Robust Covariance Matrix Estimation for Linear Panel Models with Fixed Effects

Min Seong Kim*

Department of Economics, UC San Diego

June 2010

Abstract

This paper studies the estimation of covariance matrices for linear panel models with fixed effects in the presence of heteroskedasticity and spatial and temporal correlation. We propose a bivariate kernel estimator which converges to the true covariance matrix as both n and T increase. Our estimator is flexible in the sense that it includes existing estimators as special cases, reducing to them with certain bandwidth selection. We derive the optimal bandwidth parameters based on the upper bound of the asymptotic mean square error criterion and suggest an automatic implementation procedure using a parametric plug-in method. Due to flexibility and the automatic bandwidth selection procedure, our estimator adapts to the dependence structure of data and approaches an existing estimator when the latter is favorable. This *adaptiveness* is the salient feature of our estimator. The finite sample performance of the estimator is evaluated via Monte Carlo simulation.

Keywords: Adaptiveness, Covariance matrix estimator, Dependence, Optimal bandwidth choice, Panel linear model with fixed effects, Robust standard error, Upper bound of asymptotic mean squared error.

JEL Classification Number: C13, C14, C23

*Email: msk003@ucsd.edu. I am sincerely grateful to my advisor Yixiao Sun for his invaluable guidance and support. I thank Patrik Guggenberger, Hiroaki Kaido, Philip Neary, Andres Santos, Hal White and seminar participants at UCSD. Any comment or suggestion is appreciated.

1 Introduction

This paper considers the estimation of covariance matrices for linear panel models with fixed effects in the presence of heteroskedasticity and spatial and temporal correlation. As economic data is potentially heterogeneous and correlated in unknown ways across individuals and time, the robust estimation of covariance matrices is an important issue and has gained increasing attention in panel data analysis; See, for example, Betrand, Duflo and Mullainathan (2004) and Petersen (2009).

One typical approach to construct a robust estimator in a panel setting is to use a clustered covariance matrix estimator (CCE) which is proposed by Arellano (1987). Arellano (1987) shows its consistency as $n \to \infty$ with T fixed in the presence of heteroskedasticity and serial correlation. Kèzdi (2003) explores the properties of the CCE in fixed effects panel models and Wooldridge (2003) provides a brief review on this estimator. Hansen (2007) demonstrates that this estimator remains consistent as $n, T \to \infty$ and that it converges to a limiting distribution when $T \to \infty$ with n fixed. A second method to address this problem is to apply time series HAC estimation (Newey and West, 1987) with the cross-sectional averages of orthogonality conditions as suggested by Driscoll and Kraay (1998, DK hereafter). Since their estimator is based on cross-sectional averages of orthogonality conditions, it is robust to very general forms of spatial dependence. They establish its consistency with $T \to \infty$ whether n is fixed or goes to infinity under mixing conditions which restrict the dependence of data. Gonçlaves (2008) examines the properties of the DK estimator in linear panel models with fixed effects. Another approach considered in this paper is to employ spatial HAC estimation with serial averages of orthogonality condition. This is symmetric to DK estimation. Spatial HAC estimation is first proposed by Conley (1996, 1999) and Kelejian and Procha (2007, KP hereafter) state its feasibility to extend to panel models with fixed T.[1]

While those estimators are widely used in panel models, each of them faces a critical restriction. In the case of the CCE, its asymptotic properties rely heavily on the condition that data from different clusters are uncorrelated. In practice, however, this restriction may not be satisfied due to interactions among economic agents (e.g. competition and spillover effect). Even moderate spatial dependence may lead to the substantial bias of the estimator and hence size distortion in the inference.[2] For the DK estimator, large T asymptotics are used and it will not perform well given small T even with large n. This is because the DK estimator does not control the variation from the spatial dimension with downweighting or truncation so that its variance is not reduced as n increases. For the KP estimator, in contrast, large n asymptotics are used and it performs well only when n is large enough regardless of the size of T.

We propose a bivariate kernel estimator, which overcomes these issues. Our estimator is consistent under spatial dependence in contrast to CCE, and its rate of convergence relies on both n and T, which contrasts from the DK and KP estimators. Our study shows that under some regularity conditions our estimator has a faster

[1] Other robust estimators in a panel set-up include heteroskedasticity-robust standard errors (White, 1980; Stock and Watson, 2008) and Fama-MacBeth procedure (Fama and MacBeth, 1973).

[2] Cameron, Gelbach, and Miller (2006) and Thompson (2009) address this problem by clustering on time and spatial dimensions simultaneously. While this method allows for both serial and spatial correlations, observations on different individuals in different time are assumed to be uncorrelated (Peterson, 2009).

convergence rate than the DK and KP estimators unless either T or n is much larger than the other. Provided units are located on a lattice and the Bartlett kernel is used, our estimator is more efficient than the DK estimator if $T = o(n^{3/2})$ and than the KP estimator if $n = o(T^4)$. If we use the second order kernels, such as the Parzen and Tukey kernels, the conditions are more generous, $T = o(n^{5/2})$ and $n = o(T^6)$.

In order to utilize the kernel in the spatial dimension, we employ an *economic distance* which measures the decaying pattern of spatial dependence. The natural assumption is made that the covariance of random variables at locations i and j is a decreasing function of the economic distance between them, d_{ij}. The idea of using economic distance to characterize spatial dependence is common in the spatial econometrics literature. See, for example, Conley (1996, 1999), Pinkse, Slade and Brett (2002), KP, and Kim and Sun (2009).

As our estimator is based on the bivariate kernel, it is flexible in the sense that it includes the existing estimators as special cases, reducing to each of them with certain bandwidth selection. We prove that if d_n increases fast enough, then our estimator with the flat top kernel reduces to the DK estimator. Similarly, if we increase d_T fast enough, then our estimator with the flat top kernel is shown to reduce to the KP estimator. On the other hand, if we choose d_n to be close to zero, our estimator approaches to the CCE.

We derive an optimal bandwidth selection procedure using the upper bound of the asymptotic mean square error (AMSE*) criterion. While the asymptotic mean square error (AMSE) criterion has been the conventional approach in choosing an optimal bandwidth parameter since Andrews (1991) and Newey and West (1994), it is not tractable in the panel setting. In contrast, our criterion is very intuitive and clearly shows the trade-off between the bias and variance of the estimator. It also effectively controls the MSE in terms of its upper bound. It is interesting to note that persistence of a process in each dimension affects both $d_T^\star$ and $d_n^\star$, the optimal bandwidth parameters in the time and spatial dimensions respectively, but in the opposite direction. That is, for example, if a process becomes persistent in the spatial dimension, our procedure does not only increase $d_n^\star$ but also reduces $d_T^\star$. Following Andrews (1991), we provide a parametric plug-in method for the automatic implementation of the optimal bandwidth parameters. We consider four different types of spatio-temporal parametric models in Anselin (2001) for this method.

Due to flexibility and the automatic bandwidth selection procedure, our estimator adapts to the dependence structure of data. It approaches an existing estimator when the latter is favorable. This *adaptiveness* is the salient feature of our estimator. It enables our estimation procedure to be safely used without any knowledge of the dependence structure. This is confirmed by our Monte Carlo study.

The paper that is most closely related to ours is Kim and Sun (2009) who present asymptotic properties of the spatial HAC estimator. They use the linear data representation as in this paper and employ the asymptotic truncated MSE criterion to select the bandwidth parameter. This paper in turn can be traced back to Andrews (1991) and the literature on spectral density estimation (e.g. Parzen, 1957; Hannan, 1970; Robinson, 2007). We extend Kim and Sun (2009) to the panel setting. This extension is both important and nontrivial as, in the panel setting, our bivariate kernel estimation method delivers the aforementioned nice properties and the conventional AMSE criterion is not tranctable.

The remainder of the paper is as follows. Section 2 describes the data representation we consider and introduces our estimator. Section 3 establishes the consistency, the rate of convergence, and the AMSE of the estimator. Section 4 derives asymptotically optimal sequences of fixed bandwidth parameters and proposes their data-dependent implementation. Section 5 derives the asymptotic properties of the existing estimators, i.e. the DK estimator, the CCE and the extension of the KP estimator and demonstrates flexibility. Section 6 illustrates adaptiveness of our estimator. Section 7 presents Monte Carlo simulation results. Section 8 concludes.

2 Panel model and representation

In this paper, we consider a standard linear panel regression model with fixed effects:

$$Y_{it} = X_{it}'\beta + \alpha_i + f_t + u_{it}, \tag{1}$$

where X_{it} and β are p-vectors and α_i and f_t denote scalar individual and time effects respectively. When X_{it} is correlated with α_i and f_t, we may use a fixed effects estimation approach. Let $\bar{Z}_i = T^{-1}\sum_{t=1}^{T} Z_{it}$, $\bar{Z}_t = n^{-1}\sum_{i=1}^{n} Z_{it}$ and $\bar{Z} = (nT)^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T} Z_{it}$. We also define $\tilde{Z}_{it} = Z_{it} - \bar{Z}_i - \bar{Z}_t + \bar{Z}$. Then, the fixed effects estimator, $\hat{\beta}$, is as follows:

$$\hat{\beta} = \left(\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}\tilde{X}_{it}'\right)^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}\tilde{Y}_{it}.$$

Under some regularity conditions, the asymptotic distribution of $\hat{\beta}$ is

$$\left(B_{nT}J_{nT}B_{nT}'\right)^{-\frac{1}{2}}\sqrt{nT}\left(\hat{\beta}-\beta_0\right) \xrightarrow{d} N\left(0, I_p\right) \text{ as } n, T \longrightarrow \infty,$$

where $B_{nT} = \left((nT)^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T} E\left[\tilde{X}_{it}\tilde{X}_{it}'\right]\right)^{-1}$ and $J_{nT} =\text{var}\left((nT)^{-1/2}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}u_{it}\right)$. Since B_{nT} is straightforward to estimate, our central interest is robust estimation of J_{nT} in the presence of an unknown form of heteroskedasticity and spatial and temporal correlation of $\tilde{X}_{it}u_{it}$. Letting $V_{(i,t)} = \tilde{X}_{it}u_{it}$, J_{nT} can be rewritten as

$$\begin{aligned} J_{nT} &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} E\left[V_{(i,t)}V_{(j,s)}'\right] \\ &:= \frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\Gamma_{(it,js)}. \end{aligned} \tag{2}$$

In this paper, we propose a bivariate kernel HAC estimator which is given as

$$\hat{J}_{nT} = \frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)\hat{V}_{(i,t)}\hat{V}_{(j,s)}', \tag{3}$$

where $\hat{V}_{(i,t)} = \tilde{X}_{it}(\tilde{Y}_{it} - \tilde{X}_{it}'\hat{\beta})$ is the estimate of $V_{(i,t)}$ and $K(\cdot)$ is a real-valued kernel function. d_{ij} and d_{ts} denote the distance measures in the spatial and time dimensions and d_n and d_T are their bandwidth choices. We provide further explanation on them later.

We employ the linear transformation of nTp common innovations to represent the process of $V_{(i,t)}$ as follows:

$$V_{(i,t)} = \tilde{R}_{(i,t)}\tilde{\varepsilon}, \tag{4}$$

where

$$\tilde{R}_{(i,t)} = \begin{bmatrix} \left(\tilde{r}^{(1)}_{(it,1,1)}, \tilde{r}^{(1)}_{(it,2,1)}, \ldots, \tilde{r}^{(1)}_{(it,n,T)} \right) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(\tilde{r}^{(p)}_{(it,1,1)}, \tilde{r}^{(p)}_{(it,2,1)}, \ldots, \tilde{r}^{(p)}_{(it,n,T)} \right) \end{bmatrix} \tag{5}$$

is a $p \times nTp$ block diagonal matrix with unknown elements and $\tilde{\varepsilon} = \left(\left(\tilde{\varepsilon}^{(1)}\right)', \ldots, \left(\tilde{\varepsilon}^{(p)}\right)' \right)'$ in which $\tilde{\varepsilon}^{(c)} = \left(\tilde{\varepsilon}^{(c)}_{(1,1)}, \ldots, \tilde{\varepsilon}^{(c)}_{(n,1)}, \tilde{\varepsilon}^{(c)}_{(1,2)}, \ldots, \tilde{\varepsilon}^{(c)}_{(n,T)} \right)'$. As in Kim and Sun (2009), we assume that

$$\text{var}\left(\tilde{\varepsilon}^{(c)}\right) = \sigma_{cc} I_{nT}, \ \text{cov}\left(\tilde{\varepsilon}^{(c)}, \tilde{\varepsilon}^{(d)}\right) = \sigma_{cd} I_{nT}$$

and

$$\text{var}\left(\tilde{\varepsilon}\right) = \Sigma \otimes I_{nT} \ \text{with} \ \Sigma = \left(\sigma_{cd}\right),$$

where $c, d = 1, \ldots, p$ and $\otimes$ denotes the Kronecker product. While the process exhibited in (4) is restricted to a linear array, it allows nonstationarity and unconditional heteroskedasticity of $V_{(i,t)}$. This is in sharp contrast to conventional spectral density estimation literature in which covariance stationarity is generally assumed. Stationarity seems to be a very strong assumption especially in the spatial dimension because a spatial process is nonstationary simply if each unit has different numbers of neighbors. This representation treats time and spatial dimensions in a symmetric way and includes many spatio-temporal parametric models which we use for the automatic bandwidth selection procedure presented in section 4.

Let $R_{(i,t)} := \tilde{R}_{(i,t)} \left(\Sigma^{1/2} \otimes I_{nT} \right)$ and $\varepsilon := \left(\varepsilon_1, \ldots, \varepsilon_l, \ldots, \varepsilon_{nTp} \right)' = \left(\Sigma^{-1/2} \otimes I_{nT} \right) \tilde{\varepsilon}$. Then,

$$V_{(i,t)} = R_{(i,t)}\varepsilon \ \text{and} \ \text{var}\left(\varepsilon\right) = I_{nTp}. \tag{6}$$

The matrix $R_{(i,t)}$ can be written more explicitly as

$$\begin{aligned} R_{(i,t)} &:= \begin{bmatrix} \left(r^{(1)}_{(i,t),1} \quad \cdots \quad r^{(1)}_{(i,t),nTp} \right) \\ \vdots \\ \left(r^{(p)}_{(i,t),1} \quad \cdots \quad r^{(p)}_{(i,t),nTp} \right) \end{bmatrix} \\ &= \begin{bmatrix} \sigma^{11} \left(\tilde{r}^{(1)}_{(it,1,1)} \quad \cdots \quad \tilde{r}^{(1)}_{(it,n,T)} \right) & \cdots & \sigma^{1p} \left(\tilde{r}^{(1)}_{(it,1,1)} \quad \cdots \quad \tilde{r}^{(1)}_{(it,n,T)} \right) \\ \vdots & \ddots & \vdots \\ \sigma^{p1} \left(\tilde{r}^{(p)}_{(it,1,1)} \quad \cdots \quad \tilde{r}^{(p)}_{(it,n,T)} \right) & \cdots & \sigma^{pp} \left(\tilde{r}^{(p)}_{(it,1,1)} \quad \cdots \quad \tilde{r}^{(p)}_{(it,n,T)} \right) \end{bmatrix} \end{aligned}$$

where σ^{cd} denotes the (c,d)-th element of $\Sigma^{1/2}$. We make the following assumption on ε_l.

Assumption 1 *For all* $l = 1, \ldots, nTp$, $\varepsilon_l \overset{i.i.d.}{\sim} (0,1)$ *with* $E\left[\varepsilon_l^4\right] \leq c_E$ *for some constant* $c_E < \infty$.

Under Assumption 1, the covariance matrix of $V_{(i,t)}$ and $V_{(j,s)}$ is given by

$$\Gamma_{(it,js)} := \left(\gamma_{(it,js)}^{(cd)}\right) = E\left[V_{(i,t)} V_{(j,s)}'\right] = R_{(i,t)} R_{(j,s)}', \tag{7}$$

where the (c,d)-th element of $\Gamma_{(it,js)}$ is denoted by $\gamma_{(it,js)}^{(cd)}$. Accordingly, (2) can be restated as

$$J_{nT} = \frac{1}{nT} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{T} R_{(i,t)} R_{(j,s)}', \tag{8}$$

and the (c,d)-th element of J_{nT} is

$$J_{nT}(c,d) = \frac{1}{nT} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{T} \left(\sum_{l=1}^{nTp} r_{(i,t),l}^{(c)} r_{(j,s),l}^{(d)} \right). \tag{9}$$

Assumption 2 *For all* $l = 1, ..., nTp$, $c = 1, ..., p$, n *and* T $\sum_{i=1}^{n} \sum_{t=1}^{T} \left| r_{(i,t),l}^{(c)} \right| < c_R$ *for some constant* c_R, $0 < c_R < \infty$.

Assumption 3 *There exist* $q_1, q_2 > 0$ *such that*

$$\frac{1}{nT} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{s=1}^{T} \sum_{t=1}^{T} \left\| \Gamma_{(it,js)} \right\| d_{ij}^{q_1} < \infty \text{ and } \frac{1}{nT} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{s=1}^{T} \sum_{t=1}^{T} \left\| \Gamma_{(it,js)} \right\| d_{ts}^{q_2} < \infty$$

for all n, T *and where* $\|A\|$ *denotes the Euclidean norm of matrix* A.

Assumptions 2 and 3 impose the conditions on the persistence of the process. If $|\sigma^{cd}| \leq C$ for a constant $C > 0$, then Assumption 2 holds if $\sum_{i=1}^{n} \sum_{t=1}^{T} |\tilde{r}_{(it,j,s)}^{(d)}| < c_R/C$. Since $|\tilde{r}_{(it,j,s)}^{(d)}|$ can be regarded as the (absolute) change of $V_{(i,t)}^{(d)}$ in response to one unit change in $\tilde{\varepsilon}_{(j,s)}^{(d)}$, the summability condition requires that the aggregate response be finite. The condition holds trivially if the set $\{\tilde{r}_{(it,j,s)}^{(d)}, i = 1, ..., n$ and $t = 1, ..., T\}$ has only a finite number of nonzero elements. In this case, the dependence induced by the innovation $\tilde{\varepsilon}_{(j,s)}^{(d)}$ are limited to a finite number of units. Assumption 3 states that $\Gamma_{(it,js)}$ decays to zero fast as d_{ij} and d_{ts} increase so that the two summability conditions holds. By (9), these conditions are equivalent to

$$\frac{1}{nT} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{T} \left| \sum_{a=1}^{n} \sum_{b=1}^{T} \tilde{r}_{(it,a,b)}^{(c)} \tilde{r}_{(js,a,b)}^{(d)} \right| d_{ij}^{q_1} < \infty, \tag{10}$$

$$\frac{1}{nT} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{T} \left| \sum_{a=1}^{n} \sum_{b=1}^{T} \tilde{r}_{(it,a,b)}^{(c)} \tilde{r}_{(js,a,b)}^{(d)} \right| d_{ts}^{q_2} < \infty \tag{11}$$

for all $c, d = 1, \ldots, p$. (10) and (11) imply that if two locations (i, t) and (j, s) are temporally or spatially distant, $\left(\tilde{r}^{(c)}_{(it,1,1)}, \ldots, \tilde{r}^{(c)}_{(it,n,T)}\right)$ and $\left(\tilde{r}^{(c)}_{(js,1,1)}, \ldots, \tilde{r}^{(c)}_{(js,n,T)}\right)$ become nearly orthogonal so that the two summability conditions hold. Assumption 3 enables us to truncate the sum of $\Gamma_{(it,js)}$ and downweigh the summand without incurring much bias.

As stated above, d_{ts} and d_{ij} are the distances in the time and spatial dimensions respectively. d_{ij} is called *economic distance* and widely adapted in spatial econometrics literature. Whereas it is natural to define $d_{ts} = |t - s|$, what is used to measure d_{ij} differs with applications. Geographic distance may be one of the most common measure for economic distance (e.g., Bester, Conley, Hansen and Vogelsang, 2009; Barrios, Diamond, Imbens and Kolesar, 2010), but another measure can also be considered, e.g. transportation cost (Conley and Ligon, 2000) and similarity of input and output structure (Chen and Conley, 2001; Conley and Dupor, 2003). The key property of d_{ij} is that it characterizes the decaying pattern of the spatial dependence. That is, the dependence between $V_{(i,t)}$ and $V_{(j,s)}$ decreases in absolute value as d_{ij} increases for any t and s. In addition, d_{ij} satisfies the properties of distance in a metric space: (i) $d_{ij} \geq 0$, (ii) $d_{ii} = 0$, (iii) $d_{ij} = d_{ji}$, and (iv) $d_{ij} \leq d_{ik} + d_{kj}$. Nonetheless, the symmetry condition (iii) may not hold for some candidates of economic distance. Conley and Ligon (2000), for example, notice that transportation costs among countries violate this condition if tariff barriers are asymmetric. In such a case adjustment should be made.[3] This adjustment should not affect the asymptotic properties of our estimator from a perspective of the measurement error problem as explained below.

Economic distance data available to empirical researchers potentially contains measurement errors and the results in this paper can be generalized to the case when d_{ij} is error contaminated. Following Kim and Sun (2009), we can show that our asymptotic results are valid as long as (i) the measurement error is independent of ε_l; (ii) its proportion to d_n converges to zero as d_n increases; and (iii) the first summability condition in Assumption 3 holds with an error contaminated distance measure. In this paper, however, we do not consider measurement errors for simplicity.

Let

$$\ell_{i,n} = \sum_{j=1}^{n} 1\{d_{ij} \leq d_n\} \text{ and } \ell_n = n^{-1} \sum_{i=1}^{n} \ell_{i,n}.$$

$\ell_{i,n}$ is the number of pseudo-neighbors that unit i has and ℓ_n is the average number of pseudo-neighbors. Here we use the terminology "pseudo-neighbor" in order to differentiate it from the common usage of "neighbor" in spatial modeling. We maintain the following assumption on the number of pseudo-neighbors.

Assumption 4 *For all $i = 1, \ldots, n$, $\ell_{i,n} \leq C\ell_n$ for some constant C.*

Assumption 4 allows the units to be irregularly located but rules out the case that they are concentrated only in some limited area while other area is scarce. To be

[3] In Conley and Ligon (2000), the asymmetry of transportation costs is adjusted by using the minimum cost between two countries.

symmetric, we also define

$$\ell_{t,T} = \sum_{s=1}^{T} 1\{d_{ts} \leq d_T\} \text{ and}$$
$$\ell_T = T^{-1} \sum_{t=1}^{T} \ell_{t,T} = 2d_T + 1 - \frac{d_T(d_T+1)}{T}.$$

In order to obtain the properties of the estimator in Theorem 1 below, it is important to control the boundary effects. That is, the effects of the units in the boundary should become negligible as the sample size increases, so that the asymptotic properties of the estimator depend only on the behavior of the units in the nonboundary. We define

$$E_n := \{i : \ell_{i,n} = \ell_n + o(\ell_n)\},\ n_1 = \sum_{i=1}^{n} 1\{i \in E_n\},\ n_2 = n - n_1$$
$$E_T := \{t : \ell_{t,T} = \ell_T + o(\ell_T)\},\ T_1 = \sum_{t=1}^{n} 1\{t \in E_T\} \text{ and } T_2 = T - T_1.$$

E_n and E_T represent the sets of units in nonboundary in the spatial and time dimensions, and n_1 and n_2 denote the numbers of units in the nonboundary and boundary of the spatial dimension respectively. The number of units in the boundary relies on choice of bandwidth. If d_n and d_T become larger, E_n and E_T shrink, and vice versa. Therefore, we can mitigate the boundary effects by raising d_n and d_T slowly as n and T increase. Provided that n_2/n and T_2/T are $o(1)$, the boundary effects are asymptotically negligible. This is not very restrictive in general. Suppose that units are regularly spaced on a lattice in $\mathbb{R}^2$. Then, the conditions hold with $\ell_n/n = o(1)$ and $d_T/T = o(1)$.

3 Asymptotic properties of the estimator

This section presents the consistency conditions, the rate of convergence, and the AMSE of the estimator. We begin by introducing the assumption on the kernel used in the estimator.

Assumption 5 *(i) The kernel $K : R \to [0,1]$ satisfies $K(0) = 1, K(x) = K(-x), K(x) = 0$ for $|x| \geq 1$. (ii) For all $x_1, x_2 \in R$ there is a constant, $c_L < 0$, such that*

$$|K(x_1) - K(x_2)| \leq c_L |x_1 - x_2|.$$

(iii) $\ell_n^{-1} \sum_{j=1}^{n} K^2\left(\frac{d_{ij}}{d_n}\right) \to \bar{\mathcal{K}}_1$ for all $i \in E_n$.

Examples of kernels which satisfy Assumptions 5 (i) and (ii) are the Bartlett, Tukey-Hanning and Parzen kernels. The quadratic spectral (QS) kernel does not satisfy Assumption 5 (i) because it does not truncate. We may generalized our results to include the QS kernel but this requires a considerable amount of work. Assumption

5 (iii) is more of an assumption on the distribution of the units. In the case of a 2-dimensional lattice, we have

$$\bar{\mathcal{K}}_1 = \frac{1}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}} K^2\left(\sqrt{x^2+y^2}\right)dydx = \int_{-1}^{1} K^2(r)dr$$

In finite samples, we may use

$$\bar{\mathcal{K}}_n = (n\ell_n)^{-1}\sum_{i=1}^{n}\sum_{j=1}^{n} K^2\left(\frac{d_{ij}}{d_n}\right)$$

for $\bar{\mathcal{K}}_1$. For the kernel in time dimension, we define $\bar{\mathcal{K}}_2 = \int_{-1}^{1} K^2(r)dr$.

The asymptotic variance of $\hat{J}_{nT}$ depends on J which is the limit value of J_{nT}.

$$J := \lim_{n\to\infty}\lim_{T\to\infty} J_{nT} = \lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\Gamma_{(it,js)}.$$

Assumption 6 *For $i \in E_n$ and $t \in E_T$,*

$$\lim_{n\to\infty}\lim_{T\to\infty} var\left(\frac{1}{\sqrt{\ell_n\ell_T}}\sum_{j:d_{ij}\leq d_n}\sum_{s:d_{ts}\leq d_T} V_{(j,s)}\right) = J.$$

Assumption 6 states that each covariance matrix which is locally defined in the nonboundary converges to the limiting value of J_{nT}. This assumption is related to covariance stationarity but weaker. It is implied by covariance stationarity but it can hold even though covariance stationarity is violated. It is also robust to irregularity of location of each unit in the spatial dimension in the sense that it is the property of the averages. This assumption is similar to the homogeneity assumption in Bester, Hansen and Conley (2009). They assume that the covariance matrix defined in each group converges to the same limit.

The asymptotic bias of $\hat{J}_{nT}$ is determined by the smoothness of the kernel at zero and the rates of decaying of the spatial and temporal dependences. Define

$$K_{q_0} = \lim_{x\to 0}\frac{1-K(x)}{|x|^{q_0}}, \qquad \text{for } q_0 \in [0,\infty).$$

and let $q = \max\{q_0 : K_{q_0} < \infty\}$ be the *Parzen characteristic exponent* of $K(x)$. The magnitude of q reflects the smoothness of $K(x)$ at $x = 0$. Under the assumption that $q \leq q_i$ with $i = 1, 2$, we define

$$b_1^{(q)} = \lim_{n\to\infty}\lim_{T\to\infty} b_n^{(q)}, \text{ where } b_n^{(q)} = \frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\Gamma_{(it,js)}d_{ij}^q,$$

$$b_2^{(q)} = \lim_{n\to\infty}\lim_{T\to\infty} b_T^{(q)}, \text{ where } b_T^{(q)} = \frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\Gamma_{(it,js)}d_{ts}^q.$$

Next we introduce additional assumptions required to obtain the asymptotic properties of $\hat{J}_{nT}$.

Assumption 7 *(i)* $\sqrt{nT}\left(\hat{\beta}-\beta_0\right)=O_p(1)$. *(ii)* $(nT)^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{t=1}^{T}u_{it}=O_p(1)$ *and* $(nT)^{-\frac{1}{2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}u_{it}=O_p(1)$.

Assumption 7 is rather standard. It excludes the case of strong spatial dependence.

We define the MSE as

$$MSE\left(\frac{nT}{\ell_n d_T},\hat{J}_{nT},S\right)=\frac{nT}{\ell_n d_T}E\left[\mathrm{vec}(\hat{J}_{nT}-J_{nT})'S\mathrm{vec}(\hat{J}_{nT}-J_{nT})\right],$$

where S is some $p^2\times p^2$ weighting matrix and $\mathrm{vec}(\cdot)$ is the column by column vectorization function. We also define $\tilde{J}_{nT}$ as the pseudo-estimator that is identical to $\hat{J}_{nT}$ but is based on the true parameter, β_0, instead of $\hat{\beta}$. That is,

$$\tilde{J}_{nT}=\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)V_{(i,t)}V_{(j,s)}'.$$

Under the assumptions above, the effect of using $\hat{\beta}$ instead of β_0 on the asymptotic property is $o_p(1)$. Therefore, we use $\tilde{J}_{nT}$ to analyze the asymptotic properties of $\hat{J}_{nT}$.

Assumption 8 *For* $i=1,\ldots,p$ $E|\hat{\beta}_i|^2<\infty$, *where* $\hat{\beta}_i$ *is the* i^{th} *element of* $\hat{\beta}$.

Assumption 8 rules out the case when $\hat{\beta}$ has an infinite second moment (Mariano, 1972; Kinal, 1980) which causes the underlying estimation error to dominate the MSE.[4]

Assumption 9 $S_{nT}\xrightarrow{p}S$ *for a positive definite matrix S.*

Let tr denote the trace function and $\mathbb{K}_{pp}$ the $p^2\times p^2$ commutation matrix. Under the assumptions above, we have the following theorem.

Theorem 1 *Suppose that Assumptions 1-6 hold,* $d_n,d_T\to\infty$, $n_2(d_n)=o(n)$, $d_T=o(T)$ *and* $\ell_n d_T=o(nT)$.

(a) $\lim_{n\to\infty}\lim_{T\to\infty}\frac{nT}{\ell_n d_T}var\left(vec\tilde{J}_{nT}\right)=\bar{\mathcal{K}}_1\bar{\mathcal{K}}_2(I_{pp}+\mathbb{K}_{pp})(J\otimes J)$.

(b) *Let* $k_{nT}=d_T/d_n$ *and* $k_{nT}\to k>0$ *as* $n,T\to\infty$. *Then,* $\lim_{n\to\infty}\lim_{T\to\infty}d_n^q(E\tilde{J}_{nT}-J_{nT})=-K_q\left(b_1^{(q)}+\frac{1}{k^q}b_2^{(q)}\right)$

(c) *If Assumption 7 holds and* $d_n^{2q}\ell_n d_T/nT\to\tau\in(0,\infty)$, *then* $\sqrt{\frac{nT}{\ell_n d_T}}\left(\hat{J}_{nT}-J_{nT}\right)=O_p(1)$ *and* $\sqrt{\frac{nT}{\ell_n d_T}}\left(\hat{J}_{nT}-\tilde{J}_{nT}\right)=o_p(1)$.

(d) *Under the conditions of part (c), Assumptions 8 and 9,*

$$\begin{aligned}&\lim_{n\to\infty}\lim_{T\to\infty}MSE\left(\frac{nT}{\ell_n d_T},\hat{J}_{nT},S_{nT}\right)\\&=\lim_{n\to\infty}\lim_{T\to\infty}MSE\left(\frac{nT}{\ell_n d_T},\tilde{J}_{nT},S\right)\\&=\frac{1}{\tau}K_q^2 vec\left(b_1^{(q)}+\frac{1}{k^q}b_2^{(q)}\right)'Svec\left(b_1^{(q)}+\frac{1}{k^q}b_2^{(q)}\right)+\bar{\mathcal{K}}_1\bar{\mathcal{K}}_2 tr\left(S(I+\mathbb{K}_{pp})(J\otimes J)\right).\end{aligned}$$

[4] Instead of introducing Assumption 8, we can consider asymptotic truncated MSE as Andrews (1991) and Kim and Sun (2009).

Proofs are given in the appendix. For each element of $\tilde{J}_{nT}$, the asymptotic variance in Theorem 1 (a) is rewritten as

$$\begin{aligned}&\lim_{n\to\infty}\lim_{T\to\infty}\frac{nT}{\ell_n d_T}\mathrm{cov}\left(\tilde{J}_{nT}\left(c_1,d_1\right),\tilde{J}_{nT}\left(c_2,d_2\right)\right)\\&=\bar{\mathcal{K}}_1\bar{\mathcal{K}}_2\left[J\left(c_1,c_2\right)J\left(d_1,d_2\right)+J\left(c_1,d_2\right)J\left(d_1,c_2\right)\right].\end{aligned}$$

Theorem 1 (a) and (b) show that the asymptotic variance and bias of $\tilde{J}_{nT}$ depend on the choice of d_n and d_T. When we enlarge d_n and/or d_T, the bias decreases while the variance increases and vice versa.

The second part of Theorem 1 (c) states that, in comparison with the variance term in part (a), the effect of using $\hat{V}_{(i,t)}$ instead of $V_{(i,t)}$ in the construction of $\hat{J}_{nT}$ is of a smaller order. Therefore, the rate of convergence is obtained by balancing the variance and the squared bias. Accordingly, $\ell_n d_T = o(nT)$ is the condition for the consistency of $\hat{J}_{nT}$ and its rate of convergence is $\sqrt{nT/\ell_n d_T}$ $(= O(d_n^q))$. It is also required that $T_2\left(d_T\right)=o\left(T\right)$ and $n_2\left(d_n\right)=o\left(n\right)$ to control the effects of boundary. If we assume that $\ell_n = O(d_n^\eta)$ for some $\eta > 0$, then the rate of convergence with $d_n^{2q}\ell_n d_T/nT \to \tau \in (0,\infty)$ is rewritten as $(nT)^{q/(2q+\eta+1)}$.

4 Optimal bandwidth choice and its automatic implementation

This section presents sequences of optimal fixed bandwidth parameters in the sense of minimizing the upper bound of AMSE of $\hat{J}_{nT}$ and proposes the automatic implementation procedure. Let

$$\begin{aligned}B_{11}&:=\mathrm{vec}\left(b_1^{(q)}\right)'S_{nT}\mathrm{vec}\left(b_1^{(q)}\right),\\B_{22}&:=\mathrm{vec}\left(b_2^{(q)}\right)'S_{nT}\mathrm{vec}\left(b_2^{(q)}\right),\\B_{12}&:=\mathrm{vec}\left(b_1^{(q)}\right)'S_{nT}\mathrm{vec}\left(b_2^{(q)}\right).\end{aligned}$$

Then, up to smaller order terms

$$AMSE = K_q^2\left(\frac{B_{11}}{d_n^{2q}}+2\frac{B_{12}}{d_n^q d_T^q}+\frac{B_{22}}{d_T^{2q}}\right)+\frac{\ell_n d_T}{nT}\bar{\mathcal{K}}_1\bar{\mathcal{K}}_2\mathrm{tr}\left[S_{nT}\left(I_{pp}+\mathbb{K}_{pp}\right)\left(J\otimes J\right)\right] \tag{12}$$

$$\leq 2K_q^2\left(\frac{B_{11}}{d_n^{2q}}+\frac{B_{22}}{d_T^{2q}}\right)+\frac{\ell_n d_T}{nT}\bar{\mathcal{K}}_1\bar{\mathcal{K}}_2\mathrm{tr}\left[S_{nT}\left(I_{pp}+\mathbb{K}_{pp}\right)\left(J\otimes J\right)\right] \tag{13}$$

$$:= AMSE^*,$$

where we have used the Cauchy inequality

$$2\frac{B_{12}}{d_n^q d_T^q}\leq\frac{B_{11}}{d_n^{2q}}+\frac{B_{22}}{d_T^{2q}}.$$

AMSE* can be regarded as AMSE in the worst case:

$$AMSE^* = \max_{(b_1,b_2)\in\mathfrak{B}} AMSE,$$

where

$$\mathfrak{B} = \left\{ (b_1, b_2) : \text{vec}\left(b_1^{(q)}\right)' S_{nT} \text{vec}\left(b_1^{(q)}\right) = B_{11}, \right.$$
$$\left. \text{vec}\left(b_2^{(q)}\right)' S_{nT} \text{vec}\left(b_2^{(q)}\right) = B_{22} \right\}.$$

We select $(d_n^\star, d_T^\star)$ to minimize AMSE*:

$$\begin{aligned}(d_n^\star, d_T^\star) &= \arg\min_{d_n, d_T} AMSE^* \\ &= \arg\min_{d_n, d_T} 2K_q^2 \left(\frac{B_{11}}{d_n^{2q}} + \frac{B_{22}}{d_T^{2q}} \right) + \frac{\ell_n d_T}{nT} Q, \end{aligned} \tag{14}$$

where $Q = \text{tr}\left[S_{nT}(I_{pp} + \mathbb{K}_{pp})(J \otimes J)\right]$. Here, we use the AMSE* instead of the AMSE as a criterion. Actually the AMSE has been commonly considered in HAC estimation literature since Andrews (1991) and Newey and West (1994). In our setup, though, this criterion is intractable to apply for an automatic bandwidth selection procedure. Suppose $B_{12} = -\sqrt{B_{11}B_{22}}$. This may occur in the case that we are interested in only one component of β. In this case, the first order bias term can be cancelled out with $d_n^q/d_T^q = \sqrt{B_{11}/B_{22}}$. In practice, however, it is not feasible because B_{11}/B_{22} is an unknown value and estimating this ratio generates a bias. Due to this bias, we cannot eliminate the first order bias term to achieve the faster rate of convergence. We may derive an bandwidth parameters by considering this estimation bias, but such a choice is not feasible since the estimation bias is very hard to estimate. In contrast, the AMSE* criterion is simple to implement because the trade-off between the bias and variance terms is straightforward. This criterion also effectively controls the MSE in terms of the upper bound.

(14) shows that persistence in one dimension affects both $d_n^\star$ and $d_T^\star$ but in the opposite direction. For example, if a process becomes spatially persistent, $d_n^\star$ is raised to address the increase of the bias which comes from the usage of kernel in the spatial dimension. But, the increase of $d_n^\star$, at the same time, magnifies the variance term. Therefore, in order to minimize AMSE*, $d_T^\star$ is decreased to moderate the increase of asymptotic variance. Figure 1 also illustrates the relation of $d_n^\star$ and $d_n^\star$ with dependence structure. The two graphs are the level curves of $d_n^\star$ and $d_T^\star$ as functions of λ and ρ which determine the spatial and serial persistence respectively in the following DGP:

$$V_t = \lambda V_{t-1} + u_t,\ u_t = \rho W_n u_t + \varepsilon_t \text{ and } \varepsilon_t \sim (0, I_n),$$

where V_t, u_t and ε_t are n-vectors such as $V_t = \left(V_{(1,t)}, V_{(2,t)}, \ldots, V_{(n,t)}\right)'$ and W_n is a spatial weight matrix. They indicate that $d_n^\star$ increases as spatial dependence increases or serial dependence decreases and that $d_T^\star$ increases as serial dependence grows or spatial dependence is reduced.

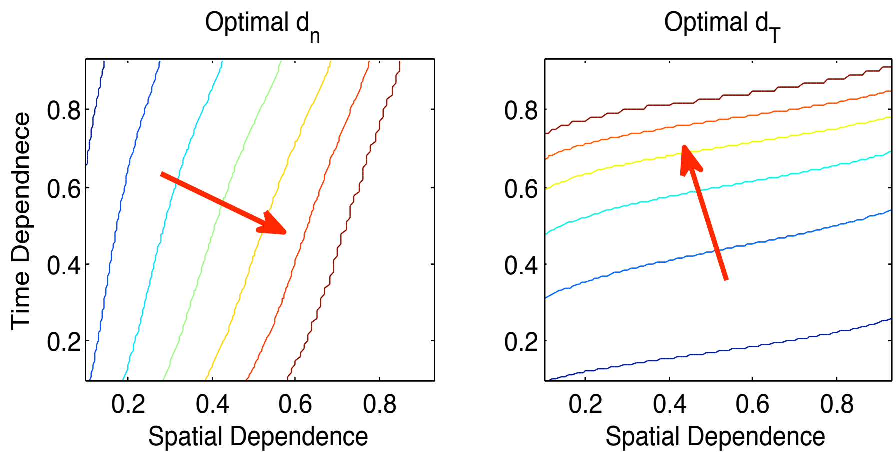


Figure 1 – Level curves of $d_n^\star$ and $d_n^\star$ as functions of Spatial and Time Dependences

In some cases, it is possible to approximate ℓ_n with a function of d_n. For example, if units are on a two dimensional lattice, $\ell_n = \pi d_n^2$ would be reasonable characterization. By using the specification of $\ell_n = \alpha_n d_n^\eta$, (14) can be rewritten as

$$d_n^\star = \left(\frac{B_{11}}{\eta B_{22}}\right)^{\frac{1}{2q(2q+\eta+1)}} \left(\frac{4qK_q^2 B_{11}}{\eta\alpha_n \bar{\mathcal{K}}_1 \bar{\mathcal{K}}_2 Q} nT\right)^{\frac{1}{2q+\eta+1}}, \tag{15}$$

$$d_T^\star = \left(\frac{\eta B_{22}}{B_{11}}\right)^{\frac{\eta}{2q(2q+\eta+1)}} \left(\frac{4qK_q^2 B_{22}}{\alpha_n \bar{\mathcal{K}}_1 \bar{\mathcal{K}}_2 Q} nT\right)^{\frac{1}{2q+\eta+1}}, \tag{16}$$

and

$$\frac{d_T^\star}{d_n^\star} = \left(\frac{\eta B_{22}}{B_{11}}\right)^{\frac{1}{2q}}.$$

Corollary 1 *Suppose Assumptions 1-9 hold. Assume that $\ell_n = \alpha_n d_n^\eta$ for some $\eta > 0$, $\alpha_n = \alpha + o(1)$. Then, for any sequence of bandwidth parameters $\{d_n, d_T\}$ such that $d_n^{2q+\eta} d_T / nT \to \tau \in (0, \infty)$, $\{d_n^\star, d_T^\star\}$ is preferred in the sense that*

$$\lim_{n\to\infty}\lim_{T\to\infty}\left[\max_{(b_1,b_2)\in\mathfrak{B}} MSE\left((nT)^{\frac{2q}{2q+\eta+1}}, \hat{J}_{nT}(d_n,d_T), S_{nT}\right) - \max_{(b_1,b_2)\in\mathfrak{B}} MSE\left((nT)^{\frac{2q}{2q+\eta+1}}, \hat{J}_{nT}(d_n^\star,d_T^\star), S_{nT}\right)\right] \geq 0.$$

The inequality is strict unless $d_n = d_n^\star + o\left((nT)^{1/(2q+\eta+1)}\right)$ and $d_T = d_T^\star + o\left((nT)^{1/(2q+\eta+1)}\right)$.

As (15) and (16) are the functions of unknown values B_{11}, B_{22} and Q, they need to be estimated for implementation with given data in a parametric (Andrews, 1991) or nonparametric way (e.g. Newey and West, 1994). In this paper, we suggest a parametric plug-in method. We consider the following four different spatio-temporal parametric models which are demonstrated in Anselin (2001).

$$V_{(i,t)}^{(c)} = \rho_c \left[W_n^{(c)} V_{t-1}^{(c)} \right]_i + \tilde{\varepsilon}_{(i,t)}^{(c)}, \tag{17}$$

$$V_{(i,t)}^{(c)} = \lambda_c V_{(i,t-1)}^{(c)} + \rho_c \left[W_n^{(c)} V_{t-1}^{(c)} \right]_i + \tilde{\varepsilon}_{(i,t)}^{(c)} \tag{18}$$

$$V_{(i,t)}^{(c)} = \lambda_c V_{(i,t-1)}^{(c)} + \phi_c \left[W_n^{(c)} V_t^{(c)} \right]_i + \tilde{\varepsilon}_{(i,t)}^{(c)} \tag{19}$$

$$V_{(i,t)}^{(c)} = \lambda_c V_{(i,t-1)}^{(c)} + \phi_c \left[W_n^{(c)} V_t^{(c)} \right]_i + \rho_c \left[W_n^{(c)} V_{t-1}^{(c)} \right]_i + \tilde{\varepsilon}_{(i,t)}^{(c)} \tag{20}$$

where $\tilde{\varepsilon}_{(i,t)}^{(c)} \overset{i.i.d}{\sim} (0, \sigma_{cc})$ and $[W_n^{(c)} V_t^{(c)}]_i$ is the i^{th} element of vector $W_n^{(c)} V_t^{(c)}$. The spatial weight matrix $W_n^{(c)}$ is determined a priori and by convention it is row-standardized and its diagonal elements are zero.

For an illustrative purpose, let's consider the model in (17). It can be rewritten in a recursive way as follows:

$$\begin{aligned}
V_1^{(c)} &= \rho_c W_n^{(c)} V_0^{(c)} + I_n \tilde{\varepsilon}_1^{(c)} \\
V_2^{(c)} &= \rho_c^2 \left(W_n^{(c)} \right)^2 V_0^{(c)} + \rho_c W_n^{(c)} \tilde{\varepsilon}_1^{(c)} + I_n \tilde{\varepsilon}_2^{(c)} \\
&\vdots \\
V_T^{(c)} &= \rho_c^T \left(W_n^{(c)} \right)^T V_0^{(c)} + \rho_c^{T-1} \left(W_n^{(c)} \right)^{T-1} \tilde{\varepsilon}_1^{(c)} + \rho_c^{T-2} \left(W_n^{(c)} \right)^{T-2} \tilde{\varepsilon}_2^{(c)} + \ldots + I_n \tilde{\varepsilon}_T^{(c)}
\end{aligned}$$

By imposing the initial condition of $V_0 = 0$, ρ_c is estimated by OLS with $\hat{V}_t^{(c)} = (\hat{V}_{(1,t)}^{(c)}, \ldots, \hat{V}_{(n,t)}^{(c)})'$. We define

$$\hat{\tilde{R}}_{ts}^{(c)} = \begin{cases} I_n, & \text{if } t - s = 0 \\ \left(\hat{\rho}_c W_n^{(c)} \right)^{t-s}, & \text{if } t - s > 0 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\hat{\tilde{R}}_{(i,t)}^{(c)} = \left[\hat{\tilde{R}}_{t1,i}^{(c)}, \hat{\tilde{R}}_{t2,i}^{(c)}, \ldots, \hat{\tilde{R}}_{tT,i}^{(c)} \right], \tag{21}$$

where $\hat{\tilde{R}}_{ts,i}^{(c)}$ is the i-th row of $\hat{\tilde{R}}_{ts}^{(c)}$. Consequently, we approximate J, $b_1^{(q)}$ and $b_2^{(q)}$ by

$$\hat{J}(c,d) = \frac{\hat{\sigma}_{cd}}{nT} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{T} \left(\hat{\tilde{R}}_{(i,t)}^{(c)} \right) \left(\hat{\tilde{R}}_{(j,s)}^{(d)} \right)', \tag{22}$$

$$\hat{b}_1^{(q)}(c,d) = \frac{\hat{\sigma}_{cd}}{nT} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{T} \left(\hat{\tilde{R}}_{(i,t)}^{(c)} \right) \left(\hat{\tilde{R}}_{(j,s)}^{(d)} \right)' d_{ij}^q, \tag{23}$$

$$\hat{b}_2^{(q)}(c,d) = \frac{\hat{\sigma}_{cd}}{nT} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{T} \left(\hat{\tilde{R}}_{(i,t)}^{(c)} \right) \left(\hat{\tilde{R}}_{(j,s)}^{(d)} \right)' d_{ts}^q, \tag{24}$$

where

$$\hat{\sigma}_{cd} = \frac{1}{n(T-1)} \left(\hat{\varepsilon}^{(c)}\right)' \left(\hat{\varepsilon}^{(d)}\right),$$

$\hat{\varepsilon}^{(c)} = ((\hat{\varepsilon}_1^{(c)})', ..., (\hat{\varepsilon}_T^{(c)})')'$, $\hat{\varepsilon}_1^{(c)} = \hat{V}_1^{(c)}$ and $\hat{\varepsilon}_t^{(c)} = \hat{V}_t^{(c)} - \hat{\rho}_c W_n^{(c)} \hat{V}_{t-1.}^{(c)}$ for $t \geq 2$. Substituting these estimators into (14) for the true parameters we obtain the data dependent bandwidth parameters, $(\hat{d}_n, \hat{d}_T)$ as follows:

$$\left(\hat{d}_n, \hat{d}_T\right) = \underset{d_n, d_T}{\arg\min}\ 2K_q^2 \left(\frac{\hat{B}_{11}}{d_n^{2q}} + \frac{\hat{B}_{22}}{d_T^{2q}}\right) + \frac{\ell_n d_T}{nT} \hat{Q}. \tag{25}$$

where

$$\begin{aligned}
\hat{B}_{11} &= \text{vec}\left(\hat{b}_1^{(q)}\right)' S_{nT} \text{vec}\left(\hat{b}_1^{(q)}\right), \\
\hat{B}_{22} &= \text{vec}\left(\hat{b}_2^{(q)}\right)' S_{nT} \text{vec}\left(\hat{b}_2^{(q)}\right), \\
\hat{Q} &= \bar{\mathcal{K}}_1 \bar{\mathcal{K}}_2 \text{tr}\left[S_{nT}(I + \mathbb{K}_{pp})(\hat{J} \otimes \hat{J})\right].
\end{aligned}$$

Correspondingly, using the specification of $\ell_n = \alpha_n d_n^{\eta}$, delivers the explicit forms of $\hat{d}_n$ and $\hat{d}_T$ as

$$\hat{d}_n = \left(\frac{\hat{B}_{11}}{\eta \hat{B}_{22}}\right)^{\frac{1}{2q}} \left(\frac{4qK_q^2 \hat{B}_{11}}{\eta \alpha_n \bar{\mathcal{K}}_1 \bar{\mathcal{K}}_2 \hat{Q}} nT\right)^{\frac{1}{2q+\eta+1}}, \tag{26}$$

$$\hat{d}_T = \left(\frac{\eta \hat{B}_{22}}{\hat{B}_{11}}\right)^{\frac{\eta}{2q}} \left(\frac{4qK_q^2 \hat{B}_{22}}{\alpha_n \bar{\mathcal{K}}_1 \bar{\mathcal{K}}_2 \hat{Q}} nT\right)^{\frac{1}{2q+\eta+1}}. \tag{27}$$

Since the models in (18), (19) and (20) are restated as

$$\begin{aligned}
V_{(i,t)}^{(c)} &= \left[\left(\lambda_c I_n + \rho_c W_n^{(c)}\right) V_{t-1}^{(c)}\right]_i + \tilde{\varepsilon}_{it}^{(c)}, \\
V_{(i,t)}^{(c)} &= \left[\lambda_c \left(I_n - \phi_c W_n^{(c)}\right)^{-1} V_{t-1}^{(c)}\right]_i + \left[\left(I_n - \phi_c W_n^{(c)}\right)^{-1} \tilde{\varepsilon}_t^{(c)}\right]_i, \\
V_{(i,t)}^{(c)} &= \left[\left(I_n - \phi_c W_n^{(c)}\right)^{-1} \left(\lambda_c I_n + \rho_c W_n^{(c)}\right) V_{t-1}^{(c)}\right]_i + \left[\left(I_n - \phi_c W_n^{(c)}\right)^{-1} \tilde{\varepsilon}_t^{(c)}\right]_i,
\end{aligned}$$

we can derive the data dependent bandwidth parameters with these models in similar procedures as (17). While the OLS estimator is consistent for (18), it is not for ϕ_c in (19) and (20) due to endogeneity of $[W_n^{(c)} V_t^{(c)}]_i$. For these models, we can achieve consistency using QMLE as follows:

$$\left(\hat{\lambda}_c, \hat{\phi}_c, \hat{\rho}_c, \hat{\sigma}_{cc}\right) = \underset{\lambda_c, \phi_c, \rho_c, \sigma_{cc}}{\arg\min} \frac{1}{2} \ln \sigma_{cc} - \frac{1}{n} \ln \left| I_n - \phi_c W_n^{(c)} \right| + \frac{1}{2\sigma_{cc}} \frac{1}{nT} \sum_{t=1}^{T} \left(\hat{\varepsilon}_t^{(c)}\right)' \left(\hat{\varepsilon}_t^{(c)}\right).$$

See Yu, de Jong and Lee (2008) for detail. In fact, however, a simple OLS estimator can be also used for (19) and (20). If (19) and (20) are the true data generating process, then the OLS estimator is inconsistent while the QML estimator is consistent. Since the parametric models are most likely to be mis-specified, the QML estimator is not necessarily preferred. In addition, as argued by Andrews (1991), good performance of the estimator only requires $(\hat{d}_n, \hat{d}_T)$ to be near the optimal bandwidth values and not to be precisely equal to them. Furthermore, OLS estimation has much computational advantage. Our simulation results confirm this.

5 Asymptotic properties of CCE, DK and KP estimators

In this section, we examine the asymptotic properties of the CCE, DK and KP estimators based on our data representation in (4) and (6). We also derive the optimal bandwidth parameters for DK and KP estimators using the AMSE criterion.

5.1 CCE

The CCE is defined as

$$\begin{aligned}\hat{J}_{nT}^{A} &:= \frac{1}{n}\sum_{i=1}^{n}\hat{V}_{Ti}\hat{V}_{Ti}' \\ &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\hat{V}_{(i,t)}\hat{V}_{(i,s)}'. \end{aligned} \tag{28}$$

The critical condition for this estimator to be consistent is that each variable from two different units (or clusters) is uncorrelated, i.e. $EV_{(i,t)}V_{(j,s)}' = 0$ if $i \neq j$. Under this condition, $\hat{J}_{nT}^{A}$ is robust to heteroskedasticity and a general form of serial correlation. Our data representation accommodates spatial independence by imposing the following restriction.

Assumption 10 $\tilde{r}_{(it,j,s)} = 0$ *if* $i \neq j$.

Assumption 10 implies that the elements of $R_{(i,t)}$ which are corresponding to $\tilde{r}_{(it,j,s)}$ with $i \neq j$ are also zero. Under Assumption 10,

$$\begin{aligned}J_{nT} &= \frac{1}{n}\sum_{i=1}^{n}E\left[V_{Ti}V_{Ti}'\right] \\ &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}E\left[V_{(i,t)}V_{(i,s)}'\right] \\ &:= J_{nT}^{A},\end{aligned}$$

where $V_{Ti} = T^{-1/2}\sum_{t=1}^{T}V_{(i,t)}$.

Assumption 11 *For all i,*

$$\begin{aligned}\lim_{T\to\infty} var\left(V_{Ti}\right) &= \lim_{n\to\infty}\lim_{T\to\infty} var\left(\frac{1}{\sqrt{nT}}\sum_{j=1}^{n}\sum_{s=1}^{T}V_{(j,s)}\right)\\ &= \lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{n}\sum_{j=1}^{n} var\left(\frac{1}{\sqrt{T}}\sum_{s=1}^{T}V_{(j,s)}\right)\\ &= J.\end{aligned}$$

Assumption 11 implies homoskedasticity of V_{Ti}, under which we can derive the asymptotic variance of $\tilde{J}^{A}_{nT}$ in Theorem 2 (b) below.

Theorem 2 *Suppose that Assumptions 1, 2, 10 and 11 hold.*

(a) $n \cdot var\left(vec\tilde{J}^{A}_{nT}\right) = O\left(1\right).$

(b) $\lim_{n\to\infty}\lim_{T\to\infty} n \cdot var\left(vec\tilde{J}^{A}_{nT}\right) = \left(I_{pp} + \mathbb{K}_{pp}\right)\left(J\otimes J\right).$

(c) *If Assumption 7 holds, then* $\sqrt{n}\left(\hat{J}^{A}_{nT} - J^{A}_{nT}\right) = O_p(1)$ *and* $\sqrt{n}\left(\hat{J}^{A}_{nT} - \tilde{J}^{A}_{nT}\right) = o_p(1)$.

Proofs are given in the appendix. Theorem 2 (a) and (c) imply $\sqrt{n}$-convergence of $\hat{J}^{A}_{nT}$ as $n, T \to \infty$, which is the same result as Hansen (2007).

5.2 DK estimator

The DK estimator is based on the time series HAC estimation method with cross-sectional averages. Let $V_{nt} = n^{-1/2}\sum_{i=1}^{n} V_{(i,t)}$. Then, the estimator is defined as

$$\begin{aligned}\hat{J}^{DK}_{nT} &:= \frac{1}{T}\sum_{t=1}^{T}\sum_{s=1}^{T} K\left(\frac{d_{ts}}{d_T}\right)\hat{V}_{nt}\hat{V}'_{ns}\\ &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} K\left(\frac{d_{ts}}{d_T}\right)\hat{V}_{(i,t)}\hat{V}'_{(j,s)},\end{aligned} \tag{29}$$

where $\hat{V}_{nt}$ is the estimator of V_{nt}. For the asymptotic properties, we introduce the following assumptions instead of Assumptions 3 and 6.

Assumption 12 *There exists $q_d > 0$ such that*

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{T}\sum_{t=1}^{T}\left\|\Gamma_{(it,js)}\right\| d_{ts}^{q_d} < \infty$$

for all n, T.

Assumption 13 *For $t \in E_T$,*

$$\lim_{n\to\infty}\lim_{T\to\infty} var\left(\frac{1}{\sqrt{\ell_T}}\sum_{s:d_{ts}\le d_T} V_{ns}\right) = \lim_{n\to\infty}\lim_{T\to\infty} var\left(\frac{1}{\sqrt{n\ell_T}}\sum_{j=1}^{n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}\right)$$
$$= J.$$

Compared with Assumption 3, Assumption 12 is sufficient for $\hat{J}_{nT}^{DK}$ because $\hat{J}_{nT}^{DK}$ is not involved with the bias caused by the usage of a kernel in the spatial dimension. Theorem 3 below states the asymptotic properties of $\hat{J}_T^{DK}$.

Theorem 3 *Suppose that Assumptions 1, 2, 5(i) and (ii), 12 and 13 hold, and $d_T \to \infty$, $d_T = o(T)$.*

(a) $\lim_{n\to\infty}\lim_{T\to\infty}\frac{T}{d_T} var\left(vec\tilde{J}_{nT}^{DK}\right) = \bar{\mathcal{K}}_2(I_{pp}+\mathbb{K}_{pp})(J\otimes J)$.

(b) $\lim_{n\to\infty}\lim_{T\to\infty} d_T^q(E\tilde{J}_{nT}^{DK} - J_{nT}) = -K_q b_2^{(q)}$

(c) *If Assumption 7 holds and $d_T^{2q+1}/T \to \tau \in (0,\infty)$, then $\sqrt{\frac{T}{d_T}}\left(\hat{J}_{nT}^{DK} - J_{nT}\right) = O_p(1)$ and $\sqrt{\frac{T}{d_T}}\left(\hat{J}_{nT}^{DK} - \tilde{J}_{nT}^{DK}\right) = o_p(1)$.*

(d) *Under the conditions of part (c) and Assumption 9,*

$$\begin{aligned}&\lim_{n\to\infty}\lim_{T\to\infty} MSE\left(\frac{T}{d_T}, \hat{J}_{nT}^{DK}, S_{nT}\right)\\ &= \lim_{n\to\infty}\lim_{T\to\infty} MSE\left(\frac{T}{d_T}, \tilde{J}_{nT}^{DK}, S\right)\\ &= \frac{1}{\tau}K_q^2\left(vec\ b_2^{(q)}\right)' S\left(vec\ b_2^{(q)}\right) + \bar{\mathcal{K}}_2 tr\left[S(I_{pp}+\mathbb{K}_{pp})(J\otimes J)\right].\end{aligned}$$

Proofs are given in the appendix. Theorem 3 (a) and (b) imply that the consistency condition is $d_T \to \infty$ and $d_T = o(T)$. The rate of convergence obtained by balancing the variance and the squared bias is $T^{q/(2q+1)}$. The optimal bandwidth parameter of $\hat{J}_{nT}^{DK}$ based on AMSE is

$$d_T^{DK} = \left(\frac{2qK_q^2 B_{22}}{\bar{\mathcal{K}}_2 Q} T\right)^{\frac{1}{2q+1}}. \tag{30}$$

In contrast to the optimal bandwidth parameters in $\hat{J}_{nT}$, the effect of spatial dependence on d_T^{DK} is not obvious because it affects $b_2^{(q)}$ and J in the same direction. We can use plug-in methods for its implementation.

5.3 KP estimator

For the purpose of symmetry, we can also consider the usage of spatial HAC estimation with the averages across time especially when n is large. The KP estimator with the serial averages is

$$\begin{aligned}\hat{J}_{nT}^{KP} &:= \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} K\left(\frac{d_{ij}}{d_n}\right)\hat{V}_{Ti}\hat{V}_{Tj}' \\ &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} K\left(\frac{d_{ij}}{d_n}\right)\hat{V}_{(i,t)}\hat{V}_{(j,s)}'. \qquad (31)\end{aligned}$$

where $\hat{V}_{Ti}$ is the estimate of $V_{Ti} = T^{-1/2}\sum_{t=1}^{T} V_{(i,t)}$.

Assumption 14 *There exists $q_1 > 0$ such that*

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{T}\sum_{t=1}^{T}\left\|\Gamma_{(it,js)}\right\| d_{ij}^{q_1} < \infty$$

for all n, T.

Assumption 15 *For $i \in E_n$,*

$$\begin{aligned}\lim_{n\to\infty}\lim_{T\to\infty} var\left(\frac{1}{\sqrt{\ell_n}}\sum_{j:d_{ij}\le d_n} V_{Ti}\right) &= \lim_{n\to\infty}\lim_{T\to\infty} var\left(\frac{1}{\sqrt{\ell_n T}}\sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T} V_{(j,s)}\right) \\ &= J.\end{aligned}$$

Theorem 4 below states the asymptotic properties of $\hat{J}_{nT}^{KP}$.

Theorem 4 *Suppose that Assumptions 1, 2, 4, 5, 14 and 15 hold, $n_2/n \to 0$, $\ell_n, d_n \to \infty$ and $\ell_n/n \to 0$.*

(a) $\lim_{n\to\infty}\lim_{T\to\infty}\frac{n}{\ell_n} var\left(vec\tilde{J}_{nT}^{KP}\right) = \bar{\mathcal{K}}_1(I_{pp} + \mathbb{K}_{pp})(J\otimes J)$.

(b) $\lim_{n\to\infty}\lim_{T\to\infty} d_n^q(E\tilde{J}_{nT}^{KP} - J_{nT}) = -K_q b_1^{(q)}$

(c) *If Assumption 7 holds and $d_n^{2q}\ell_n/n \to \tau \in (0,\infty)$, then $\sqrt{\frac{n}{\ell_n}}\left(\hat{J}_{nT}^{KP} - J_{nT}\right) = O_p(1)$ and $\sqrt{\frac{n}{\ell_n}}\left(\hat{J}_{nT}^{KP} - \tilde{J}_{nT}^{KP}\right) = o_p(1)$.*

(d) *Under the conditions of part (c) and Assumption 9,*

$$\begin{aligned}&\lim_{n\to\infty}\lim_{T\to\infty} MSE\left(\frac{n}{\ell_n}, \hat{J}_{nT}^{KP}, S_{nT}\right) \\ &= \lim_{n\to\infty}\lim_{T\to\infty} MSE\left(\frac{n}{\ell_n}, \tilde{J}_{nT}^{KP}, S\right) \\ &= \frac{1}{\tau}K_q^2 vec\left(b_1^{(1)}\right)' Svec\left(b_1^{(q)}\right) + \bar{\mathcal{K}}_1 tr\left[S(I_{pp} + \mathbb{K}_{pp})(J\otimes J)\right].\end{aligned}$$

Proofs are given in the appendix. If we can characterize $\ell_n = \alpha_n d_n^\eta$, the optimal bandwidth based on the AMSE criterion is

$$d_n^{\star KP} = \left(\frac{2qK_q^2 B_{11}}{\alpha_n \bar{\mathcal{K}}_1 Q} n \right)^{\frac{1}{2q+\eta}}. \tag{32}$$

5.4 Comparison with $\hat{J}_{nT}$

(1) Efficiency

Comparison of the rate of convergence among the estimators shows that under regularity conditions $\hat{J}_{nT}$ is more efficient than $\hat{J}_{nT}^{DK}$ and $\hat{J}_{nT}^{KP}$ unless either n or T increases much faster than the other. More specifically, under Assumptions 1-9, $\hat{J}_{nT}$ achieves the faster convergence rate than $\hat{J}_{nT}^{DK}$ if $T = o(n^{(2q+1)/\eta})$ and than $\hat{J}_{nT}^{KP}$ if $n = o(T^{2q+\eta})$.

The idea of kernel methods is attempting to reduce the variance at the expense of producing the bias of the estimator. The bias can be controlled by raising the bandwidths as the sample size increases. As $\hat{J}_{nT}$ employs kernels in both the dimensions, we can choose the sequence of $(\hat{d}_n, \hat{d}_T)$ to reduce the variance without incurring much bias so that the gain from decreasing the variance dominates the expense of the bias.

(2) Flexibility

$\hat{J}_{nT}$ is flexible in the sense that it includes the existing estimators considered in this paper as special cases and thus can reduce to each of them with certain bandwidth selection (with a flat-top kernel for the DK and KP estimators). In order to illustrate flexibility, we introduce the generalized CCE and the class of flat-top kernels considered here. Let

$$\hat{J}_{nT}^{GA} = \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{T} K\left(\frac{d_{ts}}{d_T}\right) \hat{V}_{(i,t)} \hat{V}'_{(i,s)}$$

and

$$\mathscr{K}_F = \left\{ K(\cdot) : K(x) = \begin{cases} 1 & \text{if } |x| \leq c \\ g(x) & \text{otherwise} \end{cases}, \quad \text{where } c \leq 1 \text{ and } g : R \to [0,1] \right\}.$$

$\hat{J}_{nT}^{GA}$ includes the CCE as a special case with $K(x) = 1$ for all x. A typical flat-top kernel in $\mathscr{K}_F$ is the trapezoidal kernel in which $g(x) = \max\{(|x|-1)/(c-1), 0\}$ and the truncated kernel is an extreme case with $c = 1$. We also define

$$\ell_{i,n}^{(c)} = \sum_{j=1}^{n} 1\{d_{ij} \leq c \cdot d_n\},\ \ell_n^{(c)} = n^{-1} \sum_{i=1}^{n} \ell_{i,n}^{(c)},$$
$$\ell_{t,T}^{(c)} = \sum_{s=1}^{T} 1\{d_{ts} \leq c \cdot d_T\} \text{ and } \ell_T^{(c)} = T^{-1} \sum_{t=1}^{T} \ell_{t,T}^{(c)}.$$

The following proposition states the asymptotic equivalence of $\hat{J}_{nT}$ to the existing estimators with certain sequences of d_n and d_T.

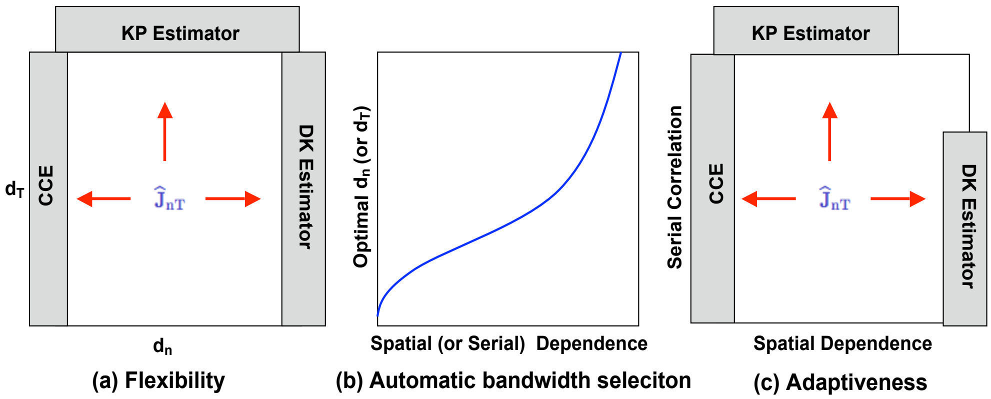


Figure 2 – Adaptiveness of $\hat{J}_{nT}$

Proposition 5 **(a)** *Suppose that Assumption 5 (i) holds and $d_n \to 0$ as $n \to \infty$. Then, $\hat{J}_{nT} - \hat{J}_{nT}^{GA} = o_p(1)$.*

(b) *Let $K(\cdot) \in \mathscr{K}_F$. If $\ell_n^{(c)}/n \to 1$ as $n \to \infty$, then $\hat{J}_{nT} - \hat{J}_{nT}^{DK} = o_p(1)$.*

(c) *Let $K(\cdot) \in \mathscr{K}_F$. If $\ell_T^{(c)}/n \to 1$ as $T \to \infty$, then $\hat{J}_{nT} - \hat{J}_{nT}^{KP} = o_p(1)$.*

Proofs are given in the appendix.

6 Adaptiveness of $\hat{J}_{nT}$

In this section, we study the adaptiveness of our estimator. While $\hat{J}_{nT}$ tends to be efficient under some mild conditions, there are certain dependence structures under which one of the existing estimators is efficient. For example, if a process is spatially highly persistent $\hat{J}_{nT}^{DK}$ is efficient because the bias caused by the usage of the kernel in the spatial dimension exceeds the variance reduction. With the same reason, $\hat{J}_{nT}^{KP}$ is efficient if a process is serially highly persistent. The CCE is also efficient in the absence of spatial dependence given $T = o(n^{(\eta+1)/2q})$. Nevertheless, $\hat{J}_{nT}$ can still be safely used under these dependence structures because $\hat{J}_{nT}$ approaches an efficient estimator by adapting the dependence structure of data. This adaptiveness is the novel advantage of our estimation method. As illustrated in Figure 2, adaptiveness comes from the flexibility and the automatic bandwidth selection procedure. In case that a process is spatially highly persistent, the automatic bandwidth selection procedure yields large $\hat{d}_n$ so that $\hat{J}_{nT}$ gets close to $\hat{J}_{nT}^{DK}$. With the symmetric way, $\hat{J}_{nT}$ approaches to $\hat{J}_{nT}^{KP}$ if a process is serially highly persistent. In the absence of spatial dependence, $\hat{d}_n$ tends to be close to zero so that $\hat{J}_{nT}$ approaches to the CCE.

We notice that the class of flat-top kernels in $\mathscr{K}_F$ cannot be used to obtain adaptiveness. Since the flat-top kernels are of the infinite order, they do not generate the asymptotic bias term. Hence, we cannot attain the optimal bandwidth parameters using the trade-off between the bias and variance. On the contrary, $\hat{J}_{nT}$ with the finite order kernels do not completely reduce to $\hat{J}_{nT}^{DK}$ and $\hat{J}_{nT}^{KP}$ with large d_n and d_T, getting close to them though. This may be regarded as a small loss of flexibility in our estimator to achieve adaptiveness.

7 Monte Carlo simulation

In this section, we provide some simulation evidence on the finite sample performance of our estimator based on the parametric plug-in procedure that minimizes AMSE*. We compare the performance of $\hat{J}_{nT}$ with $\hat{J}_{nT}^{DK}$, $\hat{J}_{nT}^{A}$ and $\hat{J}_{nT}^{KP}$ based on the optimal bandwidth selection procedures. We evaluate the estimators using the RMSE criterion and the true confidence levels of the 95% confidence intervals (CIs). We also examine the robustness of our bandwidth choice procedure to the mis-specification of the approximating parametric model and to the measurement errors in economic distance.

We assume a lattice structure, in which each unit is located on a square grid of integers over time. The data generating processes considered here are:

$$
\begin{array}{llll}
\text{DGP1:} & Y_{it} = \beta_0 + u_{it} & \beta_0 = 0 & \\
& u_t = \lambda u_{t-1} + \varepsilon_t, & \varepsilon_t = \left(I + \theta M_1 + \theta^2 M_2\right) v_t, & v_t \sim N(0, I_n); \\
\text{DGP2:} & Y_{it} = X_{it}'\beta_0 + \alpha_i + f_t + u_{it}, & \beta_0 = \alpha_i = f_t = 0 & \\
& X_t = \lambda X_{t-1} + \nu_t, & \nu_t = \left(I + \theta M_1 + \theta^2 M_2\right) \eta_t, & \eta_t \sim N(0, I_n); \\
& u_t = \lambda u_{t-1} + \varepsilon_t, & \varepsilon_t = \left(I + \theta M_1 + \theta^2 M_2\right) v_t, & v_t \sim N(0, I_n).
\end{array}
$$

$X_t = (X_{1t}, \ldots, X_{nt})'$ and $u_t = (u_{1t}, \ldots, u_{nt})'$ are modeled as AR(1) processes with the autoregressive parameter λ to generate the serial correlation of data. X_t and u_t are also spatially dependent as $\varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{nt})'$ and $\nu_t = (\nu_{1t}, \ldots, \nu_{nt})'$ follow spatial MA(2) processes with parameter θ. The values considered for the model parameters λ and θ are 0, 0.3, 0.6 and 0.9. For M_1 and M_2, each (i, j)-th element $m_{ij}^{(1)}$ and $m_{ij}^{(2)}$ is

$$
m_{ij}^{(1)} = \begin{cases} 1, & \text{if } d_{ij} = 1 \text{ or } \sqrt{2} \\ 0, & \text{otherwise} \end{cases} \quad \text{and } m_{ij}^{(2)} = \begin{cases} 1, & \text{if } d_{ij} = 2,\ \sqrt{5} \text{ or } 2\sqrt{2} \\ 0, & \text{otherwise} \end{cases}.
$$

DGP1 is used for RMSE criterion and the DGP2 is for the true confidence levels of the 95% CIs. While the DGP2 includes the time and individual effects and β_0 is estimated with the fixed effects estimator, these effects are absent in the DGP1 for easy calculation of the RMSE and hence we estimate β_0 with the simple OLS $(= (nT)^{-1} \sum_{i=1}^{n} \sum_{t=1}^{T} y_{it})$. In order to obtain the data dependent bandwidth parameters, we use the AR(1) and spatial AR(1) models with the cross-sectional and serial averages of the moment conditions respectively for $\hat{J}_{nT}^{DK}$ and $\hat{J}_{nT}^{KP}$. For $\hat{J}_{nT}$, we use the parametric model in (19). We take W_n to be the contiguity matrix in which unit i and j are neighbors if $d_{ij} \leq 1$. Note that the approximating parametric models for $\hat{J}_{nT}^{KP}$ and $\hat{J}_{nT}$ are mis-specified whereas the AR(1) model for $\hat{J}_{nT}^{DK}$ is correctly specified. We estimate parameters in (19) with OLS as well as QMLE which is consistent. To find $(d_n^\star, d_T^\star)$ and $(\hat{d}_n, \hat{d}_T)$, we use the formulas in (15), (16), (26) and (27) with $\eta = 2$ and $\ell_n = \pi d_n^2$.

We consider three different sample sizes; (i) small T and n; $T = 15, n = 49$ (7×7), (ii) large T and small n; $T = 100, n = 49$, and (iii) small T and large n; $T = 15, n = 225$ (15×15). The ranges of d_n we consider are from 1 to 9 for the 7×7 lattice and 1 to 20 for the 15×15 lattice. For d_T, it varies from 1 to T. We use the Parzen kernel, which is defined as:

$$K(x) = \begin{cases} 1 - 6x^2 + 6|x|^3, & \text{for } 0 \leq |x| \leq 1/2, \\ 2(1-|x|)^3, & \text{for } 1/2 < |x| \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

We also allow for the case with measurement errors in economic distance. The error contaminated distance, d_{ij}^* is generated as follows. If $d_{ij} < 2$, then d_{ij} is observed without a measurement error. But if $d_{ij} \geq 2$, then we observe d_{ij}^* which is generated from the following process:

$$d_{ij}^* = d_{ij} + e_{ij}, \qquad e_{ij} = \begin{cases} -1 & w.p.\ 1/3 \\ 0 & w.p.\ 1/3 \\ 1 & w.p.\ 1/3 \end{cases}$$

We use PHAC, CCE, DK and KP to denote $\hat{J}_{nT}$, $\hat{J}_{nT}^A$, $\hat{J}_{nT}^{DK}$, $\hat{J}_{nT}^{KP}$ with data dependent bandwidth parameters using OLS estimation. The subscripts, o and Q indicate respectively the estimator with infeasible optimal bandwidth parameters (PHAC$_o$, DK$_o$) and the one with data dependent bandwidth parameters using the QML estimator (PHAC$_Q$, KP$_Q$). PHAC$_e$ denotes PHAC based on error contaminated distance d_{ij}^*.

Table 1 presents the ratio of the RMSE to J_{nT} of $\hat{J}_{nT}$ and that of $\hat{J}_{nT}^{DK}$ evaluated at the data dependent bandwidth estimators (PHAC, DK) and at infeasible optimal bandwidth parameters (PHAC$_o$, DK$_o$). First, we can see that PHAC is superior to DK when spatial dependence is absent or weak while DK performs better when spatial dependence is high. This is also illustrated in Figure 3. Second, the increase of n reduces only the ratio of PHAC while increasing T improves the performance of both the estimators. Actually, improvement of PHAC by increasing T from 15 to 100 is not significantly obtained in contrast to PHAC$_o$. This can be regarded as a loss from misspecification of our approximating parametric model. Finally, we find that AMSE* criterion tends to effectively control the RMSE of $\hat{J}_{nT}$. PHAC$_o$ yields very reliable results regardless of the sample size and dependence structure except $\theta = \lambda = 0$.

Table 2 reports the empirical coverage probabilities (ECP) of 95% CIs associated with the different covariance estimators; PHAC, PHAC$_Q$, PHAC$_e$, DK, CCE and KP$_Q$. In general the inference with $\hat{J}_{nT}$ is reasonably accurate and it performs better as either n or T increases. However, the inference with $\hat{J}_{nT}$ suffers from a severe size distortion when both θ and λ are very high. For example, the ECP when $\lambda = \theta = 0.9$ with $T = 15$ and $n = 49$ is 51.7%. When a process is highly persistent in both the dimensions, $\hat{J}_{nT}$ is seriously downward biased since it does not capture high dependence well even if we choose large bandwidths. This is because $\hat{J}_{nT}$ is constructed with the estimated residuals not the true disturbance. Our asymptotic theory does not consider the effect of demeaning on the estimator. The downward bias of $\hat{J}_{nT}$ induces the Cls to be tight, which causes normal approximation to yield a serious size distortion. This is analogous to the time series case, see for example, Sun, Phillips and Jin (2008) and Sun and Phillips (2008).

Table 2 illustrates the difference of the performance between the OLS and QML estimators for our plug-in methods. The results shows that the PHAC performs reasonably well as the PHAC_Q except a few cases when both λ and θ are high and the ECPs based on both the estimators get close to each other as n increases. This is predicted in Section 4. When the approximating parametric model is mis-specified, OML estimator is not necessarily preferred. Table 2 also shows how measurement errors affect the performance of $\hat{J}_{nT}$ using the bandwidth choice we suggest. As economic distance data is likely to be error contaminated in practice, robustness of $\hat{J}_{nT}$ to measurement errors is a highly desirable property. Comparing PHAC_e with PHAC, we find that none of them is consistently more accurate than the other. Especially, the difference become trivial as n increases, which is consistent with our intuition. Comparisons between PHAC and PHAC_Q and between PHAC and PHAC_e are also exhibited in Figure 4.

In Table 2, we can also see the performance of the alternative estimators CCE, DK and KP_Q. For CCE, the results show that it performs better than the other estimators when there is no spatial dependence. Given $\theta = 0$, CCE yields very precise ECPs even when λ is extremely high, and the true confidence levels of CCE improves as n increases from 49 to 225. In contrast, when there exists even moderate spatial dependence, the hypothesis testing based on CCE suffers from a severe size distortion, which is not improved with increasing n. For DK, we find that it is robust to spatial dependence and therefore performs better than the other estimators when θ is very high. The table also shows that true confidence level of DK becomes more accurate as T increases. However, when λ is very high DK does not provide reliable CIs regardless of the degree of spatial dependence. For KP_Q, Table 2 shows that it is robust to serial correlation and performs well compared to the other estimators when n is large. However, it does not yield reliable CIs when θ is very high. All these results are consistent with the asymptotic results in Section 5.

Comparisons of PHAC with CE, DK and KP confirm the adaptiveness of our estimator with the automatic bandwidth selection procedure. Even though the performance of PHAC is not exactly as good as that of an existing estimator when the latter is efficient under a certain dependence structure, PHAC yields reasonably reliable CIs and the difference of the performance becomes smaller as the sample size increases. When $\theta = 0$, CCE is slightly superior to PHAC especially with large λ, but the difference gets smaller as n or T increases. When $\theta = 0.9$ the difference of performance between PHAC and DK gets smaller when $n = 225$ unless $\lambda = 0.9$. We can also see that given $\lambda = 0.9$ the performance of PHAC is as good as KP_Q when $T = 100$ except when θ is large. The comparisons between PHAC and the other estimators are also well exhibited in Figure 5.

8 Conclusion

In this paper we propose a bivariate kernel covariance matrix estimator in a linear panel model with fixed effects which is consistent in the presence of heteroskedasticity and spatial and temporal correlation. This estimator is flexible in the sense that it includes existing estimators as special cases and can reduce to them with certain choice of bandwidth. We derive optimal bandwidth parameters based on the upper bound of AMSE and suggest the automatic implementation using a parametric plug-in method. Due to flexibility and the automatic bandwidth selection procedure, our estimator adapts to the dependence structure of data and approaches an existing estimator when the latter is favorable. This adaptiveness enable our estimator to be safely used without the knowledge of dependence structure.

Since Kiefer and Vogelsang (2002, 2005), robust inference based on nonstandard limiting distribution, so called fixed-b asymptotics, has gained increasing attention. In a panel set-up, Vogelsang (2008), Bester, Conley and Hansen (2009) and Ibragimov and Müller (2009) propose robust inference methods in the spirit of the fixed-b asymptotics. Sun, Phillips and Jin (2008) and Sun and Phillips (2008) provide the optimal bandwidth selection procedure using a criterion that is most suited for hypothesis testing. It is interesting to extend the fixed-b asymptotics with the hypothesis testing based bandwidth selection method to the panel set-up.

Figure 3 – RMSE/Estimand with $\hat{J}_{nT}$ and $\hat{J}_{nT}^{DK}$ (DGP1)

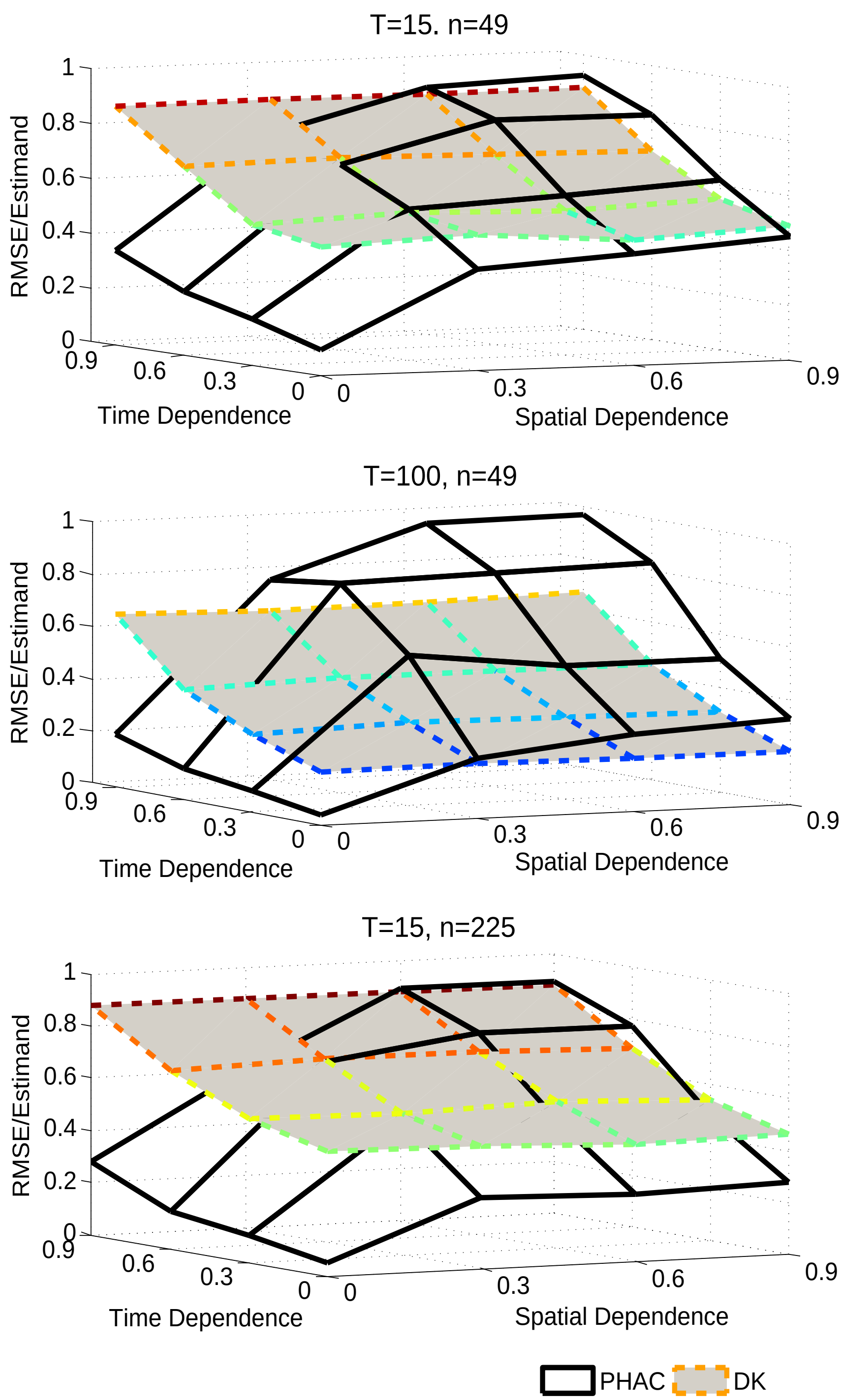
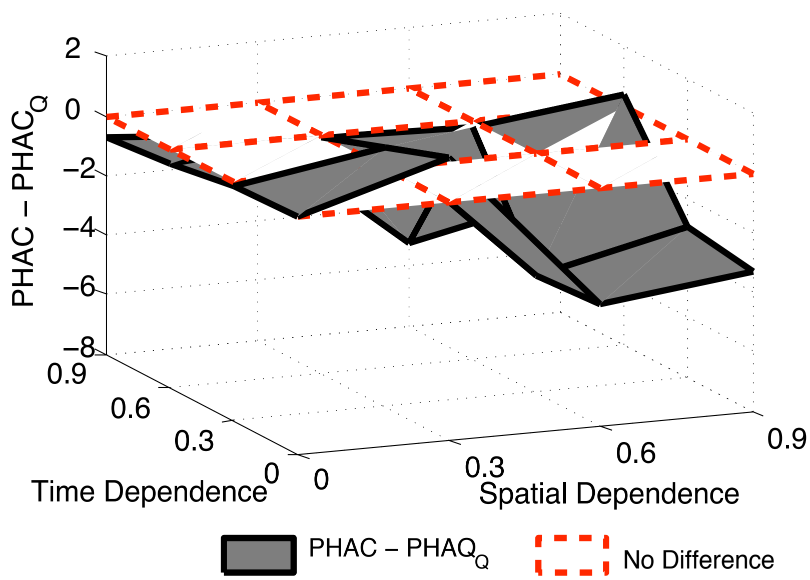


Figure 4 – Effects of QML Estimators for Parametric Plug-in Methods and Measurement Errors on Coverage Probabilities of Nominal 95% CIs Constructed with $\hat{J}_{nT}$: PHAC-PHAC$_Q$ and PHAC-PHAC$_e$ (DGP2, T=15, n=49)



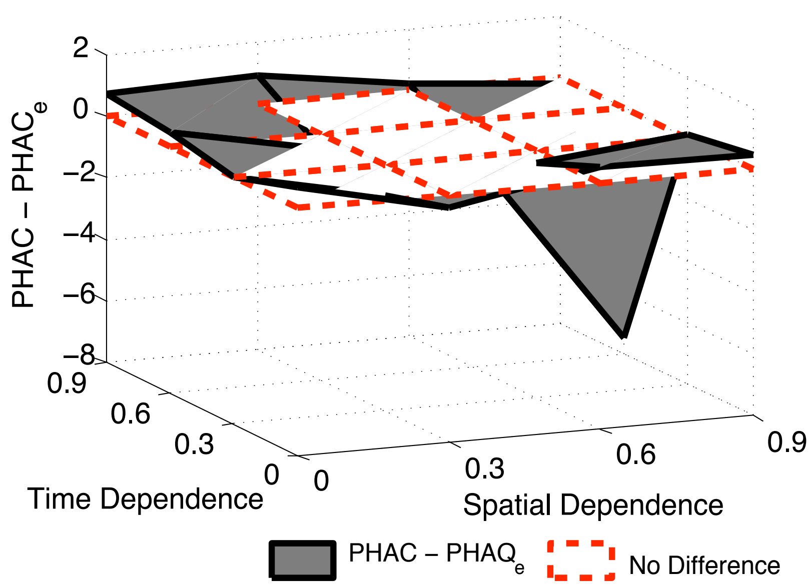


Figure 5 – Emprical Coverage Probabilities of Nominal 95% CIs
PHAC vs. CCE (DGP2)

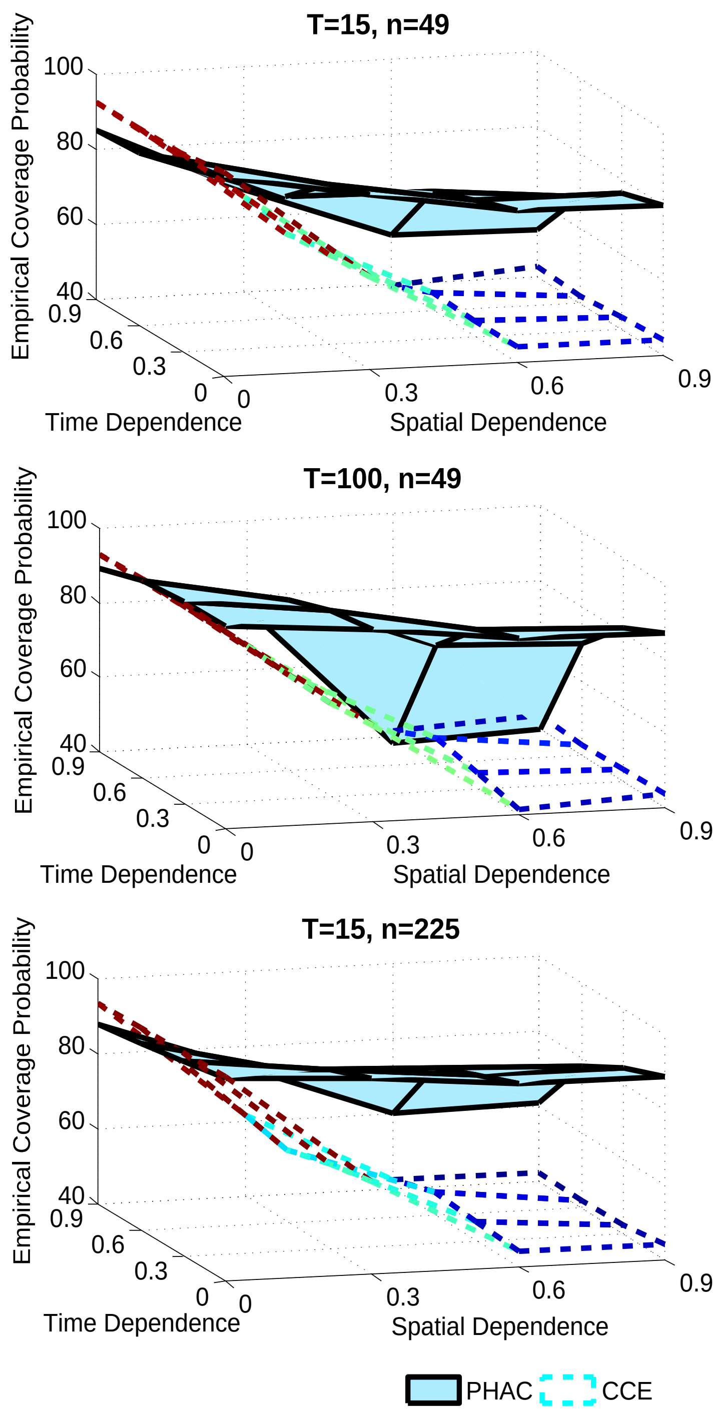


Figure 5 – Emprical Coverage Probabilities of Nominal 95% CIs - Continued
PHAC vs. DK Estimator (DGP2)

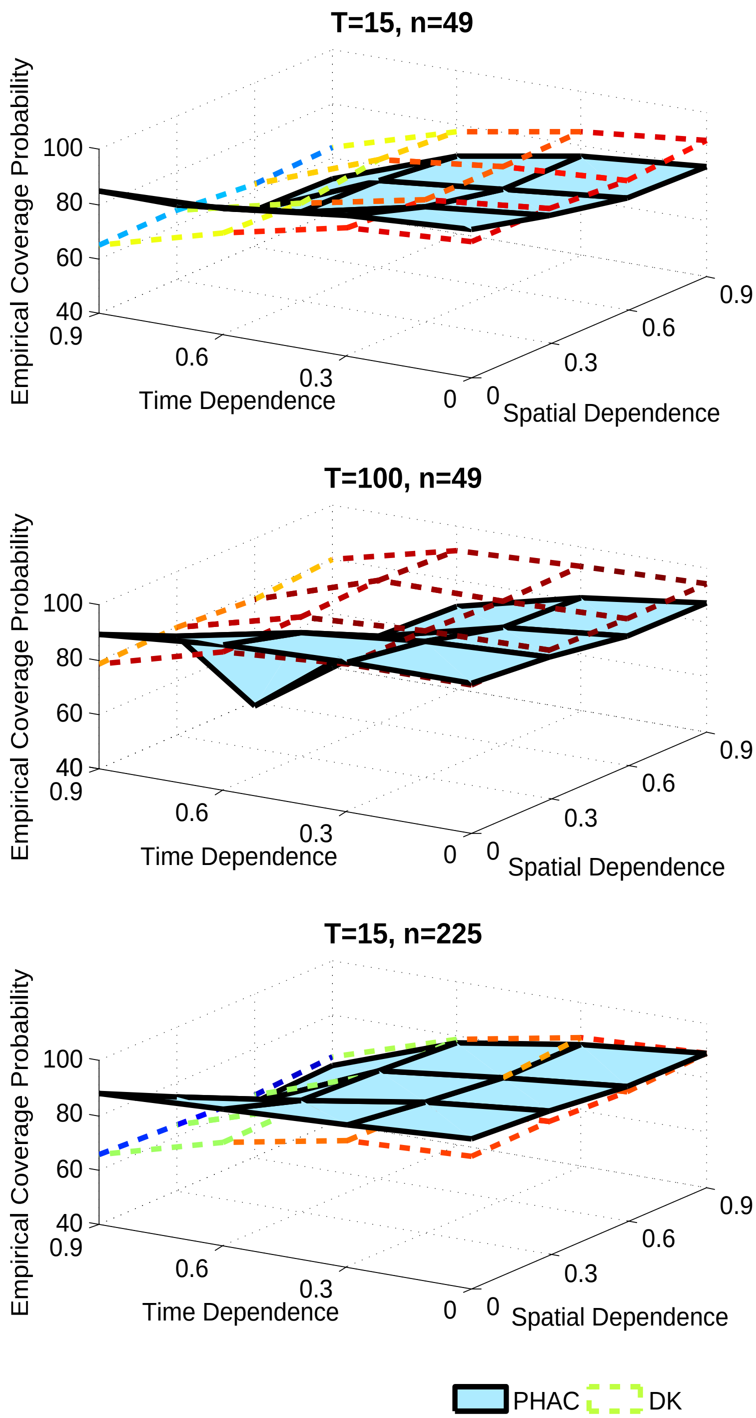


Figure 5 – Emprical Coverage Probabilities of Nominal 95% CIs - Continued
PHAC vs. KP Estimator (DGP2)

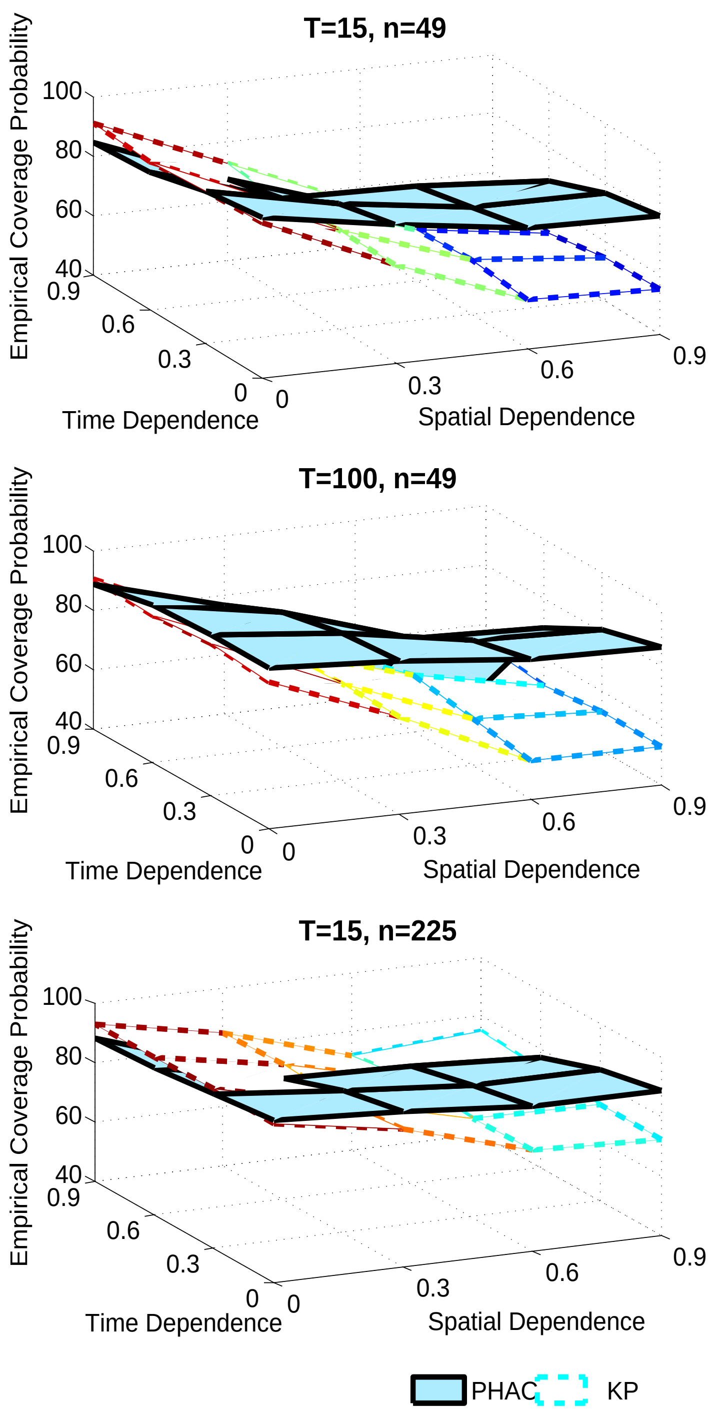


Table 1: RMSE/Estimand with $\hat{J}_{nT}$ and $\hat{J}_{nT}^{DK}$ - DGP1

	λ	θ					λ	θ			
		0.0	0.3	0.6	0.9			0.0	0.3	0.6	0.9
T=15, n=49											
PHAC	0.0	0.10	0.37	0.41	0.45	DK	0.0	0.48	0.50	0.46	0.49
	0.3	0.17	0.55	0.58	0.62		0.3	0.52	0.54	0.53	0.55
	0.6	0.23	0.68	0.82	0.82		0.6	0.69	0.70	0.69	0.69
	0.9	0.35	0.75	0.90	0.93		0.9	0.88	0.88	0.88	0.89
PHAC$_o$	0.0	0.42	0.30	0.36	0.39	DK$_o$	0.0	0.36	0.37	0.37	0.36
	0.3	0.14	0.51	0.54	0.58		0.3	0.57	0.59	0.58	0.59
	0.6	0.18	0.63	0.68	0.69		0.6	0.74	0.76	0.74	0.74
	0.9	0.31	0.72	0.78	0.82		0.9	0.89	0.89	0.89	0.90
T=100, n=49											
PHAC	0.0	0.04	0.23	0.29	0.32	DK	0.0	0.20	0.21	0.20	0.20
	0.3	0.08	0.57	0.51	0.51		0.3	0.30	0.32	0.31	0.30
	0.6	0.12	0.80	0.82	0.83		0.6	0.42	0.44	0.44	0.44
	0.9	0.20	0.77	0.96	0.96		0.9	0.66	0.65	0.66	0.67
PHAC$_o$	0.0	0.43	0.22	0.28	0.31	DK$_o$	0.0	0.14	0.14	0.14	0.14
	0.3	0.13	0.34	0.40	0.42		0.3	0.30	0.32	0.32	0.31
	0.6	0.14	0.41	0.46	0.49		0.6	0.43	0.45	0.45	0.45
	0.9	0.15	0.59	0.64	0.66		0.9	0.70	0.68	0.70	0.70
T=15, n=225											
PHAC	0.0	0.05	0.27	0.26	0.28	DK	0.0	0.48	0.47	0.45	0.46
	0.3	0.10	0.52	0.49	0.48		0.3	0.55	0.54	0.56	0.54
	0.6	0.14	0.69	0.77	0.77		0.6	0.68	0.70	0.70	0.69
	0.9	0.28	0.61	0.89	0.89		0.9	0.88	0.88	0.88	0.88
PHAC$_o$	0.0	0.42	0.23	0.23	0.24	DK$_o$	0.0	0.36	0.38	0.34	0.35
	0.3	0.07	0.37	0.42	0.43		0.3	0.60	0.59	0.61	0.59
	0.6	0.11	0.46	0.49	0.52		0.6	0.73	0.74	0.75	0.74
	0.9	0.28	0.57	0.61	0.64		0.9	0.89	0.89	0.89	0.89

PHAC and DK are based on data dependent bandwidth parameters.
PHAC$_o$ and DK$_o$ are based on the optimal bandwidth parameters.
PHAC uses the OLS estimator for the parameters in the plug-in models.

Table 2: Empirical Coverage Probabilities of Nominal 95% CIs
Constructed Using Alternative Covariance Estiamtors - DGP2

	λ	θ					λ	θ			
		0.0	0.3	0.6	0.9			0.0	0.3	0.6	0.9
T=15, n=49											
PHAC	0.0	93.7	87.0	80.6	80.1	DK	0.0	89.4	89.3	87.4	89.3
	0.3	91.1	82.5	76.5	76.2		0.3	87.0	84.6	84.9	85.0
	0.6	86.1	72.4	71.8	68.6		0.6	77.4	75.9	79.7	77.4
	0.9	85.1	67.5	53.3	52.7		0.9	64.8	65.5	62.7	63.6
PHAC$_Q$	0.0	93.7	85.5	84.5	83.4	CCE	0.0	94.9	64.7	44.2	44.1
	0.3	91.2	81.8	80.6	79.1		0.3	93.0	63.9	44.1	42.9
	0.6	86.6	72.5	72.0	68.2		0.6	92.9	62.4	44.5	41.9
	0.9	85.8	68.6	58.5	56.8		0.9	92.8	65.3	39.5	42.8
PHAC$_e$	0.0	93.0	87.4	80.1	79.6	KP$_Q$	0.0	92.3	72.9	56.7	55.2
	0.3	91.1	83.5	76.9	76.1		0.3	91.1	73.3	58.5	54.0
	0.6	85.6	72.9	72.9	76.1		0.6	89.8	70.7	57.0	51.3
	0.9	84.4	66.6	53.1	52.9		0.9	91.5	73.0	54.2	52.4
T=100, n=49											
PHAC	0.0	94.8	91.9	87.3	86.8	DK	0.0	94.1	94.5	93.7	94.1
	0.3	94.6	90.0	82.9	81.2		0.3	93.7	93.6	92.5	93.0
	0.6	93.0	85.7	71.7	70.1		0.6	90.8	91.2	92.2	90.6
	0.9	89.1	76.6	38.5	40.2		0.9	78.3	79.9	77.2	80.0
PHAC$_Q$	0.0	95.1	91.9	88.1	87.6	CCE	0.0	93.5	66.0	41.4	43.4
	0.3	94.7	90.2	86.6	85.3		0.3	94.1	65.1	44.2	43.0
	0.6	93.0	87.9	81.1	76.7		0.6	93.6	65.6	46.6	42.8
	0.9	89.3	74.6	38.6	40.2		0.9	93.2	66.5	41.7	43.7
PHAC$_e$	0.0	94.1	91.1	87.5	89.6	KP$_Q$	0.0	89.8	72.8	53.5	52.7
	0.3	93.6	90.1	82.6	84.6		0.3	91.4	73.1	56.0	53.4
	0.6	94.7	81.5	75.7	76.0		0.6	89.7	72.3	59.2	50.7
	0.9	91.7	75.2	41.4	38.1		0.9	90.4	71.1	53.2	53.2
T=15, n=225											
PHAC	0.0	94.6	92.5	88.9	88.9	DK	0.0	88.1	88.7	87.5	89.2
	0.3	92.2	87.8	84.4	84.3		0.3	86.2	86.2	84.5	86.8
	0.6	89.8	80.8	79.5	77.6		0.6	78.0	78.2	78.5	78.6
	0.9	88.2	74.4	60.3	61.1		0.9	65.9	64.4	62.7	64.3
PHAC$_Q$	0.0	94.7	92.2	89.8	90.0	CCE	0.0	95.3	63.3	44.5	44.4
	0.3	92.1	87.4	85.6	86.6		0.3	94.4	62.7	45.2	42.1
	0.6	90.0	83.8	78.1	76.9		0.6	94.3	59.4	46.0	41.7
	0.9	88.3	78.3	69.3	70.2		0.9	93.7	61.9	42.4	42.3
PHAC$_e$	0.0	94.7	92.3	88.3	88.9	KP$_Q$	0.0	93.5	86.1	73.9	72.5
	0.3	92.0	87.4	83.1	83.5		0.3	94.0	84.0	73.2	72.7
	0.6	89.4	79.9	77.7	76.2		0.6	93.0	85.7	75.1	73.1
	0.9	88.2	73.3	60.1	61.0		0.9	93.2	85.1	72.0	74.9

PHAC$_e$ uses OLS estimation for the parameters in plug-in models.
PHAC$_e$ denotes $\hat{J}_{nT}$ in the presence of measurement errors.

APPENDIX

Proof of Theorem 1

For notational simplicity, we re-order the individuals and time and make new indices. For $i_{(j)} = 1, ..., \ell_{j,n}$, $d_{i_{(j)}j} \le d_n$, and for $i_{(j)} = \ell_{j+1,n}, \ldots, n$, $d_{i_{(j)}j} > d_n$. For $t_{(s)} = 1, ..., \xi_{s,T}$, $d_{t_{(s)}s} \le d_T$, and for $t_{(s)} = \xi_{s,T} + 1, \ldots, T$, $d_{t_{(s)}s} > d_T$.

(a) Asymptotic Variance

Let $\varphi_{lkcd} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{T} r_{(i,t),l}^{(c)} r_{(j,s),k}^{(d)} K\left(\frac{d_{ij}}{d_n}\right) K\left(\frac{d_{ts}}{d_T}\right)$. We have

$$
\begin{aligned}
&\frac{nT}{\ell_n d_T} cov\left(\tilde{J}_{nT}(c_1, d_1), \tilde{J}_{nT}(c_2, d_2)\right) \\
&= \frac{1}{nT\ell_n d_T} E\left[\sum_{l=1}^{nTp}\sum_{k=1}^{nTp} \varphi_{lkc_1d_1}(\varepsilon_l\varepsilon_k - E\varepsilon_l\varepsilon_k) \sum_{e=1}^{nTp}\sum_{f=1}^{nTp} \varphi_{efc_2d_2}(\varepsilon_e\varepsilon_f - E\varepsilon_e\varepsilon_f)\right] \\
&= \frac{1}{nT\ell_n d_T} E\left[\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\sum_{e=1}^{nTp}\sum_{f=1}^{nTp} \varphi_{lkc_1d_1}\varphi_{efc_2d_2}(\varepsilon_l\varepsilon_k\varepsilon_e\varepsilon_f - \varepsilon_l\varepsilon_k E\varepsilon_e\varepsilon_f - \varepsilon_e\varepsilon_f E\varepsilon_l\varepsilon_k + E\varepsilon_l\varepsilon_k E\varepsilon_e\varepsilon_f)\right] \\
&= \frac{1}{nT\ell_n d_T} E\left[\sum_{l=1}^{nTp} \varphi_{llc_1d_1}\varphi_{llc_2d_2}\left(E\varepsilon_l^4 - 3\right) + \sum_{l=1}^{nTp}\sum_{k=1}^{nTp} \varphi_{lkc_1d_1}\varphi_{lkc_2d_2} + \sum_{l=1}^{nTp}\sum_{k=1}^{nTp} \varphi_{lkc_1d_1}\varphi_{klc_2d_2}\right] \\
&:= C_{1nT} + C_{2nT} + C_{3nT},
\end{aligned}
$$

where

$$
\begin{aligned}
C_{1nT} &= \frac{1}{nT\ell_n d_T} \sum_{l=1}^{nTp} \left(E\varepsilon_l^4 - 3\right) \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} r_{(i,t),l}^{(c_1)} r_{(j,s),l}^{(d_1)} K\left(\frac{d_{ij}}{d_n}\right) K\left(\frac{d_{ts}}{d_T}\right)\right. \\
&\quad \left. \times \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T} r_{(a,u),l}^{(c_2)} r_{(b,v),l}^{(d_2)} K\left(\frac{d_{ab}}{d_n}\right) K\left(\frac{d_{uv}}{d_T}\right)\right], \\
C_{2nT} &= \frac{1}{nT\ell_n d_T} \sum_{l=1}^{nTp}\sum_{k=1}^{nTp} \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} r_{(i,t),l}^{(c_1)} r_{(j,s),k}^{(d_1)} K\left(\frac{d_{ij}}{d_n}\right) K\left(\frac{d_{ts}}{d_T}\right)\right. \\
&\quad \left. \times \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T} r_{(a,u),l}^{(c_2)} r_{(b,v),k}^{(d_2)} K\left(\frac{d_{ab}}{d_n}\right) K\left(\frac{d_{uv}}{d_T}\right)\right], \\
C_{3nT} &= \frac{1}{nT\ell_n d_T} \sum_{l=1}^{nTp}\sum_{k=1}^{nTp} \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} r_{(i,t),l}^{(c_1)} r_{(j,s),k}^{(d_1)} K\left(\frac{d_{ij}}{d_n}\right) K\left(\frac{d_{ts}}{d_T}\right)\right. \\
&\quad \left. \times \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T} r_{(a,u),k}^{(c_2)} r_{(b,v),l}^{(d_2)} K\left(\frac{d_{ab}}{d_n}\right) K\left(\frac{d_{uv}}{d_T}\right)\right].
\end{aligned}
$$

For C_{1nT},

$$
\begin{aligned}
|C_{1nT}| &\leq \frac{1}{nT\ell_n d_T}\sum_{l=1}^{nTp}\left|E\varepsilon_l^4-3\right|\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ab}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)\\
&\quad\times\left|r_{(i,t),l}^{(c_1)}r_{(j,s),l}^{(d_1)}r_{(a,u),l}^{(c_2)}r_{(b,v),l}^{(d_2)}\right|\\
&\leq \frac{1}{nT\ell_n d_T}\sum_{l=1}^{nTp}\left|E\varepsilon_l^4-3\right|\left(\sum_{i=1}^{n}\sum_{t=1}^{T}\left|r_{(i,t),l}^{(c_1)}\right|\right)\left(\sum_{j=1}^{n}\sum_{s=1}^{T}\left|r_{(j,s),l}^{(d_1)}\right|\right)\left(\sum_{a=1}^{n}\sum_{u=1}^{T}\left|r_{(a,u),l}^{(c_2)}\right|\right)\left(\sum_{b=1}^{n}\sum_{v=1}^{T}\left|r_{(b,v),l}^{(d_2)}\right|\right)\\
&\leq \frac{c_R^4}{\ell_n d_T}\frac{1}{nT}\sum_{l=1}^{nTp}\left|E\varepsilon_l^4-3\right|\leq\frac{c_R^4c_Ep}{\ell_n d_T}=o(1) \qquad \text{(A.1)}
\end{aligned}
$$

using Assumptions 1 and 2.

C_{2nT} can be restated as

$$
\begin{aligned}
&\frac{1}{nT\ell_n d_T}\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}r_{(i,t),l}^{(c_1)}r_{(j,s),k}^{(d_1)}r_{(a,u),l}^{(c_2)}r_{(b,v),k}^{(d_2)}\\
&\times K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ab}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)\\
&=\frac{1}{nT\ell_n d_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ab}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)\\
&\times\left(\sum_{l=1}^{nTp}r_{(i,t),l}^{(c_1)}r_{(a,u),l}^{(c_2)}\right)\left(\sum_{k=1}^{nTp}r_{(j,s),k}^{(d_1)}r_{(b,v),k}^{(d_2)}\right)\\
&=\frac{1}{nT\ell_n d_T}\sum_{i=1}^{n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a=1}^{n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s_{(t)}=1}^{\xi_{t,T}}\sum_{u=1}^{T}\sum_{v_{(u)}=1}^{\xi_{u,T}}K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\\
&\times\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}. \qquad \text{(A.2)}
\end{aligned}
$$

In order to consider the boundary effects, we decompose C_{2nT} as follows:

$$
\begin{aligned}
&\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T} K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\\
&\times\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\\
&+\frac{1}{nT\ell_n d_T}\sum_{i=1}^{n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a=1}^{n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\notin E_T}\sum_{s_{(t)}=1}^{\xi_{t,T}}\sum_{u=1}^{T}\sum_{v_{(u)}=1}^{\xi_{u,T}} K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\\
&\times\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\\
&+\frac{1}{nT\ell_n d_T}\sum_{i=1}^{n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a=1}^{n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\notin E_T}\sum_{v_{(u)}=1}^{\xi_{u,T}} K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\\
&\times\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\\
&+\frac{1}{nT\ell_n d_T}\sum_{i\notin E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a=1}^{n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T} K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\\
&\times\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\\
&+\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\notin E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T} K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\\
&\times\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\\
&=D_{1nT}+D_{2nT}+D_{3nT}+D_{4nT}+D_{5,nT} \qquad \text{(A.3)}
\end{aligned}
$$

D_{1nT} is based on nonboundary units whereas the other are on boundary ones. In the following, we show that D_{1nT} converges to $\bar{\mathcal{K}}_1\bar{\mathcal{K}}_2 J(c_1,c_2)\,J(d_1,d_2)$ and the other terms become negiligible as n and T increase.

For D_{1nT}, the first step is to show that

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T} K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)\\
&\times\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\\
&=\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T} K^2\left(\frac{d_{ab_{(a)}}}{d_n}\right)K^2\left(\frac{d_{uv_{(u)}}}{d_T}\right)\\
&\times\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\\
&=\bar{\mathcal{K}}_1\bar{\mathcal{K}}_2 J(c_1,c_2)\,J(d_1,d_2), \qquad \text{(A.4)}
\end{aligned}
$$

and the next step is to prove that

$$\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)\gamma^{(c_1c_2)}_{(it,au)}$$

$$\times\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}$$

$$=\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)$$

$$\times K\left(\frac{d_{ts_{(t)}}}{d_T}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}. \tag{A.5}$$

For (A.4), let $\gamma^{(d_1d_2)}_{(\bar{\imath}\bar{t},b_{(a)}v_{(u)})}=(\ell_n\ell_T)^{-1}\sum_{h_{(i)}=1}^{\ell_{i,n}}\sum_{w_{(t)}=1}^{\ell_T}\gamma^{(d_1d_2)}_{(h_{(i)}w_{(t)},b_{(a)}v_{(u)})}$. Then,

$$\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}$$

$$=\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(\bar{\imath}\bar{t},b_{(a)}v_{(u)})}$$

$$+\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)\gamma^{(c_1c_2)}_{(it,au)}$$

$$\times\left(\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}-\gamma^{(d_1d_2)}_{(\bar{\imath}\bar{t},b_{(a)}v_{(u)})}\right). \tag{A.6}$$

$$=L_{1nT}+L_{2nT} \tag{A.7}$$

L_{1nT} is rewritten as

$$\frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t\in E_T}\sum_{u\in E_T}\gamma^{(c_1c_2)}_{(it,au)}\left(\frac{1}{\ell_n\ell_T}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\right)$$

$$\times\left(\frac{1}{\ell_n}\sum_{j=1}^{n}K^2\left(\frac{d_{ij}}{d_n}\right)\right)\left(\frac{1}{d_T}\sum_{s=1}^{T}K^2\left(\frac{d_{ts}}{d_T}\right)\right)$$

$$=\frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t\in E_T}\sum_{u\in E_T}\gamma^{(c_1c_2)}_{(it,au)}\frac{1}{\ell_n\ell_T}cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s:d_{ts}\le d_T}V^{(d_1)}_{(j,s)},\sum_{b:d_{ab}\le d_n}\sum_{v:d_{uv}\le d_T}V^{(d_2)}_{(b,v)}\right)$$

$$\times\left(\frac{1}{\ell_n}\sum_{j=1}^{n}K^2\left(\frac{d_{ij}}{d_n}\right)\right)\left(\frac{1}{d_T}\sum_{s=1}^{T}K^2\left(\frac{d_{ts}}{d_T}\right)\right)$$

$$:=G_{1nT}+G_{2nT}$$

where

$$G_{1nT} = \frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t\in E_T}\sum_{u\in E_T}\gamma_{(it,au)}^{(c_1c_2)} 1\left\{d_{ia}\le c_n, d_{ut}\le c_T\right\}$$
$$\times \frac{1}{\ell_n\ell_T} cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n}\sum_{v:d_{uv}\le d_T} V_{(b,v)}^{(d_2)}\right)$$
$$\times\left(\frac{1}{\ell_n}\sum_{j=1}^{n} K^2\left(\frac{d_{ij}}{d_n}\right)\right)\left(\frac{1}{d_T}\sum_{s=1}^{T} K^2\left(\frac{d_{ts}}{d_T}\right)\right)$$

and

$$G_{2nT} = \frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t\in E_T}\sum_{u\in E_T}\gamma_{(it,au)}^{(c_1c_2)}\left[1\left\{d_{ia}> c_n, d_{ut}> c_T\right\}\right.$$
$$\left.+1\left\{d_{ia}> c_n, d_{ut}\le c_T\right\}+1\left\{d_{ia}\le c_n, d_{ut}> c_T\right\}\right]$$
$$\times \frac{1}{\ell_n\ell_n} cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n}\sum_{v:d_{uv}\le d_T} V_{(b,v)}^{(d_2)}\right)$$
$$\times\left(\frac{1}{\ell_n}\sum_{j=1}^{n} K^2\left(\frac{d_{ij}}{d_n}\right)\right)\left(\frac{1}{d_T}\sum_{s=1}^{T} K^2\left(\frac{d_{ts}}{d_T}\right)\right)$$
$$= o(1)$$

as $c_n, c_T\to\infty$.

It suffices to consider G_{1nT}. When $d_{ia}\le c_n$ and $d_{tu}\le c_T$, we have

$$cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n}\sum_{v:d_{uv}\le d_T} V_{(b,v)}^{(d_2)}\right)$$
$$= cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{j:d_{ij}\le d_n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_2)}\right)$$
$$+ cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n}\sum_{v:d_{uv}\le d_T} V_{(b,v)}^{(d_2)} - \sum_{b:d_{ib}\le d_n}\sum_{v:d_{tv}\le d_T} V_{(b,v)}^{(d_2)}\right)$$

but

$$\sum_{b:d_{ab}\le d_n}\sum_{v:d_{uv}\le d_T} V_{(b,v)}^{(d_2)} - \sum_{b:d_{ib}\le d_n}\sum_{v:d_{tv}\le d_T} V_{(b,v)}^{(d_2)}$$
$$= \sum_{b:d_{ab}\le d_n}\sum_{v:d_{uv}\le d_T} V_{(b,v)}^{(d_2)} - \sum_{b:d_{ab}\le d_n}\sum_{v:d_{tv}\le d_T} V_{(b,v)}^{(d_2)}$$
$$+ \sum_{b:d_{ab}\le d_n}\sum_{v:d_{tv}\le d_T} V_{(b,v)}^{(d_2)} - \sum_{b:d_{ib}\le d_n}\sum_{v:d_{tv}\le d_T} V_{(b,v)}^{(d_2)}$$
$$= \sum_{b:d_{ab}\le d_n}\sum_{v:d_{uv}\le d_T, d_{tv}>d_T} V_{(b,v)}^{(d_2)} + \sum_{b:d_{ab}\le d_n, d_{ib}>d_n}\sum_{v:d_{tv}\le d_T} V_{(b,v)}^{(d_2)}$$

Now $d_{ab} \le d_n$ and $d_{ia} \le c_n$ implies that $d_{bi} \le d_n + c_n$. As the result,

$$\begin{aligned}
&\frac{1}{\ell_n \ell_T} \left| cov \left(\sum_{j:d_{ij}\le d_n} \sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n, d_{ib}>d_n} \sum_{v:d_{tv}\le d_T} V_{(b,v)}^{(d_2)} \right) \right| \\
&= \frac{1}{\ell_n \ell_T} \left| \sum_{j:d_{ij}\le d_n} \sum_{b:d_{ab}\le d_n, d_{ib}>d_n} \sum_{s:d_{ts}\le d_T} \sum_{v:d_{tv}\le d_T} EV_{(j,s)}^{(d_1)} V_{(b,v)}^{(d_2)} \right| \\
&\le \frac{1}{\ell_n \ell_T} \sum_{j:d_{ij}\le d_n} \sum_{b:d_n<d_{ib}\le d_n+c_n} \sum_{s:d_{ts}\le d_T} \sum_{v:d_{tv}\le d_T} \left| EV_{(j,s)}^{(d_1)} V_{(b,v)}^{(d_2)} \right| \\
&= o(1),
\end{aligned}$$

by choosing c_n such that $\sum_{b=1}^{n} 1\{d_{ib} \le d_n + c_n\} \le \bar{C}\ell_n$ for all i and for some constant $\bar{C}$. Similarly,

$$\frac{1}{\ell_n \ell_T} \left| cov \left(\sum_{j:d_{ij}\le d_n} \sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n} \sum_{v:d_{uv}\le d_T, d_{tv}>d_T} V_{(b,v)}^{(d_2)} \right) \right| = o(1).$$

Hence

$$\begin{aligned}
&\frac{1}{\ell_n \ell_T} cov \left(\sum_{j:d_{ij}\le d_n} \sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n} \sum_{v:d_{uv}\le d_T} V_{(b,v)}^{(d_2)} \right) \\
&= \frac{1}{\ell_n \ell_T} cov \left(\sum_{j:d_{ij}\le d_n} \sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{j:d_{ij}\le d_n} \sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_2)} \right) + o\left(1\right) \qquad \text{(A.8)}
\end{aligned}$$

where $o(1)$ term holds uniformly over i and t.

Now under Assumption 6, we have

$$\begin{aligned}
G_{1nT} &= \frac{1}{nT} \sum_{i\in E_n} \sum_{a\in E_n} \sum_{t\in E_T} \sum_{u\in E_T} \gamma_{(it,au)}^{(c_1 c_2)} \\
&\times \frac{1}{\ell_n \ell_T} cov \left(\sum_{j:d_{ij}\le d_n} \sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{j:d_{ij}\le d_n} \sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_2)} \right) \\
&\times \left(\frac{1}{\ell_n} \sum_{j=1}^{n} K^2 \left(\frac{d_{ij}}{d_n} \right) \right) \left(\frac{1}{d_T} \sum_{s_{(t)}=1}^{T} K^2 \left(\frac{d_{ts}}{d_T} \right) \right) (1 + o(1)) \\
&\to \bar{\mathcal{K}}_1 \bar{\mathcal{K}}_2 J\left(c_1, c_2\right) J(d_1, d_2).
\end{aligned}$$

by choosing d_n and d_T such that $n_1/n \to 1$ and $T_1/T \to 1$.

For L_{2nT} in (A.7), the first step is to show

$$\begin{aligned}
&\frac{1}{nT\ell_n d_T} \sum_{i\in E_n} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{a\in E_n} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{t\in E_T} \sum_{s_{(t)}=1}^{\ell_T} \sum_{u\in E_T} \sum_{v_{(u)}=1}^{\ell_T} \left(K^2 \left(\frac{d_{ij_{(i)}}}{d_n} \right) K^2 \left(\frac{d_{ts_{(t)}}}{d_T} \right) \right. \\
&\left. - K^2 \left(\frac{d_{ia}}{d_n} \right) K^2 \left(\frac{d_{tu}}{d_T} \right) \right) \gamma_{(it,au)}^{(c_1 c_2)} \left(\gamma_{(j_{(i)} s_{(t)}, b_{(a)} v_{(u)})}^{(d_1 d_2)} - \gamma_{(\bar{\imath}\bar{t}, b_{(a)} v_{(u)})}^{(d_1 d_2)} \right) = o(1), \qquad \text{(A.9)}
\end{aligned}$$

and the second step is to prove

$$\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T} K^2\left(\frac{d_{ia}}{d_n}\right)K^2\left(\frac{d_{tu}}{d_T}\right)$$
$$\times\,\gamma_{(it,au)}^{(c_1c_2)}\left(\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v_{(u)})}^{(d_1d_2)}\right)=o(1). \qquad \text{(A.10)}$$

For (A.9),

$$\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}\left(K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)\right.$$
$$\left.-K^2\left(\frac{d_{ia}}{d_n}\right)K^2\left(\frac{d_{tu}}{d_T}\right)\right)\gamma_{(it,au)}^{(c_1c_2)}\left(\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v_{(u)})}^{(d_1d_2)}\right)$$
$$\leq\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma_{(it,au)}^{(c_1c_2)}\right|\left|\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v_{(u)})}^{(d_1d_2)}\right|$$
$$=\frac{1}{nT}\sum_{(i,a)\in\mathcal{F}_1}\sum_{(t,u)\in\mathcal{G}_1}\left|\gamma_{(it,au)}^{(c_1c_2)}\right|\left(\frac{1}{\ell_n d_T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v_{(u)})}^{(d_1d_2)}\right|\right)$$
$$+\frac{1}{nT}\sum_{(i,a)\in\mathcal{F}_2}\sum_{(t,u)\in\mathcal{G}_1}\left|\gamma_{(it,au)}^{(c_1c_2)}\right|\left(\frac{1}{\ell_n d_T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v_{(u)})}^{(d_1d_2)}\right|\right)$$
$$+\frac{1}{nT}\sum_{(i,a)\in\mathcal{F}_1}\sum_{(t,u)\in\mathcal{G}_2}\left|\gamma_{(it,au)}^{(c_1c_2)}\right|\left(\frac{1}{\ell_n d_T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v_{(u)})}^{(d_1d_2)}\right|\right)$$
$$+\frac{1}{nT}\sum_{(i,a)\in\mathcal{F}_2}\sum_{(t,u)\in\mathcal{G}_2}\left|\gamma_{(it,au)}^{(c_1c_2)}\right|\left(\frac{1}{\ell_n d_T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v_{(u)})}^{(d_1d_2)}\right|\right)$$
$$=M_{1nT}+M_{2nT}+M_{3nT}+M_{4nT},$$

where

$$\mathcal{F}_1=\{(i,a):d_{ia}\leq f_n\ \&\ i,a\in E_n\},$$
$$\mathcal{F}_2=\{(i,a):d_{ia}>f_n\ \&\ i,a\in E_n\},$$
$$\mathcal{G}_1=\{(t,u):d_{tu}\leq g_T\ \&\ t,u\in E_T\},$$

and

$$\mathcal{G}_2=\{(t,u):d_{tu}>g_T\ \&\ t,u\in E_T\},$$

in which $f_n/d_n=O(1)$ and $g_T/d_T=O(1)$.

For M_{1nT}, we obtain

$$
\begin{aligned}
M_{1nT} \leq & \left(\frac{\ell_n d_T}{nT}\right) \frac{1}{\ell_n d_T} \sum_{(i,a)\in\mathcal{F}_1} \sum_{(t,u)\in\mathcal{G}_1} \left|\gamma_{(it,au)}^{(c_1c_2)}\right| \left(\frac{1}{\ell_n d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{s_{(t)}=1}^{\ell_T} \sum_{v_{(u)}=1}^{\ell_T} \left|\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\right| \right. \\
& \left. + \frac{1}{\ell_n^2 \ell_T d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{s_{(t)}=1}^{\ell_T} \sum_{h_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{w_{(t)}=1}^{\ell_T} \sum_{v_{(u)}=1}^{\ell_T} \left|\gamma_{(h_{(i)}w_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\right| \right) \\
& = O\left(\frac{\ell_n d_T}{nT}\right).
\end{aligned}
$$

For M_{2nT},

$$
\begin{aligned}
M_{2nT} \leq & \left(\frac{d_T}{T}\right) \frac{1}{n d_T} \sum_{(i,a)\in\mathcal{F}_2} \sum_{(t,u)\in\mathcal{G}_1} \left|\gamma_{(it,au)}^{(c_1c_2)}\right| \left(\frac{1}{\ell_n d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{s_{(t)}=1}^{\ell_T} \sum_{v_{(u)}=1}^{\ell_T} \left|\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\right| \right. \\
& \left. + \frac{1}{\ell_n^2 \ell_T d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{s_{(t)}=1}^{\ell_T} \sum_{h_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{w_{(t)}=1}^{\ell_T} \sum_{v_{(u)}=1}^{\ell_T} \left|\gamma_{(h_{(i)}w_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\right| \right) \\
& = O\left(\frac{d_T}{T}\right).
\end{aligned}
$$

It is straightforward that $M_{3nT} = O\left(\ell_n/n\right)$.

For M_{4nT},

$$
\begin{aligned}
M_{4nT} \leq & \frac{1}{f_n^q} \frac{1}{nT} \sum_{(i,a)\in\mathcal{F}_2} \sum_{(t,u)\in\mathcal{G}_2} \left|\gamma_{(it,au)}^{(c_1c_2)}\right| d_{ia}^q \\
& \times \left(\frac{1}{\ell_n d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{s_{(t)}=1}^{\ell_T} \sum_{v_{(u)}=1}^{\ell_T} \left|\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\right| \right. \\
& \left. + \frac{1}{\ell_n^2 \ell_T d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{s_{(t)}=1}^{\ell_T} \sum_{h_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{w_{(t)}=1}^{\ell_T} \sum_{v_{(u)}=1}^{\ell_T} \left|\gamma_{(h_{(i)}w_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\right| \right) \\
& = O\left(f_n^{-q}\right).
\end{aligned}
$$

By choosing f_n and g_T such that $f_n = O(d_n)$ and $g_T = O(d_T)$, we obtain

$$
M_{1nT} = o(1),\ M_{2nT} = o(1),\ M_{3nT} = o(1) \text{ and } M_{4nT} = o(1).
$$

Therefore, (A.9) holds.

The next step is to show (A.10).

$$
\begin{aligned}
&\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T} K^2\left(\frac{d_{ia}}{d_n}\right)K^2\left(\frac{d_{tu}}{d_T}\right)\\
&\times \gamma_{(it,au)}^{(c_1c_2)}\left(\gamma_{\left(j_{(i)}s_{(t)},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}-\gamma_{\left(\bar{\imath}\bar{t},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}\right)\\
&=\frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t\in E_T}\sum_{u\in E_T}K^2\left(\frac{d_{ia}}{d_n}\right)K^2\left(\frac{d_{tu}}{d_T}\right)\gamma_{(it,au)}^{(c_1c_2)}\\
&\times\left(\frac{1}{\ell_n d_T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left(\gamma_{\left(j_{(i)}s_{(t)},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}-\gamma_{\left(\bar{\imath}\bar{t},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}\right)\right)\\
&=o\left(1\right),
\end{aligned}
$$

because

$$
\begin{aligned}
&\frac{1}{\ell_n d_T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left(\gamma_{\left(j_{(i)}s_{(t)},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}-\gamma_{\left(\bar{\imath}\bar{t},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}\right)\\
&=\frac{\ell_T}{d_T}\left[\frac{1}{\ell_n\ell_T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\gamma_{\left(j_{(i)}s_{(t)},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}\right.\\
&\left.-\left(\frac{1}{\ell_n\ell_T}\right)^2\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{h_{(i)}=1}^{\ell_{i,n}}\sum_{w_{(t)}=1}^{\ell_T}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{v_{(u)}=1}^{\ell_T}\gamma_{\left(h_{(i)}w_{(t)},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}\right]\\
&=\frac{\ell_T}{d_T}\left[\frac{1}{\ell_n\ell_T}\sum_{j:d_{ij}\leq d_n}\sum_{b:d_{ab}\leq d_n}\sum_{s:d_{ts}\leq d_T}\sum_{v:d_{uv}\leq d_T}\gamma_{(js,bv)}^{(d_1d_2)}\right.\\
&\left.-\left(\frac{1}{\ell_n\ell_T}+o\left(1\right)\right)\sum_{h:d_{ih}\leq d_n}\sum_{b:d_{ab}\leq d_n}\sum_{w:d_{tw}\leq d_T}\sum_{v:d_{uv}\leq d_T}\gamma_{(hw,bv)}^{(d_1d_2)}\right]\\
&=\frac{\ell_T}{d_T}\left[\frac{1}{\ell_n\ell_T}\sum_{j:d_{ij}\leq d_n}\sum_{b:d_{ib}\leq d_n}\sum_{s:d_{ts}\leq d_T}\sum_{v:d_{tv}\leq d_T}\gamma_{(js,bv)}^{(d_1d_2)}\right.\\
&\left.-\left(\frac{1}{\ell_n\ell_T}+o\left(1\right)\right)\sum_{h:d_{ih}\leq d_n}\sum_{b:d_{ib}\leq d_n}\sum_{w:d_{tw}\leq d_T}\sum_{v:d_{tv}\leq d_T}\gamma_{(hw,bv)}^{(d_1d_2)}+o\left(1\right)\right]\\
&\to 0
\end{aligned}
$$

by (A.8) and Assumption 6. Therefore, $L_{1nT}=o\left(1\right)$ and $L_{2nT}=(1)$, which complete the proof of (A.4).

By a symmetric argument, we obtain the result that

$$
\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{ab_{(a)}}}{d_n}\right)K^2\left(\frac{d_{uv}}{d_T}\right)
$$
$$
\times\,\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}
$$
$$
=\bar{\mathcal{K}}_1\bar{\mathcal{K}}_2 J\left(c_1,c_2\right)J\left(d_1,d_2\right),
$$

which completes the proof of (A.4).

The next step is to prove (A.5). In view of previous derivations, it suffices to show that

$$
\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}\left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)\right.
$$
$$
\left.-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}
$$
$$
=0. \tag{A.11}
$$

$$
\frac{1}{nT\ell_n d_T}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}\left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)\right.
$$
$$
\left.-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}
$$
$$
=\frac{1}{nT\ell_n d_T}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_1}\sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_1}\left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2
$$
$$
\times\,\gamma_{(it,au)}^{(c_1c_2)}\gamma_{\left(j_{(i)}s_{(t)},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}
$$
$$
+\frac{1}{nT\ell_n d_T}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_2}\sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_1}\left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2
$$
$$
\times\,\gamma_{(it,au)}^{(c_1c_2)}\gamma_{\left(j_{(i)}s_{(t)},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}
$$
$$
+\frac{1}{nT\ell_n d_T}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_1}\sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_2}\left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2
$$
$$
\times\,\gamma_{(it,au)}^{(c_1c_2)}\gamma_{\left(j_{(i)}s_{(t)},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}
$$
$$
+\frac{1}{nT\ell_n d_T}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_2}\sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_2}\left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2
$$
$$
\times\,\gamma_{(it,au)}^{(c_1c_2)}\gamma_{\left(j_{(i)}s_{(t)},b_{(a)}v_{(u)}\right)}^{(d_1d_2)}
$$
$$
:=F_{1nT}+F_{2nT}+F_{3nT}+F_{4nT},
$$

where

$$\begin{aligned}
\mathcal{I}_1 &= \left\{ \left(i, j_{(i)}, a, b_{(a)}\right) : \left| d_{ij_{(i)}} - d_{ab_{(a)}} \right| \le 2c_n \ \& \ i, a \in E_n \right\}, \\
\mathcal{I}_2 &= \left\{ \left(i, j_{(i)}, a, b_{(a)}\right) : \left| d_{ij_{(i)}} - d_{ab_{(a)}} \right| > 2c_n \ \& \ i, a \in E_n \right\}, \\
\mathcal{J}_1 &= \left\{ \left(t, s_{(t)}, u, v_{(u)}\right) : \left| d_{ts_{(t)}} - d_{uv_{(u)}} \right| \le 2c_T \ \& \ t, u \in E_T \right\},
\end{aligned}$$

and

$$\mathcal{J}_2 = \left\{ \left(t, s_{(t)}, u, v_{(u)}\right) : \left| d_{ts_{(t)}} - d_{uv_{(u)}} \right| > 2c_T \ \& \ t, u \in E_T \right\}.$$

For F_{1nT}, we have

$$\begin{aligned}
F_{1nT} \le & \left| \frac{1}{nT\ell_n d_T} \sum_{\left(i,j_{(i)},a,b_{(a)}\right) \in \mathcal{I}_1} \sum_{\left(t,s_{(t)},u,v_{(u)}\right) \in \mathcal{J}_1} \left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right) K\left(\frac{d_{ts_{(t)}}}{d_T}\right) \right.\right. \\
& \left.\left. - K\left(\frac{d_{ab_{(a)}}}{d_n}\right) K\left(\frac{d_{uv_{(u)}}}{d_T}\right) \right]^2 \gamma^{(c_1c_2)}_{(it,au)} \gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)}, b_{(a)}v_{(u)})} \right| \\
= & \left| \frac{1}{nT\ell_n d_T} \sum_{\left(i,j_{(i)},a,b_{(a)}\right) \in \mathcal{I}_1} \sum_{\left(t,s_{(t)},u,v_{(u)}\right) \in \mathcal{J}_1} \left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right) \left(K\left(\frac{d_{ts_{(t)}}}{d_T}\right) - K\left(\frac{d_{uv_{(u)}}}{d_T}\right) \right) \right.\right. \\
& \left.\left. + K\left(\frac{d_{uv_{(u)}}}{d_T}\right) \left(K\left(\frac{d_{ij_{(i)}}}{d_n}\right) - K\left(\frac{d_{ab_{(a)}}}{d_n}\right) \right) \right]^2 \gamma^{(c_1c_2)}_{(it,au)} \gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)}, b_{(a)}v_{(u)})} \right| \\
\le & \left| \frac{2}{nT\ell_n d_T} \sum_{\left(i,j_{(i)},a,b_{(a)}\right) \in \mathcal{I}_1} \sum_{\left(t,s_{(t)},u,v_{(u)}\right) \in \mathcal{J}_1} \left[K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right) \left(K\left(\frac{d_{ts_{(t)}}}{d_T}\right) - K\left(\frac{d_{uv_{(u)}}}{d_T}\right) \right)^2 \right.\right. \\
& \left.\left. + K^2\left(\frac{d_{uv_{(u)}}}{d_T}\right) \left(K\left(\frac{d_{ij_{(i)}}}{d_n}\right) - K\left(\frac{d_{ab_{(a)}}}{d_n}\right) \right)^2 \right] \gamma^{(c_1c_2)}_{(it,au)} \gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)}, b_{(a)}v_{(u)})} \right| \\
\le & \frac{2c_L^2}{nT\ell_n d_T} \sum_{\left(i,j_{(i)},a,b_{(a)}\right) \in \mathcal{I}_1} \sum_{(t,s(t),u,v(u)) \in \mathcal{J}_1} \left[\left(\frac{d_{ts_{(t)}}}{d_T} - \frac{d_{uv_{(u)}}}{d_T} \right)^2 + \left(\frac{d_{ij_{(i)}}}{d_n} - \frac{d_{ab_{(a)}}}{d_n} \right)^2 \right] \\
& \times \left| \gamma^{(c_1c_2)}_{(it,au)} \gamma^{(c_1d_2)}_{(j_{(i)}s_{(t)}, b_{(a)}v_{(u)})} \right| \\
\le & 8c_L^2 \left(\frac{c_T^2}{d_T^2} + \frac{c_n^2}{d_n^2} \right) \left(\frac{1}{nT} \sum_{i \in E_n} \sum_{a \in E_n} \sum_{t \in E_T} \sum_{u \in E_T} \left| \gamma^{(c_1c_2)}_{(it,au)} \right| \right) \\
& \times \left(\frac{1}{\ell_n d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{s_{(t)}=1}^{\ell_T} \sum_{v_{(u)}=1}^{\ell_T} \left| \gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)}, b_{(a)}v_{(u)})} \right| \right) \\
= & O\left(\frac{c_T^2}{d_T^2}\right) + O\left(\frac{c_n^2}{d_n^2}\right),
\end{aligned} \tag{A.12}$$

since under Assumption 3

$$\frac{1}{\ell_n d_T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\right| = \frac{1}{\ell_n d_T}\sum_{j:d_{ij}\le d_n}\sum_{b:d_{ab}\le d_n}\sum_{s:d_{ts}\le d_T}\sum_{v:d_{uv}\le d_T}\left|\gamma^{(d_1d_2)}_{(js,bv)}\right|$$

$$\le \frac{1}{\ell_n d_T}\sum_{\substack{j:d_{ij}\le d_n \\ \text{or } d_{aj}\le d_n}}\sum_{\substack{b:d_{ib}\le d_n \\ \text{or } d_{ab}\le d_n}}\sum_{\substack{s:d_{ts}\le d_T \\ \text{or } d_{us}\le d_T}}\sum_{\substack{v:d_{tv}\le d_T \\ \text{or } d_{uv}\le d_T}}\left|\gamma^{(d_1d_2)}_{(js,bv)}\right|$$

$$= O(1). \tag{A.13}$$

For F_{2nT} we note that if $\left|d_{ij_{(i)}} - d_{ab_{(a)}}\right| > 2c_n$, then either $d_{ia} > c_n$ or $d_{j_{(i)}b_{(a)}} > c_n$. Otherwise, if both $d_{ia} \le c_n$ and $d_{j_{(i)}b_{(a)}} \le c_n$, then

$$d_{ij_{(i)}} - d_{ab_{(a)}} \le d_{ia} + d_{ab_{(a)}} + d_{b_{(a)}j_{(i)}} - d_{ab_{(a)}} \le 2c_n,$$

$$d_{ij_{(i)}} - d_{ab_{(a)}} \ge d_{ij_{(i)}} - d_{ia} - d_{ij_{(i)}} - d_{j_{(i)}b_{(a)}} \ge -2c_n.$$

These two inequalities imply that $\left|d_{ij_{(i)}} - d_{ab_{(a)}}\right| \le 2c_n$, a contradiction. Without the loss of generality, we assume that $d_{ia} > c_n$ for $\left(i, j_{(i)}, a, b_{(a)}\right) \in \mathcal{I}_2$. In this case

$$\begin{aligned}
F_{2nT} \le &\Bigg|\frac{1}{nT\ell_n d_T}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_2}\sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_1}\Bigg[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right) \\
&-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\Bigg]^2\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\Bigg| \\
\le &\Bigg|\frac{2}{nT\ell_n d_T}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_2}\sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_1}\Bigg[K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)\left(K\left(\frac{d_{ts_{(t)}}}{d_T}\right)-K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right)^2 \\
&+K^2\left(\frac{d_{uv_{(u)}}}{d_T}\right)\left(K\left(\frac{d_{ij_{(i)}}}{d_n}\right)-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)\right)^2\Bigg]\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s,b_{(a)}v)}\Bigg| \\
\le &\frac{8c_L^2}{nT\ell_n d_T}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_2}\sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_1}\left(\frac{c_T}{d_T}\right)^2\left|\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\right| \\
&+\frac{8}{nT\ell_n d_T}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_2}\sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_1}(d_{ia})^q\left|\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\right|(d_{ia})^{-q} \\
\le &\frac{8}{(c_n)^q}\left(\frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t\in E_T}\sum_{u\in E_T}(d_{ia})^q\left|\gamma^{(c_1c_2)}_{(it,au)}\right|\right)\left(\frac{1}{\ell_n d_T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\right|\right) \\
&+O\left(\frac{c_T^2}{d_T^2}\right) \\
= &O\left(\frac{1}{c_n^q}\right)+O\left(\frac{c_T^2}{d_T^2}\right).
\end{aligned}$$

With the similar procedure, we can show that $F_{3nT} = O\left(c_T^{-q}\right) + O\left(c_n^2/d_n^2\right)$.

For F_{4nT},

$$\begin{aligned}
F_{4nT} &\leq \left| \frac{1}{nT\ell_n d_T} \sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_2} \sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_2} \left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right) K\left(\frac{d_{ts_{(t)}}}{d_T}\right) \right.\right. \\
&\quad \left.\left. -K\left(\frac{d_{ab_{(a)}}}{d_n}\right) K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2 \gamma^{(c_1c_2)}_{(it,au)} \gamma^{(d_1d_2)}_{(j_{(i)}s,b_{(a)}v)} \right| \\
&\leq \frac{4}{nT\ell_n d_T} \sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_2} \sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_2} (d_{ia})^q \left| \gamma^{(c_1c_2)}_{(it,au)} \gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})} \right| (d_{ia})^{-q} \\
&\leq \frac{4}{(c_n)^q} \left(\frac{1}{nT} \sum_{i\in E_n} \sum_{a\in E_n} \sum_{t\in E_T} \sum_{u\in E_T} (d_{ia})^q \left| \gamma^{(c_1c_2)}_{(it,au)} \right| \right) \left(\frac{1}{\ell_n d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{s_{(t)}=1}^{\ell_T} \sum_{v_{(u)}=1}^{\ell_T} \left| \gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})} \right| \right) \\
&= O\left(\frac{1}{c_n^q}\right).
\end{aligned}$$

By choosing c_n and c_T such that $c_n, c_T \to \infty$ but $c_n/d_n, c_T/d_T \to 0$, we have

$$F_{1nT} = o\left(1\right)\ ,\ F_{2nT} = o(1),\ F_{3nT} = o(1) \text{ and } F_{4nT} = o(1)$$

and (A.5) is proved.

Next, we show that D_{2nT} is $o(1)$. For D_{2nT},

$$\begin{aligned}
&\frac{1}{nT\ell_n d_T} \sum_{i=1}^{n} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{a=1}^{n} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{t\notin E_T} \sum_{s_{(t)}=1}^{\xi_{t,T}} \sum_{u=1}^{T} \sum_{v_{(u)}=1}^{\xi_{u,T}} K\left(\frac{d_{ij_{(i)}}}{d_n}\right) K\left(\frac{d_{ab_{(a)}}}{d_n}\right) K\left(\frac{d_{ts_{(t)}}}{d_T}\right) K\left(\frac{d_{uv_{(u)}}}{d_T}\right) \\
&\times \gamma^{(c_1c_2)}_{(it,au)} \gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})} \\
&\leq \left| \frac{1}{nT\ell_n d_T} \sum_{i=1}^{n} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{a=1}^{n} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{t\notin E_T} \sum_{s_{(t)}=1}^{\xi_{t,T}} \sum_{u=1}^{T} \sum_{v_{(u)}=1}^{\xi_{u,T}} K\left(\frac{d_{ij_{(i)}}}{d_n}\right) K\left(\frac{d_{ab_{(a)}}}{d_n}\right) K\left(\frac{d_{ts_{(t)}}}{d_T}\right) K\left(\frac{d_{uv_{(u)}}}{d_T}\right) \right. \\
&\left. \times \gamma^{(c_1c_2)}_{(it,au)} \gamma^{(d_1d_2)}_{(j_{(i)}s,b_{(a)}v_{(u)})} \right| \\
&\leq \frac{1}{nT} \sum_{i=1}^{n} \sum_{a=1}^{n} \sum_{t\notin E_T} \sum_{u=1}^{T} \left| \gamma^{(c_1c_2)}_{(it,au)} \right| \left(\frac{1}{\ell_n d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{s_{(t)}=1}^{\xi_{t,T}} \sum_{v_{(u)}=1}^{\xi_{u,T}} \left| \gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})} \right| \right) \\
&\leq \frac{1}{nT} \sum_{i=1}^{n} \sum_{a=1}^{n} \sum_{t\notin E_T} \sum_{u=1}^{T} \left| \gamma^{(c_1c_2)}_{(it,au)} \right| \left(\frac{1}{\ell_n d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{s_{(t)}=1}^{\ell_T} \sum_{v_{(u)}=1}^{\ell_T} \left| \gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})} \right| \right) \\
&= o(1), \qquad\qquad \text{(A.14)}
\end{aligned}$$

as $T_2/T \to 0$. The last inequality holds by $\xi_{t,T} \le \ell_T$ for all t by definition. With the same procedure, we can show D_{3nT} are $o(1)$. We also show that $D_{4,nT}$ is $o(1)$ in a similar way.

$$
\begin{aligned}
&\frac{1}{nT\ell_n d_T}\sum_{i\notin E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a=1}^{n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T} K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\\
&\times \gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\\
&\le \left|\frac{1}{nT\ell_n d_T}\sum_{i\notin E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a=1}^{n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T} K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)K\left(\frac{d_{ts_{(t)}}}{d_T}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right.\\
&\times \gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\\
&\le \frac{1}{nT}\sum_{i\notin E_n}\sum_{a=1}^{n}\sum_{t\in E_T}\sum_{u\in E_T}\left|\gamma^{(c_1c_2)}_{(it,au)}\right|\left(\frac{1}{\ell_n d_T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\right|\right)\\
&= o(1) \qquad\qquad \text{(A.15)}
\end{aligned}
$$

by choosing the sequence of d_n in a way that $n_2/n \to 0$ as $n \to \infty$. We can also show $D_{5,nT}$ in the symmetric way.

With the same procedure, it is straightforward that

$$\lim_{n\to\infty}\lim_{T\to\infty} C_{3nT} = \bar{\mathcal{K}}_1\bar{\mathcal{K}}_2 J\left(c_1,d_2\right)J\left(c_2,d_1\right).$$

Therefore,

$$\lim_{n\to\infty}\lim_{T\to\infty}\frac{nT}{\ell_n d_T}\operatorname{cov}\left(\tilde{J}_{nT}\left(c_1,d_1\right),\tilde{J}_{nT}\left(c_2,d_2\right)\right) = \bar{\mathcal{K}}_1\bar{\mathcal{K}}_2(J\left(c_1,c_2\right)J\left(d_1,d_2\right)+J\left(c_1,d_2\right)J\left(c_2,d_1\right)).$$

In terms of matrix form,

$$\lim_{n\to\infty}\lim_{T\to\infty}\frac{nT}{\ell_n d_T}\operatorname{var}\left(\operatorname{vec}\left(\tilde{J}_{nT}\right)\right) = \bar{\mathcal{K}}_1\bar{\mathcal{K}}_2(I_{pp}+\mathbb{K}_{pp})\left(J\otimes J\right),$$

where $J = [J\left(c,d\right)]$, $c,d = 1,\ldots,p$.

(b) Asymptotic Bias

Let $d_T = k_{nT}d_n$ and $k_{nT} = k + o(1)$ where $k > 0$. We have

$$
\begin{aligned}
d_n^q\left(E\tilde{J}_{nT} - J_{nT}\right) &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\Gamma_{(it,js)}\left[(d_{ij})^q\frac{K\left(\frac{d_{ij}}{d_n}\right)-1}{\left(\frac{d_{ij}}{d_n}\right)^q} + \left(\frac{d_{ts}}{k_{nT}}\right)^q\frac{K\left(\frac{d_{ts}}{d_T}\right)-1}{\left(\frac{d_{ts}}{d_T}\right)^q}\right.\\
&\left. + (d_{ij})^q\left(\frac{d_{ts}}{d_T}\right)^q\frac{\left(K\left(\frac{d_{ij}}{d_n}\right)-1\right)\left(K\left(\frac{d_{ts}}{d_T}\right)-1\right)}{\left(\frac{d_{ij}}{d_n}\right)^q\left(\frac{d_{ts}}{d_T}\right)^q}\right]\\
&= -K_q b_1^{(q)} - \frac{1}{k^q}K_q b_2^{(q)} + o(1).
\end{aligned}
$$

Therefore, $\lim_{n\to\infty}\lim_{T\to\infty} d_n^q(\tilde{J}_{nT} - J_{nT}) = -K_q b_1^{(q)} - \frac{1}{k^q}K_q b_2^{(q)}$.

(c) $\sqrt{\frac{nT}{\ell_n d_T}}\left(\hat{J}_{nT}-J_{nT}\right)=O_p(1)$ **and** $\sqrt{\frac{nT}{\ell_n d_T}}\left(\hat{J}_{nT}-\tilde{J}_{nT}\right)=o_p(1)$

By (a) and (b), the first part of (c) is implied by the second part. Therefore, it suffices to show that $\sqrt{\frac{nT}{\ell_n d_T}}\left(\hat{J}_{nT}-\tilde{J}_{nT}\right)=o_p(1)$. This holds if and only if $\sqrt{\frac{nT}{\ell_n d_T}}\left(b'\hat{J}_{nT}b-b'\tilde{J}_{nT}b\right)=o_p(1)$ for any $b\in\mathbf{R}^p$. In consequence, we can consider the case that J_{nT} is a scalar random variable without loss of generality.

$$
\begin{aligned}
&\sqrt{\frac{nT}{\ell_n d_T}}\left(\hat{J}_{nT}-\tilde{J}_{nT}\right)\\
=&\sqrt{\frac{nT}{\ell_n d_T}}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)\left[\hat{V}_{(i,t)}\hat{V}'_{(j,s)}-V_{(i,t)}V'_{(j,s)}\right]\\
=&\left(\sqrt{nT}\left(\hat{\beta}-\beta_0\right)\right)^2\sqrt{\frac{\ell_n d_T}{nT}}\frac{1}{\ell_n d_T nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{it}^2\tilde{X}_{js}^2\\
&-2\sqrt{nT}\left(\hat{\beta}-\beta_0\right)\sqrt{\frac{\ell_n d_T}{nT}}\frac{1}{\ell_n d_T\sqrt{nT}}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{js}^2\tilde{X}_{it}\tilde{u}_{it}\\
&-\frac{2}{\sqrt{\ell_n d_T nT}}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{it}u_{it}\tilde{X}_{js}\left(\bar{u}_j+\bar{u}_s-\bar{u}\right)\\
&+\frac{1}{\sqrt{\ell_n d_T nT}}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{it}\tilde{X}_{js}\left(\bar{u}_i+\bar{u}_t-\bar{u}\right)\left(\bar{u}_j+\bar{u}_s-\bar{u}\right)\\
=&H_{1nT}+H_{2nT}+H_{3nT}+H_{4nT}.
\end{aligned}
$$

For H_{1nT}, it is $o_p(1)$ under Assumption 7(i) if

$$
\frac{1}{\ell_n d_T nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{it}^2\tilde{X}_{js}^2=O_p(1).
$$

$$
\begin{aligned}
&\frac{1}{\ell_n d_T nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{it}^2\tilde{X}_{js}^2\\
=&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}^2\left(\frac{1}{\ell_n d_T}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{js}^2\right)\\
\leq&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}^2\left(\frac{\ell_{i,n}\xi_{t,T}}{\ell_n d_T}\right)\left(\frac{1}{\ell_{i,n}\xi_{t,T}}\sum_{j:d_{ij}\leq d_n}\sum_{s:d_{ts}\leq d_T}\tilde{X}_{js}^2\right)
\end{aligned}
$$

By Assumption 7(ii), we obtain

$$
\begin{aligned}
P\left(\frac{1}{\ell_{i,n}\xi_{t,T}}\sum_{j:d_{ij}\leq d_n}\sum_{s:d_{ts}\leq d_T}\tilde{X}_{js}^2>\Delta\right)\leq&\frac{1}{\Delta\ell_{i,n}\xi_{t,T}}\sum_{j:d_{ij}\leq d_n}\sum_{s:d_{ts}\leq d_T}E\tilde{X}_{js}^2\\
&\to 0
\end{aligned}
$$

as $\Delta \to \infty$, which implies $(\ell_n d_T)^{-1} \sum_{j:d_{ij} \leq d_n} \sum_{s:d_{ts} \leq d_T} \tilde{X}_{it}^2 = O_p(1)$ uniformly. Using the same procedure, $(nT)^{-1} \sum_{i=1}^n \sum_{t=1}^T \tilde{X}_{it}^2 = O_p(1)$. As $\ell_{i,n} \xi_{t,T} / \ell_n d_T = O(1)$ due to Assumption 4 and $\xi_{t,T} \leq \ell_T$, $H_{1nT} = o_p(1)$.

For $H_{2nT,}$ it can be rewritten as

$$-2\sqrt{nT}\left(\hat{\beta} - \beta_0\right) \sqrt{\frac{\ell_n d_T}{nT}} \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T \tilde{X}_{it} \tilde{u}_{it} \left(\frac{1}{\ell_n d_T} \sum_{j=1}^n \sum_{s=1}^T K\left(\frac{d_{ij}}{d_n}\right) K\left(\frac{d_{ts}}{d_T}\right) \tilde{X}_{js}^2 \right)$$

and the part in the parenthesis is $O_p(1)$ uniformly as shown above.

$$\begin{aligned} \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T \tilde{X}_{it} \tilde{u}_{it} &= \frac{1}{\sqrt{nT}} \sum_{i=1}^n \sum_{t=1}^T \tilde{X}_{it} u_{it} \\ &= O_p(1) \end{aligned}$$

under Assumption 7(iv), which implies that $H_{2nT} = o_p(1)$.

For $H_{3nT,}$ we need to show that for all i and t

$$\frac{1}{\sqrt{\ell_n d_T}} \sum_{j=1}^n \sum_{s=1}^T K\left(\frac{d_{ij}}{d_n}\right) K\left(\frac{d_{ts}}{d_T}\right) \tilde{X}_{js} \left(\bar{u}_j + \bar{u}_s - \bar{u}\right) = o_p(1). \tag{A.16}$$

As $\bar{u}_j + \bar{u}_s - \bar{u} = o_p(1)$ uniformly, it suffice to show

$$(\ell_n d_T)^{-1/2} \sum_{j=1}^n \sum_{s=1}^T K\left(\frac{d_{ij}}{d_n}\right) K\left(\frac{d_{ts}}{d_T}\right) \tilde{X}_{js} = O_p(1).$$

$$\begin{aligned} &P\left(\left| \frac{1}{\sqrt{\ell_n d_T}} \sum_{j=1}^n \sum_{s=1}^T K\left(\frac{d_{ij}}{d_n}\right) K\left(\frac{d_{ts}}{d_T}\right) \tilde{X}_{js} \right| > \Delta \right) \\ &\leq \frac{1}{\Delta^2 \ell_n d_T} E\left[\sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{s_{(t)}=1}^{\xi_{t,T}} K\left(\frac{d_{ij_{(i)}}}{d_n}\right) K\left(\frac{d_{ts_{(t)}}}{d_T}\right) \tilde{X}_{j_{(i)} s_{(t)}} \right]^2 \\ &\leq \frac{2}{\Delta^2 \ell_n d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{s_{(t)}=1}^{\xi_{t,T}} E\left[\tilde{X}_{j_{(i)} s_{(t)}}\right]^2 \\ &\to 0, \end{aligned}$$

as $\Delta \to \infty$ under Assumption 7(ii). Therefore, H_{3nT} is also $o_p(1)$. With the similar procedures, we can show that H_{4nT} is $o_p(1)$.

As a result, $\sqrt{\frac{nT}{\ell_n d_T}} \left(\hat{J}_{nT} - \tilde{J}_{nT}\right) = o_p(1)$.

(d) AMSE

The first equality holds by Theorem 1(c). For the last equality of Theorem 1(d), since

$$\frac{nT}{\ell_n d_T} = \frac{d_n^{2q}}{d_n^{2q} \ell_n d_T / nT} = \frac{d_n^{2q}}{\tau + o(1)},$$

we have

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty} MSE\left(\frac{nT}{\ell_n d_T}, \tilde{J}_{nT}, S_{nT}\right) \\
&= \lim_{n\to\infty}\lim_{T\to\infty}\frac{nT}{\ell_n d_T}\ \text{vec}\ \left(E\tilde{J}_{nT}-J_{nT}\right)' S_{nT}\ \text{vec}\left(E\tilde{J}_{nT}-J_{nT}\right) + \lim_{n\to\infty}\lim_{T\to\infty}\frac{nT}{\ell_n d_T}\bar{\mathcal{K}}_1\bar{\mathcal{K}}_2 \text{tr}\left(S_{nT}\text{var}(\text{vec}\ \tilde{J}_{nT})\right) \\
&= \frac{1}{\tau}K_q^2\ \text{vec}\left(b_1^{(q)}+\frac{1}{k^q}b_2^{(q)}\right)' S\ \text{vec}\left(b_1^{(q)}+\frac{1}{k^q}b_2^{(q)}\right) + \bar{\mathcal{K}}_1\bar{\mathcal{K}}_2 tr\left[S(I_{pp}+K_{pp})(J\otimes J)\right],
\end{aligned}
$$

where the last equality holds by Theorem 1(a) and (b).

Proof of Corollary 1

Letting $k_{nT}=d_T/d_n$ and $k_{nT}\to k$ as $n,T\to\infty$. By Theorem 1(d), we obtain

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty}\max_{(b_1,b_2)\in\mathfrak{B}} MSE\left((nT)^{\frac{2q}{2q+\eta+1}}, \hat{J}_{nT}(d_n,d_T), S_{nT}\right) \\
&= \lim_{n\to\infty}\lim_{T\to\infty}(nT)^{\frac{2q}{2q+\eta+1}}\left(\frac{2K_q^2}{d_n^{2q}}\left(B_{11}+\frac{B_{22}}{k_{nT}^{2q}}\right)+\frac{\ell_n d_T}{nT}Q\right) \\
&= \lim_{n\to\infty}\lim_{T\to\infty}(\alpha_n k_{nT})^{\frac{2q}{2q+\eta+1}}\left(\frac{\alpha_n k_{nT} d_n^{2q+\eta+1}}{nT}\right)^{\frac{\eta+1}{2q+\eta+1}}\left(\frac{2K_q^2}{d_n^{2q}\ell_n d_T/nT}\left(B_{11}+\frac{B_{22}}{k_{nT}^{2q}}\right)+Q\right) \\
&= (\alpha k)^{\frac{2q}{2q+\eta+1}}\tau^{\frac{\eta+1}{2q+\eta+1}}\left(\frac{2K_q^2}{\tau}\left(B_{11}+\frac{B_{22}}{k^{2q}}\right)+Q\right),
\end{aligned}
$$

It is straightforward to show that this is uniquely minimized over $\tau\in(0,\infty)$ by

$$
\tau^\star = \frac{4qK_q^2\left(B_{11}+\frac{B_{22}}{(k^\star)^{2q}}\right)}{(\eta+1)\,Q} \text{ and } k^\star = \left(\frac{2\,(2q+\eta)\,K_q^2 B_{22}}{2K_q^2 B_{11}+Q\tau^\star}\right)^{\frac{1}{2q}},
$$

provided S is psd. Therefore,

$$
\tau^\star = \frac{4qK_q^2 B_{11}}{\eta Q} \text{ and } k^\star = \left(\frac{\eta B_{22}}{B_{11}}\right)^{\frac{1}{2q}}
$$

and that a sequence $\{(d_n,d_T)\}$ satisfies $d_n^{2q}\ell_n d_T/nT\to\tau^\star$ if and only if $d_n=d_n^\star+o\left((nT)^{1/(2q+\eta+1)}\right)$ and $d_T=d_T^\star+o\left((nT)^{1/(2q+\eta+1)}\right)$.

Proof of Theorem 2

$\hat{J}_{nT}^{DK}=\hat{J}_{nT}$ with $K\left(d_{ij}/d_n\right)=1$ for all i and j, which implies $\ell_n=\ell_{i,n}=n$ for all i and $E_n=\{1,\ldots,n\}$. Then, Theorem 3 is proved with the exactly same procedure with the proof of Theorem 1.

(a) Asymptotic Variance

Let $\varphi_{lkcd} = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} r_{(i,t),l}^{(c)} r_{(j,s),k}^{(d)} K\left(\frac{d_{ts}}{d_T}\right)$. As in the proof of Theorem 1, We have

$$\frac{T}{d_T} cov\left(\tilde{J}_{nT}(c_1,d_1), \tilde{J}_{nT}(c_2,d_2)\right) := C_{1nT} + C_{2nT} + C_{3nT},$$

where

$$\begin{aligned}
C_{1nT} &= \frac{1}{n^2 T d_T}\sum_{l=1}^{nTp}\left(E\varepsilon_l^4 - 3\right)\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} r_{(i,t),l}^{(c_1)} r_{(j,s),l}^{(d_1)} K\left(\frac{d_{ts}}{d_T}\right)\right. \\
&\quad \left.\times \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T} r_{(a,u),l}^{(c_2)} r_{(b,v),l}^{(d_2)} K\left(\frac{d_{uv}}{d_T}\right)\right], \\
C_{2nT} &= \frac{1}{n^2 T d_T}\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} r_{(i,t),l}^{(c_1)} r_{(j,s),k}^{(d_1)} K\left(\frac{d_{ts}}{d_T}\right)\right. \\
&\quad \left.\times \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T} r_{(a,u),l}^{(c_2)} r_{(b,v),k}^{(d_2)} K\left(\frac{d_{uv}}{d_T}\right)\right], \\
C_{3nT} &= \frac{1}{n^2 T d_T}\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} r_{(i,t),l}^{(c_1)} r_{(j,s),k}^{(d_1)} K\left(\frac{d_{ts}}{d_T}\right)\right. \\
&\quad \left.\times \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T} r_{(a,u),k}^{(c_2)} r_{(b,v),l}^{(d_2)} K\left(\frac{d_{uv}}{d_T}\right)\right].
\end{aligned}$$

In the previous proof we show that under Assumptions 1 and 2

$$|C_{1nT}| = o(1). \tag{A.17}$$

As we have shown in the previous proof, C_{2nT} can be restated as

$$\frac{1}{n^2 T d_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s_{(t)}=1}^{\xi_{t,T}}\sum_{u=1}^{T}\sum_{v_{(u)}=1}^{\xi_{u,T}} K\left(\frac{d_{ts_{(t)}}}{d_T}\right) K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(js_{(t)},bv_{(u)})}^{(d_1d_2)}. \tag{A.18}$$

In order to consider the boundary effects, we decompose C_{2nT} as follows:

$$\begin{aligned}
&\frac{1}{n^2 T d_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T} K\left(\frac{d_{ts_{(t)}}}{d_T}\right) K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(js_{(t)},bv_{(u)})}^{(d_1d_2)} \\
&+ \frac{1}{n^2 T d_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\notin E_T}\sum_{s_{(t)}=1}^{\xi_{t,T}}\sum_{u=1}^{T}\sum_{v_{(u)}=1}^{\xi_{u,T}} K\left(\frac{d_{ts_{(t)}}}{d_T}\right) K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(js_{(t)},bv_{(u)})}^{(d_1d_2)} \\
&+ \frac{1}{n^2 T d_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\notin E_T}\sum_{v_{(u)}=1}^{\xi_{u,T}} K\left(\frac{d_{ts_{(t)}}}{d_T}\right) K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(js_{(t)},bv_{(u)})}^{(d_1d_2)} \\
&= D_{1nT} + D_{2nT} + D_{3nT}.
\end{aligned} \tag{A.19}$$

D_{1nT} is based on nonboundary units whereas the other are on boundary ones. In the following, we show that D_{1nT} converges to $\bar{\mathcal{K}}_2 J(c_1,c_2) J(d_1,d_2)$ and the other terms become negiligible as n and T increase.

For D_{1nT}, the first step is to show that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\\
&=\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{uv_{(u)}}}{d_T}\right)\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\\
&=\bar{\mathcal{K}}_2 J(c_1,c_2)\,J(d_1,d_2)\,, \qquad\qquad \text{(A.20)}
\end{aligned}$$

and the next step is to prove that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}\\
&=\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K\left(\frac{d_{ts_{(t)}}}{d_T}\right)K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\\
&\times\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}. \qquad\qquad \text{(A.21)}
\end{aligned}$$

For (A.21), let $\gamma^{(d_1d_2)}_{(\bar{\imath}\bar{t},bv_{(u)})}=(n\ell_T)^{-1}\sum_{h=1}^{n}\sum_{w_{(t)}=1}^{\ell_T}\gamma^{(d_1d_2)}_{(hw_{(t)},bv_{(u)})}$. Then,

$$\begin{aligned}
&\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}\\
&=\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{\left(\bar{\imath}\bar{t},bv_{(u)}\right)}\\
&+\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)\gamma^{(c_1c_2)}_{(it,au)}\\
&\times\left(\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}-\gamma^{(d_1d_2)}_{\left(\bar{\imath}\bar{t},bv_{(u)}\right)}\right). \qquad\qquad \text{(A.22)}\\
&=L_{1nT}+L_{2nT} \qquad\qquad \text{(A.23)}
\end{aligned}$$

L_{1nT} is rewritten as

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{t\in E_T}\sum_{u\in E_T}\gamma_{(it,au)}^{(c_1c_2)}\left(\frac{1}{n\ell_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\gamma_{(js_{(t)},bv_{(u)})}^{(d_1d_2)}\right)\left(\frac{1}{d_T}\sum_{s=1}^{T}K^2\left(\frac{d_{ts}}{d_T}\right)\right)$$
$$=\frac{1}{nT}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{t\in E_T}\sum_{u\in E_T}\gamma_{(it,au)}^{(c_1c_2)}\frac{1}{n\ell_T}cov\left(\sum_{j=1}^{n}\sum_{s:d_{ts}\leq d_T}V_{(j,s)}^{(d_1)},\sum_{b=1}^{n}\sum_{v:d_{uv}\leq d_T}V_{(b,v)}^{(d_2)}\right)$$
$$\times\left(\frac{1}{d_T}\sum_{s=1}^{T}K^2\left(\frac{d_{ts}}{d_T}\right)\right)$$
$$:=G_{1nT}+G_{2nT}$$

where

$$G_{1nT}=\frac{1}{nT}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{t\in E_T}\sum_{u\in E_T}\gamma_{(it,au)}^{(c_1c_2)}1\left\{d_{ut}\leq c_T\right\}$$
$$\times\frac{1}{n\ell_T}cov\left(\sum_{j=1}^{n}\sum_{s:d_{ts}\leq d_T}V_{(j,s)}^{(d_1)},\sum_{b=1}^{n}\sum_{v:d_{uv}\leq d_T}V_{(b,v)}^{(d_2)}\right)\left(\frac{1}{d_T}\sum_{s=1}^{T}K^2\left(\frac{d_{ts}}{d_T}\right)\right)$$

and

$$G_{2nT}=\frac{1}{nT}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{t\in E_T}\sum_{u\in E_T}\gamma_{(it,au)}^{(c_1c_2)}\left[1\left\{d_{ut}>c_T\right\}+1\left\{d_{ut}\leq c_T\right\}+1\left\{d_{ut}>c_T\right\}\right]$$
$$\times\frac{1}{n\ell_T}cov\left(\sum_{j=1}^{n}\sum_{s:d_{ts}\leq d_T}V_{(j,s)}^{(d_1)},\sum_{b=1}^{n}\sum_{v:d_{uv}\leq d_T}V_{(b,v)}^{(d_2)}\right)\left(\frac{1}{d_T}\sum_{s=1}^{T}K^2\left(\frac{d_{ts}}{d_T}\right)\right)$$
$$=o(1)$$

as $c_T\to\infty$.

It suffices to consider G_{1nT}. When $d_{tu}\leq c_T$, we have

$$cov\left(\sum_{j=1}^{n}\sum_{s:d_{ts}\leq d_T}V_{(j,s)}^{(d_1)},\sum_{b=1}^{n}\sum_{v:d_{uv}\leq d_T}V_{(b,v)}^{(d_2)}\right)$$
$$=cov\left(\sum_{j=1}^{n}\sum_{s:d_{ts}\leq d_T}V_{(j,s)}^{(d_1)},\sum_{j=1}^{n}\sum_{s:d_{ts}\leq d_T}V_{(j,s)}^{(d_2)}\right)$$
$$+cov\left(\sum_{j=1}^{n}\sum_{s:d_{ts}\leq d_T}V_{(j,s)}^{(d_1)},\sum_{b=1}^{n}\sum_{v:d_{uv}\leq d_T}V_{(b,v)}^{(d_2)}-\sum_{b=1}^{n}\sum_{v:d_{tv}\leq d_T}V_{(b,v)}^{(d_2)}\right)$$

but

$$\sum_{b=1}^{n}\sum_{v:d_{uv}\leq d_T}V_{(b,v)}^{(d_2)}-\sum_{b=1}^{n}\sum_{v:d_{tv}\leq d_T}V_{(b,v)}^{(d_2)}=\sum_{b=1}^{n}\sum_{v:d_{uv}\leq d_T,d_{tv}>d_T}V_{(b,v)}^{(d_2)}$$

Now $d_{uv} \le d_T$ and $d_{tu} \le c_T$ implies $d_{tv} \le d_T + c_T$. As a result,

$$\begin{aligned}
&\frac{1}{n\ell_T}\left|cov\left(\sum_{j=1}^{n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{b=1}^{n}\sum_{v:d_{uv}\le d_T, d_{tv}>d_T} V_{(b,v)}^{(d_2)}\right)\right| \\
&= \frac{1}{n\ell_T}\left|\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s:d_{ts}\le d_T}\sum_{v:d_{uv}\le d_T, d_{tv}>d_T} EV_{(j,s)}^{(d_1)}V_{(b,v)}^{(d_2)}\right| \\
&\le \frac{1}{n\ell_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s:d_{ts}\le d_T}\sum_{v:d_T\le d_{tv}\le d_T+c_T}\left|EV_{(j,s)}^{(d_1)}V_{(b,v)}^{(d_2)}\right| \\
&= o(1),
\end{aligned}$$

by choosing c_T such that $\sum_{v=1}^{T} 1\{d_{tv} \le d_T + c_T\} \le \bar{C}\ell_T$ for all t and for some constant $\bar{C}$. Hence

$$\begin{aligned}
&\frac{1}{n\ell_T}cov\left(\sum_{j=1}^{n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{b=1}^{n}\sum_{v:d_{uv}\le d_T} V_{(b,v)}^{(d_2)}\right) \\
&= \frac{1}{n\ell_T}cov\left(\sum_{j=1}^{n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{j=1}^{n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_2)}\right) + o\left(1\right) \qquad \text{(A.24)}
\end{aligned}$$

where $o(1)$ term holds uniformly over t. Now under Assumption 13, we have

$$\begin{aligned}
G_{1nT} &= \frac{1}{nT}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{t\in E_T}\sum_{u\in E_T}\gamma_{(it,au)}^{(c_1c_2)} \\
&\times \frac{1}{n\ell_T}cov\left(\sum_{j=1}^{n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_1)}, \sum_{j=1}^{n}\sum_{s:d_{ts}\le d_T} V_{(j,s)}^{(d_2)}\right)\left(\frac{1}{d_T}\sum_{s_{(t)}=1}^{T} K^2\left(\frac{d_{ts}}{d_T}\right)\right)(1+o(1)) \\
&\to \bar{\mathcal{K}}_2 J\left(c_1,c_2\right)J(d_1,d_2).
\end{aligned}$$

by choosing d_T such that $T_1/T \to 1$.

For L_{2nT} in (A.23), the first step is to show

$$\begin{aligned}
&\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}\left(K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right) - K^2\left(\frac{d_{tu}}{d_T}\right)\right) \\
&\times \gamma_{(it,au)}^{(c_1c_2)}\left(\gamma_{(js_{(t)},bv_{(u)})}^{(d_1d_2)} - \gamma_{\left(\bar{\imath}\bar{t},bv_{(u)}\right)}^{(d_1d_2)}\right) = o(1), \qquad \text{(A.25)}
\end{aligned}$$

and the second step is to prove

$$\begin{aligned}
&\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T} K^2\left(\frac{d_{tu}}{d_T}\right) \\
&\times \gamma_{(it,au)}^{(c_1c_2)}\left(\gamma_{(js_{(t)},bv_{(u)})}^{(d_1d_2)} - \gamma_{\left(\bar{\imath}\bar{t},bv_{(u)}\right)}^{(d_1d_2)}\right) = o(1). \qquad \text{(A.26)}
\end{aligned}$$

For (A.25),

$$\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}\left(K^2\left(\frac{d_{ts_{(t)}}}{d_T}\right)-K^2\left(\frac{d_{tu}}{d_T}\right)\right)$$
$$\times\,\gamma^{(c_1c_2)}_{(it,au)}\left(\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}-\gamma^{(d_1d_2)}_{\left(\bar{\imath}\bar{t},bv_{(u)}\right)}\right)$$
$$\leq\frac{1}{nT\ell_n d_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma^{(c_1c_2)}_{(it,au)}\right|\left|\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}-\gamma^{(d_1d_2)}_{\left(\bar{\imath}\bar{t},bv_{(u)}\right)}\right|$$
$$=\frac{1}{nT}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{(t,u)\in\mathcal{G}_1}\left|\gamma^{(c_1c_2)}_{(it,au)}\right|\left(\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}-\gamma^{(d_1d_2)}_{\left(\bar{\imath}\bar{t},bv_{(u)}\right)}\right|\right)$$
$$+\frac{1}{nT}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{(t,u)\in\mathcal{G}_2}\left|\gamma^{(c_1c_2)}_{(it,au)}\right|\left(\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}-\gamma^{(d_1d_2)}_{\left(\bar{\imath}\bar{t},bv_{(u)}\right)}\right|\right)$$
$$=M_{1nT}+M_{2nT},$$

where

$$\mathcal{G}_1=\left\{(t,u):d_{tu}\leq g_T\ \&\ t,u\in E_T\right\},$$

and

$$\mathcal{G}_2=\left\{(t,u):d_{tu}>g_T\ \&\ t,u\in E_T\right\},$$

in which $g_T/d_T=O(1)$.

For M_{1nT}, we obtain

$$M_{1nT}\leq\left(\frac{d_T}{T}\right)\frac{1}{nd_T}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{(t,u)\in\mathcal{G}_1}\left|\gamma^{(c_1c_2)}_{(it,au)}\right|\left(\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}\right|\right.$$
$$\left.+\frac{1}{n^2\ell_T d_T}\sum_{j=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{h=1}^{n}\sum_{b=1}^{n}\sum_{w_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma^{(d_1d_2)}_{(hw_{(t)},bv_{(u)})}\right|\right)$$
$$=O\left(\frac{d_T}{T}\right).$$

For M_{2nT},

$$
\begin{aligned}
M_{2nT} \leq& \left(\frac{d_T}{T}\right)\frac{1}{g_T^q}\frac{1}{nd_T}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{(t,u)\in\mathcal{G}_2}\left|\gamma_{(it,au)}^{(c_1c_2)}\right| d_{tu}^q \\
&\times\left(\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}^{(d_1d_2)}\right|\right.\\
&\left.+\frac{1}{n^2\ell_T d_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{h=1}^{n}\sum_{b=1}^{n}\sum_{w_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma_{(hw_{(t)},bv_{(u)})}^{(d_1d_2)}\right|\right)\\
&= O\left(\frac{d_T}{T}\right).
\end{aligned}
$$

By choosing g_T such that $g_T = O(d_T)$, we obtain

$$M_{1nT} = o(1) \text{ and } M_{2nT} = o(1).$$

Therefore, (A.25) holds.

The next step is to show (A.26).

$$
\begin{aligned}
&\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{tu}}{d_T}\right)\gamma_{(it,au)}^{(c_1c_2)}\\
&\times\left(\gamma_{(js_{(t)},bv_{(u)})}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},bv_{(u)})}^{(d_1d_2)}\right)\\
&=\frac{1}{nT}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{t\in E_T}\sum_{u\in E_T}K^2\left(\frac{d_{tu}}{d_T}\right)\gamma_{(it,au)}^{(c_1c_2)}\\
&\times\left(\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left(\gamma_{(js_{(t)},bv_{(u)})}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},bv_{(u)})}^{(d_1d_2)}\right)\right)\\
&= o\left(1\right),
\end{aligned}
$$

because

$$
\begin{aligned}
&\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left(\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}-\gamma^{(d_1d_2)}_{\left(\bar{i}\bar{t},bv_{(u)}\right)}\right)\\
&=\frac{\ell_T}{d_T}\left[\frac{1}{n\ell_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}\right.\\
&\left.-\left(\frac{1}{n\ell_T}\right)^2\sum_{j=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{h=1}^{n}\sum_{w_{(t)}=1}^{\ell_T}\sum_{b=1}^{n}\sum_{v_{(u)}=1}^{\ell_T}\gamma^{(d_1d_2)}_{(hw_{(t)},bv_{(u)})}\right]\\
&=\frac{\ell_T}{d_T}\left[\frac{1}{n\ell_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s:d_{ts}\le d_T}\sum_{v:d_{uv}\le d_T}\gamma^{(d_1d_2)}_{(js,bv)}\right.\\
&\left.-\frac{1}{n\ell_T}\sum_{h=1}^{n}\sum_{b=1}^{n}\sum_{w:d_{tw}\le d_T}\sum_{v:d_{uv}\le d_T}\gamma^{(d_1d_2)}_{(hw,bv)}\right]\\
&=\frac{\ell_T}{d_T}\left[\frac{1}{n\ell_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s:d_{ts}\le d_T}\sum_{v:d_{tv}\le d_T}\gamma^{(d_1d_2)}_{(js,bv)}\right.\\
&\left.-\frac{1}{n\ell_T}\sum_{h=1}^{n}\sum_{b=1}^{n}\sum_{w:d_{tw}\le d_T}\sum_{v:d_{tv}\le d_T}\gamma^{(d_1d_2)}_{(hw,bv)}+o\left(1\right)\right]\\
&\to 0
\end{aligned}
$$

by (A.24) and Assumption 13. Therefore, $L_{1nT}=o\left(1\right)$ and $L_{2nT}=(1)$, which complete the proof of (A.20).

By a symmetric argument, we obtain the result that

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}K^2\left(\frac{d_{uv}}{d_T}\right)\\
&\times\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}\\
&=\bar{\mathcal{K}}_2J\left(c_1,c_2\right)J\left(d_1,d_2\right),
\end{aligned}
$$

which completes the proof of (A.20).

The next step is to prove (A.5). In view of previous derivations, it suffices to show that

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}\left[K\left(\frac{d_{ts_{(t)}}}{d_T}\right)-K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2\\
&\times\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}\\
&=0. \qquad\qquad (\text{A.27})
\end{aligned}
$$

$$\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t\in E_T}\sum_{s_{(t)}=1}^{\ell_T}\sum_{u\in E_T}\sum_{v_{(u)}=1}^{\ell_T}\left[K\left(\frac{d_{ts_{(t)}}}{d_T}\right)-K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}$$
$$=\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_1}\left[K\left(\frac{d_{ts_{(t)}}}{d_T}\right)-K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{\left(js_{(t)},bv_{(u)}\right)}$$
$$+\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_2}\left[K\left(\frac{d_{ts_{(t)}}}{d_T}\right)-K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{\left(js_{(t)},bv_{(u)}\right)}$$
$$:=F_{1nT}+F_{2nT},$$

where

$$\mathcal{J}_1=\left\{\left(t,s_{(t)},u,v_{(u)}\right):\left|d_{ts_{(t)}}-d_{uv_{(u)}}\right|\leq 2c_T\ \&\ t,u\in E_T\right\},$$

and

$$\mathcal{J}_2=\left\{\left(t,s_{(t)},u,v_{(u)}\right):\left|d_{ts_{(t)}}-d_{uv_{(u)}}\right|> 2c_T\ \&\ t,u\in E_T\right\}.$$

For F_{1nT}, we have

$$F_{1nT}\leq\left|\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{\left(t,s_{(t)},u,v_{(u)}\right)\in\mathcal{J}_1}\left[K\left(\frac{d_{ts_{(t)}}}{d_T}\right)-K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\right]^2\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\right|$$
$$\leq\frac{c_L^2}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{(t,s(t),u,v(u))\in\mathcal{J}_1}\left(\frac{d_{ts_{(t)}}}{d_T}-\frac{d_{uv_{(u)}}}{d_T}\right)^2\left|\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(c_1d_2)}_{(j_{(i)}s_{(t)},b_{(a)}v_{(u)})}\right|$$
$$\leq 4c_L^2\left(\frac{c_T^2}{d_T^2}\right)\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{t\in E_T}\sum_{u\in E_T}\left|\gamma^{(c_1c_2)}_{(it,au)}\right|\right)\left(\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}\right|\right)$$
$$=O\left(\frac{c_T^2}{d_T^2}\right),\tag{A.28}$$

since under Assumption 12

$$\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s_{(t)}=1}^{\ell_T}\sum_{v_{(u)}=1}^{\ell_T}\left|\gamma^{(d_1d_2)}_{(js_{(t)},bv_{(u)})}\right|$$
$$=\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{s:d_{ts}\leq d_T}\sum_{v:d_{uv}\leq d_T}\left|\gamma^{(d_1d_2)}_{(js,bv)}\right|$$
$$\leq\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{b=1}^{n}\sum_{\substack{s:d_{ts}\leq d_T\\ \text{or }d_{us}\leq d_T}}\sum_{\substack{v:d_{tv}\leq d_T\\ \text{or }d_{uv}\leq d_T}}\left|\gamma^{(d_1d_2)}_{(js,bv)}\right|$$
$$=O(1).\tag{A.29}$$

For F_{2nT} we note that if $\left|d_{ts_{(t)}}-d_{uv_{(u)}}\right|>2c_T$, then either $d_{tu}>c_T$ or $d_{s_{(t)}v_{(u)}}>c_T$. Otherwise, if both $d_{tu}\leq c_T$ and $d_{s_{(t)}v_{(u)}}\leq c_T$, then

$$d_{ts_{(t)}}-d_{uv_{(u)}}\leq d_{tu}+d_{uv_{(u)}}+d_{u_{(v)}s_{(t)}}-d_{uv_{(u)}}\leq 2c_T,$$

and

$$d_{ts_{(t)}} - d_{uv_{(u)}} \geq d_{ts_{(t)}} - d_{tu} - d_{ts_{(t)}} - d_{s_{(t)}v_{(u)}} \geq -2c_T.$$

These two inequalities imply that $\left| d_{ts_{(t)}} - d_{uv_{(u)}} \right| \leq 2c_T$, a contradiction. Without the loss of generality, we assume that $d_{tu} > c_T$ for $\left(t, s_{(t)}, u, v_{(u)}\right) \in \mathcal{J}_2$. In this case

$$\begin{aligned}
F_{2nT} &\leq \left| \frac{1}{n^2 T d_T} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{\left(t, s_{(t)}, u, v_{(u)}\right) \in \mathcal{J}_2} \left[K\left(\frac{d_{ts_{(t)}}}{d_T}\right) - K\left(\frac{d_{uv_{(u)}}}{d_T}\right) \right]^2 \gamma_{(it,au)}^{(c_1 c_2)} \gamma_{(j_{(i)} s_{(t)}, b_{(a)} v_{(u)})}^{(d_1 d_2)} \right| \\
&\leq \frac{4}{n^2 T d_T} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{\left(t, s_{(t)}, u, v_{(u)}\right) \in \mathcal{J}_2} (d_{tu})^q \left| \gamma_{(it,au)}^{(c_1 c_2)} \gamma_{(j s_{(t)}, b v_{(u)})}^{(d_1 d_2)} \right| (d_{tu})^{-q} \\
&\leq \frac{8}{(c_T)^q} \left(\frac{1}{nT} \sum_{i=1}^{n} \sum_{a=1}^{n} \sum_{t \in E_T} \sum_{u \in E_T} (d_{tu})^q \left| \gamma_{(it,au)}^{(c_1 c_2)} \right| \right) \left(\frac{1}{n d_T} \sum_{j=1}^{n} \sum_{b=1}^{n} \sum_{s_{(t)}=1}^{\ell_T} \sum_{v_{(u)}=1}^{\ell_T} \left| \gamma_{(j s_{(t)}, b v_{(u)})}^{(d_1 d_2)} \right| \right) \\
&= O\left(c_T^{-q}\right).
\end{aligned}$$

By choosing c_T such that $c_T \to \infty$ but $c_T/d_T \to 0$, we have

$$F_{1nT} = o\left(1\right) \text{ and } F_{2nT} = o(1)$$

and (A.21) is proved.

Next, we show that D_{2nT} is $o(1)$. For D_{2nT},

$$\begin{aligned}
&\frac{1}{n^2 T d_T} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{t \notin E_T} \sum_{s_{(t)}=1}^{\xi_{t,T}} \sum_{u=1}^{T} \sum_{v_{(u)}=1}^{\xi_{u,T}} K\left(\frac{d_{ts_{(t)}}}{d_T}\right) K\left(\frac{d_{uv_{(u)}}}{d_T}\right) \gamma_{(it,au)}^{(c_1 c_2)} \gamma_{(j s_{(t)}, b v_{(u)})}^{(d_1 d_2)} \\
&\leq \left| \frac{1}{n^2 T d_T} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{a=1}^{n} \sum_{b=1}^{n} \sum_{t \notin E_T} \sum_{s_{(t)}=1}^{\xi_{t,T}} \sum_{u=1}^{T} \sum_{v_{(u)}=1}^{\xi_{u,T}} K\left(\frac{d_{ts_{(t)}}}{d_T}\right) K\left(\frac{d_{uv_{(u)}}}{d_T}\right) \gamma_{(it,au)}^{(c_1 c_2)} \gamma_{(js, b v_{(u)})}^{(d_1 d_2)} \right| \\
&\leq \frac{1}{nT} \sum_{i=1}^{n} \sum_{a=1}^{n} \sum_{t \notin E_T} \sum_{u=1}^{T} \left| \gamma_{(it,au)}^{(c_1 c_2)} \right| \left(\frac{1}{n d_T} \sum_{j=1}^{n} \sum_{b=1}^{n} \sum_{s_{(t)}=1}^{\xi_{t,T}} \sum_{v_{(u)}=1}^{\xi_{u,T}} \left| \gamma_{(j s_{(t)}, b v_{(u)})}^{(d_1 d_2)} \right| \right) \\
&\leq \frac{1}{nT} \sum_{i=1}^{n} \sum_{a=1}^{n} \sum_{t \notin E_T} \sum_{u=1}^{T} \left| \gamma_{(it,au)}^{(c_1 c_2)} \right| \left(\frac{1}{n d_T} \sum_{j=1}^{n} \sum_{b=1}^{n} \sum_{s_{(t)}=1}^{\ell_T} \sum_{v_{(u)}=1}^{\ell_T} \left| \gamma_{(j s_{(t)}, b v_{(u)})}^{(d_1 d_2)} \right| \right) \\
&= o(1), \qquad \text{(A.30)}
\end{aligned}$$

as $d_T/T \to 0$. The last inequality holds by $\xi_{t,T} \leq \ell_T$ for all t by definition. With the same procedure, D_{3nT} are $o(1)$. Thus, $\lim_{n\to\infty} C_{2nT} = \bar{\mathcal{K}}_2 J\left(c_1, c_2\right) J\left(d_1, d_2\right).$

With the same procedure, it is straightforward that

$$\lim_{n\to\infty} \lim_{T\to\infty} C_{3nT} = \bar{\mathcal{K}}_2 J\left(c_1, d_2\right) J\left(c_2, d_1\right).$$

Therefore,

$$\lim_{n\to\infty} \lim_{T\to\infty} \frac{T}{d_T} \mathrm{cov}\left(\tilde{J}_{nT}^{DK}\left(c_1, d_1\right), \tilde{J}_{nT}\left(c_2, d_2\right) \right) = \bar{\mathcal{K}}_2 (J\left(c_1, c_2\right) J\left(d_1, d_2\right) + J\left(c_1, d_2\right) J\left(c_2, d_1\right)).$$

In terms of matrix form,

$$\lim_{n\to\infty}\lim_{T\to\infty}\frac{T}{d_T}\operatorname{var}\left(\operatorname{vec}\left(\tilde{J}_{nT}^{DK}\right)\right)=\bar{\mathcal{K}}_2(I_{pp}+\mathbb{K}_{pp})\left(J\otimes J\right),$$

where $J=[J\left(c,d\right)]$, $c,d=1,\ldots,p$.

(b) Asymptotic Bias

We have

$$\begin{aligned}d_T^q\left(E\tilde{J}_{nT}^{DK}-J_{nT}\right)&=\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\Gamma_{(it,js)}\left(d_{ts}\right)^q\frac{K\left(\frac{d_{ts}}{d_T}\right)-1}{\left(\frac{d_{ts}}{d_T}\right)^q}\\&=-K_qb_2^{(q)}+o(1).\end{aligned}$$

Therefore, $\lim_{n\to\infty}\lim_{T\to\infty}d_T^q(\tilde{J}_{nT}^{DK}-J_{nT})=-K_qb_2^{(q)}$.

(c) $\sqrt{\frac{T}{d_T}}\left(\hat{J}_{nT}^{DK}-J_{nT}\right)=O_p(1)$ **and** $\sqrt{\frac{T}{d_T}}\left(\hat{J}_{nT}^{DK}-\tilde{J}_{nT}^{DK}\right)=o_p(1)$

It suffices to show that $\sqrt{\frac{T}{d_T}}\left(\hat{J}_{nT}^{DK}-\tilde{J}_{nT}^{DK}\right)=o_p(1)$. We can consider the case that J_{nT} is a scalar random variable without loss of generality.

$$\begin{aligned}&\sqrt{\frac{T}{d_T}}\left(\hat{J}_{nT}^{DK}-\tilde{J}_{nT}^{DK}\right)\\=&\sqrt{\frac{T}{d_T}}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\left[\hat{V}_{(i,t)}\hat{V}_{(j,s)}'-V_{(i,t)}V_{(j,s)}'\right]\\=&\left(\sqrt{nT}\left(\hat{\beta}-\beta_0\right)\right)^2\sqrt{\frac{d_T}{T}}\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{it}^2\tilde{X}_{js}^2\\&-2\sqrt{nT}\left(\hat{\beta}-\beta_0\right)\sqrt{\frac{d_T}{T}}\frac{1}{nd_T\sqrt{nT}}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{js}^2\tilde{X}_{it}\tilde{u}_{it}\\&-\frac{2}{\sqrt{n^2Td_T}}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{it}u_{it}\tilde{X}_{js}\left(\bar{u}_j+\bar{u}_s-\bar{u}\right)\\&+\frac{1}{\sqrt{n^2Td_T}}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{it}\tilde{X}_{js}\left(\bar{u}_i+\bar{u}_t-\bar{u}\right)\left(\bar{u}_j+\bar{u}_s-\bar{u}\right)\\=&H_{1nT}+H_{2nT}+H_{3nT}+H_{4nT}.\end{aligned}$$

For H_{1nT}, it is $o_p(1)$ under Assumption 7(i) if

$$\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{it}^2\tilde{X}_{js}^2=O_p(1).$$

$$
\begin{aligned}
&\frac{1}{n^2Td_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{it}^2\tilde{X}_{js}^2\\
=&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}^2\left(\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{js}^2\right)\\
\leq&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}^2\left(\frac{\xi_{t,T}}{d_T}\right)\left(\frac{1}{n\xi_{t,T}}\sum_{j=1}^{n}\sum_{s:d_{ts}\leq d_T}\tilde{X}_{js}^2\right)
\end{aligned}
$$

By Assumption 7(ii), we obtain

$$
\begin{aligned}
P\left(\frac{1}{n\xi_{t,T}}\sum_{j=1}^{n}\sum_{s:d_{ts}\leq d_T}\tilde{X}_{js}^2>\Delta\right)&\leq\frac{1}{\Delta n\xi_{t,T}}\sum_{j=1}^{n}\sum_{s:d_{ts}\leq d_T}E\tilde{X}_{js}^2\\
&\to 0
\end{aligned}
$$

as $\Delta\to\infty$, which implies $(nd_T)^{-1}\sum_{j=1}^{n}\sum_{s:d_{ts}\leq d_T}\tilde{X}_{it}^2=O_p(1)$ uniformly. Using the same procedure, $(nT)^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}^2=O_p(1)$. As $\xi_{t,T}/d_T=O(1)$ due to Assumption 4 and $\xi_{t,T}\leq\ell_T$, $H_{1nT}=o_p(1)$.

For $H_{2nT,}$ it can be rewritten as

$$
-2\sqrt{nT}\left(\hat{\beta}-\beta_0\right)\sqrt{\frac{d_T}{T}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}\tilde{u}_{it}\left(\frac{1}{nd_T}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{js}^2\right)
$$

and the part in the parenthesis is $O_p(1)$ uniformly as shown above.

$$
\begin{aligned}
\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}\tilde{u}_{it}&=\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}u_{it}\\
&=O_p(1)
\end{aligned}
$$

under Assumption 7(iv), which implies that $H_{2nT}=o_p(1)$.

For $H_{3nT,}$ we need to show that for all i and t

$$
\frac{1}{\sqrt{nd_T}}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{js}\left(\bar{u}_j+\bar{u}_s-\bar{u}\right)=o_p(1). \tag{A.31}
$$

As $\bar{u}_j+\bar{u}_s-\bar{u}=o_p(1)$ uniformly, it suffice to show $(nd_T)^{-1/2}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{js}=$

$O_p(1)$ for (A.31).

$$
\begin{aligned}
&P\left(\left|\frac{1}{\sqrt{nd_T}}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ts}}{d_T}\right)\tilde{X}_{js}\right|>\Delta\right)\\
&\le \frac{1}{\Delta^2 nd_T}E\left[\sum_{j=1}^{n}\sum_{s_{(t)}=1}^{\xi_{t,T}}K\left(\frac{d_{ts_{(t)}}}{d_T}\right)\tilde{X}_{js_{(t)}}\right]^2\\
&\le \frac{2}{\Delta^2 nd_T}\sum_{j=1}^{n}\sum_{s_{(t)}=1}^{\xi_{t,T}}E\left[\tilde{X}_{js_{(t)}}\right]^2\\
&\to 0,
\end{aligned}
$$

as $\Delta\to\infty$ under Assumption 7(ii). Therefore, H_{3nT} is also $o_p(1)$. With the similar procedures, we can show that H_{4nT} is $o_p(1)$.

As a result, $\sqrt{\frac{T}{d_T}}\left(\hat{J}_{nT}^{DK}-\tilde{J}_{nT}^{DK}\right)=o_p(1)$.

(d) Asymptotic MSE

The first equality holds by Theorem 2(c). For the last equality of Theorem 2(d), since

$$
\frac{T}{d_T}=\frac{d_T^{2q}}{d_T^{2q+1}/T}=\frac{d_T^{2q}}{\tau+o(1)},
$$

we have

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty}MSE\left(\frac{T}{d_T},\tilde{J}_{nT}^{DK},S_{nT}\right)\\
&=\lim_{n\to\infty}\lim_{T\to\infty}\frac{T}{d_T}\text{ vec }\left(E\tilde{J}_{nT}^{DK}-J_{nT}\right)'S_{nT}\text{ vec}\left(E\tilde{J}_{nT}^{DK}-J_{nT}\right)+\lim_{n\to\infty}\lim_{T\to\infty}\frac{T}{d_T}\bar{\mathcal{K}}_2\mathrm{tr}\left(S_{nT}\mathrm{var}(\text{vec }\tilde{J}_{nT}^{DK})\right)\\
&=\frac{1}{\tau}K_q^2\text{ vec }\left(b_2^{(q)}\right)'S\text{ vec }\left(b_2^{(q)}\right)+\bar{\mathcal{K}}_2 tr\left[S(I_{pp}+K_{pp})(J\otimes J)\right],
\end{aligned}
$$

where the last equality holds by Theorem 2 (a) and (b).

Proof of Theorem 3

(a) $\sqrt{n}\left(\tilde{J}_{nT}^{A}-J_{nT}^{A}\right)=O\left(1\right)$ and (b) Asymptotic variance

Let $\varphi_{lkcd}=\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}r_{(i,t),l}^{(c)}r_{(i,s),k}^{(d)}$. We have

$$
\begin{aligned}
&n\cdot cov\left(\tilde{J}_{nT}\left(c_1,d_1\right),\tilde{J}_{nT}\left(c_2,d_2\right)\right)\\
&=\frac{1}{nT^2}E\left[\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\varphi_{lkc_1d_1}(\varepsilon_l\varepsilon_k-E\varepsilon_l\varepsilon_k)\sum_{e=1}^{nTp}\sum_{f=1}^{nTp}\varphi_{efc_2d_2}(\varepsilon_e\varepsilon_f-E\varepsilon_e\varepsilon_f)\right]\\
&=\frac{1}{nT^2}E\left[\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\sum_{e=1}^{nTp}\sum_{f=1}^{nTp}\varphi_{lkc_1d_1}\varphi_{efc_2d_2}(\varepsilon_l\varepsilon_k\varepsilon_e\varepsilon_f-\varepsilon_l\varepsilon_kE\varepsilon_e\varepsilon_f-\varepsilon_e\varepsilon_fE\varepsilon_l\varepsilon_k+E\varepsilon_l\varepsilon_kE\varepsilon_e\varepsilon_f)\right]\\
&=\frac{1}{nT^2}E\left[\sum_{l=1}^{nTp}\varphi_{llc_1d_1}\varphi_{llc_2d_2}\left(E\varepsilon_l^4-3\right)+\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\varphi_{lkc_1d_1}\varphi_{lkc_2d_2}+\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\varphi_{lkc_1d_1}\varphi_{klc_2d_2}\right]\\
&:=C_{1nT}+C_{2nT}+C_{3nT},
\end{aligned}
$$

where

$$
\begin{aligned}
C_{1nT}&=\frac{1}{nT^2}\sum_{l=1}^{nTp}\left(E\varepsilon_l^4-3\right)\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}r_{(i,t),l}^{(c_1)}r_{(i,s),l}^{(d_1)}\sum_{a=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T}r_{(a,u),l}^{(c_2)}r_{(a,v),l}^{(d_2)}\right],\\
C_{2nT}&=\frac{1}{nT^2}\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}r_{(i,t),l}^{(c_1)}r_{(i,s),k}^{(d_1)}\sum_{a=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T}r_{(a,u),l}^{(c_2)}r_{(a,v),k}^{(d_2)}\right],\\
C_{3nT}&=\frac{1}{nT^2}\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}r_{(i,t),l}^{(c_1)}r_{(j,s),k}^{(d_1)}\sum_{a=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T}r_{(a,u),k}^{(c_2)}r_{(a,v),l}^{(d_2)}\right].
\end{aligned}
$$

In the proof of Theorem 1, we show that $C_{1nT}=o\left(1\right)$ under Assumptions 1 and 2. Thus, in order to show (a), we need to show that $C_{2nT}=O\left(1\right)$ and $C_{3nT}=O\left(1\right)$. For asymptotic variance, we show that $\lim_{n\to\infty}\lim_{T\to\infty}C_{2nT}=J\left(c_1,c_2\right)J\left(d_1,d_2\right)$ and $\lim_{n\to\infty}\lim_{T\to\infty}C_{3nT}=J\left(c_1,d_2\right)J\left(c_2,d_1\right)$.

C_{2nT} can be restated as

$$
\begin{aligned}
&\frac{1}{nT^2}\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}r_{(i,t),l}^{(c_1)}r_{(i,s),k}^{(d_1)}r_{(a,u),l}^{(c_2)}r_{(a,v),k}^{(d_2)}\\
&=\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(is,av)}^{(d_1d_2)}\\
&=\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\gamma_{(it,iu)}^{(c_1c_2)}\gamma_{(is,iv)}^{(d_1d_2)}
\end{aligned}
\qquad\text{(A.32)}
$$

The last equality holds because $\gamma^{(cd)}_{(it,au)} = 0$ if $i \neq a$. (A.32) is rewritten as

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}\sum_{u=1}^{T}\gamma^{(c_1c_2)}_{(it,iu)}\right)\left(\frac{1}{T}\sum_{s=1}^{T}\sum_{v=1}^{T}\gamma^{(d_1d_2)}_{(is,iv)}\right)\\
&=\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{l=1}^{nTp}\sum_{t=1}^{T}r^{(c_1)}_{(i,t),l}\sum_{u=1}^{T}r^{(c_2)}_{(i,u),l}\right)\left(\frac{1}{T}\sum_{l=1}^{nTp}\sum_{s=1}^{T}r^{(d_1)}_{(i,s),l}\sum_{v=1}^{T}r^{(d_2)}_{(i,v),l}\right)\\
&=O(1),
\end{aligned}$$

where the last equality holds by Assumptions 2 and 10. By symmetry, $C_{3nT} = O(1)$, which implies $\sqrt{n}$-convergence.

Let $\gamma^{(d_1d_2)}_{(\bar{\imath}s,\bar{\imath}v)} = n^{-1}\sum_{j=1}^{n}\gamma^{(d_1d_2)}_{(js,jv)}$. Then,

$$\begin{aligned}
&\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\gamma^{(c_1c_2)}_{(it,iu)}\gamma^{(d_1d_2)}_{(is,iv)}\\
&=\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\gamma^{(c_1c_2)}_{(it,iu)}\gamma^{(d_1d_2)}_{(\bar{\imath}s,\bar{\imath}v)}\\
&+\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\gamma^{(c_1c_2)}_{(it,iu)}\left(\gamma^{(d_1d_2)}_{(is,iv)}-\gamma^{(d_1d_2)}_{(\bar{\imath}s,\bar{\imath}v)}\right)\\
&=L_{1nT}+L_{2nT}.
\end{aligned}\tag{A.33}$$

L_{1nT} is rewritten as

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{u=1}^{T}\gamma^{(c_1c_2)}_{(it,iu)}\left(\frac{1}{T}\sum_{s=1}^{T}\sum_{v=1}^{T}\gamma^{(d_1d_2)}_{(\bar{\imath}s,\bar{\imath}v)}\right)\\
&=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{u=1}^{T}\gamma^{(c_1c_2)}_{(it,iu)}\left(\frac{1}{nT}\sum_{j=1}^{n}\sum_{s=1}^{T}\sum_{v=1}^{T}\gamma^{(d_1d_2)}_{(js,kv)}\right)\\
&\to J(c_1,c_2)\,J(d_1,d_2).
\end{aligned}\tag{A.34}$$

as $n, T\to\infty$.

For L_{2nT} in (A.33),

$$\begin{aligned}
&\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\gamma^{(c_1c_2)}_{(it,iu)}\left(\gamma^{(d_1d_2)}_{(is,iv)}-\gamma^{(d_1d_2)}_{(\bar{\imath}s,\bar{\imath}v)}\right)\\
&=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{u=1}^{T}\gamma^{(c_1c_2)}_{(it,iu)}\frac{1}{T}\sum_{s=1}^{T}\sum_{v=1}^{T}\left(\gamma^{(d_1d_2)}_{(is,iv)}-\gamma^{(d_1d_2)}_{(\bar{\imath}s,\bar{\imath}v)}\right)\\
&=o(1)
\end{aligned}\tag{A.35}$$

because

$$\begin{aligned}
\frac{1}{T}\sum_{s=1}^{T}\sum_{v=1}^{T}\left(\gamma^{(d_1d_2)}_{(is,iv)}-\gamma^{(d_1d_2)}_{(\bar{\imath}s,\bar{\imath}v)}\right)&=\frac{1}{T}\sum_{s=1}^{T}\sum_{v=1}^{T}\gamma^{(d_1d_2)}_{(is,iv)}-\frac{1}{nT}\sum_{j=1}^{n}\sum_{s=1}^{T}\sum_{v=1}^{T}\gamma^{(d_1d_2)}_{(js,jv)}\\
&\to 0
\end{aligned}$$

for all i under Assumption 11. From (A.34) and (A.35)

$$\lim_{n\to\infty}\lim_{T\to\infty} C_{2nT} = J\left(c_1,c_2\right)J\left(d_1,d_2\right).$$

With the same procedure, it is straightforward that

$$\lim_{n\to\infty}\lim_{T\to\infty} C_{3nT} = J\left(c_1,d_2\right)J\left(c_2,d_1\right).$$

Therefore,

$$\lim_{n\to\infty}\lim_{T\to\infty} n\cdot \mathrm{cov}\left(\tilde{J}^A_{nT}\left(c_1,d_1\right),\tilde{J}^A_{nT}\left(c_2,d_2\right)\right) = \left(J\left(c_1,c_2\right)J\left(d_1,d_2\right)+J\left(c_1,d_2\right)J\left(c_2,d_1\right)\right).$$

In terms of matrix form,

$$\lim_{n\to\infty}\lim_{T\to\infty} n\cdot \mathrm{var}\left(\mathrm{vec}\left(\tilde{J}^A_{nT}\right)\right) = \left(I_{pp}+\mathbb{K}_{pp}\right)\left(J\otimes J\right),$$

where $J=\left[J\left(c,d\right)\right]$, $c,d=1,\ldots,p$.

(c) $\sqrt{\frac{nT}{\ell_n d_T}}\left(\hat{J}_{nT}-J_{nT}\right)=O_p(1)$ **and** $\sqrt{\frac{nT}{\ell_n d_T}}\left(\hat{J}_{nT}-\tilde{J}_{nT}\right)=o_p(1)$

By (a) and (b), the first part of (c) is implied by the second part. Therefore, it suffices to show that $\sqrt{n}\left(\hat{J}^A_{nT}-\tilde{J}^A_{nT}\right)=o_p(1)$. In consequence, we can consider the case that J_{nT} is a scalar random variable without loss of generality.

$$\begin{aligned}
\sqrt{n}\left(\hat{J}^A_{nT}-\tilde{J}^A_{nT}\right) =& \sqrt{n}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\left[\hat{V}_{(i,t)}\hat{V}'_{(i,s)}-V_{(i,t)}V'_{(i,s)}\right]\\
=& \left(\sqrt{nT}\left(\hat{\beta}-\beta_0\right)\right)^2\frac{1}{\sqrt{nT}}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{X}^2_{it}\tilde{X}^2_{is}\\
&-2\sqrt{nT}\left(\hat{\beta}-\beta_0\right)\frac{1}{\sqrt{T}}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{X}^2_{is}\tilde{X}_{it}\tilde{u}_{it}\\
&-\frac{2}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{X}_{it}u_{it}\tilde{X}_{is}\left(\bar{u}_i+\bar{u}_s-\bar{u}\right)\\
&+\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{X}_{it}\tilde{X}_{is}\left(\bar{u}_i+\bar{u}_t-\bar{u}\right)\left(\bar{u}_i+\bar{u}_s-\bar{u}\right)\\
=& H_{1nT}+H_{2nT}+H_{3nT}+H_{4nT}.
\end{aligned}$$

For H_{1nT}, it is $o_p(1)$ under Assumption 7(i) if

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{X}^2_{it}\tilde{X}^2_{is}=O_p(1).$$

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\tilde{X}^2_{it}\tilde{X}^2_{is}=\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}^2_{it}\left(\frac{1}{T}\sum_{s=1}^{T}\tilde{X}^2_{is}\right)$$

By Assumption 7(ii), we obtain

$$
\begin{aligned}
P\left(\frac{1}{T}\sum_{s=1}^{T}\tilde{X}_{is}^{2}>\Delta\right) &\leq \frac{1}{\Delta T}\sum_{s=1}^{T}E\tilde{X}_{is}^{2} \\
&\rightarrow 0
\end{aligned}
$$

as $\Delta\rightarrow\infty$, which implies $T^{-1}\sum_{s=1}^{T}\tilde{X}_{is}^{2}=O_{p}\left(1\right)$ uniformly. Using the same procedure, $\left(nT\right)^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}^{2}=O_{p}\left(1\right)$ and therefore $H_{1nT}=o_{p}(1)$.

For $H_{2nT,}$ it can be rewritten as

$$
-2\sqrt{nT}\left(\hat{\beta}-\beta_{0}\right)\frac{1}{\sqrt{n}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}\tilde{u}_{it}\left(\frac{1}{T}\sum_{s=1}^{T}\tilde{X}_{is}^{2}\right)
$$

and the part in the parenthesis is $O_{p}\left(1\right)$ uniformly as shown above.

$$
\begin{aligned}
\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}\tilde{u}_{it} &= \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}u_{it} \\
&= O_{p}\left(1\right)
\end{aligned}
$$

under Assumption 7(iv), which implies that $H_{2nT}=o_{p}(1)$.

For $H_{3nT,}$ we need to show that for all i and t

$$
\frac{1}{\sqrt{T}}\sum_{s=1}^{T}\tilde{X}_{is}\left(\bar{u}_{i}+\bar{u}_{s}-\bar{u}\right)=o_{p}\left(1\right). \tag{A.36}
$$

As $\bar{u}_{j}+\bar{u}_{s}-\bar{u}=o_{p}\left(1\right)$ uniformly, it suffice to show $T^{-1/2}\sum_{s=1}^{T}\tilde{X}_{is}=O_{p}\left(1\right)$ uniformly for (A.36).

$$
\begin{aligned}
P\left(\left|\frac{1}{\sqrt{T}}\sum_{s=1}^{T}\tilde{X}_{is}\right|>\Delta\right) &\leq \frac{1}{\Delta^{2}T}E\left[\sum_{s=1}^{T}\tilde{X}_{is}\right]^{2} \\
&\leq \frac{2}{\Delta^{2}T}\sum_{s=1}^{T}E\left[\tilde{X}_{is}\right]^{2} \\
&\rightarrow 0,
\end{aligned}
$$

as $\Delta\rightarrow\infty$ under Assumption 7(ii). Therefore, H_{3nT} is also $o_{p}(1)$. With the similar procedures, we can show that H_{4nT} is $o_{p}(1)$. As a result, $\sqrt{n}\left(\hat{J}_{nT}^{A}-\tilde{J}_{nT}^{A}\right)=o_{p}(1)$.

Proof of Theorem 4

(a) Asymptotic Variance

Let $\varphi_{lkcd} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{t=1}^{T} \sum_{s=1}^{T} r_{(i,t),l}^{(c)} r_{(j,s),k}^{(d)} K\left(\frac{d_{ij}}{d_n}\right)$. We have

$$
\begin{aligned}
&\frac{n}{\ell_n} cov\left(\tilde{J}_{nT}^{KP}(c_1,d_1), \tilde{J}_{nT}^{KP}(c_2,d_2)\right) \\
&= \frac{1}{nT^2\ell_n} E\left[\sum_{l=1}^{nTp}\sum_{k=1}^{nTp} \varphi_{lkc_1d_1}(\varepsilon_l\varepsilon_k - E\varepsilon_l\varepsilon_k) \sum_{e=1}^{nTp}\sum_{f=1}^{nTp} \varphi_{efc_2d_2}(\varepsilon_e\varepsilon_f - E\varepsilon_e\varepsilon_f)\right] \\
&= \frac{1}{nT^2\ell_n} E\left[\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\sum_{e=1}^{nTp}\sum_{f=1}^{nTp} \varphi_{lkc_1d_1}\varphi_{efc_2d_2}(\varepsilon_l\varepsilon_k\varepsilon_e\varepsilon_f - \varepsilon_l\varepsilon_k E\varepsilon_e\varepsilon_f - \varepsilon_e\varepsilon_f E\varepsilon_l\varepsilon_k + E\varepsilon_l\varepsilon_k E\varepsilon_e\varepsilon_f)\right] \\
&= \frac{1}{nT^2\ell_n} E\left[\sum_{l=1}^{nTp} \varphi_{llc_1d_1}\varphi_{llc_2d_2}\left(E\varepsilon_l^4 - 3\right) + \sum_{l=1}^{nTp}\sum_{k=1}^{nTp} \varphi_{lkc_1d_1}\varphi_{lkc_2d_2} + \sum_{l=1}^{nTp}\sum_{k=1}^{nTp} \varphi_{lkc_1d_1}\varphi_{klc_2d_2}\right] \\
&:= C_{1nT} + C_{2nT} + C_{3nT},
\end{aligned}
$$

where

$$
\begin{aligned}
C_{1nT} &= \frac{1}{nT^2\ell_n} \sum_{l=1}^{nTp} \left(E\varepsilon_l^4 - 3\right) \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} r_{(i,t),l}^{(c_1)} r_{(j,s),l}^{(d_1)} K\left(\frac{d_{ij}}{d_n}\right)\right. \\
&\quad \left.\times \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T} r_{(a,u),l}^{(c_2)} r_{(b,v),l}^{(d_2)} K\left(\frac{d_{ab}}{d_n}\right)\right], \\
C_{2nT} &= \frac{1}{nT^2\ell_n} \sum_{l=1}^{nTp}\sum_{k=1}^{nTp} \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} r_{(i,t),l}^{(c_1)} r_{(j,s),k}^{(d_1)} K\left(\frac{d_{ij}}{d_n}\right)\right. \\
&\quad \left.\times \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T} r_{(a,u),l}^{(c_2)} r_{(b,v),k}^{(d_2)} K\left(\frac{d_{ab}}{d_n}\right)\right], \\
C_{3nT} &= \frac{1}{nT^2\ell_n} \sum_{l=1}^{nTp}\sum_{k=1}^{nTp} \left[\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T} r_{(i,t),l}^{(c_1)} r_{(j,s),k}^{(d_1)} K\left(\frac{d_{ij}}{d_n}\right)\right. \\
&\quad \left.\times \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{u=1}^{T}\sum_{v=1}^{T} r_{(a,u),k}^{(c_2)} r_{(b,v),l}^{(d_2)} K\left(\frac{d_{ab}}{d_n}\right)\right].
\end{aligned}
$$

As shown in the proof of Theorem 1, $C_{1nT} = o(1)$ under Assumptions 1 and 2. C_{2nT} can be restated as

$$
\frac{1}{nT^2\ell_n} \sum_{i=1}^{n} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{a=1}^{n} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T} K\left(\frac{d_{ij_{(i)}}}{d_n}\right) K\left(\frac{d_{ab_{(a)}}}{d_n}\right) \gamma_{(it,au)}^{(c_1c_2)} \gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}. \tag{A.37}
$$

In order to consider the boundary effects, we decompose C_{2nT} as follows:

$$\begin{aligned}
&\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}\\
&+\frac{1}{nT^2\ell_n}\sum_{i\notin E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a=1}^{n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}\\
&+\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\notin E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}\\
&=D_{1nT}+D_{2nT}+D_{3nT} \qquad \text{(A.38)}
\end{aligned}$$

In the following, we show that D_{1nT} converges to $\bar{\mathcal{K}}_1 J\left(c_1,c_2\right)J\left(d_1,d_2\right)$ and the other terms become negiligible as n and T increase.

For D_{1nT}, the first step is to show that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}\\
&=\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K^2\left(\frac{d_{ab_{(a)}}}{d_n}\right)\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}\\
&=\bar{\mathcal{K}}_1 J\left(c_1,c_2\right)J\left(d_1,d_2\right), \qquad \text{(A.39)}
\end{aligned}$$

and the next step is to prove that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}\\
&=\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K\left(\frac{d_{ij_{(i)}}}{d_n}\right)K\left(\frac{d_{ab_{(a)}}}{d_n}\right)\\
&\times\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}. \qquad \text{(A.40)}
\end{aligned}$$

For (A.39), let $\gamma^{(d_1d_2)}_{(\bar{\imath}\bar{t},b_{(a)}v_{(u)})} = (\ell_n T)^{-1}\sum_{h_{(i)}=1}^{\ell_{i,n}}\sum_{w=1}^{T}\gamma^{(d_1d_2)}_{(h_{(i)}w,b_{(a)}v)}$. Then,

$$
\begin{aligned}
&\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{(j_{(i)}s,b_{(a)}v)}\\
&=\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)\gamma^{(c_1c_2)}_{(it,au)}\gamma^{(d_1d_2)}_{\left(\bar{\imath}\bar{t},b_{(a)}v\right)}\\
&+\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)\gamma^{(c_1c_2)}_{(it,au)}\\
&\times\left(\gamma^{(d_1d_2)}_{(j_{(i)}s,b_{(a)}v)}-\gamma^{(d_1d_2)}_{\left(\bar{\imath}\bar{t},b_{(a)}v\right)}\right). \qquad (A.41)\\
&=L_{1nT}+L_{2nT} \qquad (A.42)
\end{aligned}
$$

L_{1nT} is rewritten as

$$
\begin{aligned}
&\frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t=1}^{T}\sum_{u=1}^{T}\gamma^{(c_1c_2)}_{(it,au)}\left(\frac{1}{\ell_nT}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s=1}^{T}\sum_{v=1}^{T}\gamma^{(d_1d_2)}_{(j_{(i)}s,b_{(a)}v)}\right)\left(\frac{1}{\ell_n}\sum_{j=1}^{n}K^2\left(\frac{d_{ij}}{d_n}\right)\right)\\
&=\frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t=1}^{T}\sum_{u=1}^{T}\gamma^{(c_1c_2)}_{(it,au)}\frac{1}{\ell_nT}cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T}V^{(d_1)}_{(j,s)},\sum_{b:d_{ab}\le d_n}\sum_{v=1}^{T}V^{(d_2)}_{(b,v)}\right)\\
&\times\left(\frac{1}{\ell_n}\sum_{j=1}^{n}K^2\left(\frac{d_{ij}}{d_n}\right)\right)\\
&:=G_{1nT}+G_{2nT}
\end{aligned}
$$

where

$$
\begin{aligned}
G_{1nT}&=\frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t=1}^{T}\sum_{u=1}^{T}\gamma^{(c_1c_2)}_{(it,au)}1\left\{d_{ia}\le c_n\right\}\\
&\times\frac{1}{\ell_nT}cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T}V^{(d_1)}_{(j,s)},\sum_{b:d_{ab}\le d_n}\sum_{v=1}^{T}V^{(d_2)}_{(b,v)}\right)\\
&\times\left(\frac{1}{\ell_n}\sum_{j=1}^{n}K^2\left(\frac{d_{ij}}{d_n}\right)\right)
\end{aligned}
$$

and

$$G_{2nT} = \frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t=1}^{T}\sum_{u=1}^{T}\gamma_{(it,au)}^{(c_1c_2)}\left[1\left\{d_{ia} > c_n\right\} + 1\left\{d_{ia} \le c_n\right\}\right]$$
$$\times \frac{1}{\ell_n T} cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n}\sum_{v=1}^{T} V_{(b,v)}^{(d_2)}\right)\left(\frac{1}{\ell_n}\sum_{j=1}^{n} K^2\left(\frac{d_{ij}}{d_n}\right)\right)$$
$$= o(1)$$

as $c_n \to \infty$.

It suffices to consider G_{1nT}. When $d_{ia} \le c_n$ we have

$$cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n}\sum_{v=1}^{T} V_{(b,v)}^{(d_2)}\right)$$
$$= cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T} V_{(j,s)}^{(d_1)}, \sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T} V_{(j,s)}^{(d_2)}\right)$$
$$+ cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n}\sum_{v=1}^{T} V_{(b,v)}^{(d_2)} - \sum_{b:d_{ib}\le d_n}\sum_{v=1}^{T} V_{(b,v)}^{(d_2)}\right)$$

but

$$\sum_{b:d_{ab}\le d_n}\sum_{v=1}^{T} V_{(b,v)}^{(d_2)} - \sum_{b:d_{ib}\le d_n}\sum_{v=1}^{T} V_{(b,v)}^{(d_2)} = \sum_{b:d_{ab}\le d_n, d_{ib}>d_n}\sum_{v=1}^{T} V_{(b,v)}^{(d_2)}$$

Now $d_{ab} \le d_n$ and $d_{ia} \le c_n$ implies that $d_{bi} \le d_n + c_n$. As the result,

$$\frac{1}{\ell_n T}\left|cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n, d_{ib}>d_n}\sum_{v=1}^{T} V_{(b,v)}^{(d_2)}\right)\right|$$
$$= \frac{1}{\ell_n T}\left|\sum_{j:d_{ij}\le d_n}\sum_{b:d_{ab}\le d_n, d_{ib}>d_n}\sum_{s=1}^{T}\sum_{v=1}^{T} EV_{(j,s)}^{(d_1)}V_{(b,v)}^{(d_2)}\right|$$
$$\le \frac{1}{\ell_n T}\sum_{j:d_{ij}\le d_n}\sum_{b:d_n<d_{ib}\le d_n+c_n}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|EV_{(j,s)}^{(d_1)}V_{(b,v)}^{(d_2)}\right|$$
$$= o(1),$$

by choosing c_n such that $\sum_{b=1}^{n} 1\{d_{ib} \le d_n + c_n\} \le \bar{C}\ell_n$ for all i and for some constant $\bar{C}$. Hence

$$\frac{1}{\ell_n T} cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T} V_{(j,s)}^{(d_1)}, \sum_{b:d_{ab}\le d_n}\sum_{v=1}^{T} V_{(b,v)}^{(d_2)}\right)$$
$$= \frac{1}{\ell_n T} cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T} V_{(j,s)}^{(d_1)}, \sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T} V_{(j,s)}^{(d_2)}\right) + o\left(1\right) \qquad \text{(A.43)}$$

where $o(1)$ term holds uniformly over i and t.

Now using Assumption 15, we have

$$\begin{aligned}
G_{1nT} &= \frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t=1}^{T}\sum_{u=1}^{T}\gamma_{(it,au)}^{(c_1c_2)} \\
&\times \frac{1}{\ell_n T} cov\left(\sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T}V_{(j,s)}^{(d_1)}, \sum_{j:d_{ij}\le d_n}\sum_{s=1}^{T}V_{(j,s)}^{(d_2)}\right) \\
&\times \left(\frac{1}{\ell_n}\sum_{j=1}^{n}K^2\left(\frac{d_{ij}}{d_n}\right)\right)(1+o(1)) \\
&\to \bar{\mathcal{K}}_1 J\left(c_1,c_2\right)J(d_1,d_2)
\end{aligned}$$

if we choose the sequence of d_n such that $n_1/n \to 1$.

For L_{2nT} in (A.42), the first step is to show

$$\begin{aligned}
&\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\left(K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)-K^2\left(\frac{d_{ia}}{d_n}\right)\right) \\
&\times \gamma_{(it,au)}^{(c_1c_2)}\left(\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v)}^{(d_1d_2)}\right)=o(1), \qquad \text{(A.44)}
\end{aligned}$$

and the second step is to prove

$$\begin{aligned}
&\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K^2\left(\frac{d_{ia}}{d_n}\right) \\
&\times \gamma_{(it,au)}^{(c_1c_2)}\left(\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v)}^{(d_1d_2)}\right)=o(1). \qquad \text{(A.45)}
\end{aligned}$$

For (A.44),

$$\begin{aligned}
&\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\left(K^2\left(\frac{d_{ij_{(i)}}}{d_n}\right)-K^2\left(\frac{d_{ia}}{d_n}\right)\right) \\
&\times \gamma_{(it,au)}^{(c_1c_2)}\left(\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v)}^{(d_1d_2)}\right) \\
&\le \frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(it,au)}^{(c_1c_2)}\right|\left|\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v)}^{(d_1d_2)}\right| \\
&= \frac{1}{nT}\sum_{(i,a)\in\mathcal{F}_1}\sum_{t=1}^{T}\sum_{u=1}^{T}\left|\gamma_{(it,au)}^{(c_1c_2)}\right|\left(\frac{1}{\ell_n T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v)}^{(d_1d_2)}\right|\right) \\
&+ \frac{1}{nT}\sum_{(i,a)\in\mathcal{F}_2}\sum_{t=1}^{T}\sum_{u=1}^{T}\left|\gamma_{(it,au)}^{(c_1c_2)}\right|\left(\frac{1}{\ell_n T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v)}^{(d_1d_2)}\right|\right) \\
&= M_{1nT}+M_{2nT},
\end{aligned}$$

where

$$\mathcal{F}_1 = \{(i,a) : d_{ia} \le f_n \ \&\ i,a \in E_n\},$$

and

$$\mathcal{F}_2 = \{(i,a) : d_{ia} > f_n \ \&\ i,a \in E_n\},$$

in which $f_n/d_n = O(1)$.

For M_{1nT}, we obtain

$$\begin{aligned} M_{1nT} \le& \left(\frac{\ell_n}{n}\right)\frac{1}{\ell_n T}\sum_{(i,a)\in\mathcal{F}_1}\sum_{t=1}^{T}\sum_{u=1}^{T}\left|\gamma_{(it,au)}^{(c_1c_2)}\right|\left(\frac{1}{\ell_n T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}\right|\right. \\ &\left.+\frac{1}{\ell_n^2T^2}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{s_{(t)}=1}^{\ell_T}\sum_{h_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{w=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(h_{(i)}w,b_{(a)}v)}^{(d_1d_2)}\right|\right) \\ &= O\left(\frac{\ell_n}{n}\right). \end{aligned}$$

For M_{2nT},

$$\begin{aligned} M_{2nT} \le& \frac{1}{f_n^q}\frac{1}{nT}\sum_{(i,a)\in\mathcal{F}_2}\sum_{t=1}^{T}\sum_{u=1}^{T}\left|\gamma_{(it,au)}^{(c_1c_2)}\right| d_{ia}^q \\ &\times\left(\frac{1}{\ell_n T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}\right|\right. \\ &\left.+\frac{1}{\ell_n^2T^2}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{s=1}^{T}\sum_{h_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{w=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(h_{(i)}w,b_{(a)}v)}^{(d_1d_2)}\right|\right) \\ &= O\left(f_n^{-q}\right). \end{aligned}$$

By choosing f_n such that $f_n = O(d_n)$, we obtain

$$M_{1nT} = o(1) \text{ and } M_{2nT} = o(1).$$

Therefore, (A.9) holds.

The next step is to show (A.10).

$$\begin{aligned} &\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K^2\left(\frac{d_{ia}}{d_n}\right) \\ &\times\gamma_{(it,au)}^{(c_1c_2)}\left(\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v)}^{(d_1d_2)}\right) \\ &=\frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t=1}^{T}\sum_{u=1}^{T}K\left(\frac{d_{ia}}{d_n}\right)\gamma_{(it,au)}^{(c_1c_2)} \\ &\times\left(\frac{1}{\ell_n T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s=1}^{T}\sum_{v=1}^{T}\left(\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}-\gamma_{(\bar{\imath}\bar{t},b_{(a)}v)}^{(d_1d_2)}\right)\right) \\ &= o(1), \end{aligned}$$

because

$$
\begin{aligned}
&\frac{1}{\ell_n T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s=1}^{T}\sum_{v=1}^{T}\left(\gamma^{(d_1 d_2)}_{(j_{(i)}s,b_{(a)}v)}-\gamma^{(d_1 d_2)}_{(\bar{\imath}\bar{t},b_{(a)}v)}\right)\\
&=\Bigg[\frac{1}{\ell_n T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s=1}^{T}\sum_{v=1}^{T}\gamma^{(d_1 d_2)}_{(j_{(i)}s,b_{(a)}v)}\\
&-\left(\frac{1}{\ell_n T}\right)^2\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{s=1}^{T}\sum_{h_{(i)}=1}^{\ell_{i,n}}\sum_{w=1}^{T}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{v=1}^{T}\gamma^{(d_1 d_2)}_{(h_{(i)}w,b_{(a)}v)}\Bigg]\\
&=\frac{1}{\ell_n T}\sum_{j:d_{ij}\le d_n}\sum_{b:d_{ab}\le d_n}\sum_{s=1}^{T}\sum_{v=1}^{T}\gamma^{(d_1 d_2)}_{(js,bv)}\\
&-\left(\frac{1}{\ell_n T}+o(1)\right)\sum_{h:d_{ih}\le d_n}\sum_{b:d_{ab}\le d_n}\sum_{w=1}^{T}\sum_{v=1}^{T}\gamma^{(d_1 d_2)}_{(hw,bv)}\\
&=\frac{1}{\ell_n T}\sum_{j:d_{ij}\le d_n}\sum_{b:d_{ib}\le d_n}\sum_{s=1}^{T}\sum_{v=1}^{T}\gamma^{(d_1 d_2)}_{(js,bv)}\\
&-\left(\frac{1}{\ell_n T}+o(1)\right)\sum_{h:d_{ih}\le d_n}\sum_{b:d_{ib}\le d_n}\sum_{w=1}^{T}\sum_{v=1}^{T}\gamma^{(d_1 d_2)}_{(hw,bv)}+o(1)\\
&\to 0
\end{aligned}
$$

by (A.43) and Assumption 15. Therefore, $L_{1nT}=o(1)$ and $L_{2nT}=(1)$, which complete the proof of (A.39).

By a symmetric argument, we obtain the result that

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}K^2\left(\frac{d_{ab_{(a)}}}{d_n}\right)\\
&\times\gamma^{(c_1 c_2)}_{(it,au)}\gamma^{(d_1 d_2)}_{(j_{(i)}s,b_{(a)}v)}\\
&=\bar{\mathcal{K}}_1 J(c_1,c_2)\,J(d_1,d_2)\,,
\end{aligned}
$$

which completes the proof of (A.39).

The next step is to prove (A.40). In view of previous derivations, it suffices to show that

$$
\begin{aligned}
&\lim_{n\to\infty}\lim_{T\to\infty}\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\Bigg[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)\\
&-K\left(\frac{d_{uv_{(u)}}}{d_T}\right)\Bigg]^2\gamma^{(c_1 c_2)}_{(it,au)}\gamma^{(d_1 d_2)}_{(j_{(i)}s,b_{(a)}v)}\\
&=0. \qquad\qquad (A.46)
\end{aligned}
$$

$$\frac{1}{nT^2\ell_n}\sum_{i\in E_n}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{a\in E_n}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)\right]^2\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}$$

$$=\frac{1}{nT^2\ell_n}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_1}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)\right]^2\gamma_{(it,au)}^{(c_1c_2)}\gamma_{\left(j_{(i)}s,b_{(a)}v\right)}^{(d_1d_2)}$$

$$+\frac{1}{nT^2\ell_n}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)\right]^2\gamma_{(it,au)}^{(c_1c_2)}\gamma_{\left(j_{(i)}s,b_{(a)}v\right)}^{(d_1d_2)}$$

$$:=F_{1nT}+F_{2nT},$$

where

$$\mathcal{I}_1=\left\{\left(i,j_{(i)},a,b_{(a)}\right):\left|d_{ij_{(i)}}-d_{ab_{(a)}}\right|\le 2c_n\ \&\ i,a\in E_n\right\},$$

and

$$\mathcal{I}_2=\left\{\left(i,j_{(i)},a,b_{(a)}\right):\left|d_{ij_{(i)}}-d_{ab_{(a)}}\right|> 2c_n\ \&\ i,a\in E_n\right\}.$$

For F_{1nT}, we have

$$\begin{aligned}
F_{1nT}&\le\left|\frac{1}{nT^2\ell_n}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_1}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right)-K\left(\frac{d_{ab_{(a)}}}{d_n}\right)\right]^2\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}\right|\\
&\le\frac{c_L^2}{nT\ell_nd_T}\sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_1}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\left(\frac{d_{ts_{(t)}}}{d_T}-\frac{d_{uv_{(u)}}}{d_T}\right)^2\left|\gamma_{(it,au)}^{(c_1c_2)}\gamma_{(j_{(i)}s,b_{(a)}v)}^{(c_1d_2)}\right|\\
&\le 4c_L^2\left(\frac{c_n^2}{d_n^2}\right)\left(\frac{1}{nT}\sum_{i\in E_n}\sum_{a\in E_n}\sum_{t=1}^{T}\sum_{u=1}^{T}\left|\gamma_{(it,au)}^{(c_1c_2)}\right|\right)\\
&\times\left(\frac{1}{\ell_nT}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}\right|\right)\\
&=O\left(\frac{c_n^2}{d_n^2}\right),
\end{aligned}\tag{A.47}$$

since under Assumption 14

$$\begin{aligned}
&\frac{1}{\ell_nT}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{b_{(a)}=1}^{\ell_{a,n}}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)}\right|\\
&=\frac{1}{\ell_nT}\sum_{j:d_{ij}\le d_n}\sum_{b:d_{ab}\le d_n}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(js,bv)}^{(d_1d_2)}\right|\\
&\le\frac{1}{\ell_nT}\sum_{\substack{j:d_{ij}\le d_n\\ \text{or }d_{aj}\le d_n}}\sum_{\substack{b:d_{ib}\le d_n\\ \text{or }d_{ab}\le d_n}}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(js,bv)}^{(d_1d_2)}\right|\\
&=O(1).
\end{aligned}\tag{A.48}$$

For F_{2nT} we note that if $\left|d_{ij_{(i)}} - d_{ab_{(a)}}\right| > 2c_n$, then either $d_{ia} > c_n$ or $d_{j_{(i)}b_{(a)}} > c_n$, which explained in Proof 1. Without the loss of generality, we assume that $d_{ia} > c_n$ for $\left(i, j_{(i)}, a, b_{(a)}\right) \in \mathcal{I}_2$. In this case

$$
\begin{aligned}
F_{2nT} &\leq \left| \frac{1}{nT^2\ell_n} \sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_2} \sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T} \left[K\left(\frac{d_{ij_{(i)}}}{d_n}\right) - K\left(\frac{d_{ab_{(a)}}}{d_n}\right)\right]^2 \gamma_{(it,au)}^{(c_1c_2)} \gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)} \right| \\
&\leq \frac{4c_L^2}{nT^2\ell_n} \sum_{\left(i,j_{(i)},a,b_{(a)}\right)\in\mathcal{I}_2} \sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T} (d_{ia})^q \left| \gamma_{(it,au)}^{(c_1c_2)} \gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)} \right| (d_{ia})^{-q} \\
&\leq \frac{4c_L^2}{(c_n)^q} \left(\frac{1}{nT} \sum_{i\in E_n} \sum_{a\in E_n} \sum_{t=1}^{T}\sum_{u=1}^{T} (d_{ia})^q \left| \gamma_{(it,au)}^{(c_1c_2)} \right| \right) \left(\frac{1}{\ell_n d_T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{s=1}^{T}\sum_{v=1}^{T} \left| \gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)} \right| \right) \\
&= O\left(\frac{1}{c_n^q}\right).
\end{aligned}
$$

By choosing c_n such that $c_n \to \infty$ but $c_n/d_n \to 0$, we have

$$
F_{1nT} = o\left(1\right) \text{ and } F_{2nT} = o(1)
$$

and (A.40) is proved.

Next, we show that D_{2nT} is $o(1)$.

$$
\begin{aligned}
&\frac{1}{nT^2\ell_n} \sum_{i\notin E_n} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{a=1}^{n} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T} K\left(\frac{d_{ij_{(i)}}}{d_n}\right) K\left(\frac{d_{ab_{(a)}}}{d_n}\right) \gamma_{(it,au)}^{(c_1c_2)} \gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)} \\
&\leq \left| \frac{1}{nT^2\ell_n} \sum_{i\notin E_n} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{a=1}^{n} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T} K\left(\frac{d_{ij_{(i)}}}{d_n}\right) K\left(\frac{d_{ab_{(a)}}}{d_n}\right) \gamma_{(it,au)}^{(c_1c_2)} \gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)} \right| \\
&\leq \frac{1}{nT} \sum_{i\notin E_n} \sum_{a=1}^{n} \sum_{t=1}^{T}\sum_{u=1}^{T} \left| \gamma_{(it,au)}^{(c_1c_2)} \right| \left(\frac{1}{\ell_n T} \sum_{j_{(i)}=1}^{\ell_{i,n}} \sum_{b_{(a)}=1}^{\ell_{a,n}} \sum_{s=1}^{T}\sum_{v=1}^{T} \left| \gamma_{(j_{(i)}s,b_{(a)}v)}^{(d_1d_2)} \right| \right) \\
&= o(1) \qquad\qquad (\text{A.49})
\end{aligned}
$$

by choosing the sequence of d_n in a way that $n_2/n \to 0$ as $n \to \infty$. We can also show $D_{5,nT}$ in the symmetric way.

With the same procedure, it is straightforward that

$$
\lim_{n\to\infty} C_{3nT} = \bar{\mathcal{K}}_1 J\left(c_1, d_2\right) J\left(c_2, d_1\right).
$$

Therefore,

$$
\lim_{n\to\infty}\lim_{T\to\infty} \frac{n}{\ell_n} \mathrm{cov}\left(\tilde{J}_{nT}^{KP}\left(c_1, d_1\right), \tilde{J}_{nT}^{KP}\left(c_2, d_2\right)\right) = \bar{\mathcal{K}}_1 (J\left(c_1, c_2\right) J\left(d_1, d_2\right) + J\left(c_1, d_2\right) J\left(c_2, d_1\right)).
$$

In terms of matrix form,

$$
\lim_{n\to\infty}\lim_{T\to\infty} \frac{n}{\ell_n} \mathrm{var}\left(\mathrm{vec}\left(\tilde{J}_{nT}^{KP}\right)\right) = \bar{\mathcal{K}}_1 (I_{pp} + \mathbb{K}_{pp})\left(J \otimes J\right),
$$

where $J = \left[J\left(c, d\right)\right]$, $c, d = 1, \ldots, p$.

(b) Asymptotic Bias

We have

$$d_n^q\left(E\tilde{J}_{nT}^{KP}-J_{nT}\right)=\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\Gamma_{(it,js)}\left[(d_{ij})^q\frac{K\left(\frac{d_{ij}}{d_n}\right)-1}{\left(\frac{d_{ij}}{d_n}\right)^q}\right]$$

$$=-K_q b_1^{(q)}+o(1).$$

Therefore, $\lim_{n\to\infty}\lim_{T\to\infty}d_n^q(\tilde{J}_{nT}^{KP}-J_{nT})=-K_q b_1^{(q)}$.

(c) $\sqrt{\frac{n}{\ell_n}}\left(\hat{J}_{nT}^{KP}-J_{nT}\right)=O_p(1)$ **and** $\sqrt{\frac{nT}{\ell_n d_T}}\left(\hat{J}_{nT}^{KP}-\tilde{J}_{nT}^{KP}\right)=o_p(1)$

It suffices to show that $\sqrt{\frac{n}{\ell_n}}\left(\hat{J}_{nT}^{KP}-\tilde{J}_{nT}^{KP}\right)=o_p(1)$. We can consider the case that J_{nT} is a scalar random variable without loss of generality.

$$\begin{aligned}
&\sqrt{\frac{n}{\ell_n}}\left(\hat{J}_{nT}^{KP}-\tilde{J}_{nT}^{KP}\right)\\
=&\sqrt{\frac{n}{\ell_n}}\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\left[\hat{V}_{(i,t)}\hat{V}_{(j,s)}'-V_{(i,t)}V_{(j,s)}'\right]\\
=&\left(\sqrt{nT}\left(\hat{\beta}-\beta_0\right)\right)^2\sqrt{\frac{\ell_n}{n}}\frac{1}{\ell_n nT^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\tilde{X}_{it}^2\tilde{X}_{js}^2\\
&-2\sqrt{nT}\left(\hat{\beta}-\beta_0\right)\sqrt{\frac{\ell_n}{n}}\frac{1}{\ell_n T\sqrt{n}}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\tilde{X}_{js}^2\tilde{X}_{it}\tilde{u}_{it}\\
&-\frac{2}{T\sqrt{\ell_n n}}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\tilde{X}_{it}u_{it}\tilde{X}_{js}\left(\bar{u}_j+\bar{u}_s-\bar{u}\right)\\
&+\frac{1}{T\sqrt{\ell_n n}}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\tilde{X}_{it}\tilde{X}_{js}\left(\bar{u}_i+\bar{u}_t-\bar{u}\right)\left(\bar{u}_j+\bar{u}_s-\bar{u}\right)\\
=&H_{1nT}+H_{2nT}+H_{3nT}+H_{4nT}.
\end{aligned}$$

For H_{1nT}, it is $o_p(1)$ under Assumption 7(i) if

$$\frac{1}{\ell_n nT^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\tilde{X}_{it}^2\tilde{X}_{js}^2=O_p(1).$$

$$
\begin{aligned}
&\frac{1}{\ell_n n T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\tilde{X}_{it}^2\tilde{X}_{js}^2\\
=&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}^2\left(\frac{1}{\ell_n T}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\tilde{X}_{js}^2\right)\\
\leq&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}^2\left(\frac{\ell_{i,n}}{\ell_n}\right)\left(\frac{1}{\ell_{i,n}T}\sum_{j:d_{ij}\leq d_n}\sum_{s=1}^{T}\tilde{X}_{js}^2\right)
\end{aligned}
$$

By Assumption 7(ii), we obtain

$$
\begin{aligned}
P\left(\frac{1}{\ell_{i,n}T}\sum_{j:d_{ij}\leq d_n}\sum_{s=1}^{T}\tilde{X}_{js}^2>\Delta\right)&\leq\frac{1}{\Delta\ell_{i,n}T}\sum_{j:d_{ij}\leq d_n}\sum_{s=1}^{T}E\tilde{X}_{js}^2\\
&\to 0
\end{aligned}
$$

as $\Delta\to\infty$, which implies $(\ell_n T)^{-1}\sum_{j:d_{ij}\leq d_n}\sum_{s=1}^{T}\tilde{X}_{it}^2=O_p(1)$ uniformly. Using the same procedure, $(nT)^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}^2=O_p(1)$. As $\ell_{i,n}/\ell_n=O(1)$ due to Assumption 4, $H_{1nT}=o_p(1)$.

For $H_{2nT,}$ it can be rewritten as

$$
-2\sqrt{nT}\left(\hat{\beta}-\beta_0\right)\sqrt{\frac{\ell_n}{n}}\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}\tilde{u}_{it}\left(\frac{1}{\ell_n T}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\tilde{X}_{js}^2\right)
$$

and the part in the parenthesis is $O_p(1)$ uniformly as shown above.

$$
\begin{aligned}
\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}\tilde{u}_{it}&=\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\tilde{X}_{it}u_{it}\\
&=O_p(1)
\end{aligned}
$$

under Assumption 7(iv), which implies that $H_{2nT}=o_p(1)$.

For $H_{3nT,}$ we need to show that for all i and t

$$
\frac{1}{\sqrt{\ell_n T}}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\tilde{X}_{js}\left(\bar{u}_j+\bar{u}_s-\bar{u}\right)=o_p(1). \tag{A.50}
$$

As $\bar{u}_j+\bar{u}_s-\bar{u}=o_p(1)$ uniformly, it suffice to show $(\ell_n T)^{-1/2}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\tilde{X}_{js}=$

$O_p(1)$ for (A.16).

$$
\begin{aligned}
& P\left(\left|\frac{1}{\sqrt{\ell_n T}}\sum_{j=1}^{n}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)\tilde{X}_{js}\right|>\Delta\right) \\
& \leq \frac{1}{\Delta^2\ell_n T}E\left[\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{s=1}^{T}K\left(\frac{d_{ij_{(i)}}}{d_n}\right)\tilde{X}_{j_{(i)}s}\right]^2 \\
& \leq \frac{2}{\Delta^2\ell_n T}\sum_{j_{(i)}=1}^{\ell_{i,n}}\sum_{s=1}^{T}E\left[\tilde{X}_{j_{(i)}s}\right]^2 \\
& \to 0,
\end{aligned}
$$

as $\Delta \to \infty$ under Assumption 7 (ii). Therefore, H_{3nT} is also $o_p(1)$. With the similar procedures, we can show that H_{4nT} is $o_p(1)$.

As a result, $\sqrt{\frac{n}{\ell_n}}\left(\hat{J}_{nT}-\tilde{J}_{nT}\right)=o_p(1)$.

(d) Asymptotic MSE

The first equality holds by Theorem 1(c). For the last equality of Theorem 1(d), since

$$
\frac{n}{\ell_n}=\frac{d_n^{2q}}{d_n^{2q}\ell_n/n}=\frac{d_n^{2q}}{\tau+o(1)},
$$

we have

$$
\begin{aligned}
& \lim_{n\to\infty}\lim_{T\to\infty}MSE\left(\frac{n}{\ell_n},\tilde{J}_{nT}^{KP},S_{nT}\right) \\
& =\lim_{n\to\infty}\lim_{T\to\infty}\frac{n}{\ell_n}\text{ vec }\left(E\tilde{J}_{nT}^{KP}-J_{nT}\right)' S_{nT}\text{ vec}\left(E\tilde{J}_{nT}^{KP}-J_{nT}\right)+\lim_{n\to\infty}\lim_{T\to\infty}\frac{n}{\ell_n}\bar{\mathcal{K}}_1\text{tr}\left(S_{nT}\text{var}(\text{vec }\tilde{J}_{nT}^{KP})\right) \\
& =\frac{1}{\tau}K_q^2\text{ vec }\left(b_1^{(q)}\right)' S\text{ vec }\left(b_1^{(q)}\right)+\bar{\mathcal{K}}_1 tr\left[S(I_{pp}+K_{pp})(J\otimes J)\right],
\end{aligned}
$$

where the last equality holds by Theorem 1(a) and (b).

Proof of Proposition 5

(a) $\hat{J}_{nT}-\hat{J}_{nT}^{GA}=o_p(1)$ **if** $d_n\to 0$ **as** $n\to\infty$.

From Theorem 1 (c), $\hat{J}_{nT}-\tilde{J}_{nT}=o_p(1)$ and similarly $\hat{J}_{nT}^{GA}-\tilde{J}_{nT}^{GA}=o_p(1)$. Therefore, it is enough to show that

$$
\tilde{J}_{nT}(c,d)-\tilde{J}_{nT}^{GA}(c,d)=o_p(1), \tag{A.51}
$$

if $d_n\to 0$ as $n\to\infty$.

By Chebyshev's inequality, for any Δ,

$$
\begin{aligned}
& P\left(\left|\tilde{J}_{nT}(c,d)-\tilde{J}_{nT}^{GA}(c,d)\right|>\Delta\right) \\
& \leq \frac{1}{\Delta^2}E\left(\tilde{J}_{nT}(c,d)-\tilde{J}_{nT}^{GA}(c,d)\right)^2 \\
& =\frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{i\neq j}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{a\neq b}\sum_{u=1}^{T}\sum_{s=1}^{T}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ab}}{d_n}\right)K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right) \\
& \times E\left[V_{(i,t)}^{(c)}V_{(j,s)}^{(d)}V_{(a,u)}^{(c)}V_{(b,v)}^{(d)}\right] \\
& =\frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{s=1}^{T}\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\sum_{e=1}^{nTp}\sum_{f=1}^{nTp}\sum_{i\neq j}\sum_{a\neq b}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ab}}{d_n}\right) \\
& \times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)r_{(i,t),l}^{(c)}r_{(j,s),k}^{(d)}r_{(a,u),e}^{(c)}r_{(b,v),f}^{(d)}E\left[\varepsilon_l\varepsilon_k\varepsilon_e\varepsilon_f\right] \\
& =\tilde{C}_{1nT}+\tilde{C}_{2nT}+\tilde{C}_{3nT}+\tilde{C}_{4nT},
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{C}_{1nT} &= \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{s=1}^{T}\sum_{l=1}^{nTp}\sum_{i\neq j}\sum_{a\neq b}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ab}}{d_n}\right) \\
&\quad \times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)r_{(i,t),l}^{(c)}r_{(j,s),l}^{(d)}r_{(a,u),l}^{(c)}r_{(b,v),l}^{(d)}\left(E\varepsilon_l^4-3\right) \\
\tilde{C}_{2nT} &= \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{s=1}^{T}\sum_{l=1}^{nTp}\sum_{e=1}^{nTp}\sum_{i\neq j}\sum_{a\neq b}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ab}}{d_n}\right) \\
&\quad \times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)r_{(i,t),l}^{(c)}r_{(j,s),l}^{(d)}r_{(a,u),e}^{(c)}r_{(b,v),e}^{(d)} \\
\tilde{C}_{3nT} &= \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{s=1}^{T}\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\sum_{i\neq j}\sum_{a\neq b}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ab}}{d_n}\right) \\
&\quad \times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)r_{(i,t),l}^{(c)}r_{(j,s),k}^{(d)}r_{(a,u),l}^{(c)}r_{(b,v),k}^{(d)} \\
\tilde{C}_{4nT} &= \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{s=1}^{T}\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\sum_{i\neq j}\sum_{a\neq b}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ab}}{d_n}\right) \\
&\quad \times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)r_{(i,t),l}^{(c)}r_{(j,s),k}^{(d)}r_{(a,u),k}^{(c)}r_{(b,v),l}^{(d)}.
\end{aligned}
$$

For $\tilde{C}_{1nT}$,

$$\left|\tilde{C}_{1nT}\right|\leq\left|C_{1nT}\right|=o(1),$$

as shown in (A.1).

For $\tilde{C}_{2nT}$,

$$\begin{aligned}\tilde{C}_{2nT} &= \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{s=1}^{T}\sum_{i\neq j}\sum_{a\neq b}K\left(\frac{d_{ij}}{d_n}\right)K\left(\frac{d_{ab}}{d_n}\right)\\ &\times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)\gamma^{(cd)}_{(it,js)}\gamma^{(cd)}_{(au,bv)}\\ &\leq \frac{1}{\Delta^2}\left(\frac{1}{nT}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{i\neq j}K\left(\frac{d_{ij}}{d_n}\right)\left|\gamma^{(cd)}_{(it,js)}\right|\right)^2\\ &\to 0\end{aligned}$$

as $d_n \to 0$ because $K\left(d_{ij}/d_n\right) \to 0$ for all $i \neq j$.

With the similar procedures, we can show that $\tilde{C}_{3nT} \to 0$ and $\tilde{C}_{4nT} \to 0$.Therefore, (A.51) holds.

(b) $\hat{J}_{nT} - \hat{J}^{DK}_{nT} = o_p(1)$ **if** $\ell^{(c)}_n/n \to 1$ **as** $n \to \infty$.

From Theorem 2 (c), $\hat{J}^{DK}_{nT} - \tilde{J}^{DK}_{nT} = o_p(1)$. Therefore, it is enough to show that

$$\tilde{J}_{nT}\left(c,d\right) - \tilde{J}^{DK}\left(c,d\right) = o_p(1), \tag{A.52}$$

if $\ell^{(c)}_n/n \to 1$ as $n \to \infty$.

By Chebyshev's inequality, for any Δ,

$$\begin{aligned}&P\left(\left|\tilde{J}_{nT}\left(c,d\right) - \tilde{J}^{DK}\left(c,d\right)\right| > \Delta\right)\\ &\leq \frac{1}{\Delta^2}E\left(\tilde{J}_{nT}\left(c,d\right) - \tilde{J}^{DK}\left(c,d\right)\right)^2\\ &= \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{u=1}^{T}\sum_{s=1}^{T}\left(K\left(\frac{d_{ij}}{d_n}\right)-1\right)\left(K\left(\frac{d_{ab}}{d_n}\right)-1\right)\\ &\times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)E\left[V^{(c)}_{(i,t)}V^{(d)}_{(j,s)}V^{(c)}_{(a,u)}V^{(d)}_{(b,v)}\right]\\ &= \check{C}_{1nT} + \check{C}_{2nT} + \check{C}_{3nT} + \check{C}_{4nT},\end{aligned}$$

where

$$\check{C}_{1nT} = \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{s=1}^{T}\sum_{l=1}^{nTp}\left(K\left(\frac{d_{ij}}{d_n}\right)-1\right)\left(K\left(\frac{d_{ab}}{d_n}\right)-1\right)$$
$$\times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)r^{(c)}_{(i,t),l}r^{(d)}_{(j,s),l}r^{(c)}_{(a,u),l}r^{(d)}_{(b,v),l}\left(E\varepsilon_l^4-3\right)$$

$$\check{C}_{2nT} = \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{s=1}^{T}\sum_{l=1}^{nTp}\sum_{e=1}^{nTp}\left(K\left(\frac{d_{ij}}{d_n}\right)-1\right)\left(K\left(\frac{d_{ab}}{d_n}\right)-1\right)$$
$$\times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)r^{(c)}_{(i,t),l}r^{(d)}_{(j,s),l}r^{(c)}_{(a,u),e}r^{(d)}_{(b,v),e}$$

$$\check{C}_{3nT} = \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{s=1}^{T}\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\left(K\left(\frac{d_{ij}}{d_n}\right)-1\right)\left(K\left(\frac{d_{ab}}{d_n}\right)-1\right)$$
$$\times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)r^{(c)}_{(i,t),l}r^{(d)}_{(j,s),k}r^{(c)}_{(a,u),l}r^{(d)}_{(b,v),k}$$

$$\check{C}_{4nT} = \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{s=1}^{T}\sum_{l=1}^{nTp}\sum_{k=1}^{nTp}\left(K\left(\frac{d_{ij}}{d_n}\right)-1\right)\left(K\left(\frac{d_{ab}}{d_n}\right)-1\right)$$
$$\times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)r^{(c)}_{(i,t),l}r^{(d)}_{(j,s),k}r^{(c)}_{(a,u),k}r^{(d)}_{(b,v),l}.$$

We can show that $\check{C}_{1nT} = o(1)$ using the procedure in (A.1).
For $\check{C}_{2nT}$,

$$\begin{aligned}
\check{C}_{2nT} &= \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{s=1}^{T}\left(K\left(\frac{d_{ij}}{d_n}\right)-1\right)\left(K\left(\frac{d_{ab}}{d_n}\right)-1\right)\\
&\times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)\gamma^{(cd)}_{(it,js)}\gamma^{(cd)}_{(au,bv)}\\
&\le \frac{1}{\Delta^2}\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\left(K\left(\frac{d_{ij}}{d_n}\right)-1\right)\left|\gamma^{(cd)}_{(it,js)}\right|\right)^2\\
&\le \frac{1}{\Delta^2}\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}1\left\{\frac{d_{ij}}{d_n}>c\right\}d_{ij}^{-q}\left|\gamma^{(cd)}_{(it,js)}\right|d_{ij}^{q}\right)^2\\
&\le \frac{1}{\Delta^2}\left(\frac{1}{nT}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}1\left\{\frac{d_{ij}}{d_n}>c\right\}(c\cdot d_n)^{-q}\left|\gamma^{(cd)}_{(it,js)}\right|d_{ij}^{q}\right)^2\\
&\to 0
\end{aligned}$$

For $\check{C}_{3nT}$,

$$\begin{aligned}
\check{C}_{3nT} &= \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\left(K\left(\frac{d_{ij}}{d_n}\right)-1\right)\left(K\left(\frac{d_{ab}}{d_n}\right)-1\right) \\
&\quad\times K\left(\frac{d_{ts}}{d_T}\right)K\left(\frac{d_{uv}}{d_T}\right)\gamma_{(it,au)}^{(cc)}\gamma_{(js,bv)}^{(dd)} \\
&\leq \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}\left(K\left(\frac{d_{ij}}{d_n}\right)-1\right)\left(K\left(\frac{d_{ab}}{d_n}\right)-1\right)\left|\gamma_{(it,au)}^{(cc)}\gamma_{(js,bv)}^{(dd)}\right| \\
&\leq \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}1\left\{\frac{d_{ij}}{d_n}>c\right\}1\left\{\frac{d_{ab}}{d_n}>c\right\}\left|\gamma_{(it,au)}^{(cc)}\gamma_{(js,bv)}^{(dd)}\right| \\
&= \frac{1}{\Delta^2}\frac{1}{n^2T^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{t=1}^{T}\sum_{s=1}^{T}\sum_{u=1}^{T}\sum_{v=1}^{T}1\left\{\frac{d_{ij}}{d_n}>c\right\}1\left\{\frac{d_{ab}}{d_n}>c\right\}1\left\{\frac{d_{ia}}{d_n}\leq c\right\}1\left\{\frac{d_{jb}}{d_n}\leq c\right\} \\
&\quad\times\left|\gamma_{(it,au)}^{(cc)}\gamma_{(js,bv)}^{(dd)}\right|+o\left(1\right) \\
&= \frac{1}{\Delta^2}\frac{1}{nT}\sum_{i=1}^{n}\sum_{a:d_{ia}/d_n\leq c}\sum_{t=1}^{T}\sum_{u=1}^{T}\left|\gamma_{(it,au)}^{(cc)}\right|\left(\frac{1}{nT}\sum_{j:d_{ij}/d_n>c}\sum_{b:d_{jb}/d_n\leq c}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(js,bv)}^{(dd)}\right|\right)+o\left(1\right).
\end{aligned}$$

Assumption 7 implies that for all i $\ell_{i,n}^{(c)}\leq C\ell_n^{(c)}$ with some constant C. Thus, if $\ell_n^{(c)}/n\to$ 1,then

$$\begin{aligned}
&\frac{1}{nT}\sum_{j:d_{ij}/d_n>c}\sum_{b:d_{jb}/d_n\leq c}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(js,bv)}^{(dd)}\right| \\
&= \frac{n-\ell_n^{(c)}}{n}\frac{1}{\left(n-\ell_n^{(c)}\right)T}\sum_{j:d_{ij}/d_n>c}\sum_{b:d_{jb}/d_n\leq c}\sum_{s=1}^{T}\sum_{v=1}^{T}\left|\gamma_{(js,bv)}^{(dd)}\right| \\
&\to 0,
\end{aligned}$$

which implies $\check{C}_{3nT}\to 0$ as $n,T\to\infty$. With the same procedure, we can show that $\check{C}_{4nT}=o\left(1\right)$. Therefore, we complete the proof.

(c) $\hat{J}_{nT}-\hat{J}_{nT}^{KP}=o_p(1)$ **if** $\ell_T^{(c)}/T\to 1$ **as** $T\to\infty$.

The proof is similar to that of (b).

References

Andrews, D. (1991). Heteroskedasticity and autocorrelation consistent covariance matrix estimation. *Econometrica*, 59(3):817–858.

Anselin, L. (2001). Spatial econometrics. In: Baltagi, Badi H. (Ed.), A companion to theoretical econometrics.

Arellano, M. (1987). Computing Robust Standard Errors for Within Group Estimators. *Oxford bulletin of Economics and Statistics*, 49:431–434.

Barrios, T., Diamond, R., Imbens, G., and Kolesar, M. (2010). Clustering, Spatial Correlations and Randomization Inference. *NBER Working Paper.*

Bertrand, M., Duflo, E., and Mullainathan, S. (2004). How Much Should We Trust Differences-in-Differences Estimates?*. *Quarterly Journal of Economics*, 119(1):249–275.

Bester, A., Conley, T., Hansen, C., and Vogelsang, T. (2008a). Fixed-b asymptotics for spatially dependent robust nonparametric covariance matrix estimators. *Working paper, Michigan State University.*

Bester, C., Conley, T., and Hansen, C. (2008b). Inference with dependent data using cluster covariance estimators. *Working paper, Grad. School Bus., Univ. Chicago.*

Cameron, A., Gelbach, J., and Miller, D. (2009). Robust Inference with Multi-way Clustering. *Working Papers, UC Davis.*

Chen, X. and Conley, T. (2001). A new semiparametric spatial model for panel time series. *Journal of Econometrics*, 105(1):59–83.

Conley, T. (1999). GMM Estimation with Cross Sectional Dependence. *Journal of Econometrics*, 92(1):1–45.

Conley, T. and Dupor, B. (2003). A spatial analysis of sectoral complementarity. *Journal of Political Economy*, 111(2):311–352.

Conley, T. and Ligon, E. (2002). Economic distance and cross-country spillovers. *Journal of Economic Growth*, 7(2):157–187.

Driscoll, J. and Kraay, A. (1998). Consistent Covariance Matrix Estimation with Spatially Dependent Panel Data. *Review of Economics and Statistics*, 80(4):549–560.

Fama, E. and MacBeth, J. (1973). Risk, return, and equilibrium: Empirical tests. *Journal of Political Economy*, 81(3).

Gonçalves, S. (2008). The moving blocks bootstrap for panel linear regression models with individual fixed effects. *Working paper, Universite de Montreal.*

Hannan, E. (1970). *Multiple Time Series.* Wiley New York.

Hansen, C. (2007). Asymptotic properties of a robust variance matrix estimator for panel data when T is large. *Journal of Econometrics*, 141(2):597–620.

Ibragimov, R. and Müller, U. t-Statistic Based Correlation and Heterogeneity Robust Inference. *Journal of Business and Economic Statistics*, (0):1–16.

Kelejian, H. and Prucha, I. (2007). HAC Estimation in a Spatial Framework. *Journal of Econometrics*, 140(1):131–154.

Kézdi, G. Robust Standard Error Estimation In Fixed-Effects Panel Models. *Mimeo.*

Kiefer, N. and Vogelsang, T. (2002). Heteroskedasticity-Autocorrelation Robust Testing Using Bandwidth Equal to Sample Size. *Econometric Theory*, 18(06):1350–1366.

Kiefer, N. and Vogelsang, T. (2005). A New Asymptotic Theory for Heteroskedasticity-Autocorrelation Robust Tests. *Econometric Theory*, 21(06):1130–1164.

Kim, M. and Sun, Y. (2009). Spatial heteroskedasticity and autocorrelation consistent estimation of covariance matrix. *Working paper, UCSD.*

Kinal, T. (1980). The existence of moments of k-class estimators. *Econometrica*, 48(1):241–249.

Mariano, R. (1972). The existence of moments of the ordinary least squares and two-stage least squares estimators. *Econometrica*, 40(4):643–652.

Newey, W. and West, K. (1987). A Simple, Positive Semi-Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix. *Econometrica*, 55:703–708.

Newey, W. and West, K. (1994). Automatic Lag Selection in Covariance Matrix Estimation. *Review of Economic Studies*, 61:631–631.

Parzen, E. (1957). On Consistent Estimates of the Spectrum of a Stationary Time Series. *Ann. Math. Statist*, 28:329–348.

Petersen, M. (2009). Estimating standard errors in finance panel data sets: Comparing approaches. *Review of Financial Studies*, 22(1):435.

Pinkse, J., Slade, M., and Brett, C. (2002). Spatial Price Competition: A Semiparametric Approach. *Econometrica*, 70(3):1111–1153.

Politis, D. (2005). Higher-order accurate, positive semi-definite estimation of large-sample covariance and spectral density matrices. *Working paper, USCD.*

Robinson, P. (2007). Nonparametric spectrum estimation for spatial data. *Journal of Statistical Planning and Inference*, 137(3):1024–1034.

Stock, J. and Watson, M. (2008). Heteroskedasticity-Robust Standard Errors for Fixed Effects Panel Data Regression. *Econometrica*, 76(1):155–174.

Sun, Y. and Phillips, P. (2008). Bandwidth Choice for Interval Estimation in GMM Regression. *Unpublished paper: Department of Economics, USCD.*

Sun, Y., Phillips, P., and Jin, S. (2008). Optimal Bandwidth Selection in Heteroskedasticity-Autocorrelation Robust Testing. *Econometrica*, 76(1):175–194.

Thompson, S. (2006). Simple formulas for standard errors that cluster by both firm and time. *Working paper, Harvard University.*

Vogelsang, T. (2008). Heteroskedasticity, Autocorrelation, and Spatial Correlation Robust Inference in Linear Panel Models with Fixed-Effects. *Working paper, Department of Economics, Michigan State University.*

White, H. (1980). A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity. *Econometrica: Journal of the Econometric Society*, 48(4):817–838.

Wooldridge, J. (2003). Cluster-sample methods in applied econometrics. *American Economic Review*, 93(2):133–138.

Yu, J., de Jong, R., and Lee, L. (2008). Quasi-maximum likelihood estimators for spatial dynamic panel data with fixed effects when both n and T are large. *Journal of Econometrics*, 146(1):118–134.