Panel Threshold Regression Models with Endogenous Threshold Variables

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Abstract

This paper extends the panel threshold regression of Hansen (1999) and Kourtellos, Stengos and Tan (2007) to allow for endogeneity of the threshold variable. We consider the static linear panels with fixed effect. The modified concentrated two-stage least square methods that are based on inverse Mills ratio bias correction terms are proposed to estimate threshold parameters. Our estimators are consistent even though individual effects have any correlation with regressors. Monte Carlo simulations are performed to investigate the finite sample properties of the proposed estimators.

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1 Introduction

One of the most popular topics in nonlinear regression models is the threshold-type regression model. Threshold models are attractive because they can allow for more flexible regression functional forms by splitting data with certain unknown threshold values. Although there are several previous studies on the statistical inference of threshold models, only a few papers investigate statistical inference under endogenous threshold values. Tong (1983) first proposes threshold regression models for time series data. Hansen (1999) extends the threshold regression to static panel data structure and derives the corresponding asymptotic theory for threshold parameters and regression slopes. Hansen (2000) further shows the asymptotic properties of concentrated least square estimators for threshold and coefficient parameters. Caner and Hansen (2004) turn to consider endogenous regressors in threshold regression models with exogenous thresholds. They use instrument variable methods to obtain the consistent coefficient estimators. However, all above-mentioned papers virtually impose the assumption of the threshold variables being exogenous.

To the best of our knowledge, the first paper to investigate threshold regression with endogenous threshold variables is Kourtellos, Stengos and Tan (below KST, 2007). They consider the threshold model with endogenous threshold values under cross section data. When threshold variables are considered endogenous, previous methods cannot obtain accurate threshold parameters, i.e., splitting data based on inaccurate threshold variables will result in estimation bias in coefficients. KST (2007) interpret the endogenous threshold variables in terms of the traditional sample selection bias. First, they decompose the error terms into two parts: one is correlated with the endogenous threshold variable, and the other is not. Second, they use a two-stage least square procedure proposed by Heckman (1979) to rectify the endogeneity of the threshold variables. Although KST (2007) use the same technology as the general sample selection model to correct bias, there is one main difference between the threshold regression model with endogenous threshold and a typical sample selection model. For a sample selection model, whether the sample is observed hinges on the first stage selection function, while the entire sample can be observed in the threshold regression model with endogenous thresholds. KST (2007) utilize a similar two-step approach in sample selection models to obtain the bias correction parts.

Although KST (2007) propose a bias correction method to estimate threshold regression models with endogenous regressors, their method is in particular developed for cross-sectional data. Some recent research regarding the issue of investment and growth convergence often use threshold panel data models, but threshold variables may be correlated with independent variables. For instance, Hansen (1999) explores the relationship between investment and financial constraints. Hansen (1999) uses the lagged one period of ratio of long- term debt to assets for the threshold variable, but the lagged one period of ratio of long- term debt may be correlated with one of the regressors, e.g., the lagged one period ratio of cash flow to assets. This correlation can cause inconsistency in traditional panel threshold regressions with exogenous threshold variables. Kremer, Bick and Nautz (2008) consider the relationship between economic Growth and inflation. They adopt the five-year average of the annual percentage change of the CPI index for threshold variable. Although the five-year average CPI index is a good variable for modeling the threshold effect, still CPI index is possibly correlated with the growth rate of monetary supply. Therefore, a panel threshold model which accounts for potentially endogenous variables should be designed for those empirical research.

In this paper, we attempt to fill up the gap in the literature on panel threshold models by allowing for endogenous threshold variables. We will propose the two-stage bias correction methods to estimate parameters of panel threshold model with endogenous threshold variables across different specifications. Three types of panel threshold models are considered in this article: 1) both (main) panel threshold model and threshold variable equation without individual specific effects; 2) panel threshold model with individual specific effect but threshold variable equation without individual specific effect; and 3) both panel threshold model and threshold variable equation with individual specific effects. For the panel threshold model without any fixed effect, we use a bias-correction method to get the consistent estimator. Regarding the main panel threshold model with individual effect only, we use a difference transformation to eliminate fixed effect. The consistent estimators can be obtainede by a bias-correction method after the transformation. When both equations have individual specific effects, we adopt the least squares dummy variable (LSDV) approach to get the sample residuals sequence in threshold equation. With the sample residuals, we then conduct the first-difference transformation to remove the fixed effect from panel threshold equation. The bias correction method again will be used to obtain consistent estimators for threshold value and coefficient parameters.

The remainder of this paper is organized as below: In Section 2, the panel threshold models without individual effects are introduced. We will propose one consistent estimation method to handle panel threshold regression with endogenous threshold variables. In Section 3, we relax the assumptions in Section 2 and allow individual effects in main panel threshold equation. The consistent bias correction estimators will be developed for panel threshold model with fixed effect and endogenous regressors. In Section 4, we consider a more general case of the endogenous panel threshold model, the main and threshold equations will exist individual effects together. A new consistent bias correction estimation method will be suggested. Finally, some of the future research will be discussed in the last section.

2 The Panel Threshold Regression with Endogenous Threshold Models

We consider the balanced panel data throughout the paper, namely, $\{y_{it}, x_{it}, q_{it} : 1 \leq i \leq n, 1 \leq t \leq T\}$, where the subscripts i and t are indexes for individual and time, respectively. Consider the typical panel threshold model as below:

$$y_{it} = x_{it} I(q_{it} \leq \theta)\beta_1 + x_{it} I(q_{it} > \theta)\beta_2 + e_{it} \tag{1}$$

$$q_{it} = z_{it}\pi + u_{it}, \tag{2}$$

where q_{it} is an observed threshold variables, θ is an unknown threshold parameter, and z_{it} is a vector of instruments.[1]

Let us define the indicators with respect to the threshold variable q_{it}:

$$I(q_{it} \leq \theta) = \begin{cases} 1 \text{ iff } u_{it} \leq \theta - z_{it}\pi \\ 0 \text{ iff } u_{it} > \theta - z_{it}\pi \end{cases} \tag{3}$$

and

$$I(q_{it} > \theta) = \begin{cases} 0 \text{ iff } u_{it} \leq \theta - z_{it}\pi \\ 1 \text{ iff } u_{it} > \theta - z_{it}\pi \end{cases}. \tag{4}$$

We shall also make the following assumptions in this paper:

[1] Note that $z_{it} = \begin{bmatrix} z_{1it} & z_{2it} \end{bmatrix}$ contain the set of instrument variables, where $z_{2it} = x_{it}$.

Assumption 2.1. $\{y_{it}, x_{it}, q_{it}, e_{it}\}$ *is strictly stationary, ergodic.*

Assumption 2.2. $E|x_{it}|^4 < \infty$ *and* $E|e_{it}|^4 < \infty$.

Assumption 2.3. $n \to \infty$ *and* T *is fixed.*

Assumption 2.4. *For some fixed number* $G < \infty$ *and* $0 < \alpha < 1/2$, $\delta = \beta_2 - \beta_1 = n^{-\alpha}G$.

Assumption 2.5. *Let* $f_t(\theta)$ *denote the density function of* q_{it}. *Define*

$$D(\theta) = \sum_{t=1}^{T} E(Gx_{it}|q_{it} = \theta)f_t(\theta)$$

$D(\theta)$ *is continuous at* $\theta = \theta_o$.

Assumption 2.6. *Assume* $D(\theta_o) = D$, $0 < D < \infty$.

Assumption 2.7. $u_{it}|z_{it} \sim \mathcal{N}(0, 1)$

Assumption 2.8. *The joint distribution between* e_{it} *and* u_{it} *is defined as:*

$$\begin{bmatrix} e_{it} \\ u_{it} \end{bmatrix} |x_{it}, z_{it} \sim \mathcal{N}\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma_e^2 & \gamma_j \\ \gamma_j & 1 \end{bmatrix} \right), \quad j = 1, 2, \tag{5}$$

where γ_j *is the covariance between* e_{it} *and* u_{it}; $\gamma_j = \gamma_1$ *when* $q_{it} \leq \theta$ *and* $\gamma_j = \gamma_2$ *when* $q_{it} > \theta$.

Assumptions (2.1)-(2.3) are standard for fixed-effect panel data regressions. Assumptions (2.4)-(2.6) are similar assumptions that are used in Hansen (1999). The main purpose of these assumptions is to simplify the analysis and to get rid of the nuisance parameters in the distribution of threshold estimations.

Assumptions (2.7) and (2.8) impose the correlation relationship between threshold variable and panel threshold errors. Assumption (2.7) is the same as KST (2007). We are able to find the accurate functional form of the generated regressor for a two-stage bias correction estimator with Assumption (2.7). Assumption (2.8) describes the endogeneity structure of the panel threshold model. From Assumption (2.8) and KST (2007), it is useful to decompose e_{it} into two parts:

$$\begin{bmatrix} \varepsilon_{it} \\ u_{it} \end{bmatrix} = \begin{bmatrix} 1 & -\gamma_j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{it} \\ u_{it} \end{bmatrix}. \tag{6}$$

We then get the joint distribution of ε_{it} and u_{it}:

$$\begin{bmatrix} \varepsilon_{it} \\ u_{it} \end{bmatrix} | x_{it}, z_{it} \sim \mathcal{N}\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma_e^2 - \gamma_j^2 & 0 \\ 0 & 1 \end{bmatrix} \right). \tag{7}$$

Under Assumptions (2.1)-(2.8), the effect introduced by endogenous threshold variables, $\gamma_j u_{it}$ enters Equation (1) linearly.[2] When $q_{it} \leq \theta$, the conditional expectation of panel threshold model is:

$$E\left[y_{it}|x_{it}, z_{it}, q_{it} \leq \theta\right] = E\left[y_{it}|x_{it}, u_{it} \leq \theta - z_{it}\pi\right] = x_{it}\beta_1 + \lambda_1(\theta - z_{it}\pi). \tag{8}$$

When $q_{it} > \theta$, we have that:

$$E\left[y_{it}|x_{it}, z_{it}, q_{it} > \theta\right] = E\left[y_{it}|x_{it}, u_{it} > \theta - z_{it}\pi\right] = x_{it}\beta_1 + \lambda_2(\theta - z_{it}\pi), \tag{9}$$

where $\lambda_1(\theta - z_{it}\pi) = -\frac{\phi(\theta - z_{it}\pi)}{\Phi(\theta - z_{it}\pi)}$ and $\lambda_2(\theta - z_{it}\pi) = -\frac{\phi(\theta - z_{it}\pi)}{1-\Phi(\theta - z_{it}\pi)}$. $\phi(.)$ and $\Phi(.)$ denote the density and cumulated density function of a standard normal distribution. $\lambda_1(\theta - z_{it}\pi)$ and $\lambda_2(\theta - z_{it}\pi)$ are the well-known inverse Mills, which play the role of bias correction terms. We can rewrite panel threshold model with endogenous threshold variables for a single equation form:

$$y_{it} = x_{it}I(q_{it} \leq \theta)\beta_1 + x_{it}I(q_{it} > \theta)\beta_2 + \psi(q_{it}, z_{it}, \theta, \pi) + \varepsilon_{it}, \tag{10}$$

where

$$\psi(q_{it}, z_{it}, \theta, \pi) = \gamma_1\lambda_1(\theta - z_{it}\pi)I(q_{it} \leq \theta) + \gamma_2\lambda_2(\theta - z_{it}\pi)I(q_{it} > \theta).$$

When $\gamma_1 = 0$ and $\gamma_2 = 0$, the bias correction items will disappear. We get Hansen (1999) panel threshold regression for exogenous threshold models.

$$y_{it} = x_{it}I(q_{it} \leq \theta)\beta_1 + x_{it}I(q_{it} > \theta)\beta_2 + \varepsilon_{it} \tag{11}$$

Although Equation (10) looks like the error interdependence between main equation and sample selection equation, there is one important difference between panel threshold models and sample selection models. In panel threshold model with endogenous models, we can observe all variables, but in sample selection models, we only observe part of the variables.

[2] To simplify our analysis, we consider that the correlation between e_{it} and u_{it} is fixed across i and t, i.e., γ_j, in this paper. This setting can be relaxed in the future study.

Our estimation procedure proceeds in three steps: First, we estimate the parameter π in Equation (2) by ordinary least square method. [3] Second, we estimate the threshold parameter θ by minimizing a concentrated least square criterion using $\hat{\pi}$ from first stage.

$$S^{CLS}\left(\beta_i(\theta), \gamma_i(\theta), \theta\right)$$

$$= \arg \min_{\theta \in [\underline{\theta}, \overline{\theta}]} \sum_{i=1}^{N} \sum_{t=1}^{T} \left(y_{it} - x_{it} I(q_{it} \leq \theta)\beta_1 - x_{it} I(q_{it} > \theta)\beta_2 - \psi(q_{it}, z_{it}, \theta, \hat{\pi})\right)^2 \qquad (12)$$

Third, we estimate the least square coefficient parameters $\hat{\beta}_1$ and $\hat{\beta}_2$ based on the split samples implied by $\hat{\theta}$.

Corollary 1. *For concentrated two-stage least square estimator in the case of endogenous panel threshold regression defined as Equation (12), we have* $\hat{\theta}_{CLS} \xrightarrow{p} \theta$.

3 The Panel Threshold Regression with Endogenous Threshold and Fixed Effect

Now, consider the panel threshold model with time-invariant fixed effect as below:

$$y_{it} = x_{it} I(q_{it} \leq \theta)\beta_1 + x_{it} I(q_{it} > \theta)\beta_2 + \eta_i + e_{it} \qquad (13)$$

$$q_{it} = z_{it}\pi + u_{it}, \qquad (14)$$

where q_{it} is an observed threshold variables, θ is an unknown threshold parameter, z_{it} is a vector. We allow for individual effects η_i to be arbitrarily correlated with independent variables x_{it} in Equation (13). When panel threshold model exists time-invariant individual effect, the individual effect η_i must be eliminated in advance. For obtaining the consistent coefficient estimators, individual effects must cancel out before we use two-stage least square methods. We use the same idea that corrects for attrition bias in unbalanced panel data from Wooldridge (2002). We use first difference transformation to eliminate fixed effect η_i:

$$\Delta y_{it} = (x_{it} I(q_{it} \leq \theta) - x_{it-1} I(q_{it-1} \leq \theta))\beta_1 + (x_{it} I(q_{it} > \theta) - x_{it-1} I(q_{it-1} > \theta))\beta_2 + \Delta e_{it}, \qquad (15)$$

[3] Because the threshold parameters is crucial for sample splitting, we will first obtain the estimated threshold parameter $\hat{\theta}$. If we want to obtain coefficient estimators with threshold parameter together by nonlinear regression, it will take a time-consuming calculation. For simplifying our analysis, we will use two-step estimation procedure suggested by Hansen (2000) and Kourtellos, Stengos and Tan (2007).

where $\Delta y_{it} = y_{it} - y_{it-1}$ and $\Delta e_{it} = e_{it} - e_{it-1}$. To obtain the bias correction estimators, we need to reset some original assumptions:

Assumption 3.1. *$\{y_{it}, x_{it}, q_{it}, \Delta e_{it}\}$ is strictly stationary, ergodic.*

Assumption 3.2. *The joint distribution between Δe_{it} and u_{it} is defined as:*

$$\begin{bmatrix} \Delta e_{it} \\ u_{it} \end{bmatrix} | x_{it}, z_{it} \sim \mathcal{N}\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma_e^2 & \gamma_j \\ \gamma_j & 1 \end{bmatrix} \right), \tag{16}$$

where γ_j is covariance between Δe_{it} and u_{it}, $\gamma_j = \gamma_1$ when $q_{it} \leq \theta$ and $\gamma_j = \gamma_2$ when $q_{it} > \theta$.

Assumption 3.3. *: $E(\Delta x'_{it} \Delta x_{it})$ has full rank.*

Compared with the assumptions in panel threshold models without individual effect, we can find that Δe_{it} and u_{it} must have joint normal assumptions, not e_{it} and u_{it}. Under Assumptions (2.2)-(2.7) and (3.1)- (3.3) satisfied, we can decompose the residuals sequence Δe_{it} of transformed equation into two parts:

$$\begin{bmatrix} \varepsilon_{it} \\ u_{it} \end{bmatrix} \sim \begin{bmatrix} 1 & -\gamma_j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta e_{it} \\ u_{it} \end{bmatrix}, \tag{17}$$

and

$$\begin{bmatrix} \varepsilon_{it} \\ u_{it} \end{bmatrix} | x_{it}, z_{it} \sim \mathcal{N}\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} \sigma_e^2 - \gamma_j^2 & 0 \\ 0 & 1 \end{bmatrix} \right) \tag{18}$$

As the case of panel threshold model without individual effect, panel threshold model with individual effect can be written as a single equation form:

$$\begin{aligned} \Delta y_{it} = {} & (x_{it} I(q_{it} \leq \theta) - x_{it-1} I(q_{it-1} \leq \theta))\beta_1 + (x_{it} I(q_{it} > \theta) - x_{it-1} I(q_{it-1} > \theta))\beta_2 \\ & + \psi(q_{it}, z_{it}, \theta, \pi) + \epsilon_{it}, \end{aligned} \tag{19}$$

where

$$\psi(q_{it}, z_{it}, \theta, \pi) = \gamma_1 \lambda_1(\theta - z_{it}\pi) I(q_{it} \leq \theta) + \gamma_2 \lambda_2(\theta - z_{it}\pi) I(q_{it} > \theta) \tag{20}$$

When we neglect sample selection parts in panel threshold model, the threshold value $\hat{\theta}$ can not be estimated consistently. The ordinary least square estimators for first difference transformations of threshold panel data are generally inconsistent. Only at $\gamma_1 = \gamma_2 = 0$, the OLS estimators are consistent.

For getting consistent estimators, our estimation procedure proceeds in four steps: First, we execute first difference transformations for main equation to eliminate individual effect. Second, we use ordinary least square method to estimate π in Equation (2). Third, we estimate the threshold parameter θ by minimizing a concentrated least square criterion with $\hat{\pi}$ from second stage.

$$S^{CLS-FE}\left(\beta_i(\theta), \gamma_i(\theta), \theta\right)$$
$$= \arg\min_{\theta\in\left[\underline{\theta},\bar{\theta}\right]} \sum_{i=1}^{n}\sum_{t=1}^{T}\left(\Delta y_{it} - \Delta x_{it} I(q_{it} \leq \theta)\beta_1 - \Delta x_{it} I(q_{it} > \theta)\beta_2 - \psi(q_{it}, z_{it}, \theta, \pi)\right)^2 \quad (21)$$

Fourth, when $\hat{\theta}$ is acquired by concentrated least square, we use $\hat{\theta}$ to split data set and use least square method to estimate coefficient parameters.

Theorem 1. *When Assumption (2.2)-(2.7) and assumption (3.1)-(3.3) are satisfied, for concentrated two-stage least square estimator in the case of endogenous panel threshold model with fixed effect defined as Equation (21), we have* $\hat{\theta}_{CLS-FE} \xrightarrow{p} \theta$.

4 The Panel Threshold Regression with Endogenous Threshold and Fixed Effects in Main and Threshold Equations

In this section we allow both main and threshold equations with fixed effects. The panel threshold model we consider is

$$y_{it} = x_{it} I(q_{it} \leq \theta)\beta_1 + x_{it} I(q_{it} > \theta)\beta_2 + c_i + e_{it} \quad (22)$$

$$q_{it} = z_{it}\pi + \eta_i + u_{it}, \quad (23)$$

where q_{it} is an observed threshold variables, θ is an unknown threshold parameter, and z_{it} is an exogenous variable. Because the fixed effects exist in main and threshold equations, we cannot obtain consistent estimators $\hat{\beta}_1$ and $\hat{\beta}_2$ by first difference transformation for two equations. We propose a modified two stage least square method. First, we use least square dummy variable regression for threshold equation (23) to obtain $\hat{\pi}$ and $\hat{\eta}_i$. Second, the first difference transformation is used to remove individual effect for main equation. After the individual effect has been removed in main equation, we can rewrite Equation (22) and (23) as one equation.

$$\Delta y_{it} = (x_{it}I(q_{it} \leq \theta) - x_{it-1}I(q_{it-1} \leq \theta))\beta_1 + (x_{it}I(q_{it} > \theta) - x_{it-1}I(q_{it-1} > \theta))\beta_2$$

$$+\psi(q_{it}, z_{it}, \theta) + \epsilon_{it}, \tag{24}$$

where

$$\psi(q_{it}, z_{it}, \theta, \pi, \eta_i) = \gamma_1\lambda_1(\theta - z_{it}\pi - \eta_i)I(q_{it} \leq \theta) + \gamma_2\lambda_2(\theta - z_{it}\pi - \eta_i)I(q_{it} > \theta) \tag{25}$$

Because the coefficients of bias-correction equation (24) depend on threshold value θ, the consistent estimator of θ can be derived by solving the concentrated least square equation.

$$S^{CLS-DFE}(\beta_i(\theta), \gamma_i(\theta), \theta) = \arg \min_{\theta \in [\underline{\theta}, \bar{\theta}]} \sum_{i=1}^{n}\sum_{t=1}^{T}\{\Delta y_{it} - (x_{it}I(q_{it} \leq \theta) - x_{it-1}I(q_{it-1} \leq \theta))\beta_1$$

$$-(x_{it}I(q_{it} > \theta) - x_{it-1}I(q_{it-1} > \theta))\beta_2 - \psi(q_{it}, z_{it}, \hat{\eta}_i, \theta, \hat{\pi})\}^2 \tag{26}$$

When we obtain the consistent estimator of threshold parameter $\hat{\theta}$, Equation (22) can be estimated by ordinary least square method.

Corollary 2. *When Assumption (2.2)-(2.7) and (3.1)-(3.3) are satisfied, the concentrated two-stage least square estimator in the case of endogenous panel threshold model with fixed effect defined as Equation (26), we have* $\hat{\theta}_{CLS-FE} \xrightarrow{p} \theta$.

5 Monte Carlo Simulation

We conduct a Monte Carlo simulation to study the finite-sample properties of the proposed threshold panel data estimators against other threshold panel estimators in the literature. We compare our two-stage panel threshold estimators with the fixed effect panel threshold estimator by Hansen (1999).The data generation process considered in this paper is based on following panel threshold regression:

$$y_{it} = \begin{cases} \beta_1 x_{it} + c_i + e_{it} & q_{it} \leq \theta \\ \beta_2 x_{it} + c_i + e_{it} & q_{it} > \theta \end{cases}, \tag{27}$$

where the threshold value θ is set to 2. For threshold equation, we consider two cases. The first one is the threshold equation without fixed effect η_i:

$$q_{it} = 2 + 3z_{1it} + 3z_{2it} + u_{it} \tag{28}$$

Second, we allow the threshold equation with fixed effect η_i:

$$q_{it} = 2 + 3z_{1it} + 3z_{2it} + \eta_i + u_{it}, \tag{29}$$

where x_{it}, z_{1it}, and z_{2it} are identical and mutually independent standard normal distributed. The fixed effects c_i and η_i follow $i.i.d.$ $\mathcal{N}(0,1)$. For controlling the endogeneity, we use the similar method as in KST (2007) to generate $(v_{it}, u_{it}) \sim i.i.d.\ \mathcal{N}(0, I)$, and $e_{it} = \sigma_e^2(\gamma_o v_{it} + (1-\gamma_o)u_{it})/\sqrt{\gamma_o^2 + (1-\gamma_o)^2}$. So that the correlation between e_{it} and threshold variable q_{it} is given by $\gamma = \gamma_o/\sqrt{\gamma_o^2 + (1-\gamma_o)^2}$. We set $\gamma_o = (0.05, 0.5, 0.9)$. Regarding the coefficients of threshold regression, we fix $\beta_2 = 1$, and vary β_1. We set $\delta = \beta_1 - \beta_2$ and examine $\delta = (0.05, 0.5)$. Finally, we consider $n = 50, 100, 200, 500$ and $T = 10, 20$. The number of replications is $1,000$ for all cases.

Tables 1-2 show the mean square errors (MSE) of the Monte Carlo simulations of the threshold coefficients θ and the difference of slope coefficients $\delta = \beta_2 - \beta_1$ of two different estimators. First we consider the estimation of threshold value θ. No matter the γ and δ are, our panel threshold estimators perform much better than Hansen's threshold estimators under large N and small T in terms of MSE. We also find that panel threshold estimators using fixed effect transformations have serious downward biases. Second, Tables 3-4 indicate that the estimators of difference of slope coefficients estimated by fixed effect transformations are seriously upward biased. Our bias-correction coefficient estimators are consistent under different γ and θ when N goes to infinity and T fixed. In term of thresholds and slope coefficients our bias-correction estimators ourperform Hansen (1999) fixed effect estimators.

6 Conclusion

In this paper we extends the panel threshold regression of Hansen (1999) to allow endogeneity of the threshold variable. We develop a bias-correction two stage least square estimator to eliminate the endogeneity in the threshold variable. From the Monte Carlo simulations we examine the finite sample performance of our bias- correction estimators with Hansen's fixed effect estimators. Our estimators perform fairly well for a variety of parameters and (N, T) combinations.

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Appendix: Mathematical Proof

Proof of Theorem 1:

Let $y_i = \begin{pmatrix} y_{i2} \\ \vdots \\ y_{iT} \end{pmatrix}$ and $x_i = \begin{pmatrix} x_{i2} \\ \vdots \\ x_{iT} \end{pmatrix}$. Then let Y_t, X_t denote the data stacked over all individuals, $Y_t = \begin{pmatrix} y_1 \\ \vdots \\ y_i \\ \vdots \\ y_T \end{pmatrix}$ and $X_t \begin{pmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_T \end{pmatrix}$. We rewrite Equation (19) as a vector form.

$$\Delta Y = \Delta X \beta_2 + X_\gamma \delta + \lambda_2 \Lambda(\theta) + \kappa \Lambda_\theta + e, \tag{30}$$

where $\Delta Y = Y_t - Y_{t-1}$, $\Delta X = X_t - X_{t-1}$, $X_\gamma = X_t I(q_{it} \leq \theta) - X_{t-1} I(q_{it-1} \leq \theta)$, $\Lambda(\theta) = I(q_{it} \leq \theta)\lambda_1(\theta) + (1 - I(q_{it} \leq \theta))\lambda_2(\theta)$, $\Lambda_\theta = \Lambda(\theta) I(q_{it} \leq \theta)$ and $\kappa = \lambda_1 - \lambda_2$. When $\theta = \theta_0$, $\Lambda_{\theta_0} = \Lambda(0)$, $\Lambda_{\theta_0} = \Lambda_0$ and $X_{\theta_0} = X_0$. We define $\tilde{X}(\theta) = (\Delta X, \Lambda(\theta))$, $\tilde{X}_\theta = (X_\theta, \Lambda_\theta)$, $\tilde{X}^*_\theta = (\tilde{X}_\theta, (\tilde{X}_\theta - \tilde{X}_\theta))$ like KST (2007).

We define the projection matrix $\tilde{P}^*_\theta$ and orthogonal matrix $\tilde{M}^*_\theta$.

$$\tilde{P}^*_\theta = \tilde{X}^*_\theta (\tilde{X}^{*\prime}_\theta \tilde{X}^*_\theta)^{-1} \tilde{X}^{*\prime}_\theta \tag{31}$$

$$\tilde{M}^*_\theta = I - \tilde{P}^*_\theta \tag{32}$$

Equation (30) can be rewritten as:

$$\Delta Y = \tilde{X}(0)\varphi + \tilde{X}_0 \omega + e \tag{33}$$

under $\theta = \theta_o$, where $\varphi = \begin{pmatrix} \beta_2 \\ \delta \end{pmatrix}$ and $\omega = \begin{pmatrix} \lambda_2 \\ \kappa \end{pmatrix}$

Then,

$$n^{-1+2\alpha} T^{-1} (S^{CLS-FE}(\theta) - e'e)$$
$$= n^{-1+2\alpha} T^{-1} (\Delta Y' \tilde{M}^*_\theta \Delta Y - e'e)$$
$$= n^{-1+2\alpha} T^{-1} \left[(\tilde{X}(0)\varphi + \tilde{X}_0\omega + e)' \tilde{M}^*_\theta (\tilde{X}(0)\varphi + \tilde{X}_0\omega + e) - e'e \right]$$

$$= (nT)^{-1}[C_1'(\tilde{X}(0)'\tilde{X}_\theta^*)(\tilde{X}_\theta^{*\prime}\tilde{X}_\theta^*)^{-1}(\tilde{X}_\theta^{*\prime}\tilde{X}_0)C_1 + C_2'(\tilde{X}(0)'\tilde{X}_\theta^*)(\tilde{X}_\theta^{*\prime}\tilde{X}_\theta^*)^{-1}(\tilde{X}_\theta^{*\prime}\tilde{X}_0)C_2$$

$$+C_3'(\tilde{X}_0'\tilde{X}_\theta^*)(\tilde{X}_\theta^{*\prime}\tilde{X}_\theta^*)^{-1}(\tilde{X}_\theta^{*\prime}\tilde{X}_0)C_3] + o_p(1)$$

We can derive the following result:

$$n^{-1+2\alpha}T^{-1}(S^{CLS-FE}(\theta) - e'^{-1}[C'(\tilde{X}_0^{*\prime}\tilde{X}_\theta^*)(\tilde{X}_\theta^{*\prime}\tilde{X}_\theta^*)^{-1}(\tilde{X}_\theta^{*\prime}\tilde{X}_0^*)C] + o_p(1). \tag{34}$$

When $\theta \in [\theta_o, \bar{\theta}]$,

$$[C'(\frac{\tilde{X}_0^{*\prime}\tilde{X}_\theta^*}{nT})(\frac{\tilde{X}_\theta^{*\prime}\tilde{X}_\theta^*}{nT})^{-1}(\frac{\tilde{X}_\theta^{*\prime}\tilde{X}_0^*}{nT})C] \xrightarrow{p} b_1(\theta) \tag{35}$$

Under Assumptions 2.2-2.7 and 3.1-3.3 are satisfied, $\frac{d}{d\theta}b_1(\theta) > 0$. Because $b_1(\theta) > 0$ is monotonic increasing and continuous at θ_0, $b_1(\theta) > 0$ will have uniquely minimized at θ_0. Similarly, When $\theta \in [\underline{\theta}, \theta_0]$,

$$[C'(\frac{\tilde{X}_0^{*\prime}\tilde{X}_\theta^*}{nT})(\frac{\tilde{X}_\theta^{*\prime}\tilde{X}_\theta^*}{nT})^{-1}(\frac{\tilde{X}_\theta^{*\prime}\tilde{X}_0^*}{nT})C] \xrightarrow{p} b_2(\theta), \tag{36}$$

$b_2(\theta) > 0$ will have uniquely minimized at θ_0.

Under $\theta \in [\underline{\theta}, \bar{\theta}]$,

$$n^{-1+2\alpha}T^{-1}(S^{CLS-FE}(\theta) - e'e) \xrightarrow{p} b_1(\theta)I(\theta > \theta_0) + b_2(\theta)I(\theta \leq \theta_0), \tag{37}$$

where $0 < \alpha < 1/2$. We can obtain

$$\hat{\theta}_{CLS-FE} \xrightarrow{p} \theta. \tag{38}$$

Proof of Corollary 2:

Corollary 2 is a special case of Theorem 1. Please refer to the proof for Theorem 1.

Table 1: Finite-sample Performance for Threshold Value Estimation ($\beta_1 - \beta_2 = 0.05$)

	Method	$T = 10$			$T = 20$		
$\gamma = 0.05$		mean	median	MSE	mean	median	MSE
$N = 50$	pthet	-0.665	-0.036	24961.78	-0.628	0.265	25832.25
	FE	-11.345	-11.16	181688.4	-12.43	-12.257	210775.7
$N = 100$	pthet	-0.546	0.282	24626.43	-0.289	0.732	22597.33
	FE	-12.382	-12.185	209352.7	-13.285	-13.066	236098.5
$N = 200$	pthet	-0.265	0.968	23640.77	0.622	1.508	14370.53
	FE	-13.232	-13.232	234145.7	-14.115	-13.92	261835
$N = 500$	pthet	0.804	1.726	13200.37	1.57	1.961	5116.832
	FE	-14.396	-14.209	270869	-15.181	-15.034	297085.8

	Method	$T = 10$			$T = 20$		
$\gamma = 0.5$		mean	median	MSE	mean	median	MSE
$N = 50$	pthet	1.76	1.843	4180.392	1.868	1.855	2629.118
	FE	-11.388	-11.183	181967.1	-12.502	-12.336	212850.7
$N = 100$	pthet	1.984	1.982	2716.145	1.986	2.057	1914.692
	FE	-12.347	-12.188	208403	-13.316	-13.139	236969.3
$N = 200$	pthet	2.096	2.055	2163.062	1.994	2.013	1368.487
	FE	-13.213	-12.977	233551.9	-14.142	-13.948	26267.85
$N = 500$	pthet	1.892	1.946	1272.687	2.017	2.018	831.415
	FE	-14.432	-14.236	272146.5	-15.204	-15.011	297784.2

	Method	$T = 10$			$T = 20$		
$\gamma = 0.9$		mean	median	MSE	mean	median	MSE
$N = 50$	pthet	1.968	1.952	2903.662	1.94	1.97	1950.399
	FE	-11.404	-11.191	182333	-12.457	-12.226	211744.9
$N = 100$	pthet	2.087	2.061	2039.976	1.987	1.992	1346.099
	FE	-12.407	-12.201	210181.9	-13.177	-13.003	132646.9
$N = 200$	pthet	2.053	2.07	1388.162	1.98	1.991	810.065
	FE	-13.233	-13.022	233997.3	-14.21	-14.029	264870.4
$N = 500$	pthet	2.032	2.01	768.197	2.015	2.034	488.923
	FE	-14.43	-14.213	272078.7	-15.15	-14.985	296049.4

Table 2: Finite-sample Performance for Threshold Value Estimation ($\beta_1 - \beta_2 = 0.5$)

	Method	$T = 10$			$T = 20$		
$\gamma = 0.05$		mean	median	MSE	mean	median	MSE
$N = 50$	pthet	1.981	1.986	166.392	2.001	1.996	22.96
	FE	-11.51	-11.381	185594.5	-12.452	-12.28	211379.6
$N = 100$	pthet	2.001	1.997	26.769	1.996	1.996	4.78
	FE	-12.348	-12.196	208522.5	-13.35	-13.218	237957.2
$N = 200$	pthet	1.997	1.998	5.771	1.998	1.998	1.294
	FE	-13.246	-13.081	234529.1	-14.171	-13.946	263563.2
$N = 500$	pthet	1.998	1.999	0.954	1.999	1.999	0.225
	FE	-14.31	-14.088	268120.9	-15.23	-15.059	298604.3

	Method	$T = 10$			$T = 20$		
$\gamma = 0.5$		mean	median	MSE	mean	median	MSE
$N = 50$	pthet	1.956	1.974	588.074	2.01	2.005	146.177
	FE	-11.397	-11.204	182358.2	-12.519	-12.303	213241.6
$N = 100$	pthet	1.987	1.991	147.534	2	1.996	37.577
	FE	-12.403	-12.22	209992.5	-13.288	-13.061	235957.1
$N = 200$	pthet	1.994	1.995	39.786	1.997	1.999	9.798
	FE	-13.218	-13.032	233865.7	-14.061	-13.888	259966.5
$N = 500$	pthet	1.997	1.999	6.236	2	1.999	1.536
	FE	-14.353	-14.207	269418.1	-15.198	-14.978	297707.8

	Method	$T = 10$			$T = 20$		
$\gamma = 0.9$		mean	median	MSE	mean	median	MSE
$N = 50$	pthet	2.007	1.978	586.007	2.017	2.002	152.444
	FE	-11.449	-11.245	183448.1	-12.462	-12.259	211568.1
$N = 100$	pthet	2.005	2.001	163.956	2.002	1.994	35.211
	FE	-12.281	-12.117	206305.2	-13.229	-13.084	236487.9
$N = 200$	pthet	1.997	1.999	46.364	1.997	1.998	1.177
	FE	-13.332	-13.1	237440	-14.134	-13.936	262362.6
$N = 500$	pthet	1.997	1.999	6.466	2	1.999	0.147
	FE	-14.31	-14.072	268027.8	-15.18	-14.979	297017.2

Table 3: Finite-sample Performance for Main Regression Coefficient ($\beta_1 - \beta_2 = 0.05$)

	Method	$T=10$			$T=20$		
$\gamma = 0.05$		mean	median	MSE	mean	median	MSE
$N=50$	pthet	0.066	0.072	210.707	0.062	0.059	296.961
	FE	1.026	1.029	955.946	1.026	1.026	956.9
$N=100$	pthet	0.038	0.064	363.922	0.046	0.059	745.058
	FE	1.025	1.023	953.157	1.025	1.025	952.595
$N=200$	pthet	0.006	0.06	6352.047	0.045	0.054	48.168
	FE	1.026	1.026	953.414	1.025	1.025	951.868
$N=500$	pthet	0.061	0.054	89.671	0.05	0.051	0.66
	FE	1.024	1.024	949.718	1.024	1.023	949.313

	Method	$T=10$			$T=20$		
$\gamma = 0.5$		mean	median	MSE	mean	median	MSE
$N=50$	pthet	0.037	0.039	6.476	0.044	0.045	2.826
	FE	1.025	1.024	953.712	1.024	1.022	951.684
$N=100$	pthet	0.043	0.043	2.752	0.044	0.045	1.167
	FE	1.023	1.023	948.686	1.024	1.024	951.048
$N=200$	pthet	0.045	0.047	1.326	0.044	0.044	0.621
	FE	1.025	1.026	951.615	1.025	1.024	950.783
$N=500$	pthet	0.045	0.044	0.518	0.046	0.047	0.257
	FE	1.024	1.025	949.627	1.025	1.025	951.217

	Method	$T=10$			$T=20$		
$\gamma = 0.9$		mean	median	MSE	mean	median	MSE
$N=50$	pthet	0.046	0.045	5.483	0.044	0.044	2.309
	FE	1.025	1.025	953.695	1.027	1.028	959.098
$N=100$	pthet	0.045	0.043	2.51	0.043	0.042	1.236
	FE	1.024	1.025	949.796	1.024	1.025	951.172
$N=200$	pthet	0.045	0.043	1.291	0.045	0.044	0.553
	FE	1.025	1.024	951.191	1.026	1.028	953.335
$N=500$	pthet	0.046	0.045	0.486	0.046	0.046	0.234
	FE	1.025	1.026	951.089	1.025	1.025	950.986

Table 4: Finite-sample Performance for Main Regression Coefficient ($\beta_1 - \beta_2 = 0.5$)

	Method	$T=10$			$T=20$		
$\gamma = 0.05$		mean	median	MSE	mean	median	MSE
$N = 50$	pthet	0.507	0.511	4.431	0.504	0.506	1.961
	FE	1.25	1.251	567.927	1.25	1.25	566.405
$N = 100$	pthet	0.504	0.504	2.154	0.502	0.502	1.085
	FE	1.25	1.249	565.006	1.248	1.248	561.43
$N = 200$	pthet	0.502	0.502	1.067	0.501	0.501	0.528
	FE	1.25	1.25	563.493	1.249	1.25	562.564
$N = 500$	pthet	0.5	0.5	0.447	0.499	0.499	0.214
	FE	1.25	1.25	562.583	1.251	1.251	564.662

	Method	$T=10$			$T=20$		
$\gamma = 0.5$		mean	median	MSE	mean	median	MSE
$N = 50$	pthet	0.485	0.489	6.411	0.489	0.49	2.69
	FE	1.251	1.252	568.459	1.249	1.249	564.645
$N = 100$	pthet	0.492	0.495	2.632	0.495	0.496	1.273
	FE	1.25	1.25	564.915	1.251	1.252	566.119
$N = 200$	pthet	0.498	0.498	1.219	0.498	0.498	0.554
	FE	1.25	1.25	564.07	1.25	1.25	563.196
$N = 500$	pthet	0.498	0.498	0.49	0.499	0.499	0.213
	FE	1.25	1.249	562.307	1.249	1.249	561.724

	Method	$T=10$			$T=20$		
$\gamma = 0.9$		mean	median	MSE	mean	median	MSE
$N = 50$	pthet	0.481	0.483	6.197	0.487	0.487	3.039
	FE	1.249	1.25	565.868	1.252	1.25	569.449
$N = 100$	pthet	0.488	0.488	2.978	0.494	0.494	1.221
	FE	1.247	1.247	560.682	1.252	1.251	567.521
$N = 200$	pthet	0.495	0.495	1.374	0.498	0.498	0.536
	FE	1.251	1.25	564.748	1.251	1.25	565.302
$N = 500$	pthet	0.497	0.497	0.487	0.499	0.5	0.215
	FE	1.25	1.25	563.459	1.25	1.251	563.469