Testing the Order of Fractional Integration in Panel Data Models[1]

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Abstract

Maddala and Wu (1999) proposed a simple panel unit test using a Fisher type test. We propose using Percival and Walden (2006) approximate maximum likelihood estimator of the fractional integration order with a Fisher type test to create a simple panel fractional integration order test. By combining the estimator and the test with a wavestrap method to obtain critical values, we allow for cross-sectional dependence. Monte Carlo simulations in the paper show that the test has the correct size and strong power over the alternative.

1. Introduction

The literature on panel unit root test is large and growing (for a summary see *e.g.* Baltagi, 2008). Test that take into account fractional alternatives is however rare. Among the few exceptions are Andersson and Lyhagen (1999) and Chen (2008) who consider local Whittle estimators for panel

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data models. However, neither allow for cross-sectional dependence and Andersson and Lyhagen only use the estimator to improve the power of unit root test and not to consider fractional alternatives. This paper combines Percival and Walden's (2006) approximate maximum likelihood estimator of the fractional integration order with a Fisher type test to create a panel data test of the integration order. Adding a wavestrap technique to obtain critical values we can in addition allow for cross-sectional dependence.

Non-stationary time series of integration order, *d*, are highly autocorrelated. Nevertheless, Craigmile and Percival (2005) and Percival and Walden (2006) show that a discrete wavelet transform (DWT) of a non-stationary time series yields two sets of transform coefficients; one set that are approximately uncorrelated and one set that are correlated. Based on the uncorrelated transform coefficients Percival and Walden (2006) develop a simple estimator of the integration order for all real values of *d*.

Craigmile *et al.* (2005), furthermore, show that the DWT's decorrelating property is maintain when the time series contains a deterministic trend of some polynomial order. They also show that the uncorrelated transform coefficients are unaffected by deterministic components. We can therefore estimate the integration order using the PWE without having to specify if the model contains deterministic components.

Under the assumption of normally distributed shocks, the PWE estimates of the integration order are normally distributed whereby inference on the integration order is straightforward. Building on the PWE, we can create a panel test of the integration order by using the Fisher test, which combines *p*-values from independent tests to a panel test. A similar use of the Fisher test for panel unit root test has been proposed by Maddala and Wu (1999) and Choi (2001).

A shortcoming of the Fisher test is that the test assumes cross-sectional independence. The distribution of the Fisher test is unkown if the the cross-sectional units are correlated. In this case, Maddala and Wu propsed bootstrapping the test-statistic. Using the wavelet coefficients we can wavestrap the test-statistic to allow for cross-sectional dependence. Due to the DWT's decorrelating property, wavestrapping easily performed also when the panel contain non-stationary time series.

Obviously, any estimtor of the integration order can be combined with the Fisher-test and a bootstrap to create a panel test of the fractional integration order. Alternative estimator of the fractional integration order include Geweke and Porter-Hudak (1983), Robinson (1995) and Jensen (1999). These estimators approximate the spectral density function at the low frequencies using the spectrum of a power function. By taking the log of the spectral density function they can obtain an ordinary least squares estimator of the integration order.

Other alternative estimators include maximum likelihood estimators by, for example, Sowel (l992) and Chung and Baillie (1993). A complication for most estimators, OLS and maximum likelihood, is that they require variance stationary data. Nonstationary data has to be differenced such that the integration order falls within the stationary interval before estimating the integration order. This, however, requires pre-knowledge of how many times one has to difference to obtain stationary data. An additional complication is that differencing the data too many times may yield a non-stationary processes.

Velasco (1999) proposed a spectral window technique that allows for non-stationary integration orders up to 1.5. Shimotsu and Phillips (2005) develop an exact local Whittle estimator that considers all values of d similar to the approximate likelihood estimator. They also show that the estimator can be used on data with a linear trend. The estimator requires a lot of computure power. The PWE is easier to implement and more general as the underlying model may contain not just a linear trend by any polynomial trend and therefore suitable for our purposes.

Small sample properies of the panel test is evaluated using Monte Carlo Simulations. These simulations show that the test has the correct size and relatively strong power against the alternative. Interestingly the power of the test is 100% when the panel contains both non-stationary I(1) and with statinoary I(b) processes where $0 \leq b < 0.5$.

The rest of the paper is organized as follows. First we briefly introduce wavelet analysis, followed by a presentation of the approximate likelihood estimator of the integration order. This is followed by a discussion on how to perform a panel integration test and wavestrapping. At the end of the paper follows the Monte Carlo Simulations.

2. Discrete Wavelet Transform

This section include a brief introduction to wavelet analysis, for a more comprehensive presentation see *e.g.* Gencay *et al.* (2001), Percival and Walden (2006) or Crowley 2007. The discrete wavelet transform (DWT) is a complete orthonormal transform. The wavelet domain, furthermore, combines time and frequency resolution. This combination of time and frequency resolution makes it possible to analyze irregular and non-stationary time series' frequency domain properties without having to pre-filter the data or in other ways remove for example outliers or structural breaks before the series are transformed. Nevertheless, the combination of the two domains comes with a cost as well since it is impossible to combine time- and frequency-resolution and maintain the same detailed time-resolution as in the time domain or the same detailed frequency resolution as in the frequency domain. We therefore have to give up some resolution in one of the domains to gain resolution in the other domain.

To transform a series form the time domain to the wavelet domain we need a set of basis functions. Basis functions come in different shapes and with slightly differen statistical properties, and they are therefore divide these into different families. Among these families include the Haar-wavelet, the Daubechie-wavelets, Least Asymmetric wavelets to name just three. Although different in shape, all wavelet functions fulfil certain properties, among the most important properties are that they are local in time and form an orthonormal basis.

A wavelet transform can be both continous and discrete. Basis functions for the discrete transform are oftentimes interpreted as bandpass filters. Consider a generic wavelet function with high pass filter h_l and low pass filter g_l, l=1,…,L, where L is the length of the filter. All such wavelet band pass filters fulfill three properties,

$$\sum_{l=0}^{L-1} h_l = 0 \tag{1}$$

$$\sum_{l=0}^{L-1} g_l = 1. \tag{2}$$

$$\sum_{l=0}^{L-1} g_l^2 = \sum_{l=0}^{L-1} h_l^2 = 1, \tag{3}$$

and

$$\sum_{l=1}^{L-1} h_l h_{l+2n} = \sum_{l=1}^{L-1} g_l g_{l+2n} = \sum_{l=1}^{L-1} g_l h_{l+2n} = 0. \tag{4}$$

A DWT of a time series yields a set of transform coefficients, which are usually separated into smaller sets called scales. Due to the properties of the wavelet filters, transform coefficients for a particular scale represent the series in frequency domain at a given frequency band.

First scale transform coefficients are easily obtained using the high and low pass filters,

$$W_{k,1} = \sum_{l=0}^{L-1} h_l x_{2k+1-l \bmod T}\, , k = 0, \dots, \frac{T}{2} - 1, \tag{5}$$

$$V_{k,1} = \sum_{l=0}^{L-1} g_l x_{2k+1-l \bmod T}\, , k = 0, \dots, \frac{T}{2} - 1. \tag{6}$$

where $W_{k,1}$ are the high frequency transform coefficients and $V_{k,1}$ are the low frequency coefficients. It can be shown that the filter h_l filters the frequencies ¼ to ½ and g_l the frequencies 0 to ¼.

The length of the filter is most times shorter than the length of the series (*i.e.* $L<T$). Each transform coefficients is hence a function of a limited set of time observations and the DWT is therefore a local transform and not a global transform.

For more detailed frequency resolution, we can apply the wavelet filters to the first scales low frequency coefficients,

$$W_{k,2} = \sum_{l=0}^{L-1} h_l V_{2k+1-l \bmod T/2,1}, k = 0, \dots, \frac{T}{4} - 1, \quad (7)$$

$$V_{k,2} = \sum_{l=0}^{L-1} h_l V_{2k+1-l \bmod T/2,1}, k = 0, \dots, \frac{T}{4} - 1, \quad (8)$$

where the new high pass coefficients for the second scale, $W_{k,2}$, represent approximately the frequency band $\frac{1}{8}$ to $\frac{1}{4}$ and the second scale low pass filter coefficients, $V_{k,2}$, the frequency band 0 to $\frac{1}{8}$.

By continuing with this algorithm we can create J scales of transform coefficients where $J=\ln(T)/\ln(2)$. Each $W_{k,j}$ represents frequency band $\frac{1}{2^{j+1}}$ to $\frac{1}{2^j}$ and each V_{kj} the frequency band 0 to $\frac{1}{2^{j+1}}$. The DWT reduced the number of transform coefficients for each scale such that if the original sample contains T observations, there will be $T/2^j$ transform coefficients for scale j. Consequently, scale J has only one transform coefficient.

Different wavelet filters, have slightly different properties. One property that differ between different filters is how well they separate the two frequency bands (0 – ¼ and ¼ to ½). Some filters are very accurate while others have poor frequency domain properties and the frequency band decomposition is only approximate.

Beside the combination of time and frequency resolution one advantage using wavelet analysis is that we can obtain a consistent estimate of the spectrum without having to use some smoothing algorithm. The estimate of the spectrum based on the wavelet coefficients is consistent and unbiased (see Percival and Walden, 2006).

Despite all its desirable properties, the DWT also has its limitations. For example, we can implement the DWT if the sample size, T, equals 2^J observations where J is an integer. This restriction can be relaxed if the sample has a different size by data padding. Since the DWT coefficients contain time resolution, removing the coefficients affect by the padding is straightforward. The DWT's sample size restriction is therefore easily dealt with[2].

A second problem is that the wavelet filters are symmetric around a specific point in time. This implies that to employ the DWT on the early part of the sample we need information of $t = -1, -2, \ldots$ in addition, at the end of the sample we need observations $t = T,\ T + 1, \ldots$ and so on. To be able to implement the DWT on a finite sample we thus have to data pad (see *e.g.* Percival and Walden, 2006 for a detailed discussion). Those coefficients affected by data padding are in this case called boundary coefficients and they are common excluded from the analysis once the DWT has been implemented. A particular scale always contains the same number of boundary coefficients irrespective of the sample size, and increasing the sample size therefore increases the number of non-boundary coefficients, but leaves the number of boundary coefficients constant for a given scale.

The length of the wavelet filter determines the number of boundary coefficients; a short filter has few boundary coefficients and a wide filter many boundary coefficients. The shortest filter, the Haar-wavelet, has no boundary coefficients. However, a longer filter has better frequency resolution and the Haar-wavelet's frequency resolution is relatively poor. A wider filter than the Haar wavelet is therefore commonly preferred, although this comes with the cost of more boundary coefficients. There is thus a tradeoff between good frequency resolution and

[2] One can also use a maximal overlap DWT transform, which does not impose this restriction on the sample size, however, the MODWT does not have the properties we will need later in this paper to estimate the integration order. For more information, see Percival and Walden (2006).

the number of boundary coefficients. How to choose a wavelet filter commonly depends on the problem at hand. In a small sample, one chose a short filter with few boundary coefficients and in a large sample a longer filter with more accurate frequency resolution but also more boundary coefficients.

For future discussion, denote $\mathbf{W}_j$ as a vector with the high pass transform coefficients for scale j and $\mathbf{V}_j$ the corresponding vector of low pass transform coefficients for scale j. Furthermore, denote $\widetilde{\mathbf{W}}_j$ and $\widetilde{\mathbf{V}}_j$ the corresponding vectors, which excludes the boundary coefficients. Also, let $\tilde{T}$ denote the number of non-boundary coefficients.

2.1 Decorrelating Non-stationary Time Series

Assume that x_t is an integrated time series of integration order I(d), where d is a real number. If we perform a DWT on x_t and remove boundary coefficients, Craigmile and Percival (2005) have shown that as long as $L>2d$ all non-boundary transform coefficients are uncorrelated between scales, *i.e.*

$$cov\left(\widetilde{w}_{k,j}, \widetilde{w}_{l,m}\right) = 0 \text{ as long as } j \neq m, \tag{9}$$

for all integration orders. The autocorrelation structure of x_t is captured by the low pass transform coefficients and the boundary coefficients. In addition, Percival and Walden (2006) show that for integrated processes the correlation within each scale is close to zero as well, *i.e.*

$$cov\left(\widetilde{w}_{k,j}, \widetilde{w}_{l,j}\right) \approx 0 \text{ as long as } k \neq l. \tag{10}$$

The decorrelation property follows from the peroperties of the wavelet filter. Furthermore, the decorrelation within each scale is only approxmiate while the decorrelation between scales is exact, given an appropriate filter. As an illustration, the Haar-wavelet has relatively poor decorrelation property while a Daubechei (8) wavelet has quite good decorrelating properties.

Even if the original series x_t is an I(1) process such that it is highly autocorrelated, the non-boundary high pass transform coefficients from a DWT are uncorrelated between scales and the correlation within scales are approximately zero. We can therefore analyze the high pass non-boundary coefficients from the DWT as *iid*, even if the original time series is highly autocorrelated in the time domain.

In addition, Craigmile *et al.* (2005) have shown that deterministic components of some polynomial order are captured by the low frequency transform coeffiicents as well as the high frequency boundary coefficients. As long as we only analyze high frequency, non-boundary coefficients we can ignore if the original data contains deterministic components as these do not influence these transform coefficients. If we are specifically interested in testing if the data contains a trend, special bootstrapping techniques on the boundary coefficients have been developed (see *e.g.* Andreas and Trevino, 1997; Percival and Walden, 2006).

3. Long Memory Processes

Consider the integrated time series x_t, t=0,…, T-1 which data generating processes may be written as,

$$(1-L)^d x_t = \varepsilon_t \tag{11}$$

where d is the integration order and ε_t an *iid* shock (Granger and Joyeux, 1980). The auto-covariance function for x_t is,

$$\gamma(h) = \int_{-1/2}^{1/2} s(f) e^{i2\pi f h} df, \tag{12}$$

where,

$$s(f) = \frac{\sigma_\varepsilon^2}{[4\sin^2(\pi f)]^d}, \tag{13}$$

is the spectral density function (Brockwell and Davis, 1998; Percival and Walden, 2006).

All series with an integration order $-1 < d < 1$ has a stationary mean and all series with $-0.5 \leq d < 0.5$ also has a stationary variance-covariance matrix (Brockwell and Davis, 1998). Every frequencies except the zero frequency have a finite variance-covariance matrix also when $d \geq 0.5$, and the decorrelating property of the DWT ensures that we can analyze $\tilde{\mathbf{w}}_j$ as uncorrelated random variables with a finite variance covariance matrix (see *e.g.* McCoy and Walden 1996; Percival and Walden, 2006). The variance of these uncorrelated high frequency transform coefficients for scale j is given by,

$$var(\tilde{w}_{k,j}) = 2^{j+1} \int_{-1/2}^{1/2} \frac{\sigma_\varepsilon^2}{[4\sin^2(\pi f)]^d} df, \tag{14}$$

and these transform coefficients have the same distriubtion as the *iid* shocks ε_t.

Looking at (13) it is evident that the spectral density function is a function of the integration order, but the only the way the integration order affects the spectral density function is by increasing or decreasing the variance of certain frequencies in relation to the variance of the shocks. If $d=0$ the denominator in (13) is equal to one and all individual frequencies have the same variance. As d increases the variance of the low frequencies increases relative to the high frequencies. It is important to emphasize that when $d \geq 0.5$ the variance is no longer defined for $f=0$, but it is still finite for $f \neq 0$.

3.1 Estimating the Fractional Integration Order

The approximate likelihood estimator of the integration order puts no restriction on the integration order and can be used to estimate any real valued integration order. Percival and Walden (2006) show that the estimator is consistent and the estimates normally distributed for all values of d if the shocks are normally distributed.

Before we consider the approximate likelihood estimator let us consider an exact likelihood estimator for a stationary time series. Assuming that the shock term is normally distributed, we can define the exact likelihood function as,

$$L(d, \sigma_\varepsilon^2 | \mathbf{x}) = \frac{e^{\mathbf{x}'\Sigma^{-1}\mathbf{x}/2}}{(2\pi)^{T/2}|\Sigma|^{1/2}}, \tag{15}$$

$\boldsymbol{x} = [x_1, x_2, \dots, x_{T-1}]'$ a vector with the observations of x_t and $\sum$ is the covariance matrix with elements given by the autocovariance function (12). There are two parameters to be estimated the integration order d and the variance of the shocks σ_ε^2.

If $d \geq 0.5$, x_t has a nonstationary variance covariance due to the zero frequency and we cannot use the likelihood in equation (15) to estimate the integration order. However, the covariance matrix is stationary if we exclude the zero frequency. Using that the DWT yields a consistent estimate of the spectrum, and it decorrelates non-boundary coefficients we can define the approximate likelihood function based on the non-boundray coefficients;

$$\tilde{L}(d, \sigma_\varepsilon^2 | \widetilde{\mathbf{w}}) = \frac{e^{\widetilde{\mathbf{w}}'\Sigma_{\widetilde{\mathbf{w}}}^{-1}\widetilde{\mathbf{w}}/2}}{(2\pi)^{T/2}|\Sigma_{\widetilde{\mathbf{w}}}|^{1/2}}, \tag{16}$$

where, due to the decorrelation property of the DWT, the covariance matrix $\sum_{\widetilde{\mathbf{w}}}$ is a diagonal matrix. Using the decorrelation we can derive the reduced log likelihood function (for more details see Perival and Walden, 2006);

$$\tilde{l}(d|\widetilde{\mathbf{w}}) = \tilde{T} ln(\tilde{\sigma}_\varepsilon^2) + \sum_{j=1}^{J_0} \tilde{T}_j ln\left(c_j(d)\right), \tag{17}$$

where,

$$\hat{\sigma}_\varepsilon^2 = \frac{1}{\tilde{T}} \sum_{j=1}^{J_0} \frac{\widetilde{\mathbf{W}}\prime\widetilde{\mathbf{W}}}{c_j(\hat{d})}, \tag{18}$$

and

$$c_j(d) = 2^{j+1} \int_{1/2^{j+1}}^{1/2^j} \frac{\sigma_\varepsilon^2}{[4\sin^2(\pi f)]^d} df. \tag{19}$$

The reduced likelihood function is only a function of one parameter, the integration order, and the variance of the shock is given by (18). Since we only have to minimize the reduced log likelihood function over one parameter, it is computationally easy to perform.

To be able to apply the maximum likelihood estimator we have assumed that the shocks, ε_t, are normally distritbuted to be able to define the likelihood function. The estimates of the integration order are thus, asymptotically, normally distributed;

$$\hat{d} \sim N\left(d, \mathrm{var}(\hat{d})\right), \tag{20}$$

where,

$$\mathrm{var}(\hat{d}) = 2\left[\sum_{j=1}^{J_0} \tilde{T}_j g_j^2 - \tilde{T}^{-1}\left(\sum_{j=1}^{J_0} \tilde{T}_j g_j^2\right)^2\right]^{-1}, \tag{21}$$

$$g_j = -\frac{4\hat{\sigma}_\varepsilon^2}{\mathrm{var}(\mathbf{w}_j)} \int_0^{1/2} H(f) \frac{\ln(2\sin(\pi f))}{[4\sin^2(\pi f)]^d} df, \tag{22}$$

and $H_j(f)$ is the squared gain function of the wavelet filter, h_j for scale j; *i.e.*

$$H_j(f) = \left|\sum_{l=-\infty}^{\infty} h_{jl} e^{-i2\pi f l}\right|^2. \tag{23}$$

For a small sample evaluation of the PWE see Craigmile *et al.* (2005) and Percival and Walden (2006). To perform inference on the parametrs we can either use the integration order estimates' normal distribution or a likelihood-ratio test.

3.2 Panel Data Test of the Integration Order

Independent test can be combined to a joint test using a Fisher type test that combines *p*-values as proposed by Maddala and Wu (1999) for panel unit root tests. The Fisher test is exact and is thus suitable for a small cross-sectional dimension. Choi (2001) proposes a modified test to allow for a large cross-sectional dimension.

To test if all series in a panel are integrated of the same order, we can combine inference on the PWE estimates for each series of the panel indpendently using the Fisher Test (FT) or the Modifed Fisher Test (MFT). If all series independent the distribution of the tests are known, otherwise we have to wavestrap the test statistic.

Consider a panel data model with integrated processes,

$$(1-L)^{d_n} x_{nt} = \varepsilon_{nt}, \tag{24}$$

where T is the time dimension, n=1,…,N is the cross sectional dimension. Allowing for cross-sectional depedence we may assume that the off diagonal elements of the covariance matrix, Ω, are non-zero. To be able to estimate the the integration order we have to assume that ε_n is *iid* and normally distributed $N(0, \sigma_n^2)$, where $\sigma_n^2 = \mathrm{var}(\varepsilon_{nt})$.

Assume we wish to test the hypothesis that all series' are integrated of the same order δ,

$$\begin{cases} H_0: & d_n = \delta \quad \text{for all } n \\ H_1: & d_n \neq \delta \quad \text{for some or all } n \end{cases} \tag{26}$$

We define the null hypothesis in different ways; *i.e.* we can choose different values for δ. We may wish to test for a unit root by setting δ=1, or for non-stationarity by setting δ=0.5, or we can test if all series are integrated of the same order by setting δ=average(d_n).

Irrespective of how we choose δ we test the hypotesis by first estimating the integration order for each cross-sectional unit individually and testing the hypothesis for that unit. Once we have obtain an individual p-value for all units we can combine these into a panel test using the Fisher test,

$$FT = -2\sum_{n=1}^{N} \ln(p_n), \tag{27}$$

where p_n is the p-value from cross-sectional unit n of the panel. The test-statistic FT has a χ^2 distribution with 2N degrees of freedom under the null hypothesis if the cross section units are independent. Under the assumption of cross-sectional dependence, the distribution of the test is unkonwn. For panel unit root tests, Maddala and Wu proposed bootstrapping the test-statistic to obtain a critical value under such circumstances. Similarly, we can wavestrap the test-statistic to obtain a critical value.

The Fisher test in (27) is an exact test as we only require T to go to infinity. For a large panel with a large cross-sectional dimension we can use the by Choi (2001) modified Fisher-test,

$$MFT = \frac{1}{2\sqrt{N}} \sum_{n=1}^{N} (-2\ln(p_n) - 2), \quad (28)$$

which has a $N(0,1)$ distribution if we let $T \to \infty$ followed by $N \to \infty$. Also this test assumes cross-sectional independence and we must wavestrap the test-statistic when that assumption is unsatisfied.

3.3 Wavestrapping

Wavestrapping and bootstrapping are essentially the same, the only difference is that wavestrapping is performed on the DWT coefficients. For more information on wavestrapping, see *e.g.* Percival *et al.* (2001) and Percival and Walden (2006).

To wavestrap the test statistics in (27) and (28) we use the DWT's decorrelating property and re-scale the transform coefficients scale by scale such that they represent an I(δ) process. Looking at (13) we see that the integration order affects the slope of the spectral density function. If we rescale the transform coefficients for each scale we can change the slope of the denstity function and thereby create a new series with a new integration order.

Let us make the rescaling in steps[3] by first rescaling all non-boundary coefficients using the factor r_{nj},

$$\ddot{\mathbf{w}}_{nj} = \tilde{\mathbf{w}}_{nj} \times r_{nj}, \quad (29)$$

where

$$r_{nj} = \left(\int_{1/2^{j+1}}^{1/2^j} \frac{1}{[4\sin^2(\pi f)]^{\hat{d}_n}} df \right)^{-1/2}, \quad (30)$$

[3] Obviously, we can combine the two rescaling steps make it in just one step.

and $\tilde{\mathbf{w}}_{nj}$ are the transform coefficeints for cross-sectional unit n at scale j. Asymptotically the estimates of the integration order converges in probability to the true value, d_n, thus by multiplying the transform coefficients with the weights r_{nj} we rescale the coefficients such that the new coefficents, $\ddot{\mathbf{w}}_{nj}$**,** that goes in distribution to an I(0) process with the variance $2^{j+1}\sigma_n^2$ (see Slutsky, 1925).

Let us create the matrix with the rescaled coefficients,

$$\mathbf{g}_j = \left[\ddot{\mathbf{w}}_{1,j}, \ldots, \ddot{\mathbf{w}}_{N,j}\right], \tag{31}$$

where we put the transform coefficients for individual n in column n. Due to the DWT's decorrelateing property all coefficients in $\mathbf{g}_j$ are uncorrelated along the row dimension, but may be correlated along column dimension (cross-sectional dependence). If we rescale the transform coefficients a second time using the new weights,

$$p_j = \left(\int_{1/2^{j+1}}^{1/2^j} \frac{1}{[4\sin^2(\pi f)]^{\delta}} df\right)^{1/2}, \tag{32}$$

we get the coefficients

$$\bar{\bar{\mathbf{g}}}_j = \mathbf{g}_j \times p_j. \tag{33}$$

These coefficients goes in distribution to an I(δ) series. If we rescale all scales, j=1,.., J_0, we obtain a new set of transform coefficients with a spectrum that asymptotically converges to that of an I(δ) process. Furthermore, these rescalings of the transform coefficinents does not affect the transform coefficients cross-sectional correlation structure.

Given the decorrelating property of the DWT, all rows in (33) are independent of each other, but the coefficients may be correlated along the column dimension if the cross-sectional units are correlated. To simulate to series we therefore pick randomly with replacement row by row from $\bar{\bar{\mathbf{g}}}_j$. The number of nonboundary coefficients for each scale equals $\tilde{T}_j$ and we accordingly randomly pick $\tilde{T}_j$ rows from $\bar{\bar{\mathbf{g}}}_j$ and put those in the matrix $\mathbf{a}_j$. Once we have done this for all scales we have our new simulated series of integration order δ and with the same cross-sectional correlation between the transform coefficients as in the original data. Using these coefficients we can stimate the integration order, unit by unit, test the hypothesis that $d_n=\delta$ and combine the individual *p*-values to a panel data test using the Fisher test. This gives us a value of the test-statistic when the null hypothesis is true. Repeating the procedure, we can obtain the distribution of the test-statistics (27) and (28) from which we can pick our critical value.

4. Monte Carlo Experiment

The FT and the MFT's small sample properties are analyzed using a Monte Carlo study. Power and size are evaluated by testing the null that all series are integrated of the same order, *i.e.* $d_n=\delta$ where we choose δ as the average of estimated d_n.

Estimates of size are generated by assuming that all series have a unit root (are I(1)). We also considered other alternatives such as all series being I(0.5), however, the choice of integration order did not affect the results. Power estimates are obtained by splitting the panel into two groups; one group (Group 1) with a integration order that vary between 0 and 2, and one group (Group 2) that has a fixed integration order of 1. Setting the integration order of Group 2 to another value than 1 has no effect on the power of the test as it turns out that only the distance $|d_1 - d_2|$ affect the power of the test. [4]

4.1. Simulations

[4] Considering that the estimates of the integration order are normally distributed, this is an unsurprising result.

Fractionally integrated times series are simulatid using the method proposed by Percival and Walden (2006). Using this method, we first simulate a set of DWT transform coefficients of an I(0) processes and then rescale them using the same techinque as discussed in Section 3.3. Once we have rescaled them, time domain observations are obatained by taking the inverse DWT of the transform coefficients.

To simulate series we use the following the data generating process[5];

$$(1-L)^{d_n} x_{nt} = \varepsilon_{nt}. \tag{34}$$

Cross-sectional dependence is achieved by including a common factor in the error term,

$$\varepsilon_{nt} = v_{nt} + \theta_n f_t, \tag{35}$$

where $v_{nt} \sim N(0, \sigma_n^2)$ is an idiosyncratic shock, and $f_t \sim N(0,1)$ a common factor with factor loadings θ_n. The spectral density function for ε_{nt} is given by,

$$s_n(f) = \sigma_n^2 + \theta_n^2, \tag{36}$$

and the spectral density function for x_{nt} by,

$$s_{x_n}(f) = \frac{\sigma_n^2+\theta_n^2}{[4\sin^2(\pi f)]^{d_n}}. \tag{37}$$

We allow the indiosyncratc component to be heteroscedastic across cross-sectional units and draw the variance, σ_n^2, from a uniform distribution $U(1, 4)$. Factor loadings are also drawn from a unifrom distribution, $U(-2, 2)$. The common factor can thus have both a positive or a negative effect on an individual series. For each time series, furthermore, the maximum

[5] We also made simulations with linear deterministic trend. However, as previously discussed, including a trend does not affect the non-boundary coefficients we use to estimate and test the integration order. The size and the power of the test were unaffected down to the fourth decimal if we include a linear trend or not.

variance of the idiosyncratic component is 4, and for each series the maximum variance from the common factor is also 4.

Three different T are considered; 128, 256, 512[6], also four different N are considered; 5, 10 20 and 50. The memory of the series increase as we increase the integration order; *i.e.* the autocorrelation increases when d increase. It is not too difficult to distinguish between and I(0) or an I(1) processes even with a small sample, but to distinguish between say an I(0.8) and I(0.9) processes requires a fair amount of data. Having tested different sample sizes, we conclude that to be able to distinguish between more than just I(0) or I(1) we need at least 128 time observations.

All simulated transform coefficients are transformed to the time domain using the inverse DWT. A complication when taking the inverse DWT is the DWT boundary coefficients. To avoid having boundary coefficients affecting our simulations we simulate $4\times T$ observations and pick T observations from the middle of the simulated sample. Observations in the middle of the sample is unaffected by boundary coefficients and the simulated series are therfore clean from such problems[7].

Once the data has been simulated the following algorithm is used to evaluate the size and power of the test;

(i) All series are transform to the wavelet domain using a Daubechie (4)

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[6] We could have chosen other values for T, but since the DWT puts restrictions on the sample size, choosing other values would imply having to data pad, which we now avoid. Our choice of T does not affect our results; it only makes the simulation implementation easier.

[7] To transform the series to the time domain we need to specify a wavelet filter. We use a Daubechie (8) filter to simulate the series because it has good frequency domain properties.

wavelet[8,9], such that the integration order can be estimated using PWE.

(ii) The integration order is estiatmed for each time series independently.

(iii) We calculate the average integration order, $\bar{d} = N^{-1} \sum_{i=1}^{N} \hat{d}_n$.

(iv) We test for each unit independently if $d_n = \bar{d}$ and save the *p*-values. The hypothesis is tested using a likelihood-ratio test.

(v) We combine the *p*-values using the Fisher-test or the Modified Fisher-test.

(vi) The test statistics are wavestrapped[10] to obtain a critical value using a 5% significance level.

(vii) The procedure is repeated 1000 times.

The Daubechie 4 wavelet yields one boundary coefficient for scale 1 and 2 boundary coefficients each for the other scales (Percival and Walden, 2006). Because the DWT downsizes the sample as the scale increase such that when the first scale contains $T/2$ coefficients and the j^{th} scale $T/2^j$ coefficients, eventually the number of boundary coefficients equals the number transform coefficients. Therefore, even though J=7 when T=128 if we ignore that some coefficents are boundary coefficients, we have to limit the analysis to 5

[8] For our sample sizes a Daubechie (4) is reasonable length given its decorrelating properties, frequency resolution and the number of boundary coefficients. When we simulated the series we used a filter of double length, that was possible as we simulated 4*T observations. Given our sample lengths of 128, 256, 512 that wavelet is no longer practical to use.

[9] We also tried a Haar-wavelet, but that wavelet gave biased estimates of the integration order. We also tried a Daubechie (6) wavelet, which gave results similar to those presented in the paper.

[10] We use 1000 replicas in the wavestrap.

scales[11] if we are to remove boundary coefficients from the data set. As T increases to 256 and then to 512, we add first scale 6 and then also scale 7. The number of excluded coefficients are quite few and a sample of 128 observations include 115 nonboundary coefficient, increasing the sample size to 241 increases the number of nonboundary coefficients to 241 and finally when T=512 we have 495 nonboundary high frequency coefficients.

4.3 Simulation Results

Before we analyse power and size of the tets let us look at the bias and in particular the variance of the PWE estimates as these affect the properties of the panel tests. Table 1 presents bias and variance of the PWE, which have been generated assuming that all series are I(1). As can be seen in the Table, the estimates of the integration order are slightly biased when T=128; the bias is between -0.010 and -0.007. The bias disappears as T increases and the estimates are unbiased when T=512. The variance is between 0.002 and 0.013 depending on time dimension, with a decreased variation as T increases. The decline in variation is also illustrated by the fact that when T=128 95% of all estimates lay within the intervall 0.76 to 1.21. For T=512, the width of intervall is reduced to 0.91 and 1.09.

Table 2 contains estimates of the size for both the FT and the MFT for each time and cross-sectional dimension. These statistics have been generated letting that all series being I(1) [12]. The size of the test is most of the times close to the 5% level, but both the FT and MFT

[11] Scale 5 include four transform coefficients when T=128. Removing two boundary coefficients leaves us with two non-boundary coefficients that we can use to estimate the integration order from that scale.

[12] The size of the FT is 0.061 and the size of the MFT is 0.097 when T=128 and N=5. If we

are oversized when T=128. The size distortion is particularly large when N=5. For example, MFT has a size of 0.097 and the corresponding value for the FT is 0.061. Although the differences in size between the FT and the MFT, are often not as large as this example illustrates, the MFT commonly has a greater size than the FT.

That both the FT and the MFT are oversized when T=128 is unsurprisingly considering the relatively high variation in the PWE estimates. Increasing T to 256 reduces the variability of the PWE estiamtes by more than 50%, which spills over to the size of the tests, which are now closer to the 5% level. But, the FT is undersized for most combinations of N and $T \geq 256$; out of the 8 cases the FT is undersized 6 times. The MFT on the other hand, is oversized in 4 out of the 8 simulations and undersized in 4. The MFT is oversized when N is small and undersized when N is large. Excluding the cases when N=5, the size of the MFT is between 0.040 and 0.076. Similar size for the FT is between 0.039 and 0.067.

4.3.2 Simulation Results – Power

To generate power estimates we split the panel into two groups. Group 1 contains time series with an integration order, d_1, which we let vary between 0 and 2. Group 2 contains time series which integration order, d_2, is fixed at 1 at all times. For each N we look at different combinatios of N_1 and N_2, where N_1 denotes the number of time series in Group 1 and N_2 the number of time series in Group 2.

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assume that all series are I(0.5) instead of I(1), the size for the two tests are 0.062 and 0.098 respectively. For T=256, N=10, the size presented in this paper for d_1=1 is 0.046 and 0.055, if d_1=0.5 the sizes are 0.045 and 0.058.

Tables 3 to 6 and Figures 1 and 2 contain size adjusted power[13]. Each table show the results for a given cross-sectional dimension and a few selected d_1. Power for all values of d_1 are shown in Figure 1 and 2. Power of the FT is presented in Figure 1 and power of the MFT in Figure 2. Both figures are divided into six panels where each column show represent one time dimension. The first row of the figures show the tests power when Group 1 includes only one series (*i.e.* all series except one is I(1)). The second row contains panels where Group 1 and Group 2 are of equal size[14]. Power of the other combinations of N_1 and N_2 fall in between of these two combinations. The integration order of Group 1 is plotted on the *x*-axis and the *y*-axis shows how often we reject the null hypothesis that all series are integrated of the same order. When d_1 = 1 (integration order for Group 1), the figures show the size and not the power as the series are all integrated of the same order.

From these tables and figures several observations can be made. First, the power is symetric, *i.e.* we reject the null as often when d_1=d_2-z as we do when when d_1= d_2+z, where z is some real number[15]. Second, the null is rejected less often when Group 1 contains few units compared to Group 2. This can be seen in the figures by for each column, comparing the upper panel with the lower panel in each column.

[13] Non-adjusted power is not much different from size adjusted power except when T=128.

[14] When N=5, N_1=2 and N_2=3.

[15] What matters is not the integration order of Group 1 or Group 2, but the difference between the two group's integration orders. For example, if we let Group 2 be integrated I(0.5) instead of I(1) we obtain similar results as presented in this paper when the distance between d_1 and d_2 is the same as in this paper. Because those results are similar to those presented in the paper they are excluded from the paper.

Third, the FT has stronger power than the MFT although the differences between them are in generally small. The greatest difference in power is when N_1 is small compared to N_2. The power of the MFT improves as N increase which is expected considering that it is an asymptotic test.

Fourth, the tests are more powerful when T is large, which can be seen by looking at a given row in each figure. As we move to the right in the figures we see that the power functions have less spead and power is equal to 100% also for smaller differnces between d_1 and d_2. As an illustration, consider the case when N_1=1 and N_2=49, to obtain a rejection rate of more than 50%, the difference between d_1 and d_2 has to be greater than 0.45 when T=128, but the same power is achieved for T=512 when the difference is greater than 0.25.

In all cases, except when T=128, power is 100% when the panel mixes unit roots with stationary series; *i.e.* d_1<0.5. Only when T=128 and $N\geq 20$ is power below 1, otherwise the power of the test is 1 for all combiantions of stationary and non-stationary variables irrespective of the number of time observations. For example, it is sufficient that just one series is stationary when all others are unit roots for the test to reject the hypothesist that they are all integrated of the same order.

As previously mentioned, simulations not presented in the show that it is not the absolute values of d_1 and d_2 that affect the power of the test but the distance between the two. The presented results are thus more general and not dependent on d_2=1. Overall we can thus conclude that power of the test is strong when $T\geq 256$, and when Group 1 and Group 2 are of equal size.

5. Conclusions

Panel unit root test are common, but panel tests taking fractional alternatives into account are uncommon. This paper builds on Percival and Walden (2006), Maddala and Wu (1999) and Choi (2001) and develops a test of the fractional integration order. The PWE is a general estimator for all real valued integration orders and any polynomial trends. By combining these with the Fisher test and a wavestrap technique we obtain a general test for most data generating properties. Simulation resutls in the paper, show that the proposed test has the correct size and strong power. The power is especially strong when the sample contains both stationary and non-stationary variables. A short coming of the test is that it requires at least $T \geq 128$ time observations for each unit. If the sample is shorter than 128 time observations it becomes increasingly difficult to distinguish between different integration orders and the panel test has low power. However, for $T \geq 256$ the test has correct size and strong power, especially if the panel combines stationary with nonstationary processes.

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Table 1: PWE Bias and Variance			
(*T, N*)	$\hat{d}$	Bias	Variance
(128, 5)	0.990	-0.010	0.012
(128, 10)	0.992	-0.008	0.013
(128, 20)	0.992	-0.008	0.013
(128, 50)	0.993	-0.007	0.013
(256, 5)	0.993	-0.070	0.005
(256, 10)	0.999	-0.001	0.006
(256, 20)	1.000	0.000	0.005
(256, 50)	1.000	0.000	0.005
(512, 5)	1.001	0.001	0.002
(512, 10)	1.000	0.000	0.002
(512, 20)	1.001	0.001	0.002
(512, 50)	1.000	0.000	0.002

Table 2: Size								
T/N	5	10	20	50	5	10	20	50
	Fisher				*Modified Fisher*			
128	0.061	0.067	0.050	0.040	0.097	0.076	0.070	0.052
256	0.043	0.046	0.039	0.050	0.086	0.055	0.042	0.045
512	0.044	0.044	0.041	0.058	0.066	0.052	0.048	0.040

Table 3: Power, N=5							
		128		256		512	
(N_1, N_2)	$I(d_1)$	*FT*	*MFT*	*FT*	*MFT*	*FT*	*MFT*
(1, 4)	0.00	1.000	1.000	1.000	1.000	1.000	1.000
(1, 4)	0.50	0.823	0.755	0.999	0.998	1.000	1.000
(1, 4)	0.80	0.181	0.142	0.378	0.266	0.757	0.672
(1, 4)	0.90	0.088	0.101	0.083	0.074	0.175	0.121
(1, 4)	1.10	0.088	0.123	0.108	0.099	0.222	0.153
(1, 4)	1.20	0.205	0.178	0.454	0.346	0.794	0.719
(1, 4)	1.50	0.871	0.830	1.000	1.000	1.000	1.000
(1, 4)	2.00	1.000	1.000	1.000	1.000	1.000	1.000
(2, 3)	0.00	1.000	1.000	1.000	1.000	1.000	1.000
(2, 3)	0.50	0.969	0.946	1.000	1.000	1.000	1.000
(2, 3)	0.80	0.295	0.223	0.595	0.492	0.952	0.918
(2, 3)	0.90	0.101	0.113	0.137	0.089	0.314	0.226
(2, 3)	1.10	0.099	0.106	0.159	0.122	0.320	0.235
(2, 3)	1.20	0.205	0.178	0.630	0.539	0.952	0.923
(2, 3)	1.50	0.980	0.969	1.000	1.000	1.000	1.000
(2, 3)	2.00	1.000	1.000	1.000	1.000	1.000	1.000

Note: *FT*=Fisher Test; *MFT*=Modified Fisher Test

Table 4: Power, N=10

(N_1, N_2)	$I(d_1)$	T=128		T=256		T=512	
		FT	*MFT*	*FT*	*MFT*	*FT*	*MFT*
(1, 9)	0.00	1.000	1.000	1.000	1.000	1.000	1.000
(1, 9)	0.50	0.724	0.613	0.994	0.989	1.000	1.000
(1, 9)	0.80	0.163	0.122	0.277	0.188	0.637	0.514
(1, 9)	0.90	0.088	0.082	0.094	0.071	0.176	0.114
(1, 9)	1.10	0.084	0.083	0.105	0.081	0.191	0.137
(1, 9)	1.20	0.172	0.146	0.358	0.255	0.637	0.514
(1, 9)	1.50	0.848	0.786	0.998	0.996	1.000	1.000
(1, 9)	2.00	1.000	1.000	1.000	1.000	1.000	1.000
(3, 7)	0.00	1.000	1.000	1.000	1.000	1.000	1.000
(3, 7)	0.50	0.999	0.997	1.000	1.000	1.000	1.000
(3, 7)	0.80	0.376	0.284	0.763	0.668	0.990	0.979
(3, 7)	0.90	0.137	0.101	0.179	0.120	0.415	0.332
(3, 7)	1.10	0.131	0.105	0.189	0.132	0.470	0.368
(3, 7)	1.20	0.371	0.294	0.790	0.706	0.994	0.989
(3, 7)	1.50	0.996	0.993	1.000	1.000	1.000	1.000
(3, 7)	2.00	1.000	1.000	1.000	1.000	1.000	1.000
(5, 5)	0.00	1.000	1.000	1.000	1.000	1.000	1.000
(5, 5)	0.50	1.000	1.000	1.000	1.000	1.000	1.000
(5, 5)	0.80	0.465	0.420	0.876	0.809	0.997	0.995
(5, 5)	0.90	0.148	0.121	0.230	0.145	0.527	0.428
(5, 5)	1.10	0.132	0.102	0.228	0.170	0.534	0.442
(5, 5)	1.20	0.441	0.344	0.883	0.806	1.000	0.997
(5, 5)	1.50	0.999	0.999	1.000	1.000	1.000	1.000
(5, 5)	2.00	1000	1.000	1.000	1.000	1.000	1.000

Note: *FT*=Fisher Test; *MFT*=Modified Fisher Test

Table 5: Power, N=20							
		T=128		T=256		T=512	
(N_1, N_2)	$I(d_1)$	*FT*	*MFT*	*FT*	*MFT*	*FT*	*MFT*
(1, 19)	0.00	1.000	1.000	1.000	1.000	1.000	1.000
(1, 19)	0.50	0.512	0.394	1.000	1.000	1.000	1.000
(1, 19)	0.80	0.103	0.081	0.189	0.111	0.491	0.387
(1, 19)	0.90	0.066	0.071	0.061	0.045	0.117	0.068
(1, 19)	1.10	0.062	0.063	0.082	0.055	0.134	0.081
(1, 19)	1.20	0.131	0.092	0.261	0.168	0.611	0.504
(1, 19)	1.50	0.831	0.665	0.991	0.990	1.000	1.000
(1, 19)	2.00	1.000	1.000	1.000	1.000	1.000	1.000
(5, 15)	0.00	1.000	1.000	1.000	1.000	1.000	1.000
(5, 15)	0.50	1.000	1.000	1.000	1.000	1.000	1.000
(5, 15)	0.80	0.426	0.306	0.907	0.838	1.000	1.000
(5, 15)	0.90	0.110	0.088	0.234	0.147	0.605	0.482
(5, 15)	1.10	0.122	0.091	0.238	0.144	0.611	0.485
(5, 15)	1.20	0.487	0.369	0.927	0.879	1.000	1.000
(5, 15)	1.50	1.000	1.000	1.000	1.000	1.000	1.000
(5, 15)	2.00	1.000	1.000	1.000	1.000	1.000	1.000
(10, 10)	0.00	1.000	1.000	1.000	1.000	1.000	1.000
(10, 10)	0.50	1.000	1.000	1.000	1.000	1.000	1.000
(10, 10)	0.80	0.611	0.498	0.987	0.970	1.000	1.000
(10, 10)	0.90	0.140	0.103	0.341	0.241	0.776	0.665
(10, 10)	1.10	0.137	0.098	0.331	0.213	0.777	0.660
(10, 10)	1.20	0.613	0.496	0.986	0.967	1.000	1.000
(10, 10)	1.50	1.000	1.000	1.000	1.000	1.000	1.000
(10, 10)	2.00	1.000	1.000	1.000	1.000	1.000	1.000

Note: *FT*=Fisher Test; *MFT*=Modified Fisher Test

Table 6: Power, N=50							
		T=128		*T*=256		*T*=512	
(N_1, N_2)	$I(d_1)$	*FT*	*MFT*	*FT*	*MFT*	*FT*	*MFT*
(1, 49)	0.00	1.000	1.000	1.000	1.000	1.000	1.000
(1, 49)	0.50	0.516	0.386	1.000	0.987	1.000	1.000
(1, 49)	0.80	0.080	0.068	0.125	0.075	0.270	0.168
(1, 49)	0.90	0.041	0.056	0.073	0.068	0.080	0.063
(1, 49)	1.10	0.046	0.054	0.074	0.069	0.098	0.065
(1, 49)	1.20	0.086	0.064	0.200	0.075	0.378	0.258
(1, 49)	1.50	0524	0.412	0.999	0.989	1.000	1.000
(1, 49)	2.00	1.000	1.000	1.000	1.000	1.000	1.000
(10, 40)	0.00	1.000	1.000	1.000	1.000	1.000	1.000
(10, 40)	0.50	1.000	1.000	1.000	1.000	1.000	1.000
(10, 40)	0.80	0.590	0.452	1.000	1.000	1.000	1.000
(10, 40)	0.90	0.104	0.084	0.343	0.225	0.800	0.660
(10, 40)	1.10	0.166	0.120	0.298	0.183	0.812	0.662
(10, 40)	1.20	0.632	0.546	0.995	1.000	1.000	1.000
(10, 40)	1.50	1.000	1.000	1.000	1.000	1.000	1.000
(10, 40)	2.00	1.000	1.000	1.000	1.000	1.000	1.000
(25, 25)	0.00	1.000	1.000	1.000	1.000	1.000	1.000
(25, 25)	0.50	1.000	1.000	1.000	1.000	1.000	1.000
(25, 25)	0.80	0.898	0.838	1.000	1.000	1.000	1.000
(25, 25)	0.90	0.214	0.132	0.590	0.415	0.983	0.958
(25, 25)	1.10	0.212	0.176	0.568	0.415	0.987	0.967
(25, 25)	1.20	0.890	0.844	1.000	1.000	1.000	1.000
(25, 25)	1.50	1.000	1.000	1.000	1.000	1.000	1.000
(25, 25)	2.00	1.000	1.000	1.000	1.000	1.000	1.000

Note: *FT*=Fisher Test; *MFT*=Modified Fisher Test

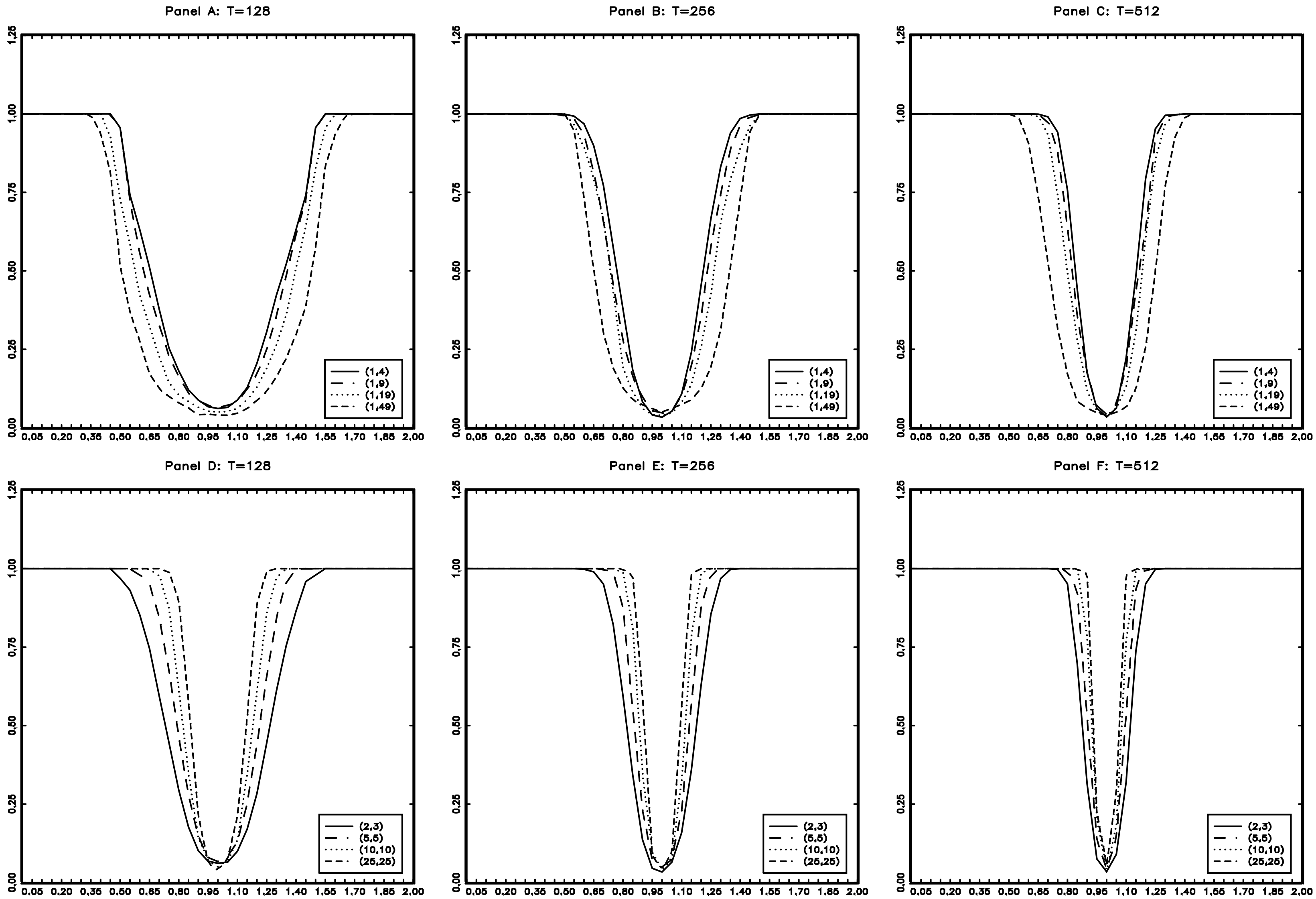

Figure 1: Power, Fisher Test

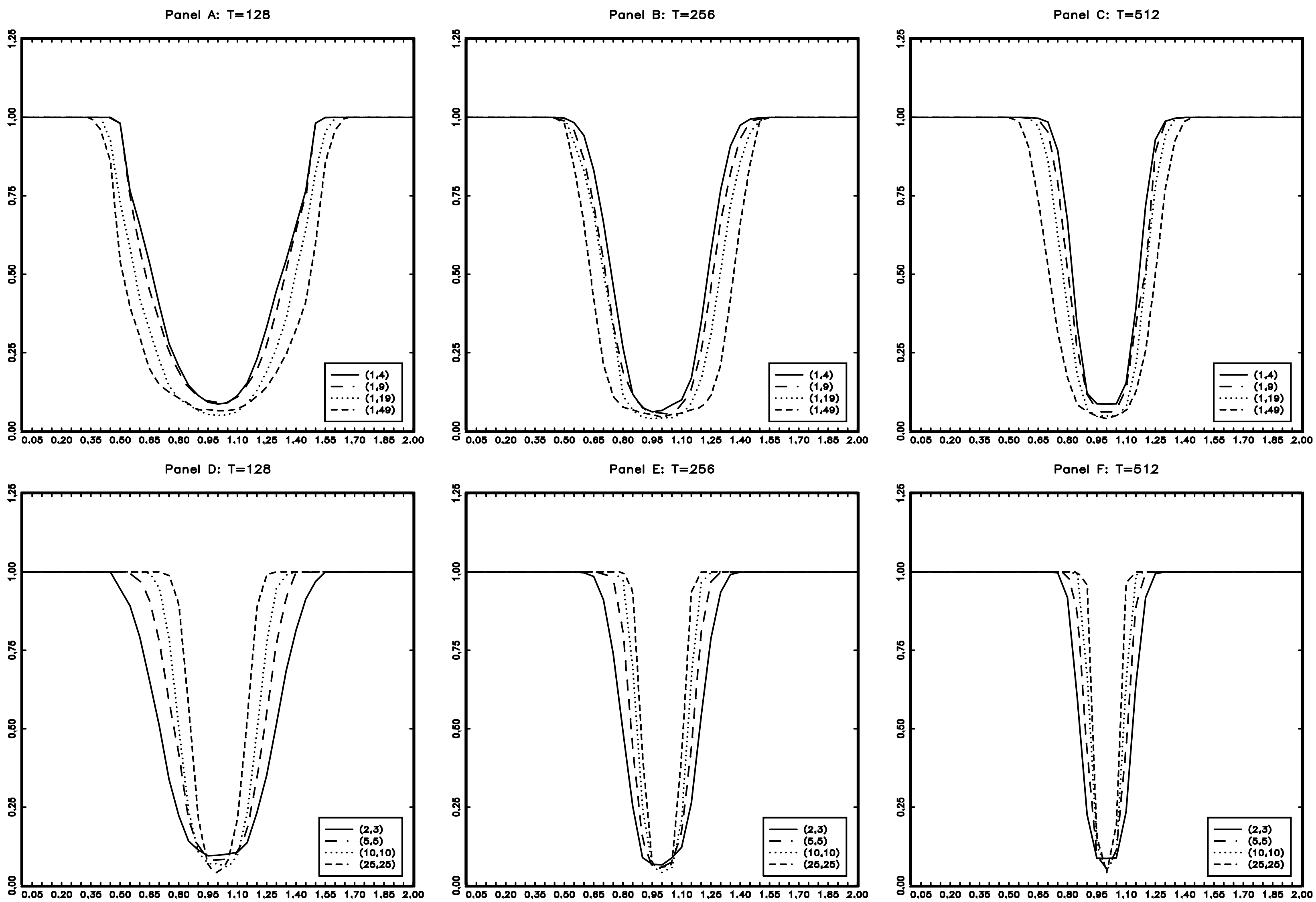

Figure 2:Power, Modified Fisher Test