Aggregate Indices and Their Corresponding Elementary Indices

Jens Mehrhoff*, Deutsche Bundesbank

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Abstract

"Which index formula at the elementary level, where no expenditure share weights are available, corresponds to a desired aggregate index?" To answer this question, this paper develops a statistical approach. It proposes a theoretical framework which makes it possible to achieve numerical equivalence of an elementary index with the Laspeyres, Paasche or Fisher price index. Depending on the price elasticity, different elementary indices should be applied to different groups of goods in order to approach the desired aggregate index as closely as possible. This is also demonstrated empirically in an application using data from German foreign trade statistics.

Keywords: Generalised Mean, Quadratic Mean, Log-Normal Distribution, Partial Adjustment Model, Price Elasticity, Foreign Trade.

JEL: C43, D12, E31, F14.

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*This paper represents the author's personal opinion and does not necessarily reflect the view of the Deutsche Bundesbank or its staff. Detailed results and descriptions of methodology are available on request from the author. Address for correspondence: Jens Mehrhoff, Statistics Department and Research Centre, Deutsche Bundesbank, Wilhelm-Epstein-Strasse 14, 60431 Frankfurt am Main, Germany, Tel: +49 69 9566-3417, Fax: +49 69 9566-2941, E-mail: jens.mehrhoff@bundesbank.de, Homepage: www.bundesbank.de. Paper prepared for the 16$^{\text{th}}$ International Conference on Panel Data in Amsterdam, 2-4 July 2010. The author gratefully acknowledges the work of the Foreign Trade Division of the German Federal Statistical Office in providing the data, and would like to thank Erwin Diewert, Bert Balk, Peter von der Lippe, Hans-Albert Leifer, Robert Kirchner, Karl-Heinz Toedter, Johannes Hoffmann and Sophia Mueller-Spahn as well as the participants at the 13$^{\text{th}}$ Conference on Measurement of Prices in Constance, 26-27 June 2008, the 11$^{\text{th}}$ Meeting of the International Working Group on Price Indices (Ottawa Group) in Neuchâtel, 27-29 May 2009, the 15$^{\text{th}}$ Young Researchers' Workshop in Merseburg, 3-4 June 2009 and the 57$^{\text{th}}$ Session of the International Statistical Institute in Durban, 16-22 August 2009 for valuable comments. All remaining errors are, of course, the author's sole responsibility.

1 Introduction

1.1 Motivation

It is customary in official statistics, although often neglected in theoretical papers, for most price indices to be calculated in two stages. At the first stage, elementary indices are calculated on the basis of prices or their relatives, without having information on quantities or expenditures. At the second stage, the aggregate index is calculated on the basis of the elementary indices from the first stage, using aggregate expenditure share weights.

In general, the question of “what should be measured?” directly yields the optimal index formula at the second stage: for measuring genuine price movements, a Laspeyres price index is used; for deflation purposes, a Paasche price index is preferred; and for the “cost of living”, a Fisher price index, among others, is the formula of choice. However, it is less clear which index formula should be used at the first stage, where no expenditure share weights are available. The existing approaches to index numbers including but not restricted to the axiomatic approach are of little guidance in choosing the elementary index corresponding to the characteristics of the index at the second stage.

The point in question is “how can the corresponding elementary index be selected?” The answer to this question is found by the proposition of a statistical approach. A single comprehensive framework, known as “generalised means”, unifies the aggregate and elementary levels. With the aid of this approach, theoretical conditions under which a particular index formula at the elementary level exactly equals the desired aggregate index are identified and empirically approximated.

The remainder of the paper is organised as follows. It continues with a review of a selection of the existing literature on elementary indices. Section 2 introduces basic concepts and approaches in index theory along with a more thorough explanation of the problem at the elementary level. Both the theoretical foundations of generalised means as well as the application to the Laspeyres, Paasche and Fisher price indices and their corresponding elementary indices are presented in detail in Section 3. The results of an empirical application using data from German foreign trade statistics are to be found in Section 4. The final section concludes.

1.2 Literature Review

After a long period of research into aggregate formulae and an almost equally long policy debate in Europe and the US on whether the Laspeyres or Fisher formula should be used for a consumer price index (cf. Boskin et al., 1996, 1998, and Schultze and Mackie, 2002), the focus of attention has recently moved more to the question of which index formula should be used at the elementary level. Nowadays, the capabilities of modern computers and the increasing coverage of data, first and foremost, through the advent of scanner data, enables statistical offices to calculate more refined price indices even at the elementary level (cf. Silver, 1995, Silver and Webb, 2002, Feenstra and Shapiro, 2003, Diewert, 2004, and Proceedings of the Meetings of the Ottawa Group).

Diewert (2004), and Diewert and Silver (2004, 2008) devote whole chapters in the CPI, PPI and XMPI manuals to elementary indices. They deal with virtually all topics that arise around the calculation of price indices at the elementary level. Theoretical issues, such as the problem of aggregation, are covered as well as practical questions, such as numerical relationships between different elementary indices. They continue by outlining the classical approaches in index theory, i.e. the axiomatic, economic, sampling and stochastic approaches (cf. Konüs, 1924, and Diewert, 1976; Eichhorn, 1978, and Diewert, 1995; Selvanathan and Prasada Rao, 1994; and Balk, 2005, 2008, respectively), and discuss the use of scanner data. Currently, there is an active ongoing discussion at Eurostat's Working Group on Harmonisation of Consumer Price Indices – more specifically, in the Task Force on Sampling – on which index formula is to be used at the elementary level (cf. EC, 2001, Section I). The Commission Regulation (EC, 1996, Article 7 in conjunction with Annex II) abandons the use of the Carli index but allows the use of either the Jevons or Dutot index. More precisely, the Carli index is not prohibited de jure but de facto as it would have to be shown that the results do not differ by more than one-tenth of a percentage point from either the Jevons or Dutot index (cf. the next-but-one paragraph for empirical evidence).

Balk (1994) discusses the index formula problem at the elementary level. He poses the question whether ratios of average prices or an average of price relatives, and which type of average, i.e. arithmetic, geometric or harmonic, should be used. Turvey (1996) addresses the same problem. He also presents empirical evidence that recalculations of elementary indices with different index formulae give significant changes in aggregate CPIs, annually by more than two percentage points, in Finland, Sweden, Canada and France. The use of unit values at the lowest level in a price index is analysed by Balk (1998), which is commonly taken for granted to be an appropriate method of aggregation for prices of homogeneous goods. He tries to answer the questions of the conditions under which a group of goods is sufficiently homogeneous to warrant the use of unit values, and if one needs to restrict the use of unit values to homogenous goods alone. In the context of foreign trade, Silver (2009) criticises the use of aggregate indices which are calculated from unit values at the elementary level. He advocates pure price indices and reveals substantial biases of customs-based unit values: they depend on the structure of quantities and hence, cannot be considered surrogates for survey-based prices.

Szulc (1989) describes the fact that biases at the elementary level are more severe than the pros and cons of the formula at the aggregate level. He finds that if one ignores the particularities of the aggregate index when calculating elementary indices, this might result in surprisingly low differences between different aggregate indices. This is because the indices at the elementary level might not be paying attention to the characteristics of the index formula at the aggregate level, in particular if the same elementary indices are used as building blocks of the aggregate index – no matter which aggregate index should be used. In his 1994 paper he presents numerical evidence for the Canadian CPI that the choice of the elementary index matters the most, particularly in the short term. Dalén (1992, 1995) discusses the impact of the choice of the wrong index formula at the elementary level in the Swedish CPI. Statistics Sweden switched over to the Carli index in January 1990. As soon as April it was replaced by a variant of the geometric index due to the well-known severe upward bias of the Carli index – of more than half a percent in these three months. Using Swedish and Finnish data, he shows in his 1998 paper that the Carli index consistently gives results which are year-on-year two index points and more larger than the Dutot and Jevons indices, while the latter two indices are fairly close to each other. Fenwick (1999) presents evidence that the UK HICP, which is based on the Jevons index at the elementary level, is annually about half a percentage point lower than the national equivalent, the Retail Prices Index, which uses a combination of the Dutot and Carli indices, only because of the different formulae. His main argument for this notable difference is the relative broad item description, leading to aggregation of highly heterogeneous items. Silver and Heravi (2007) show that the difference between the Jevons and Dutot indices is due to different variances in the observed prices at different points in time alone, i.e. these indices will differ if prices exhibit dispersion. From a hedonic regression they derive a heterogeneity-controlled Dutot index and successfully test their approach empirically with scanner data.

2 Aggregate Indices

2.1 First Principles

At the aggregate level, the target of measurement determines the index concept to be used. This is either the cost of goods (COGI) or the cost of living (COLI). In general, the former case leads to Laspeyres (1871) and Paasche (1874) price indices, while the latter results inter alia in the Fisher (1922) price index – other formulae include the Walsh (1901, 1921) and Törnqvist (1936) price indices.

The Laspeyres price index is the arithmetic mean of price relatives with base period expenditure share weights. This is the only price index which ensures the principle of pure price comparison (cf. von der Lippe, 2007). Here, p_{ib} and q_{ib} denote the price and quantity, respectively, of the i^{th} good at time $b \in \{0, t\}$.

$$P^L = \sum_{i=1}^{n} \frac{p_{it}}{p_{i0}} \frac{p_{i0}q_{i0}}{\sum_{i=1}^{n} p_{i0}q_{i0}} = \frac{\sum_{i=1}^{n} p_{it}q_{i0}}{\sum_{i=1}^{n} p_{i0}q_{i0}} \tag{1}$$

For volume measurement, one would opt for the Laspeyres quantity index Q^L, with $Q^L = V/P^P$, where V is the ratio of expenditures at times t and 0 or the value index and P^P is the Paasche price index. The Paasche price index is the harmonic mean of price relatives with current period expenditure share weights.

$$P^P = \left(\sum_{i=1}^{n} \left(\frac{p_{it}}{p_{i0}} \right)^{-1} \frac{p_{it}q_{it}}{\sum_{i=1}^{n} p_{it}q_{it}} \right)^{-1} = \frac{\sum_{i=1}^{n} p_{it}q_{it}}{\sum_{i=1}^{n} p_{i0}q_{it}} \tag{2}$$

The Fisher price index, among others, is a superlative index. It is defined as the geometric mean of the Laspeyres and Paasche price indices. This is the most famous price index approximating the change in the minimum expenditures, which preserve utility at a constant level, owing to changes in (relative) prices (cf. Allen, 1975).

$$P^F = \sqrt{\frac{\sum_{i=1}^n \frac{p_{it}}{p_{i0}} \frac{p_{i0}q_{i0}}{\sum_{i=1}^n p_{i0}q_{i0}}}{\sum_{i=1}^n \left(\frac{p_{it}}{p_{i0}}\right)^{-1} \frac{p_{it}q_{it}}{\sum_{i=1}^n p_{it}q_{it}}}} = \sqrt{P^L P^P} \tag{3}$$

Note that in the case of chain indices, the fixed base year 0 becomes the moving previous year $t-1$. The resulting indices are the so-called links which in turn are then multiplied consecutively to form the chain index: $\tilde{P}_{t,0} = P_{1,0}P_{2,1}\ldots P_{t,t-1}$ (cf. von der Lippe, 2001). Everything else remains unchanged, so that price indices with a fixed base year are discussed throughout.

2.2 Two-Staged Indices

While the Laspeyres and Paasche price indices are exactly consistent in aggregation, superlative indices, such as the Fisher price index, are approximatively consistent in aggregation (Diewert, 1978). This means that the result of a two-staged index calculation coincides (approximatively) with that of a calculation in a single stage. However, when statistical offices cannot use a quantity or expenditure-weighted formula at the first stage of the aggregation process, owing to the unavailability of this information, they have to rely on an unweighted index. Such an index might not reflect the characteristics of the index formula at the aggregate level. This elementary index bias is equally applicable to the Laspeyres and Paasche price indices as well as to the Fisher price index, no matter which unweighted index is used. A two-staged index with a non-according formula at the elementary level can lead to a different conclusion than the true price index. This is due to the fact that the elementary indices may not even be close to the desired target index. In addition, important axiomatic properties of the aggregate index might get lost. Hence, more attention should be paid to the calculation of elementary indices.

In fact, it is not possible to compensate for biases at the elementary level, one just weights them together. If it were possible to match the elementary index to the desired target index, this bias would be vanishing. Thus, the goal which is to be achieved is numerical equivalence of these two indices. When the results of the desired target index are given, it is possible to approximate the elementary index numerically. A similar exercise is undertaken by Shapiro and Wilcox (1997) for the US consumer price index. They calculate the constant elasticity of substitution (CES) price index (Lloyd, 1975, and Moulton, 1996) for various parameter values and then choose the value that most closely approximates the Törnqvist price index. This should allow the timely calculation of a price index which is virtually free of substitution bias. However, the estimation of the elasticity of substitution itself is not without severe problems. In addition, substitution bias is a phenomenon at the aggregate level, not at the elementary level. In fact, there is no elasticity of substitution for elementary indices as they depend on prices or price relatives alone. Even if their numerical method were applied at the elementary level (without the interpretation of the resulting parameter as elasticity of substitution or the like), due to its purely numerical nature, it would be impossible to ensure that price and volume measurement are internally consistent (cf. Subsection 3.1.3 for a discussion of the aforementioned technical difficulties of this methodology at the elementary level).

Here, internal consistency means that price measurement with the Laspeyres price index and volume measurement via the Paasche price index (in order to arrive at the Laspeyres quantity index) call for different elementary indices which are inter-related. Note that this is how the European Statistical System works. In contrast, it is not intended to postulate the use of a specific formula for each elementary aggregate as this is presumably not feasible for a statistical office. However, it is possible to estimate the optimal general formula and, based on this, calculate consistent elementary indices for either the Laspeyres or the Paasche price index. Thus, owing to the aforementioned limitations, the numerical optimisation methodology is not followed here but a new statistical approach using generalised means is proposed. It allows the achievement of numerical equivalence between an elementary index and a desired aggregate index based on the price elasticity alone.

3 Corresponding Elementary Indices

3.1 Theoretical Foundations

In order to achieve numerical equivalence between an elementary index and an arbitrary aggregate index, a statistical approach is developed. In Subsection 3.1.1 it is firstly demonstrated that every weighted index can be expressed one-to-one and onto as a "generalised mean", as long as the former satisfies the strict mean value property. The generalised mean represents a whole class of unweighted elementary indices, such as the Carli and Jevons indices. However, an analytical derivation of the concrete generalised mean of a weighted aggregate index is not possible without further assumptions. Hence, secondly the log-normal distribution is introduced in Subsection 3.1.2 and the generalised means – which correspond to the Laspeyres, Paasche and Fisher price indices – are related to the distribution's parameters. Although, at that stage, one would be able to numerically calculate elementary indices, corresponding to the desired aggregate ones, the present paper goes one step further and gives an economic interpretation to the parameters through a partial adjustment model in Subsection 3.1.3. Thirdly, the log-normal distribution parameters are related to the price elasticity. Finally, it is shown in the succeeding Subsections 3.2 and 3.3 that the choice of the elementary indices which correspond to the desired aggregate ones can be based on the price elasticity alone.

3.1.1 Generalised Mean

In what follows the trivial case of perfect homogeneity is neglected throughout, i.e. not all price relatives are equal: $\exists\, j \neq k \colon p_{jt}/p_{j0} \neq p_{kt}/p_{k0}$.

Definition 1. Let p_{it}/p_{i0} denote the price relative of the i^{th} good at time t, where $i = 1, 2, \ldots, n$ and $n \geq 2$. Furthermore, all price relatives are assumed to be positive real numbers, $0 < p_{it}/p_{i0} < \infty \,\forall\, i$. Then, their generalised mean of order r is defined as

$$P^{GM}(r) = \sqrt[r]{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}}\right)^{r}}. \tag{4}$$

By choosing the appropriate orders r, the resulting generalised means equal some of the most important elementary indices (cf. Table 1). Figure 1 exemplifies the typical shape of the generalised mean as a function of its argument r.

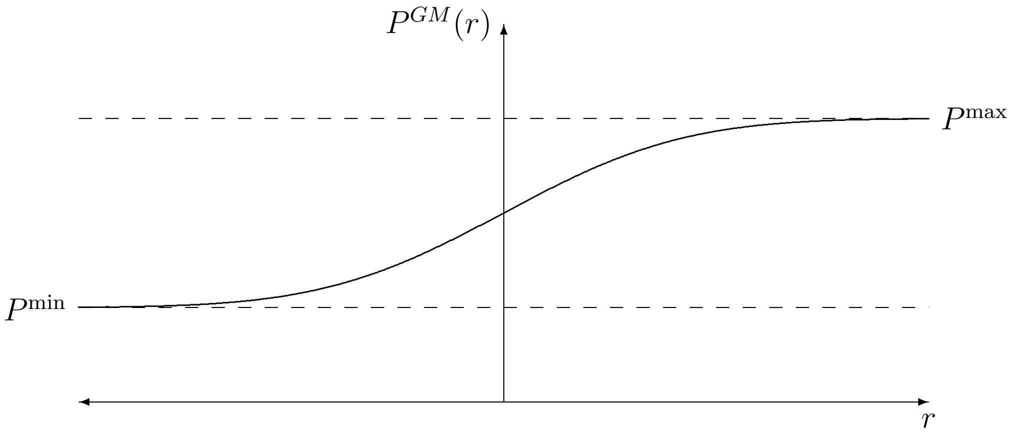

Figure 1: Generalised Mean of Price Relatives

Given this, the problem of choosing the elementary index corresponding to the Laspeyres, Paasche or Fisher price indices becomes solvable. To this end, it is shown that every *weighted* aggregate index can be written as an *unweighted* generalised mean of price relatives.

Theorem 1. *Let P_t^* be an arbitrary aggregate index that satisfies the mean value property in its strict form. Then, for each t, there exists one and only one real r_t^* for which the generalised mean is numerically equivalent:*

$$\exists!\, r_t^* \in \mathbb{R} \colon P^{GM}(r_t^*) = P_t^*. \tag{5}$$

In the remainder, the discussion is restricted to the case of the order of the generalised mean being constant over time: $r_t^* = r^* \,\forall\, t$.

Theorem 1 provides the basis for the following derivation of the corresponding elementary indices in the case of the Laspeyres, Paasche or Fisher price indices, Equations (1), (2) and (3), respectively, as desired aggregate indices. An intuitive interpretation of the theorem goes as follows. Per definitionem, the aggregate index P^* lies between the smallest and largest price relative: $P^{\min} < P^* < P^{\max}$

(Eichhorn and Voeller, 1976, propose this so-called Mean Value Test and show that all three aggregate indices satisfy it). The generalised mean $P^{GM}(r)$ covers the whole range between these two price relatives: $P^{GM}(r) : \mathbb{R} \rightarrow (P^{\min}, P^{\max})$. Moreover, it is a continuous function and hence, it has to take on the value of the aggregate index at least once. Eventually, uniqueness of the order r is secured through the proposition that not all price relatives are equal. Thus, the generalised mean is a strictly monotonic increasing function in r which ensures the existence of an inverse function (Hardy et al., 1934, discuss the generalised mean in great detail and prove its properties).

Table 1 depicts some of the most frequently used formulae at the elementary level (cf. Subsection 3.3 for the definitions of quadratic means).

Table 1: Generalised Means and Their Formulae

r	Generalised Mean	Price Index	Formula
-2	reciprocal quadratic	–	$P^{GM}(-2) = \sqrt{n/\sum_{i=1}^{n}(p_{i0}/p_{it})^2}$
-1	harmonic	Coggeshall (1887)	$P^h = n/\sum_{i=1}^{n}(p_{i0}/p_{it})$
$0^\dagger$	geometric	Jevons (1863, 1865)	$P^J = \sqrt[n]{\prod_{i=1}^{n}(p_{it}/p_{i0})}$
1	arithmetic	Carli (1764)	$P^C = \sum_{i=1}^{n}(p_{it}/p_{i0})/n$
2	quadratic	–	$P^{GM}(2) = \sqrt{\sum_{i=1}^{n}(p_{it}/p_{i0})^2/n}$

† The Jevons index is the limit of $P^{GM}(r)$ as r approaches zero.

Another very famous formula at the elementary level is the one of Dutot (1738), the ratio of arithmetic mean prices:

$$P^D = \frac{\frac{1}{n}\sum_{i=1}^{n} p_{it}}{\frac{1}{n}\sum_{i=1}^{n} p_{i0}} = \sum_{i=1}^{n} \frac{p_{it}}{p_{i0}} \frac{p_{i0}}{\sum_{i=1}^{n} p_{i0}}. \tag{6}$$

Drobisch (1871) proposes another index which is of importance at the elementary level, the ratio of unit values or the unit value index:

$$P^{UV} = \frac{\sum_{i=1}^{n} p_{it}q_{it}/\sum_{i=1}^{n} q_{it}}{\sum_{i=1}^{n} p_{i0}q_{i0}/\sum_{i=1}^{n} q_{i0}} = \sum_{i=1}^{n} \frac{p_{it}}{p_{i0}} \frac{p_{i0}q_{it}/\sum_{i=1}^{n} q_{it}}{\sum_{i=1}^{n} p_{i0}q_{i0}/\sum_{i=1}^{n} q_{i0}}. \tag{7}$$

Because the Dutot index is not an unweighted generalised mean and the assigned weights of the unit value index do not even necessarily sum up to unity, both indices will not be analysed any further here.

3.1.2 Log-Normal Distribution

The order r in Subsection 3.1.1 cannot be derived analytically without making further assumptions. Based on Theorem 2, a closed form solution is provided as to which generalised mean corresponds to a given aggregate index. Proof for this and following theorems are to be found in the Appendix.

Theorem 2. *Under weak assumptions on the underlying data generating process, which are outlined in the proof (cf. the Appendix), prices* p_{ib} *and quantities* q_{ib}*,* $b \in \{0,\, t\}$*, are jointly log-normally distributed:*

$$\begin{bmatrix} \mathbf{p_i} \\ \mathbf{q_i} \end{bmatrix} \sim \mathcal{LN}\left(\begin{bmatrix} \boldsymbol{\mu}_\mathbf{p} \\ \boldsymbol{\mu}_\mathbf{q} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{\mathbf{p},\mathbf{p}} & \boldsymbol{\Sigma}_{\mathbf{p},\mathbf{q}} \\ \boldsymbol{\Sigma}_{\mathbf{q},\mathbf{p}} & \boldsymbol{\Sigma}_{\mathbf{q},\mathbf{q}} \end{bmatrix} \right). \tag{8}$$

Upon this, an explicit formula is derived by which the order of the generalised mean can be computed directly from the log-normal distribution parameters. In Subsection 3.1.3, these distribution parameters will be linked to the price elasticity.

The assumption of a quadrivariate log-normal distribution of prices and quantities seems reasonable and predecessors are found in the literature. Moulton (1993), and Dalén (1999) use the log-normal distribution assumption for price relatives, while Silver and Heravi (2007) use it for prices in their own right. Note that the latter assumption is a generalisation of the former one. Log-normal distribution of price relatives is a direct consequence of log-normal distribution of prices. Ehemann (2007) goes one step further and assumes log-normal distribution of both prices and quantities, which will also be assumed here.

The link between the generalised mean on the one side and the log-normal distribution parameters on the other side is built in the following theorem.

Theorem 3. *The generalised mean in Equation (4) corresponds to the* r^{th} *root of the* r^{th} *raw moment of the (joint) marginal distribution of price relatives, which is also the log-normal distribution. It follows that*

$$\mathrm{E}(P^{GM}(r)) = \exp\left(\mu_{p_t} - \mu_{p_0} + r\frac{\sigma_{p_t}^2 + \sigma_{p_0}^2 - 2\sigma_{p_t,p_0}}{2} \right). \tag{9}$$

From Theorem 3, it can be seen that the Carli index ($r = 1$), unlike the Jevons index ($r \to 0$), is an increasing function of the variance of the price relatives. Hence, a mathematical argument for the upward bias of the Carli index compared with the Jevons index is given through this: the more heterogeneous the goods become at the elementary level, the higher will be the bias (cf. Subsection 5.1 for a discussion of the Carli index' upward bias).

Theorem 4 establishes the link between the Laspeyres and Paasche price indices and the log-normal distribution parameters (cf. Subsection 3.3 for the solution in the case of the Fisher price index). Moreover, Theorem 5 gives an exact expression for the generalised mean corresponding to either of the two price indices.

Theorem 4. *The Laspeyres price index corresponds to the ratio of the first raw product moment of the joint marginal distribution of current period prices and base period quantities, and the first raw product moment of the joint marginal distribution of base period prices and quantities. It turns out that*

$$\mathrm{E}(P^L) = \exp\left(\mu_{p_t} - \mu_{p_0} + \frac{\sigma_{p_t}^2 - \sigma_{p_0}^2 + 2\sigma_{p_t,q_0} - 2\sigma_{p_0,q_0}}{2}\right). \tag{10}$$

The Paasche price index' correspondence is the same as the one of the Laspeyres price index but with the difference that here there are current period quantities instead of base period ones. It becomes

$$\mathrm{E}(P^P) = \exp\left(\mu_{p_t} - \mu_{p_0} + \frac{\sigma_{p_t}^2 - \sigma_{p_0}^2 + 2\sigma_{p_t,q_t} - 2\sigma_{p_0,q_t}}{2}\right). \tag{11}$$

Theorem 5. *In order to let the generalised mean coincide in expectation with either of the two price indices, one equates* $\mathrm{E}(P^{GM}(r))$ *from Equation (9) with* $\mathrm{E}(P^L)$ *and* $\mathrm{E}(P^P)$ *from Equations (10) and (11), respectively, and then solves for* r *which yields after some algebra:*

$$\mathrm{E}(P^{GM}(r)) = \mathrm{E}(P^L) \Longleftrightarrow r_L = \frac{\sigma_{p_t}^2 - \sigma_{p_0}^2 + 2\sigma_{p_t,q_0} - 2\sigma_{p_0,q_0}}{\sigma_{p_t}^2 + \sigma_{p_0}^2 - 2\sigma_{p_t,p_0}}, \tag{12}$$

$$\mathrm{E}(P^{GM}(r)) = \mathrm{E}(P^P) \Longleftrightarrow r_P = \frac{\sigma_{p_t}^2 - \sigma_{p_0}^2 + 2\sigma_{p_t,q_t} - 2\sigma_{p_0,q_t}}{\sigma_{p_t}^2 + \sigma_{p_0}^2 - 2\sigma_{p_t,p_0}}. \tag{13}$$

3.1.3 Partial Adjustment Model

Next, the implied orders r_L and r_P of the Laspeyres and Paasche price indices, Equations (12) and (13), are connected to the price elasticity derived from a partial adjustment model as in Definition 2.

Definition 2. It is assumed that an equilibrium quantity traded for each good $i = 1, 2, \ldots, n$ and time $b \in \{0, t\}$ exists. This quantity is related to the price of the good, which, in turn, is assumed to be predetermined, and to other, strictly exogenous variables, such as time and seasonal dummies or a trend. The parameter η_i^q is a panel fixed effect, accounting for unobserved heterogeneity in the data.

$$\ln \overline{q}_{ib} = \alpha + \beta \ln p_{ib} + \mathbf{x_{ib}} \boldsymbol{\delta} + \eta_i^q \tag{14}$$

The adjustment to the equilibrium in Equation (14) is assumed to be both incomplete and erroneous. This is mirrored by the introduction of lagged quantity and an i.i.d. error term. Here, $\beta^* := (1-\rho)\beta$ denotes the effective price elasticity.

$$\begin{aligned} \ln q_{ib} &= (1-\rho) \ln \overline{q}_{ib} + \rho \ln q_{ib-1} + \varepsilon_{ib}^q \\ &= (1-\rho)\alpha + \beta^* \ln p_{ib} + \rho \ln q_{ib-1} + \mathbf{x_{ib}}(1-\rho)\boldsymbol{\delta} + [(1-\rho)\eta_i^q + \varepsilon_{ib}^q] \end{aligned} \tag{15}$$

Prices are assumed to follow a panel AR(1) process:

$$\ln p_{ib} = \gamma_0 + \gamma_1 \ln p_{ib-1} + (\eta_i^p + \varepsilon_{ib}^p). \tag{16}$$

Five remarks have to be made regarding the chosen model. First, the model is able to capture the particularities encountered with scanner data, namely price and quantity bouncing and stock building due to sales (via the parameters β^* and ρ). Second, the specification of the pseudo panel (cf. Subsection 4.1 for the structure of the data) means an implicit quality adjustment, i.e. the same quality is assumed within groups while a different quality is allowed between groups. Third, the implied cross-price elasticity in Equation (14) is zero without ruling out the possibility of substitution between goods. Fourth, the underlying equilibrium price elasticity β is attenuated by sluggish adjustment of quantities. Fifth, owing to the problem of identification with observed data on prices and quantities, the estimated effective price elasticity β^* has to be understood as being the one of the supply-demand equilibrium rather than the one of demand. As the focus of this paper is on the effective price elasticity only, it is referred to simply as the price elasticity in what follows.

Using Equations (15) and (16), the covariance matrices can be derived subject to the model parameters. The results are collected in Theorem 6.

Theorem 6. *The covariance matrices $\mathbf{\Sigma}_{\mathbf{p},\mathbf{p}}$ and $\mathbf{\Sigma}_{\mathbf{p},\mathbf{q}} = \mathbf{\Sigma}'_{\mathbf{q},\mathbf{p}}$ of the log-normal distribution as given in Equation (8) are as follows (the elements of $\mathbf{\Sigma}_{\mathbf{q},\mathbf{q}}$ do not appear in the calculation of the order r):*

$$\mathbf{\Sigma}_{\mathbf{p},\mathbf{p}} = \begin{bmatrix} \sigma^2_{p_t} & \sigma_{p_t,p_0} \\ \sigma_{p_t,p_0} & \sigma^2_{p_0} \end{bmatrix} = \sigma^2_p \begin{bmatrix} 1 & \gamma_1^t \\ \gamma_1^t & 1 \end{bmatrix}, \tag{17}$$

$$\mathbf{\Sigma}_{\mathbf{p},\mathbf{q}} = \begin{bmatrix} \sigma_{p_t,q_t} & \sigma_{p_t,q_0} \\ \sigma_{p_0,q_t} & \sigma_{p_0,q_0} \end{bmatrix} = \beta^* \sigma^2_p \begin{bmatrix} \frac{1}{1-\rho\gamma_1} & \frac{\gamma_1^t}{1-\rho\gamma_1} \\ \left(\frac{\gamma_1^t - \rho^t}{1-\frac{\rho}{\gamma_1}} + \frac{\rho^t}{1-\rho\gamma_1} \right) & \frac{1}{1-\rho\gamma_1} \end{bmatrix}. \tag{18}$$

In the derivation of Equations (17) and (18) use was made of the weak stationarity assumption, especially of stationarity in covariance.

Before turning to the final relation, it is worth revisiting the methodology of Shapiro and Wilcox (1997) and discussing the pros and cons of their approach compared to this one. One might say that the assumption of log-normal distribution is debatable and that the partial adjustment model is complex. This begs the question why this should be advantageous if even the simpler methodology is not used. The reason is that, in contrast to numerical optimisation, the new proposal allows an economic interpretation of the results. While a numerical procedure can only produce an estimate, with this method it is possible to easily judge the appropriateness of an elementary index. To reiterate, group-specific elementary indices are not suggested here but internally consistent price indices (cf. Subsection 2.2 for the meaning of internal consistency). This can be achieved solely with the aid of the new statistical approach. In any case, numerical techniques suffer from the problem of low degrees of freedom. At the elementary level one typically has only a few goods for which prices are sampled in each group which in general leads to estimates biased towards plus or minus infinity. Eventually, the violation of the assumptions of the statistical approach and the consequences are exclusively an empirical matter.

3.2 Laspeyres and Paasche Price Indices

It is shown that the solution to the problem of elementary indices that correspond to a desired aggregate index depends on the empirical correlation between prices and quantities. In particular, the order r is a function of the price elasticity alone. The succeeding theorem summarises the results for the Laspeyres and Paasche price indices (again, cf. Subsection 3.3 for the solution in the case of the Fisher price index).

Theorem 7. *Combining the equations relating the generalised mean to the log-normal distribution parameters, Equations (12) and (13), with those relating the log-normal distribution parameters to the model coefficients, Equations (17) and (18), gives the final results:*

$$r_L = -\beta^* \frac{1}{1-\rho\gamma_1} \approx -\beta^*, \tag{19}$$

$$r_P \xrightarrow{t\to\infty} \beta^* \frac{1}{1-\rho\gamma_1} \approx \beta^*. \tag{20}$$

From Theorem 7, the general results for the generalised mean are as follows. A generalised mean of order r equal to minus the price elasticity $(-\beta^*)$ yields approximately the same result as the Laspeyres price index. Hence, if the price elasticity is minus one, for example, r must equal one and the Carli index (cf. Table 1) at the elementary level will correspond to the Laspeyres price index as target index. This can be seen in the simplest form from the following example: from $q_{i0} = \bar{q}_0/p_{i0}$, where $\bar{q}_0$ is an arbitrary constant, follows a price elasticity of minus and $P^L = [\sum_{i=1}^n p_{it}(\bar{q}_0/p_{i0})]/[\sum_{i=1}^n p_{i0}(\bar{q}_0/p_{i0})] = \sum_{i=1}^n (p_{it}/p_{i0})/n = P^C$. However, if the Paasche price index should be replicated, the order of the generalised mean must equal the price elasticity, in the above example minus one. Thus, the harmonic index gives the same result and therefore, in this case it should be used at the elementary level. Unfortunately, this simple exposition works only with unrealistically restrictive assumptions about the data generating process.

3.3 Fisher Price Index

The Fisher price index is derived from the Laspeyres and Paasche price indices as their geometric mean. Owing to the symmetry of the generalised means which correspond to the Laspeyres and Paasche price indices, a quadratic mean corresponds to the Fisher price index. In Definition 3 the properties of quadratic means in general are presented.

Definition 3. A quadratic mean of price relatives of order q is defined as follows:

$$P^{QM}(q) = \left(\frac{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}}\right)^{\frac{q}{2}}}{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}}\right)^{-\frac{q}{2}}} \right)^{\frac{1}{q}} = \sqrt{P^{GM}\left(\frac{q}{2}\right) P^{GM}\left(-\frac{q}{2}\right)}. \tag{21}$$

The index defined by Equation (21) is symmetric, i.e. $P^{QM}(q) = P^{QM}(-q)$. Furthermore, it is either increasing or decreasing in $|q|$, depending on the data. Both characteristics can also be seen from Figure 2. Note that a quadratic mean of order q, $P^{QM}(q)$, should not be mistaken for the quadratic index, $P^{GM}(2)$ (cf. Table 1).

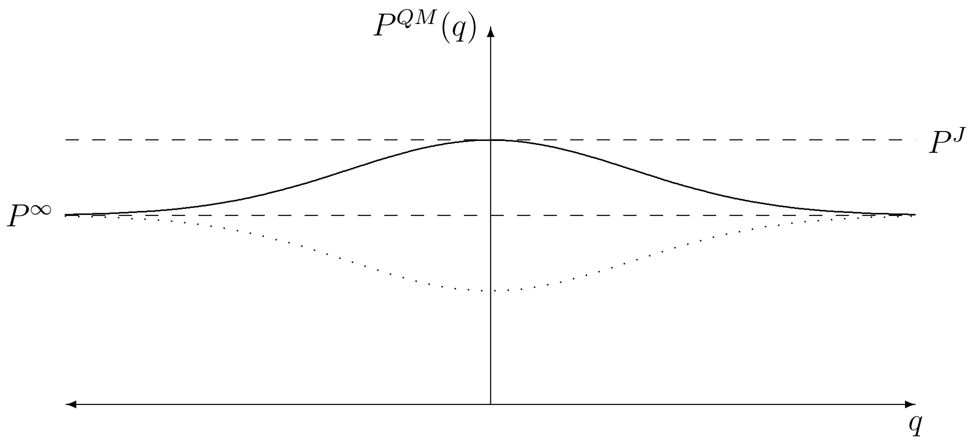

Figure 2: Quadratic Mean of Price Relatives

Dalén (1992), and Diewert (1995) show via a Taylor series expansion that all quadratic means approximate each other to the second order. However, as Hill (2006) demonstrates, the limit of $P^{QM}(q)$ if q diverges is $P^\infty = \sqrt{P^{\min}P^{\max}}$. He concludes that quadratic means are not necessarily numerically similar.

For $q \to 0$ the quadratic mean becomes the Jevons index. For $q = 1$ an index results, which was first described by Balk (2005, 2008) as the unweighted Walsh price index and independently devised by Mehrhoff (2007) as a linear approximation to the Jevons index. Hence, this index is referred to as the BMW index. Lastly, one arrives at the CSWD index (Carruthers, Sellwood and Ward, 1980, and Dalén, 1992) for $q = 2$, which is the geometric mean of the Carli and harmonic indices. Table 2 contrasts these indices.

Table 2: Quadratic Means and Their Formulae

q	Quadratic Mean	Formula
$0^\dagger$	Jevons	$P^J = \sqrt[n]{\prod_{i=1}^n (p_{it}/p_{i0})}$
1	BMW	$P^{BMW} = \sum_{i=1}^n \sqrt{(p_{it}/p_{i0})} / \sum_{i=1}^n \sqrt{(p_{i0}/p_{it})}$
2	CSWD	$P^{CSWD} = \sqrt{\sum_{i=1}^n (p_{it}/p_{i0})} / \sqrt{\sum_{i=1}^n (p_{i0}/p_{it})}$
3	cubic	$P^{QM}(3) = \sqrt[3]{\sum_{i=1}^n \sqrt{(p_{it}/p_{i0})^3}} / \sqrt[3]{\sum_{i=1}^n \sqrt{(p_{i0}/p_{it})^3}}$
4	quartic	$P^{QM}(4) = \sqrt[4]{\sum_{i=1}^n (p_{it}/p_{i0})^2} / \sqrt[4]{\sum_{i=1}^n (p_{i0}/p_{it})^2}$

† The Jevons index is the limit of $P^{QM}(q)$ as q approaches zero.

Applying the preceding definitions gives the final result which is stated in Theorem 8.

Theorem 8. *A quadratic mean of order two times the absolute price elasticity corresponds to the Fisher price index:*

$$P^F \approx \sqrt{\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}}\right)^{-\beta^*}\right)^{-\frac{1}{\beta^*}} \left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{p_{it}}{p_{i0}}\right)^{\beta^*}\right)^{\frac{1}{\beta^*}}} = P^{QM}(2|\beta^*|). \qquad (22)$$

The approximate equality in Equation (22) follows from Equations (19) and (20) in conjunction with Equation (4) – with r being equal to $\pm\beta^*$ – or (21) – with $q/2$ being equal to $|\beta^*|$.

4 Findings in Foreign Trade Statistics

4.1 Data Description

The statistical approach, developed in the preceding section, is applied to real data in this section. In other words, the methodology outlined here is illustrated by an example. At first, an overview of the data and their properties is given. The stationarity assumption, which was used in the derivation of the approach, is justified on empirical grounds. Then, the model is estimated for various goods. The results are presented in terms of the model parameters and as regards the implications for price statistics.

An application to scanner data for homogeneous goods would be suited best because information on both prices and quantities at the elementary level is necessary which scanner data would provide. Unfortunately, this kind of data are not available for the German CPI. Hence, as an empirical application, data from German foreign trade statistics are analysed as an alternative. The source of these data is the German Federal Statistical Office. At the time of frontier crossing, movements of goods in special trade are to be reported for statistical purposes; with member states of the European Union in the Intrastat system, and with non-member states via the customs' Single Administrative Document (EC, 2006). Declarations are to be made according to the Commodity Classification for Foreign Trade Statistics and consist inter alia of the goods' values and quantities, the latter generally in terms of the weights. Based on these declarations, albeit not derived from homogeneous goods, unit values are calculated at the elementary level as $\tilde{p}_{ib} = (\sum_{i=1}^{n} p_{ib}q_{ib})/(\sum_{i=1}^{n} q_{ib})$, $b \in \{0, t\}$, which, in turn, form the basis for the succeeding analysis.

Owing to the nature of collected data, their structure is repeated cross-sections rather than a panel. Repeated cross-sections arise by independent cross-sectional surveys at consecutive points in time. Unlike in price statistics, it is not ensured in foreign trade statistics that the same goods are observed over time. The coverage of the universe of goods is time-varying and it is not possible to establish a one-to-one correspondence between goods over time. In this case, Deaton (1985) suggests estimation to be performed on a pseudo panel. This is averaging the data within a cohort, where a cohort is a group of goods sharing common characteristics and every good belongs to one group and one group only which is the same over time. Here, unique transactions are aggregated at the lowest level available, that is their reporting level: the eight-digit code of the Commodity Classification. These lower level aggregates are the individual observations which are nested at the four-digit code level to form an upper level aggregate.

The data set covers 1,264 pseudo panels (nests) consisting of 12,948 groups of goods (cohorts), for exports as well as for imports, and a total of 1,839,384 observations over the period January 2000 to December 2007. Only goods measured in kilograms – these are about three-quarters of all goods – are included in the analysis. The data, unit values in €1,000 per 100 kg (hereafter "prices") and weights in 100 kg (hereafter "quantities"), are transformed into their natural logarithms. Although the goods at the elementary level are not homogeneous, they are treated as if they were for the following analysis.

4.2 Regression Results

As weak stationarity of prices and quantities is assumed in the derivation of corresponding elementary indices (cf. Subsection 3.1.3), panel unit root tests are performed prior to estimation in order to test the validity of this assumption. In particular, these are the test of Levin, Lin and Chu (LLC, 2002), Breitung (2000), Im, Pesaran and Shin (IPS, 2003), augmented Dickey and Fuller (ADF, 1979), and Phillips and Perron (PP, 1988). The first two assume a common unit root process under the null hypothesis and no unit root under the alternative. The last three, by contrast, test the null hypothesis of an individual unit root process against the alternative of some cross-sections without a unit root. The latter two tests for panel data are derived as a combination of their time series variants using the results of Fisher (1925). Included in the test specification are individual effects and individual linear trends. Lag lengths, if necessary, are selected automatically based on the Schwarz information criterion; if applicable, the spectral estimator's bandwidth is selected according to Newey and West (1994) using the Bartlett kernel.

As can be seen from Table 3, the tests show stationarity of both prices and quantities for almost all panels in exports as well as in imports. Throughout, quantities perform better than prices, and exports and imports do equally well. That not all of them are stationary is largely due to non-unity power of the tests. Thus, the issue of (co-)integration can safely be ignored for the remainder of the analysis.

Table 3: Percentages of Stationary Panels at the 5% Significance Level

	Exports		Imports	
Test	Prices	Quantities	Prices	Quantities
LLC	89.81%	94.82%	90.46%	93.07%
Breitung	84.47%	90.88%	85.66%	92.26%
IPS	93.34%	97.70%	92.50%	97.72%
ADF	93.51%	97.78%	93.07%	98.04%
PP	96.87%	98.72%	96.88%	99.36%

The price elasticity β^* is estimated in the framework of the log-linear partial adjustment model given in Equation (15) by means of dynamic panel data one-step system GMM (Arellano and Bover, 1995, and Blundell and Bond, 1998). Neither time or seasonal dummies nor a deterministic trend are included. Prices are assumed to be predetermined and are instrumented accordingly. The instrument set is collapsed in order to reduce the instrument count (cf. Roodman, 2009).

The overall results are fairly robust to different specifications of the model (inclusion of dummies or a trend), choice of instruments (limited lag depth) and estimation methods (fixed effects or difference GMM). Thus, only results which are derived from the above set-up are reported.

After adjusting for outliers, 1,246 panels in exports and 1,249 in imports remain. The distribution of the price elasticity in exports and imports can be gathered from Figure 3. The histograms show positive excess kurtosis, or leptokurtosis, for exports as well as for imports. Compared with the associated normal distribution, the peak around the mean is more pronounced, i.e. there is a higher probability of values near the mean, and the tails are fatter, i.e. there is a higher probability of extreme values. However, the distributions look both quite unimodal and symmetric. The distribution of imports lies slightly more to the right than the one of exports.

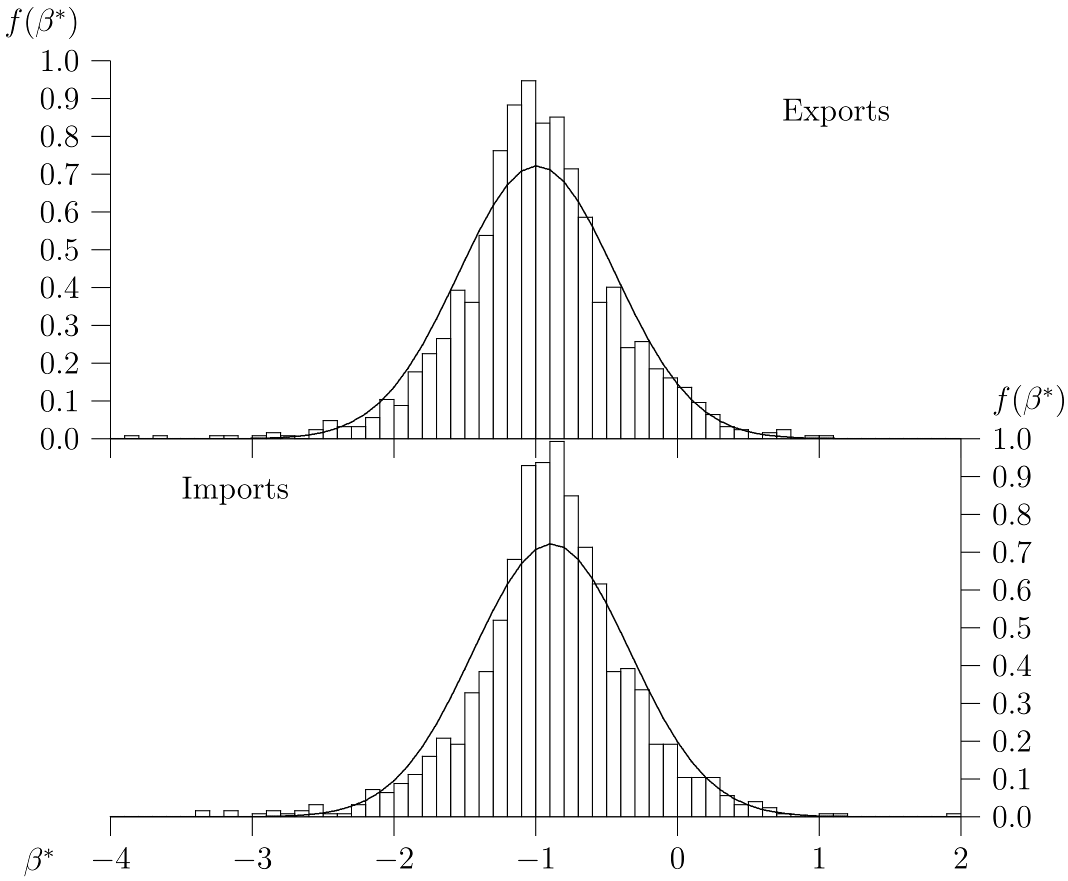

Figure 3: Density Histogram (Bin Width = 0.1) and Normal Density Plot of β^*

As a goodness-of-fit measure, a Pseudo-$R^2 = \text{Corr}^2(\ln q_{ib}, \ln \hat{q}_{ib})$ is used. This is the squared coefficient of correlation between observed and fitted values with the obvious interpretation of explained variance of a regression of observed on fitted values and an intercept.

The most important descriptive summary statistics are collected in Table 4. The average price elasticity (β^*) for exports is −0.99, ranging from −3.9 to 1.1. Adjustment to the equilibrium is strong with the adjustment parameter $(1 - \rho)$ being 0.80 on average, lying in the range from 0.0 (no adjustment) to 1.6 (over-adjustment). The goodness-of-fit measure (Pseudo-R^2) is high at 0.51 on average, covering the whole range from 0.0 to 1.0. The results for imports are almost the same with the notable difference of the average price elasticity, which is −0.89, i.e. a significant 0.1 point lower than for exports. Adjustment and goodness-of-fit are as strong as for exports. Yet results for imports are less stable than those for exports. This is due to higher heterogeneity of observed data owing to the large number of different countries from which German companies import goods.

Table 4: Descriptive Summary Statistics of the Partial Adjustment Model

	Exports			Imports		
Statistic	β^*	$1-\rho$	Pseudo-R^2	β^*	$1-\rho$	Pseudo-R^2
Mean	−0.9911	0.8014	0.5060	−0.8877	0.8071	0.5116
Variance	0.3055	0.0466	0.0775	0.3055	0.0425	0.0748
Minimum	−3.8923	−0.0157	0.0001	−3.3727	0.0491	0.0000
Maximum	1.0826	1.6344	0.9961	1.9547	1.4127	0.9850

Persistence of the process of prices given in Equation (16) is relatively low; on average, the autoregressive parameter γ_1 is 0.17 for exports and 0.19 for imports, thus rendering the simplification of Theorem 7 valid.

More important than the regression results themselves are their implications for price statistics. These are summarised in Tables 5 and 6, respectively, in terms of the proportions to which each of the elementary indices corresponds to the Laspeyres, Paasche and Fisher price indices for panels with at least two groups of goods. While the first set of figures relates to the number of panels, the second set mirrors their monetary value. The classification of the panels to elementary indices is based on rounding the orders r and q, respectively, to the closest integer. This procedure is illustrated in Figure 4.

For the Laspeyres price index as the desired aggregate index, 70% of the panels in exports and 72% in imports imply the use of the Carli index at the elementary level. This means that if one wants to calculate a Laspeyres price index at the aggregate level, the Carli index will yield approximately the same result at the elementary level in these panels (as it is shown in an example in Subsection 3.2). Regarding trade values, these figures reduce to 62% and 66%, respectively. The Jevons index performs best at the first stage in 14% of the panels in exports and 17% in imports with much higher shares with respect to trade value, i.e. 29% and 28%, respectively. In 15% of the panels in exports and 10% in imports, the quadratic index is desirable at the lower level of aggregation; trade value shares here are 7% and 5%, respectively. Shares missing to 100% reflect other indices.

Table 5: Elementary Indices Corresponding to a Laspeyres Price Index

		Panels		Trade Values	
r	Price Index	Exports	Imports	Exports	Imports
0	Jevons	14%	17%	29%	28%
1	Carli	70%	72%	62%	66%
2	quadratic	15%	10%	7%	5%

If the Paasche price index is taken as the desired aggregate index, the corresponding generalised means are inverted: instead of the Carli index, the harmonic index, and, instead of the quadratic index, the reciprocal quadratic index have to be used. As mentioned before, if the Jevons index corresponds to the Laspeyres price index, it does so for the Paasche price index, too. This is what is meant by internal consistency in Subsection 2.2.

If the desired aggregate index is chosen to be the Fisher price index, the results are as follows. The use of the Jevons index is suggested by 6% of the panels in exports and 7% in imports; with respect to trade values these shares increase to 20% and 17%, respectively. The BMW index is found to be superior in 21% of the panels in exports (trade values: 19%) and 28% (25%) in imports. 46% of the panels in exports (48%) and 44% in imports (43%) favour the CSWD index. A quadratic mean of cubic order should be used for 21% of the panels in exports (9%) and 15% in imports (12%). For a quadratic mean of quartic order the figures are 6% (3%) and 4% (2%), respectively. Again, quintic and higher orders make up shares missing to 100%.

Table 6: Elementary Indices Corresponding to a Fisher Price Index

		Panels		Trade Values	
q	Price Index	Exports	Imports	Exports	Imports
0	Jevons	6%	7%	20%	17%
1	BMW	21%	28%	19%	25%
2	CSWD	46%	44%	48%	43%
3	cubic	21%	15%	9%	12%
4	quartic	6%	4%	3%	2%

All in all, different elementary indices should be applied to each panel in order to approach the Laspeyres, Paasche or Fisher price index as closely as possible.

5 Conclusion

5.1 Summary

This paper addresses the problem of index calculation at the elementary level, where no expenditure share weights are available. The question of "which index formula at the elementary level corresponds to the characteristics of the index at the aggregate level?" is dealt with. A statistical approach is proposed which theoretically allows the achievement of numerical equivalence of an elementary index with the desired aggregate index – in this instance, the Laspeyres, Paasche or Fisher price index. Based on "generalised means" and the assumption of joint log-normal distribution of prices and quantities, it is shown that the solution depends on the price elasticity alone, which is derived from a partial adjustment model. Thus, a feasible framework is provided which aids the choice of the corresponding elementary index. The results are graphically produced in Figure 4. If, for example, the price elasticity β^* is minus one, the Carli index corresponds to the Laspeyres price index, the harmonic index to the Paasche price index and the CSWD index to the Fisher price index.

From an empirical application to German foreign trade statistics, it can be seen that the choice of the elementary index does matter. The choice itself depends on the characteristics of prices and quantities. Therefore, depending on the price elasticity, different elementary indices should be applied to each group of goods in order to approach the Laspeyres, Paasche or Fisher price index as closely as possible. While not relying on axiomatic considerations, this paper finds notable empirical differences between different elementary indices and aggregate indices formed from them. Furthermore, the results indicate that a range of elementary indices should be applied in the calculation of price indices. This is in line with the findings of other authors (cf. the review of the empirical literature in Subsection 1.2). In particular, the Carli index performs remarkably well at the elementary level of a Laspeyres price index, corresponding to a price elasticity of minus one.

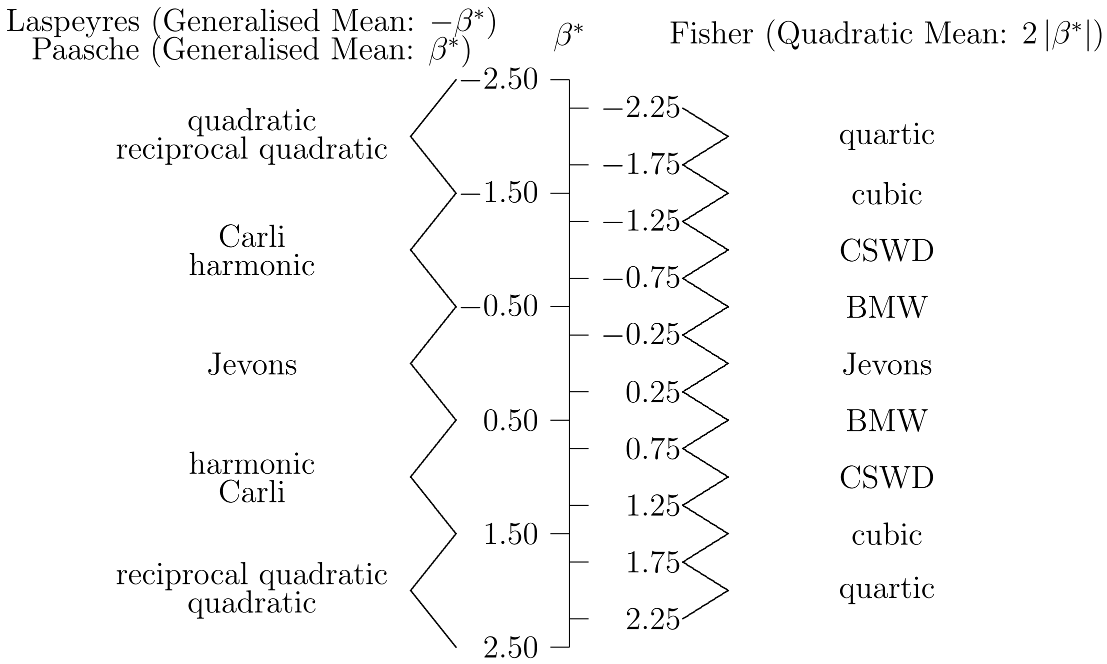

Figure 4: Overview of Corresponding Elementary Indices

Sometimes, it is argued that the Carli index is upward biased. However, this holds only in comparison with the Jevons index. Yet, the comparison in question is not with another elementary index but with the desired aggregate index. So, it may be the case that the Carli index is unbiased or even downward biased compared with the Laspeyres price index (cf. Subsection 3.1.2 for the mathematical relation).

5.2 Outlook

Two possible applications of the approach outlined in this paper arise immediately after a decision has been taken on which aggregate index is desired. Firstly, index calculation can be rendered more precise if different elementary indices are applied to each group of goods, reflecting their specific price elasticities. At least for prominent groups of goods with high expenditure shares, studies on the price elasticity should be available. This will drive down biases of official price indices. In fact, the desired aggregate index can be approximated by using appropriate elementary indices. Secondly, for different purposes – either price or volume measurement –

different elementary indices should be calculated for the same data. This means that if the Carli index is applied as the single formula at the elementary level of a Laspeyres price index, implying a price elasticity of minus one, the harmonic index must be used at the elementary level of a Paasche price index. Still, this is in contrast to the current practice as regards foreign trade in Germany, where the Carli index is used at the elementary level in both price statistics and volume measurement in national accounts. The former task is achieved via the Laspeyres price index, while the latter results in an implicit deflator in the form of the Paasche price index.

An application of this approach to scanner data in a CPI would be worthwhile. Scanner data in its most familiar form are collected at the checkouts of retail stores by the scanning of bar codes. Thus, they provide a census of all transactions rather than a sample. Furthermore, they are collected continuously and provide simultaneous information on both prices and quantities, unlike discrete surveys of prices alone. Lastly, qualitative information may be linked to scanner data, allowing for hedonic adjustment. The foreign trade application of this paper and the prospective study of scanner data are different subject matters. In foreign trade statistics, the data are intermediately aggregated and unit values are used, which are neither seasonally nor quality adjusted, rather than observed purchase prices. Disaggregate scanner data allow the calculation of unbiased price indices and hence, a more thorough analysis based on them might change the results.

Appendix: Proof of Theorems

Proof of Theorem 1. The proof is outlined in Subsection 3.1.1. For details cf. Eichhorn and Voeller (1976), and Hardy et al. (1934). □

Proof of Theorem 2. The processes of prices and quantities are assumed to have both started in the infinite past.

$$p_{i,t} = p_{i,-\infty} \cdot \ldots \cdot \frac{p_{i,0}}{p_{i,-1}} \cdot \frac{p_{i,1}}{p_{i,0}} \cdot \frac{p_{i,2}}{p_{i,1}} \cdot \ldots \cdot \frac{p_{i,t-1}}{p_{i,t-2}} \cdot \frac{p_{i,t}}{p_{i,t-1}}$$
$$q_{i,t} = q_{i,-\infty} \cdot \ldots \cdot \frac{q_{i,0}}{q_{i,-1}} \cdot \frac{q_{i,1}}{q_{i,0}} \cdot \frac{q_{i,2}}{q_{i,1}} \cdot \ldots \cdot \frac{q_{i,t-1}}{q_{i,t-2}} \cdot \frac{q_{i,t}}{q_{i,t-1}}$$

However, the period-to-period changes are not independently distributed. The sequences are assumed to satisfy a mixing condition, which implies ergodicity; hence, a central limit theorem under weak dependence becomes applicable. Thus, it follows that prices and quantities are marginally log-normally distributed. Having proven marginal log-normal distribution, it follows that they are also jointly log-normally distributed by imposing a functional relationship between prices and quantities and an autoregressive relationships within them. □

Proof of Theorem 3. The k^{th} raw moment of a log-normally distributed random variable is given by $\exp(k\mu + k^2\sigma^2/2)$. After taking natural logarithms it applies that $a \ln X \pm b \ln Y \sim \mathcal{N}(a\mu_X \pm b\mu_Y,\, a^2\sigma_X^2 + b^2\sigma_Y^2 \pm 2ab\sigma_{X,Y})$. Using this and the definition of the population counterpart of the sample generalised mean, one finds the following result.

$$\mathrm{E}(P^{GM}(r)) \rightarrow \sqrt[r]{\mathrm{E}\left(\left(\frac{p_{it}}{p_{i0}}\right)^r\right)} = \exp\left[\frac{1}{r}\left(r(\mu_{p_t} - \mu_{p_0}) + r^2\frac{\sigma_{p_t}^2 + \sigma_{p_0}^2 - 2\sigma_{p_t,p_0}}{2}\right)\right]$$

By reducing the terms, the proposition follows. □

Proof of Theorem 4. Using the definitions of the population counterparts of the sample Laspeyres and Paasche price indices, the expectations are as follows.

$$\mathrm{E}(P^L) \rightarrow \frac{\mathrm{E}(p_{it}q_{i0})}{\mathrm{E}(p_{i0}q_{i0})} = \frac{\exp\left(\mu_{p_t} + \mu_{q_0} + \frac{\sigma_{p_t}^2 + \sigma_{q_0}^2 + 2\sigma_{p_t,q_0}}{2}\right)}{\exp\left(\mu_{p_0} + \mu_{q_0} + \frac{\sigma_{p_0}^2 + \sigma_{q_0}^2 + 2\sigma_{p_0,q_0}}{2}\right)}$$

$$\mathrm{E}(P^P) \rightarrow \frac{\mathrm{E}(p_{it}q_{it})}{\mathrm{E}(p_{i0}q_{it})} = \frac{\exp\left(\mu_{p_t} + \mu_{q_t} + \frac{\sigma_{p_t}^2 + \sigma_{q_t}^2 + 2\sigma_{p_t,q_t}}{2}\right)}{\exp\left(\mu_{p_0} + \mu_{q_t} + \frac{\sigma_{p_0}^2 + \sigma_{q_t}^2 + 2\sigma_{p_0,q_t}}{2}\right)}$$

The proposition follows by reducing the terms. □

Proof of Theorem 5. The corresponding generalised means are found by solving the equations for r. □

Proof of Theorem 6. Stationarity in covariance of the processes, i.e. $0 \leq \rho < 1$ and $0 \leq \gamma_1 < 1$, implies that the covariance between any two observations depends only on the lag between them. For the covariance of logarithmic prices, it follows that it is an exponentially decreasing function.

$$\sigma_{p_\kappa,p_\ell} = \gamma_1^{|\kappa-\ell|}\sigma_p^2$$

Using the lag operator and inverting the lag polynom in the function of logarithmic quantities, it can be written as follows.

$$\ln q_{ib} = \alpha + \beta^* \sum_{\tau=0}^{\infty} \rho^\tau \ln p_{ib-\tau} + \left(\sum_{\tau=0}^{\infty} \rho^\tau \mathbf{x_{ib-\tau}}\right)(1-\rho)\boldsymbol{\delta} + \left(\eta_i^q + \sum_{\tau=0}^{\infty} \rho^\tau \varepsilon_{ib-\tau}^q\right)$$

Taking the expectation and subtracting it on both sides yields the following expression.

$$\ln q_{ib} - \mu_q = \beta^* \sum_{\tau=0}^{\infty} \rho^\tau (\ln p_{ib-\tau} - \mu_p) + \sum_{\tau=0}^{\infty} \rho^\tau \varepsilon_{ib-\tau}^q$$

Multiplying this expression with $\ln p_{i\xi} - \mu_p$ and taking the expectation results in the desired covariances.

$$\sigma_{p_\xi,q_b} = \beta^* \sum_{\tau=0}^{\infty} \rho^\tau \sigma_{p_\xi,p_{b-\tau}} = \beta^* \sigma_p^2 \sum_{\tau=0}^{\infty} \rho^\tau \gamma_1^{|\xi-(b-\tau)|}$$

Substituting the appropriate expressions for ξ and b, either 0 or t, the proposition follows by applying the formula for the sum of a geometric series. □

Proof of Theorem 7. Substituting the respective expressions into the equations directly yields the stated results. Under the stationarity in covariance assumption, the difference of (co-)variances at different points in time vanishes and approaches zero. For the generalised means corresponding to the Laspeyres and Paasche price indices, r_L and r_P, respectively, it is assumed that the product of the autoregressive parameters is sufficiently small to be negligible, i.e. the sluggishness of adjustment of quantities or the persistence of the process of prices is low: $\rho\gamma_1 \to 0$. The generalised mean corresponding to the Paasche price index is derived under the additional assumption of sufficiently large t in order for the serial correlations to converge to zero: $\rho^t \to 0$ and $\gamma_1^t \to 0$. □

Proof of Theorem 8. The proposition follows directly by reducing the terms. □

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