}\right)-d\left(\mathbf{H}\right), \tag{102}$$

$$\mathbf{0} = d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{A}}-\left(\mathbf{A}_{22}-\mathbf{H}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right), \tag{103}$$

$$\mathbf{0} = d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}}-\left(\mathbf{A}_{22}-\mathbf{H}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right), \tag{104}$$

$$\begin{aligned}\mathbf{0} = {} & \beta\widehat{\mathbf{B}}^{'}d\left(\overline{\mathbf{V}}\right)\mathbf{M}+\left(\widehat{\mathbf{Q}}+\beta\widehat{\mathbf{B}}^{'}\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right)+2\left(\mathbf{K}^{'}\mathbf{W}_{22}+\mathbf{U}_{2}^{'}+\beta\widehat{\mathbf{B}}^{'}\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right)\mathbf{F}\\ & +\left(\mathbf{W}_{12}+\mathbf{J}^{'}\mathbf{W}_{22}+\beta\widehat{\mathbf{A}}^{'}\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right)+\left(\mathbf{K}^{'}\mathbf{W}_{22}+\mathbf{U}_{2}^{'}+\beta\widehat{\mathbf{B}}^{'}\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right),\end{aligned} \tag{105}$$

$$\begin{aligned}\mathbf{0} = {} & 2\left(\widehat{\mathbf{U}}+\mathbf{F}^{'}\widehat{\mathbf{Q}}+\beta\mathbf{M}^{'}\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right)+2\left(\mathbf{W}_{12}+\mathbf{H}^{'}\mathbf{W}_{22}+\mathbf{F}^{'}\mathbf{U}_{2}^{'}+\beta\mathbf{M}^{'}\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{J}\right)\\ & +2\left(\mathbf{W}_{12}+\mathbf{H}^{'}\mathbf{W}_{22}+\mathbf{F}^{'}\mathbf{U}_{2}^{'}+\beta\mathbf{M}^{'}\mathbf{V}\mathbf{A}_{12}\right)d\left(\mathbf{K}\right)\mathbf{F}+\beta\mathbf{M}^{'}d\left(\overline{\mathbf{V}}\right)\mathbf{M}-d\left(\mathbf{V}\right).\end{aligned} \tag{106}$$

Now, using equations (103) and (104) to solve for $d\left(\mathbf{J}\right)$ and $d\left(\mathbf{K}\right)$, respectively, and substituting these expressions into equations (102), (105), and (106) produces

$$\mathbf{0} = \mathbf{K}d\left(\mathbf{F}\right)+\left(\mathbf{A}_{22}-\mathbf{H}\mathbf{A}_{12}\right)^{-1}d\left(\overline{\mathbf{H}}\right)\mathbf{M}-d\left(\mathbf{H}\right), \tag{107}$$

$$\begin{aligned}\mathbf{0} = {} & \beta\widehat{\mathbf{B}}^{'}d\left(\overline{\mathbf{V}}\right)\mathbf{M}+\left(\widehat{\mathbf{Q}}+\beta\widehat{\mathbf{B}}^{'}\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right)\\ & +\left(\mathbf{W}_{12}+\mathbf{J}^{'}\mathbf{W}_{22}+\beta\widehat{\mathbf{A}}^{'}\mathbf{V}\mathbf{A}_{12}\right)\left(\mathbf{A}_{22}-\mathbf{H}\mathbf{A}_{12}\right)^{-1}d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}}\\ & +2\left(\mathbf{K}^{'}\mathbf{W}_{22}+\mathbf{U}_{2}^{'}+\beta\widehat{\mathbf{B}}^{'}\mathbf{V}\mathbf{A}_{12}\right)\left(\mathbf{A}_{22}-\mathbf{H}\mathbf{A}_{12}\right)^{-1}d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{B}}\mathbf{F}\\ & +\left(\mathbf{K}^{'}\mathbf{W}_{22}+\mathbf{U}_{2}^{'}+\beta\widehat{\mathbf{B}}^{'}\mathbf{V}\mathbf{A}_{12}\right)\left(\mathbf{A}_{22}-\mathbf{H}\mathbf{A}_{12}\right)^{-1}d\left(\overline{\mathbf{H}}\right)\widehat{\mathbf{A}}\end{aligned} \tag{108}$$

$$\begin{aligned}\mathbf{0} = {} & 2\left(\widehat{\mathbf{U}}+\mathbf{F}^{'}\widehat{\mathbf{Q}}+\beta\mathbf{M}^{'}\mathbf{V}\widehat{\mathbf{B}}\right)d\left(\mathbf{F}\right)+\beta\mathbf{M}^{'}d\left(\overline{\mathbf{V}}\right)\mathbf{M}-d\left(\mathbf{V}\right)\\ & +2\left(\mathbf{W}_{12}+\mathbf{H}^{'}\mathbf{W}_{22}+\mathbf{F}^{'}\mathbf{U}_{2}^{'}+\beta\mathbf{M}^{'}\mathbf{V}\mathbf{A}_{12}\right)\left(\mathbf{A}_{22}-\mathbf{H}\mathbf{A}_{12}\right)^{-1}d\left(\overline{\mathbf{H}}\right)\mathbf{M},\end{aligned} \tag{109}$$

where, again, the invertibility of $(\mathbf{A}_{22} - \mathbf{HA}_{12})$ is virtually ensured by the assumption that $\mathbf{A}_{22}$ has full rank. By vectorizing and stacking equations (107)—(109) they can be written in the form

$$\mathbf{P}vec\left[d\left(\mathbf{G}\right)\right] = \mathbf{L}vec\left[d\left(\overline{\mathbf{G}}\right)\right],$$

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{I} & -\mathbf{K} & \mathbf{0} \\ \mathbf{0} & -\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right) & \mathbf{0} \\ \mathbf{0} & -2\left(\widehat{\mathbf{U}} + \mathbf{F}'\widehat{\mathbf{Q}} + \beta\mathbf{M}'\mathbf{V}\widehat{\mathbf{B}}\right) & \mathbf{I} \end{bmatrix},$$

and $\mathbf{L}$ is defined implicitly by equations (107)—(109). Because $\left(\widehat{\mathbf{Q}} + \beta\widehat{\mathbf{B}}'\mathbf{V}\widehat{\mathbf{B}}\right)$ has full rank in any symmetric Markov-perfect Stackelberg-Nash equilibrium, $\mathbf{P}$ too has full rank. The result follows. ■

E Proof of Lemma (5)

Proof. Consider equation (33). If equilibrium $\mathfrak{G}$ is locally IE-stable under private sector learning, then, $\mathbf{H}^{(s)} \to \mathbf{H}^{\mathfrak{G}}$ in the limit as $K \uparrow \infty$, which implies $\mathbf{M}^{(s)} \to \mathbf{M}^{\mathfrak{G}}$ and $\mathbf{V}^{(s)} \to \mathbf{V}^{\mathfrak{G}}$. Because equilibrium $\mathfrak{G}$ Pareto-dominates equilibrium $\mathfrak{B}$, the first-period policymaker's best response must switch from $\mathbf{F}^{\mathfrak{B}}$ to $\mathbf{F}^{\mathfrak{G}}$. On the contrary, if equilibrium $\mathfrak{G}$ is not locally IE-stable under private sector learning, then although $\mathbf{H}^{(s)}$ may converge to $\widetilde{\mathbf{H}} \neq \mathbf{H}^{\mathfrak{G}}$ in the limit as $K \uparrow \infty$, because $\widetilde{\mathbf{H}} \neq \mathbf{H}^{\mathfrak{G}}$ the first-period policymaker's best response cannot be $\mathbf{F}^{\mathfrak{G}}$. ■

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