Endogenous State Prices and the Yield Curve

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Abstract

We show, in a general equilibrium model with liquidity constraints, that state prices in a complete market are a function of the supply of liquidity by the Central Bank. In our model, derived along the lines of Dubey and Geanakoplos (1992), two agents trade goods and Arrow-Debreu securities (AD securities) to smooth consumption across periods and future states. Therefore the prices of the AD securities depend on the supply of liquidity in each state. We show that, with Von Neumann-Morgenstern utility functions, the price of an AD security (and therefore the state price or equivalently the risk-neutral probability) is inversely related to liquidity. The upshot of our argument is that agents’ expectations computed using risk-neutral probabilities give more weight in the states with higher interest rates. Thus, an upward yield curve, even though short-term interest rates are fairly stable, can be supported in equilibrium.

Keywords: cash-in-advance constraints; risk-neutral probabilities; state prices; term structure of interest rates

JEL Classification: E43; G10

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Introduction

Many financial economists have been puzzled by the fact that historical forward interest rates are on average higher than future spot interest rates. According to the expectation hypothesis, forward interest rates should reflect expectations of future spot rates, and this forms the basis of the efficiency market hypothesis. However, in the words of Shiller (1990), the forward term premium (i.e. the difference between the forward rate and the expectation of the corresponding future spot rate) has empirically been positive. Therefore, the puzzle is that forward rates have usually been higher than the historically stable interest rates, a situation that is evident in figure 1 in the period 1983-1984.

Figure 1: The term structure of interest rate in Germany, source: McCulloch data (Shiller and McCulloch (1987))

In the absence of arbitrage, with complete markets, no transaction costs, unsegmented markets, and equal tax treatment\(^1\), an increasing term structure is possible if and only if instantaneous forward rates are increasing\(^2\). The history of upward sloping term structure im-

\(^1\)The McCulloch data shown in the picture correct for differences in taxation

\(^2\)At time t, the yield of a long-term bond maturing at T is equal to the simple averages of the instantaneous forward rates (see Shiller (1990) p.640)
plies increasing forward rates, and therefore, assuming the expectation hypothesis, increasing expected future spot rates - something that does not correspond to the historical evolution of short-term interest rates.

Are rational expectations failing and bond markets inefficient then? Most of the literature on the term structure has tried to explain this puzzle by appealing to risk-preferences or liquidity. Hicks (1946) emphasized that a risk-averse investor would prefer to lend short-term if he was not given any premium on long-term lending because there is a higher risk that the prices of long-term bonds change. Lutz (1940) suggested that long-term securities are less liquid than short-term ones, where the most liquid asset is money. Finally, Modigliani and Sutch (1966) proposed the preferred habitat hypothesis - a theory that has been very influential - arguing that agents prefer to trade bonds to match asset and liability maturities. This way, the markets for long-term and short-term bonds would somehow be segmented, and therefore the link between long and short-term interest rates would break down.

In this paper, we offer a sufficient explanation for the upward term structure in an exchange economy with integrated short-term and long-term markets where the risk-premium is endogenous. We argue that the puzzle arises from a misuse of historical interest rate data when testing the rational expectations hypothesis, using subjective instead of risk-neutral probabilities.

In particular, empirical tests of rational expectations use the subjective probability measure $E_t[r_{t,m}]$ derived from equally weighted data at time $t$, where $m$ is the maturity date. Evidently this leads to forecasting errors since the risk-neutral probability measure is not used. Put differently, the difference between rational expectations and actual equilibrium values, usually defined as error terms, will not have zero mean nor be un-correlated with predictors.

What matters for an investor is the expected payoff of the security he is holding. Assume all states are equi-probable both in reality and as understood by the investor (i.e. subjective probabilities are uniform). The expected payoff commonly used in the economic literature is equal to the equal-weighted average of payoffs known for each state of nature. However, the ex-ante payoff for each state of nature is equal to the yield in this state of nature multiplied by the price of a security giving one in this state of nature and 0 everywhere else. What is missed in the usual calculus of $E_t[r_{t,m}]$ or in the calculus of any proxy used, is the importance of the price of the Arrow-Debreu (AD) security in computing the expected payoff. The strong assumption of complete markets is needed here because we want to solve for all AD securities’ prices. If the prices of AD securities were constant through all states of nature, then the historical average of spot interest rates that would proxy for rational expectations $E_t[r_{t,m}]$ would be equal to the expected interest rate $E_\pi[r_{t,m}]$ using risk-neutral probabilities $\pi$. But
if this is not the case, then there is no reason to expect the ex-ante agent’s expectation and the subjective non-weighted average to be equal.

Another way to put it is that there is no reason for the risk-neutral probabilities (i.e. the price of AD securities normalised to 1) to be equal to the actual probabilities of nature, that we assume here to be equal to the subjective probabilities chosen by an investor with perfect knowledge of the distribution of events. The difference comes from the market forces that generate equilibrium prices for AD securities. The idea that the wedge between risk-neutral and subjective probabilities may explain some financial market phenomena is not new. Heston (1993) for instance underlines how, if risk-neutral probabilities differ from subjective probabilities, volatility of security prices will be stochastic, but he does not explain where the wedge between risk-neutral and subjective probabilities comes from. In a paper that is the closest to ours, Breeden and Litzenberger (1978) show in an exchange economy that state prices are inversely related to optimal consumption choices. However, they do not model liquidity and do not extend the model to get implications on the term structure.

We show in this paper how the price of Arrow-Debreu securities, and therefore the risk-neutral probabilities, are endogenous in a cash-in-advance general equilibrium model built along the lines of Dubey and Geanakoplos (1993) and Geanakoplos and Tsomocos (2002). We need such a general equilibrium model because we want to endogenise all demands for money in order to build a term structure and the associated risk-neutral probabilities. Conditional on the existence of outside moneys, these models are able to generate proper demand for liquidity and well-defined and unique interest rates.

The main result of the paper is that states with higher interest rates (lower liquidity supplied by the Central Bank) have higher prices. The intuition goes like this. Because we model consumer’s utility with a Cobb-Douglas specification with equal weights on all states of nature, the cost of consumption is constant through all states. This cost of consumption is equal to the transaction cost to transfer money from period 0 to period 1 (i.e. the AD security price) multiplied by the value of trade in period 1 (i.e. the price of the good multiplied by the volume traded).

In a cash-in-advance constraint model, which is thought to capture the effects of constraints of liquidity on an economy, the quantity theory of money holds. Therefore, the value of trade is equal to the overall supply of liquidity (i.e. money supply from the Central Bank plus outside money). If state 1 has more liquidity than state 2, the value of trade in state 1 has to be higher than the value of trade in state 2. But, according to the lines above, this is possible only if the transaction cost for state 1 is lower than the transaction cost in state 2. Therefore,
a state with lower interest rate (higher liquidity) is also a state with lower transaction cost (i.e. lower state price and therefore lower risk-neutral probability).

In some sense, our result can be thought of as an extension of a Consumption CAPM (CCAPM), because we find that states with higher interest rate (lower activity) have higher prices, or loosely speaking (to make clear the link with the CCAPM), assets that give income in the states of nature where consumption is lower have a higher price. However, the model goes beyond the CCAPM result: first, markets are complete and so all securities can be priced; second, we model activity so that it is endogenous - in fact, activity is determined by liquidity; third, the price of a safe bond can be computed, giving us a closed-form term structure; fourth, because activity is constrained by liquidity, security prices - and in particular bond prices - are determined by liquidity, giving us an explicit value for the “liquidity premium”.

The model is an exchange economy with cash-in-advance constraints where larger money supply has the only effect of lowering transaction costs and has no other effect on production or endowments. More money supply allows a better optimum to occur because more trade is possible with lower transaction, the first best equilibrium being reached with infinite money supply. These cash-in-advance models have several drawbacks, in particular money supply is exogenously given, and the value of the interest rate depends on the existence of “outside money” (money endowed and free of any liability), a somehow controversial assumption; however this is not a problem because it seems that many other specifications would yield the same results, that we think therefore as being quite general. For instance, assume higher interest rates reduce investment, output and therefore consumption. The value of an AD security giving 1 in this state of nature will be higher because of the CCAPM argument. Therefore, risk-neutral probabilities will also be larger for states of nature with higher interest rates, so that our key intuition holds. The advantage of the cash-in-advance constraint model we are using here is that it combines in a homogeneous way the markets for money even in a finite-horizon model (and this allows us to derive the cost of high interest rates) and the markets for AD securities (that are crucially needed in any model of endogenous state prices).

The main contribution of the model is to give a proof of the existence of a “liquidity premium”, but there is a definition issue here. One can think of three effects of liquidity on bond prices. One liquidity premium would come from the costs incurred in a market where volumes and trade in an asset are small so that transaction costs are larger. A second cost is the one described by Hicks (1939):

“the imperfect ‘moneyness’ of those bills which are not money is due to their lack of general acceptability which causes the trouble of investing in them, and causes them to stand at a discount”
The third effect, the liquidity premium we have here, comes from the additional cost incurred by investors (and priced in the term structure) that an uncertain money supply will generate when liquidity is restricted (i.e. when the constraint binds, which is the assumption behind a cash-in-advance constraint model). Note that the level of money supply does not really matter: in the long run, if prices adjust to the money supply, constraints on liquidity do not have real effects - although this is not captured in our cash-in-advance constraint where the optimal supply of money would be infinite\(^3\). However, what still has effects is the variance (or risk) of liquidity. This is exactly what is captured in the model, where we show that larger liquidity risks generate higher long-term interest rates. *Stricto sensu*, our model is therefore a model of the “liquidity-risk premium”.

The model of endogenous risk-neutral probabilities has at least two consequences:

- the term structure is always upward sloping above what is expected from the expectation hypothesis because of the liquidity-risk premium.
- countries with a larger range of possible interest rates will have a higher average long-term interest rate. This means that stability of monetary policy matters.

## 1 The Model

The model is an exchange economy where no production takes place. Trade takes place between two agents who want to trade across periods (for consumption smoothing purposes) and across states (because of risk-aversion). Because cash is needed before commodity transactions, and because receipts of sells cannot be used immediately to buy commodities, agents require cash as a derived demand due to their transaction needs. Money is supplied exogenously by the Central Bank who can diminish transaction costs by increasing money supply. With lower transaction costs, more trade (i.e. more activity in this exchange economy) takes places and agents are closer to the standard General Equilibrium Pareto optimum.

The model is built around two periods, period 0 (now) and period \(f\) (future). Periods are divided into sub-periods at which the different commodity and money markets meet, as pictured in figure 2. We will explain below what each agent does in each period.

Period \(f\) endowments in goods and money, and period \(f\) supply of money are uncertain: \(n\) states of nature are possible, indexed by \(i \in \mathcal{N} = \{1, \ldots, n\}\) and with subjective probabilities all equal to \(\frac{1}{n}\). All variables indexed by 0 refer to period 0; variables indexed by \(i\) refer to period \(f\). All agent variables without subscript refer to agent \(\alpha\) (who will be the borrower), while all agent variables with subscript \(*\) refer to agent \(\beta\) (who will be the lender). The

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\(^3\)We acknowledge that a deterministic decrease in money supply may also have a short-term liquidity cost if there is some inertia
security market is complete, with all Arrow-Debreu securities available. The following tree summarizes all endowments and assets in the model.

The Central Bank provides money in \( n + 1 \) money markets:

- the short-run period-0 money market, with money supply \( M_{00} \), interest rate \( r_{00} \) and bond price \( \eta_{00} = \frac{1}{1 + r_{00}} \)

- \( \forall i \in \mathcal{N} \) the state-\( i \) money market, with money supply \( M_i \), interest rate \( r_i \) and bond price \( \eta_i = \frac{1}{1 + r_i} \)

The money supplies in the different states of nature are exogenous. Although the interest rate in each of these money markets will be deduced from demand and supply, the form of the model ensures in fact that the interest rates are just inversely proportional to the Central Bank money supply. Therefore, the interest rates for all money markets are also exogenous.
to the model. In addition to these \( n + 1 \) money markets, the two agents can trade \( n \) Arrow-Debreu securities \((AD_i)_{1 \leq i \leq n}\) that give 1 in state \( i \) and 0 in all other states \( j \neq i \). All AD securities are in zero net supply. Financial markets are complete with this structure. Knowing the exogenous interest rates, the paper will compute the AD security prices and show how they are related to interest rates.

1.1 Budget Set for Agent \( \alpha \)

Agent \( \alpha \) is endowed with goods \((e_0 = 0 ; \forall i \in \mathcal{N} ; e_i > 0)\) and with money \((m_0 > 0 ; \forall i \in \mathcal{N} ; m_i = m > 0)\). The money endowed to agents \( \alpha \) is called outside money, and it is money that the agent has received clear of any liability and that he considered as wealth. In the paper we assume that outside money is non-random, i.e. constant through the different states of nature.

In period 0, agent \( \alpha \) borrows conditional on the future states to finance consumption at time 0. He therefore sells \( q_{AD_i} \) securities at price \( \theta_i \) (i.e. borrows with repayment conditional on state \( i \)).

In state \( i \), agent \( \alpha \) has to give \( q_{AD_i} \) to agent \( \beta \) who - as we will see later - has bought the AD securities. Since \( \alpha \) cannot use yet the receipts of good \( i \) sells, he has to borrow short-term (i.e. roll over his debt), at cost \( \eta_i \). He thus borrows \( \eta_i \mu_i \), and uses this amount with his money endowment \( m \) to pay agent \( \beta \) on the \( AD_i \) market. He then uses the receipts of his state-\( i \) sells \((p_i q_i)\) to repay state-i loan \( \mu_i \).

Agent \( \alpha \) maximises inter-temporal consumption with discount factor 1 and equal weight between the \( n \)-states since the states are assumed to be equi-probable. In addition to his financial choices \( ((\mu_i)_{1 \leq i \leq n} , (q_{AD_i})_{1 \leq i \leq n}) \), he chooses how much to buy in period 0 \((b_0 \text{ noted in values})\) and how much to sell in the different states \( p_i q_i \).

In addition to his endowment, he can consume what he buys \( b_i/p_i \), but he cannot consume what he sells \( p_i q_i \). Hence, consumption in each period or state is

\[
\forall i \in \mathcal{N} \cup \{0\} \quad c_i = e_i - q_i + \frac{b_i}{p_i}
\]

Therefore, considering his endowments \((e_0 = 0 ; \forall i e_i > 0)\) and \((m_0 > 0 ; \forall i m_i = m > 0)\), agent \( \alpha \)’s maximisation programme is : (in brackets are the lagrangian multipliers)\(^4\)

\(^4\)We do not make explicit in these equations that agent \( \alpha \) can carry money over from period 0 to period \( f \) because he will in fact never choose to do so, as all constraints are binding - something we will see later.
\[
\max_{b_0, (q_i)_{i \in \mathcal{N}}, (\mu_i)_{i \in \mathcal{N}}, (q_{ADi})_{i \in \mathcal{N}}} \quad U(b_0, (q_i)_{i \in \mathcal{N}}) = \ln \left( \frac{b_0}{p_0} \right) + \frac{1}{n} \sum_{i \in \mathcal{N}} \ln(e_i - q_i) \quad \text{s.t.} \quad \tag{1}
\]

\[
b_0 \leq m_0 + \sum_{1 \leq i \leq n} \theta_i q_{ADi} \quad (\varphi_i) \quad \tag{2}
\]

\[
\forall i \in \mathcal{N} \begin{cases} q_{ADi} \leq \eta_i \mu_i + m \quad (\Psi_i) \\
\mu_i \leq p_i q_i \quad (\chi_i) \end{cases} \quad \tag{3}
\]

### 1.2 Budget Set for Agent $\beta$

Agent $\beta$ is endowed with goods \( (e_0^* > 0; \forall i \in \mathcal{N} \: e_i^* = 0) \) and with money \( (m_0^* > 0; \forall i \in \mathcal{N} \: m_i^* = m^*) \).

**In period 0**, he sells \( q_0^* \) but he will receive the cash only at the end of the period. He wants to buy AD securities \( b_{ADi}^* \) to finance consumption-smoothing in the future. For this, he has to borrow \( \eta_0 \mu_0^* \) on the period-0 short-term money market at cost \( r_{00} \). He then repays the loan with his receipts \( p_0 q_0^* \). We do not make explicit the possibility for agent $\beta$ to carry money over since he will never do so, since we will show that all constraints are binding. As a result, all the receipts from the sells of good 0 are invested in AD securities, and this means that agent $\beta$ borrows short-term as much as he will be able to repay.

**In state $i$**, he receives the revenue from the $AD_i$-security \( \frac{b_{ADi}^*}{\theta_i} \) and he uses this to buy good $i$ (in value $b_i^*$).

Agent $\beta$ maximises inter-temporal consumption with discount factor 1 and equal weight between the $n$ states. In addition to his financial choices \( (\mu_0^*, (\mu_i^*)_{1 \leq i \leq n}, (q_{ADi})_{1 \leq i \leq n}) \), he chooses how much to buy \( (b_i^* \text{ noted in values}) \) and how much to sell \( (p_0 q_0^*) \). Therefore, considering his endowments in goods \( (e_0^* > 0; \forall i \in \mathcal{N} \: e_i^* = 0) \) and in money \( (m_0^* > 0; m_i^* = m^*) \), agent $\beta$‘s maximisation programme is : (in brackets are the lagrangian multipliers to be used later)

\[
\max_{q_0^*, (b_i^*)_{1 \leq i \leq n}, \mu_0^*, (b_{ADi}^*)_{1 \leq i \leq n}} \quad U(q_0^*, (b_i^*)_{i \in \mathcal{N}}) = \ln(e_0^* - q_0^*) + \frac{1}{n} \sum_{1 \leq i \leq n} \ln \left( \frac{b_i^*}{p_i} \right) \quad \text{s.t.} \quad \tag{4}
\]

\[
\sum_{1 \leq i \leq n} b_{ADi}^* \leq m_0^* + \eta_0 \mu_0^* \quad (\varphi^*) \quad \tag{5}
\]

\[
\mu_0^* \leq p_0 q_0^* \quad (\xi^*) \quad \tag{6}
\]

\[
\forall i \in \mathcal{N} \quad b_i^* \leq \frac{b_{ADi}^*}{\theta_i} + m^* \quad (\chi_i^*) \quad \tag{7}
\]
1.3 Financial General Equilibrium

We say that \( \left(p_0, q_0^*, b_0, \eta_{00}, \mu_{00}, (p_i, q_i, b_i^*, \eta_i, \mu_i, \theta_i, b_{AD_i}^*, q_{AD_i})_{i \in \mathcal{N}} \right) \in \mathbb{R}^{8n+5}_{++} \) is a Financial General Equilibrium if and only if:

(i) \( (b_0, (q_i, \mu_i, q_{AD_i})_{i \in \mathcal{N}}) \in \text{Argmax } U \left(b_0, (q_i)_{i \in \mathcal{N}} \right) \)

(ii) \( (q_0^*, (b_i^*, \mu_{00}, b_{AD_i}^*, q_{AD_i})_{i \in \mathcal{N}}) \in \text{Argmax } U \left(q_0^*, (b_i^*)_{i \in \mathcal{N}} \right) \)

(iii) Commodity markets clear, i.e.

\[
p_0 = \frac{b_0}{q_0^*} \iff p_0 q_0^* = b_0 \quad \forall i \in \mathcal{N} \quad p_i = \frac{b_i^*}{q_i} \iff p_i q_i = b_i^*\]

(iv) Money and AD security markets clear, i.e.

\[
\mu_{00} = (1 + r_{00})M_{00} = M_{00}/\eta_{00} \quad \forall i \in \mathcal{N} \quad \mu_i = (1 + r_i)M_i = M_i/\eta_i \quad \forall i \in \mathcal{N} \quad \theta_i q_{AD_i} = b_{AD_i}^*
\]

From the previous money market conditions, we find the following relationship, common in cash-in-advance general equilibrium models:

**Proposition 0: Term Structure**

\[
\forall i \in \mathcal{N} \quad r_i M_i + r_{00} M_{00} = m_0 + m_0^* + m + m^* \quad (8)
\]

In fact, because we no money is carried over from period 0 to period \( f \), we can separate this in two equation

\[
r_{00} M_{00} = m_0 + m_0^* \quad (9)
\]

and

\[
\forall i \in \mathcal{N} \quad r_i M_i = m + m^* \quad (10)
\]

These equations allow to determine all short-term interest rates, in a way that is almost exogenous to the equilibrium model.

**Proof**

Using period-0 constraint
\[(1 + r_{00})M_{00} = \mu_{00} = p_0 q_0^* = b_0 = m_0 + \sum_{1 \leq i \leq n} \theta_i q_{AD_i} = m_0 + m_0^* + \eta_{00} \mu_{00} \quad (11)\]

\[(1 + r_{00})M_{00} = m_0 + m_0^* + M_{00} \quad (12)\]

from which the first result is obvious. Similarly

\[\forall i (1 + r_i)M_i = \mu_i = b_i^* = q_{AD_i}^* + m^* = \eta_i \mu_i + m + m^* = M_i + m + m^* \quad (13)\]

### 1.4 Agent β First Order Conditions

Denote \(\mathcal{L}\) the lagrangian formed from the maximisation problem. The first order conditions are:

\[\frac{\partial \mathcal{L}}{\partial q^*} = -1 + p_0 \xi^* = 0 \quad (14)\]

\[\forall i \in \mathcal{N} \quad \frac{\partial \mathcal{L}}{\partial b_i^*} = \frac{1}{n} \frac{1}{b_i^*} - \chi_i^* = 0 \quad (15)\]

\[\frac{\partial \mathcal{L}}{\partial \mu_{00}} = \eta_{00} \varphi^* - \xi^* = 0 \quad (16)\]

\[\frac{\partial \mathcal{L}}{\partial b_{AD_i}^*} = -\varphi^* + \frac{\chi_i^*}{\theta_i} = 0 \quad (17)\]

Note that all constraints are binding. This is easy to prove since the marginal utility of consuming \(b_i^*\) - which is never null with a logarithmic utility - is equal to \(\chi_i^* > 0\). Therefore \(\varphi^* > 0\) and \(\xi^* > 0\). The same demonstration applies for agent \(\alpha\). This result is expected in a complete market where the other financial instruments will give higher returns than money.

### 1.5 Intermediary results

Since all constraints are binding, and using good, money, and AD market equilibria, we can deduce several results that will prove useful in making the model tractable.

**Proposition 1: Quantity theory of money in period 0**

\[p_0 q_0^* = b_0 = m_0 + m_0^* + M_0\]

**Proof**

We know that
\begin{equation}
b_0 = m_0 + \sum_{1 \leq i \leq n} \theta_i q_{AD_i}
\end{equation}

and

\begin{equation}
\sum_{1 \leq i \leq n} \theta_i q_{AD_i} = \sum_{1 \leq i \leq n} b^*_{AD_i} = m^*_0 + \eta_{00} \mu_{00}^*
\end{equation}

The result is then obvious since \( \eta_{00} \mu_{00}^* = M_{00} \).

**Interpretation**

Agent \( \alpha \) is the only agent who needs cash in period 0, and he needs it only for \( b_0 \). Therefore, he will use all available cash (his cash and the cash supplied by agent \( \beta \), \( m^*_0 + M_{00} \), thanks to the AD securities) to buy \( b_0 \). This result is the quantity theory of money for period 0, since \( b_0 \) represents all trade in period 0. The same intuition mechanics works for all future states of nature, with the role of agents being inverted.

**Proposition 2: Quantity theory of money for all states \( i \)**

\( \forall i \in \mathcal{N} \quad p_i q_i = b^*_i = m + m^* + M_i \)

**Proof**

\( \forall i \in \mathcal{N} \quad p_i q_i = b^*_i = \frac{b^*_{AD_i}}{\theta_i} + m^* = q_{AD_i} + m^* = \eta_i \mu_i + m + m^* = M_i + m + m^* \) \hspace{1cm} (20)

**1.6 Solving Demand equations of agent \( \beta \)**

For any \((i, j) \in \{1, \ldots, n\}^2\), subtracting state-j to state-i application of the quantity theory of money

\( \forall i, j \in \mathcal{N} \quad b^*_i - b^*_j = M_i - M_j \) \hspace{1cm} (21)

This is just a direct application of the quantity theory of money assuming that outside money is constant through the different states of nature.

We also note from the first order conditions on \( b^*_{AD_i} \) and \( b^*_{AD_j} \) that

\( \forall i, j \in \mathcal{N} \quad \frac{\chi^*_i}{\theta_i} = \varphi^* = \frac{\chi^*_j}{\theta_j} \) \hspace{1cm} (22)

From the first order conditions on \( b^*_i \), we can then deduce that

\( \theta_i b^*_i = \theta_j b^*_j \) \hspace{1cm} (23)
This is a consequence of the Cobb-Douglas utility function with equal weight on each state (the share of expenditure over income is constant) but with state-contingent transaction costs. This makes clear how the ratio of transaction costs matters to determine trade volumes.

From these two last equations, we find that:

$$\forall i, j \in \mathcal{N}, \quad i \neq j, \quad b_i^* = \frac{\theta_j (M_i - M_j)}{\theta_j - \theta_i}$$

(24)

**Proposition 3 : Endogenous State Prices**

In an interior equilibrium (i.e. $\forall i \in \mathcal{N}, \quad q_i > 0$ and $b_i^* > 0$), states with higher interest rates correspond to higher state prices (i.e. bigger risk-neutral probabilities).

**Proof**

From the last result, since all trades are positive in an interior equilibrium:

$$\forall i, j \in \mathcal{N}, \quad i \neq j \quad \text{since} \quad b_i^* \geq 0, \quad M_i > M_j \iff \theta_j > \theta_i$$

(25)

Furthermore, we know from the section on the term structure that $M_j > M_i \iff r_j < r_i$.

Finally, from the fundamental theorem of finance, the risk neutral probability associated to state $i$ is equal to

$$\pi_i = \frac{\theta_i}{\sum_{1 \leq k \leq n} \theta_k}$$

(26)

Hence $\theta_j > \theta_i \iff \pi_j > \pi_i$

From the preceding equations we deduce the final result

$$\forall i \neq j \quad r_j > r_i \iff \pi_j > \pi_i$$

(27)

This is the main result of our model. It first shows that risk-neutral probabilities are endogenous in a cash-in-advance general equilibrium model, in contrast with arbitrage models whereby prices follow various stochastic processes and thus risk-neutral probabilities are not demand and supply dependent. Furthermore, the fact that states with higher interest rates are given higher weights yields important results for the yield curve.

The intuition of the result is straightforward. Because we model consumer’s utility with a Cobb-Douglas specification with equal weights on all states of nature, the cost of consumption is constant through all states (see equation 23). This cost of consumption is equal to the transaction cost to transfer money from period 0 to period 1 (i.e. the AD security price).
multiplied by the value of trade in period 1 (i.e. the price of the good multiplied by the volume traded). Because the quantity theory of money holds, the value of trade is equal to the overall supply of liquidity. If state 1 has more liquidity than state 2, the value of trade in state 1 has to be higher than the value of trade in state 2. But, according to equation 23, this is possible only if the transaction cost for state 1 is lower than the transaction cost in state 2. Therefore, a state with lower interest rate (higher liquidity) is also a state with lower transaction cost (i.e. lower state price and therefore lower risk-neutral probability).

1.7 Explicit solution

We can now solve explicitly for all AD security prices, a result that will allow us to compute comparative exercises.

**Proposition 4**

\[ \forall i, j \in \mathcal{N}, \quad b_{AD_i}^* - b_{AD_j}^* = (\theta_j - \theta_i)m^* \]

**Proof**

From the first order conditions:

\[ \theta_j b_j^* = \theta_i b_i^* \]  \hspace{1cm} (28)

From which we deduce

\[ b_{AD_i}^* = \theta_j b_j^* - \theta_j m^* = \theta_i b_i^* - \theta_j m^* = b_{AD_i}^* \theta_i m^* - \theta_j m^* \]

\hspace{1cm} (29)

**Proposition 5 : Solved-out state prices**

\[ \forall i \in \mathcal{N}, \quad \theta_i = \frac{1}{n} \left( \frac{\sum_{k=1}^{n} \theta_k}{m_i + M + m^*} \right) \]

**Proof**

From equation (5) and proposition 4

\[ \sum_{1 \leq k \leq n} b_{AD_k}^* = m_0^* + M_{00} \]  \hspace{1cm} (30)

\[ \forall i \in \mathcal{N}, \quad \sum_{1 \leq k \leq n} b_{AD_i}^* = \sum_{k=1}^{n} (b_{AD_i}^* + (\theta_i - \theta_k)m^*) = n\theta_i q_{AD_i} + n\theta_i m^* - m^* \sum_{k=1}^{n} \theta_k = m_0^* + M_0 \]  \hspace{1cm} (31)

Since \( q_{AD_i} = M_i + m \)
\[ n\theta_i(M_i + m + m^*) = m_0^* + M_{00} + m^* \sum_{k=1}^{n} \theta_k \]  

(32)

from which we deduce the result.

This confirms proposition 3, since a state with higher interest rate will have lower money supply and therefore higher AD-security price. The intuition of the argument suggested in propositions 3 to 5 is also robust to continuous changes in monetary policy. As the following proposition shows, the inverse relation of liquidity and state price is monotonic.

**Proposition 6 : Comparative statics result**

An increase in state-\( i \) interest rate increases the risk-neutral probability associated to this state.

**Proof**

Totally differentiating equation (32)

\[ nd\theta_i(M_i + m + m^*) + n\theta_idM_i = m^*d\theta_i + m^*\sum_{k \neq i}^{n} d\theta_k \]  

(33)

\[ d\theta_i [n(M_i + m + m^*) - m^*] = -n\theta_idM_i + m^*\sum_{k \neq j}^{n} d\theta_k \]  

(34)

Intuitively, the effect of a change in \( M_i \) on \( \theta_k \) \( (k \neq i) \) will be of a second order magnitude, and therefore \( dM_i > 0 \) implies \( d\theta_i < 0 \)

More formally, let us assume that money supply is changed only for \( i = 1 \) (we can indeed reorder the index to ensure this is is true), solving the system of differential equations presented in equation (34) means solving

\[
\begin{bmatrix}
A_1 - m^* & -m^* & \ldots & -m^* \\
-m^* & A_2 - m^* & \ldots & -m^* \\
-m^* & \ldots & A_j - m^* & -m^* \\
-m^* & \ldots & -m^* & A_n - m^*
\end{bmatrix}
\begin{bmatrix}
d\theta_1 \\
d\theta_2 \\
\vdots \\
d\theta_n
\end{bmatrix}
= -n\theta_1
\begin{bmatrix}
dM_1 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

(35)

where \( \forall j \in \mathcal{N} \quad A_j = n(M_j + m + m^*) \). We call the square matrix \( \mathcal{A}_{1,n} \) and the vectors \( d\Theta \) and \( dM \) with obvious notations. \( \mathcal{A}_{1,n} \) is clearly definite and therefore invertible. Let us write \( \mathcal{A}_{1,n} = \Delta - m^*E \), where \( \Delta \) is the diagonal matrix with diagonal elements \( A_j \) and \( E \) is the square matrix with ones everywhere \( E = (1)_{1 \leq i,j \leq n} \).
From the Sherman-Morrison-Woodbury Formula (see Golub and Van Loan, 1996), we know that
\[ \Delta_{i,n}^{-1} = \Delta^{-1} + \frac{m^*}{1 - m^*/h} \Delta^{-1} E \Delta^{-1} \] (36)
where \( h \) is the harmonic mean of available liquidity defined by
\[ \frac{1}{h} = \sum_{k \in \mathcal{N}} \frac{1}{A_k} = \frac{1}{n} \sum_{1 \leq k \leq n} \frac{1}{m + m^* + M_k} \]
As a first approximation, since liquidity is of the same order in all states of nature, we can think of \( h \approx M_1 + m + m^* \). As a result,
\[ \frac{m^*}{1 - m^*/h} \approx m^* > 0 \] (37)
It is also straightforward to see that the first column of \( \Delta^{-1} E \Delta^{-1} \) is
\[
\begin{bmatrix}
\frac{1}{A_1} \\
\frac{1}{A_2} \\
\vdots \\
\frac{1}{A_n}
\end{bmatrix}
\]
Hence, the first column of \( \Delta_{i,n}^{-1} \) is
\[
\begin{bmatrix}
c_{11} \\
c_{12} \\
\vdots \\
c_{1n}
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\frac{1}{A_1} \\
0 \\
\vdots \\
0
\end{bmatrix} + \left( \frac{m^*}{1 - m^*/h} \right) \begin{bmatrix}
\frac{1}{A_1} \\
\frac{1}{A_2} \\
\vdots \\
\frac{1}{A_n}
\end{bmatrix}
\]
Our first result is then
\[ \forall i \in \mathcal{N}, i \neq 1, \quad c_{11} >> c_{ii} > 0 \]
And since
\[ \frac{\partial \theta_1}{\partial M_1} = -n \theta_1 c_{11} < 0 \]
we showed that a decrease in state-1 liquidity increases the price of state 1.
Since \( \forall i \in \mathcal{N}, i \neq 1, \quad c_{ii} << c_{11} \) and from the definition of risk-neutral probabilities
\[ \frac{d \pi_1}{d r_1} > 0 \] (38)
1.8 Solved-out Term Structure
We compute now the risk-neutral probabilities in a closed form, which allows us to find the long-term bond price.
Proposition 7: Endogenous risk-neutral probabilities

\[ \forall i \in \mathcal{N} \quad \pi_i = \frac{1}{n} \left( \frac{h}{m + m^* + M_i} \right) \] (39)

Proof

\[ \pi_i = \frac{1}{n} \frac{m_i^* + M_{00} + m^* \sum_{k=1}^{n} \theta_k}{m + m^* + M_k} = \frac{1}{n} \frac{m_i^* + M_{00} + m^* \sum_{k=1}^{n} \theta_k}{m + m^* + M_k} \] (40)

Proposition 8: Long-term interest rate

\[ \eta_{00} = \frac{m_0 + M_{00}}{h - m_0^*} \] (41)

Proof

From equation (32),

\[ \eta_{00} = \frac{m_0 + M_{00} + m_0^* \eta_{00}}{h - m_0^*} \iff \eta_{00} \left(1 - \frac{m_0^*}{h} \right) = \frac{m_0 + M_{00}}{h} \] (43)

2 Application: Term Structure of Interest Rates

Let \( B \) a bond that would be bought in the first period at the time when the money market clears and that would mature at the time the intra-period bond of second period matures, as shown in the thick arrow of figure 4.

Figure 4: Long-term bond
A no arbitrage argument ensures that the price of such a bond is:

\[ P_B = \eta_00 \left(\sum_{i=1}^{n} \theta_i \eta_i\right) \]  

(44)

or equivalently

\[ P_B = \eta_00 \eta_0 \left(\sum_{i=1}^{n} \pi_i \eta_i\right) \]  

(45)

Note that no ones in our model needs such a bond. A more relevant bond may be the one that is bought at the time the AD market meets and would mature at the time the intra-period bond of second period matures. For this bond \( b \), a no-arbitrage arguments ensure that its price is

\[ P_b = \sum_{i=1}^{n} \theta_i \eta_i \]  

(46)

or equivalently

\[ P_b = \eta_0 \sum_{i=1}^{n} \pi_i \eta_i \]  

(47)

By approximating, we find

\[ r_b \approx r_0 + \sum_{i=1}^{n} (\pi_i r_i) = r_0 + \sum_{i=1}^{n} \left( \frac{1}{n} \left( \frac{h}{m + m^* + M_i} + \frac{m + m^*}{M_i} \right) \right) \]  

(48)

It is clear here how the long-term interest rate depends on a quadratic function of \( M_i \) and is implicitly the average of a convex function of spot rates. The first consequence of this is that the long-term interest rate is above the average of spot rates, something we define as the “liquidity-risk premium”. The second result is that a larger variance in spot rates will generate a higher “liquidity-risk premium” and long-term interest rate, so that stability of monetary policy matters in determining the equilibrium value of long-term interest rates.

**Conclusion**

In a state with low liquidity, trade has to be low, and, in order to induce consumers to have trade at a low level, the cost of transferring money to this state must be high. This transaction cost is equal to the state price. Therefore, state prices and risk-neutral probabilities are higher in states with higher interest rates. It is important to stress in conclusion that the result is due to the interaction of the money market (the quantity theory of money) with the exchange economy (the maximisation problem) and therefore cannot be found in a pure financial model. However, the real sector is not critical to the result, because the quantity theory of money ensures that the value of trade is equal to the supply of money. Ultimately, it is the risk in the
supply of money that matters to determine the risk in trade values; and because of the Von Neumann-Morgenstern Cobb-Douglas utility function, it is trade values (i.e. nominal trade as opposed to real trade) that are equalised through states. Our result is therefore more general and can be found in any model with complete market, Von Neumann-Morgenstern logarithmic utility, and some form of the quantity theory of money. Other shocks than liquidity shocks may of course also affect risk-neutral probabilities. Liquidity shocks are however crucial to understand the upward sloping term structure because two phenomena push in the same direction: first, the futures spot interest rates are affected; second the risk-neutral probabilities are modified. The interaction of these two effects pushes long-term rates above the historical average of future spot rates. And the more uncertainty in the future spot rates, the higher the long-term rates. Stability of monetary policy is therefore required to maintain flat yield curves, an intuitive consequence of the existence of a liquidity-risk premium.

Further research will extend the model to a 3-period framework where re-trading of assets is the source of asset price uncertainty, rather than an exogenously given set of AD securities, and will analyse the importance of the use of risk-neutral probabilities in tests of rational expectations.

References


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