Cointegration versus Spurious Regression and Heterogeneity in Large Panels

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December 14, 2009

Abstract

This paper provides an estimation and testing framework to identify the source(s) of lack of cointegration in large nonstationary panels. This can be determined by two non mutually exclusive causes: neglecting the presence of heterogeneity when pooling and genuine presence of $I(1)$ errors in some of the units. The paper proposes two tests that the researcher should carry out after testing for the null of cointegration: one test for the null of homogeneity (and thus presence of spuriousness due to some of the units being genuinely spurious regressions) and one for the null of genuine cointegration in all units of the panel (and thus spuriousness arising only from neglected heterogeneity). The results are derived using a combination of two estimators (one consistent, one inconsistent) of the variance of the estimated pooled slopes. Two consistent estimators, for the degree of heterogeneity and for the fraction of spurious regressions, are also studied.

JEL Codes: C23.

Keywords: Large Panels, Heterogeneity, Spurious Regression, Joint Limit.

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1 Introduction

In recent years, the literature on nonstationary panels has seen numerous developments in the area of testing for cointegration. Various tests have been proposed for either the null of no cointegration or the null of cointegration - see e.g. the recent survey by Breitung and Pesaran (2005). Examples of such tests are the ones proposed, *inter alia*, by McCoskey and Kao (1998), Pedroni (2004; see also the references therein) and Westerlund (2006). Testing for cointegration at the panel level is a frequently undertaken exercise, and it is not uncommon to find that cointegration is rejected due to some units in the panel not being cointegrated contrary to some economic theory and despite other units exhibiting a cointegration relationship. This is e.g. often the case in the PPP literature, in the Feldstein-Horioka puzzle studies, and in the growth literature. Albeit popular, cointegration tests applied to *homogeneous* panels can lead to inconclusive results. As noted by Phillips and Moon (1999, p. 1080), finding evidence of no cointegration could be due to genuine lack of cointegration in some of the panel units, but also to neglected heterogeneity. In this respect, customarily employed tests for panel cointegration (whether the null be no cointegration or presence of cointegration) are not constructive.

This paper investigates the step that a researcher should undertake after conducting a test for cointegration in a panel regression model. The main purpose of the analysis presented hereinafter is to identify and test for the possible causes for the *rejection* of cointegration; thus, the theory developed here is a complement to extant panel cointegration tests. In order to formally illustrate the research question, consider the heterogeneous nonstationary panel model

$$y_{it} = \alpha_i + \beta_i' x_{it} + u_{it}, \tag{1}$$

where $i = 1, ..., n$, $t = 1, ..., T$ and y_{it} and x_{it} are both $I(1)$ for each i. For the purposes of inference and forecasting, researchers often employ the pooled version of (1), i.e.

$$y_{it} = \alpha_i + \beta' x_{it} + v_{it}. \tag{2}$$

In the context of forecasting with stationary models, pooling is frequently done as it leads to more parsimonious models with better predictive ability - see e.g. the contributions by Baltagi and Griffin (1997), Baltagi, Griffin and Xiong (2000), Baltagi, Bresson and Pirotte (2002) and Baltagi, Bresson, Griffin and Pirotte (2003). Also, imposing the homogeneity constraint that $\beta_i = \beta$ for all i has an appealing economic interpretation when the object of interest are not the unit-specific slopes β_i but the average elasticity β, as also pointed out by Temple (1999, par. 4.1). Although it is well known that the long-run average parameter β is equal to the average slope, say $E(\beta_i)$, only under some restrictions, still estimating β is customary since it is a more robust indicator of the realtionship between y_{it} and x_{it} - see Phillips and Moon (2000). This is especially true in those cases where there is no panel cointegration, as estimates of β are $\sqrt{n}$-consistent.

Despite the possible advantages, the validity of model (2) can be hindered by two different, non mutually exclusive reasons: slope heterogeneity (i.e. $\beta_i \neq \beta$ for some i) and lack of cointegration in equation (1), which arises whenever the error term u_{it} is nonstationary. Heterogeneity means that the constraint $\beta_i = \beta$ is not valid, which could mar the forecasting ability of equation (2) in presence of large degrees of heterogeneity - see e.g. Baltagi, Bresson and Pirotte (2004) and also the simulation results in Trapani and Urga (2009). On the other hand, absence of cointegration in some (or all) of the units in model (1), could lead to wrong inference e.g. as far as the rate of convergence of the estimated β or its confidence intervals are concerned - we refer, *inter alia*, to the theory laid out in Kao (1999) and Phillips and Moon (1999). The two problems (heterogeneity and spurious regression) are in a dual relationship: on the one hand, the fact that some of the units are cointegrated and others are not is a form of heterogeneity across units per se, although not slope heterogeneity; on the other hand, pooling under heterogeneous slopes introduces a further component in the error term of model (2), namely $(\beta_i - \beta)' x_{it}$, which is $I(1)$, thereby making equation (2) a spurious panel regression where the estimate of β is $\sqrt{n}$-consistent - Phillips and Moon (1999).

Hypotheses of interest and the main results of the paper

This paper proposes two tests that can be carried out in parallel after finding evidence of no panel cointegration. The tests are for the null of genuine panel cointegration and for the null of slope homogeneity respectively, thereby making it possible to identify the source of spuriousness in the panel. As a by-product, this paper also proposes two consistent estimators, for the degree of heterogeneity across units and for the fraction of spurious regressions, say λ. The two estimators are consistent on the whole parameter space, and they use the model with variables in levels directly, (2), as opposed to models using first differences where the risk of overdifferencing is present, and where omission of the error correction term, where needed, could lead to inconsistency. The inferential theory developed here is based on combining two alternative estimators (one consistent and one inconsistent) of the variance of the estimated β in (2). In a panel unit root setting, Ng (2008) has recently developed an estimation technique for the fraction of nonstationary units; however, no inferential theory is developed for tests where the null hypothesis is the boundary case that the fraction of nonstationary units is zero. In this paper, a consistent estimator for λ is proposed and consistency is shown in all the interval $[0, 1]$, including the boundary $\lambda = 0$; in addition, no special assumptions, such as unit long run variances in the u_{it}s are required. Based on this, a test for the null of cointegration at the micro level, $H_0 : \lambda = 0$, is constructed; this has various advantages, since, as mentioned, cointegration is often the working hypothesis of interest. Results are derived jointly for $(n, T) \to \infty$, and all the asymptotics is derived allowing for cross dependence of various strength, including strong cross dependence that could arise from a factor structure in the error term.

The paper is organised as follows. Section 2 discusses the model and the main assumptions; consistent estimation of the degree of heterogeneity and of the fraction of spurious regressions is discussed in Sections 3 (further results are in 6) and the results concerning testing are in Section 4. Monte Carlo evidence is reported in Section 5; Section 7 concludes. Notation is fairly standard. Throughout the paper, $\stackrel{d}{\to}$ and $\underset{H_0}{\stackrel{d}{\to}}$ denote weak convergence and weak convergence under the null respectively, and $\stackrel{p}{\to}$ convergence in probability. Stochastic processes such as $W(r)$ on $[0, 1]$ are usually written as W, and $\bar{W}$ denotes demeaned Brownian motions; integrals such as $\int_0^1 W(r)\,dr$ are written as $\int W$. We let M_1, M_2 etc. such that $M_j < \infty$ be generic positive constants, not depending on T or n. Whenever employed, the notation $o_p(1)$ should be intended as "of higher order of magnitude than all the other terms".

2 Model and assumptions

Recall model (1), where for the sake of simplicity we consider only one regressor, x_{it}, and the pooled model (2)

$$\begin{aligned} y_{it} &= \alpha_i + \beta_i x_{it} + u_{it}, \\ y_{it} &= \alpha_i + \beta x_{it} + v_{it}. \end{aligned}$$

Let

$$u_{it} = u_{it}^{(\lambda)} d_\lambda + u_{it}^{(1-\lambda)} (1 - d_\lambda) + u_{it}^{(f)}, \tag{3}$$

where $u_{it}^{(f)}$ is assumed to be stationary and $d_\lambda = 1$ for $i = 1, ..., \lfloor n\lambda \rfloor$ and zero otherwise. We let $\Delta u_{it}^{(\lambda)} = \varepsilon_{it}^{u}$ and $u_{it}^{(1-\lambda)} = \varepsilon_{it}^{u}$, where ε_{it}^{u} is a stationary process for all i. Thus, we assume that $u_{it}^{(\lambda)}$ is non-stationary and $u_{it}^{(1-\lambda)}$ is stationary. In (2), $v_{it} = u_{it} + (\beta_i - \beta) x_{it}$. The regressor x_{it} is assumed to have the following DGP

$$x_{it} = x_{it-1} + e_{it}^{x}.$$

Let $\omega_{it} = [e_{it}^{x}, \varepsilon_{it}^{u}]'$, $\omega_{it}^{*} = \left[\omega_{it}', u_{it}^{(f)}\right]'$ and consider the following assumptions.

Assumption 1: [*cross sectional properties*] *(i)* ω_{it} is a martingale difference sequence (MDS) across i with $E(\omega_{it}) = 0$; *(ii)* there exists an invariant σ-field C independent of ω_{it} such that $E\left(u_{it}^{(f)} \middle| C \right) = 0$ and $u_{it}^{(f)}\Big| C$ is an MDS across i.
Assumption 2: [*time series properties*] *(i)* $E \left\| \omega_{it}^{*} \right| C \|^{8+\delta} < \infty$ for some $\delta > 0$; *(ii)* and an invariance principle holds for the partial sums of $\omega_{it}^{*} | C$ such that for all $r \in [0, 1]$

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor Tr \rfloor} \omega_{it}^{*} | C \overset{a.s.}{=} W_i(r) + O_p\left(\frac{1}{\sqrt{T}}\right),$$

where $W_i(r)$ is a vector Brownian motion with covariance matrix

$$\Omega_i = \begin{bmatrix} \sigma_{x,i}^2 & \cdot & \cdot \\ 0 & \sigma_{u,i}^2 & \cdot \\ 0 & 0 & \sigma_{uf,i}^2 \end{bmatrix}.$$

Assumption 3: [*heterogeneous coefficients*] *(i)* for all i, the β_is are *i.i.d.* with $E(\beta_i) = \beta$, $Var(\beta_i) = \sigma_\beta^2 \in [0, S]$ with $S < \infty$ and $E|\beta_i|^{4+\delta} < \infty$ for some $\delta > 0$; *(ii)* $\{\beta_i\}$ and $\{\omega_{it}^{*}\}$ are two mutually independent groups for every i.

The setup considered here allows for a mixed panel, where $\lfloor n\lambda \rfloor$ units are spurious regressions and the rest of the units are cointegration relationships. Since we allow for $\lambda \in [0, 1]$, we entertain the cases that *(a)* all units are cointegrated, *(b)* all units are spurious regressions, and *(c)* the panel is a mixture of cointegrated and spurious regressions (mixed panel). Similarly, allowing for $\sigma_\beta^2 \in [0, S]$ means that the results derived henceforth are valid under both homogeneity and heterogeneity.

Assumption 1 considers a general specification for the cross sectional properties of panel y_{it}. The presence of the additive, stationary component $u_{it}^{(f)}$ in (3) is considered in order to allow for various possible degrees of (additive) cross sectional dependence. Of course, $u_{it}^{(f)}$ can be i.i.d. across i, thereby ruling out any cross dependence, and in this case C is just the empty set. On the other extreme, strong cross sectional dependence as could arise from a factor model can be accommodated within this framework. Setting e.g. $u_{it}^{(f)} = \varphi_i' f_t + w_{it}$, where φ_i is a vector of loading, f_t a vector of common stationary factors and w_{it} an idiosyncratic, MDS component, each of them satisfying some standard assumptions (see e.g. Bai, 2003), one could define the σ-field $\{f_t\}_{t=1}^T$. Then $u_{it}^{(f)}$ is independent across i when conditioned upon $\{f_t\}_{t=1}^T$. Other forms of cross sectional dependence can be also considered. Making $\{\omega_{it}^*\}_{i=1}^n$ an MDS by conditioning on some σ-field is needed to prove the asymptotics hereafter; essentially, the zero mean condition in Assumption 1, together with the moment conditions in Assumption 2*(i)*, make it possible to use a Central Limit Theorem (CLT henceforth) and a Law of Large Numbers (LLN) for MDSs. A similar approach was proposed, in a cross sectional framework, by Andrews (2005), and it heavily relies on Hall and Heyde (1980). Last, note that whilst the presence of a common factor structure in u_{it} is allowed for by Assumption 1, the common factors are required to be $I(0)$. The asymptotics with $I(1)$ common factors has been discussed in Trapani (2009); however, in this context it is necessary to rule out the presence of non-stationary common factors, as these would render the error term $u_{it} \sim I(1)$ for all units, thereby leading to $\lambda = 1$ by construction - see also Ng (2008, p. 121).

Time dependence is allowed for, as long as the Functional CLT (FCLT) holds. Assuming that the remainder is bounded by $O_p\left(T^{-1/2}\right)$ is fairly harmless and it is essentially a Berry-Esseen bound in the i.i.d. case (see Serfling, 1980); see also Phillips and Moon (1999, p. 1101-1102). The bound $O_p\left(T^{-1/2}\right)$ needs not be sharp; for the purpose of the proofs, the order of magnitude of the remainder can be different, without substantially altering the main results. The moment condition in Assumption 2*(i)* is required to prove a Liapunov condition in the asymptotics below. Assumption 3 poses some restrictions on the existence of the moments of β_i, and it essentially needed in order for the β_is to satisfy a LLN and a CLT. Note that Assumption 3*(ii)* implies that the long run average parameter β is the same as the average β_i, i.e. $\beta = E(\beta_i)$.

Let $\bar{x}_{it} = x_{it} - T^{-1}\sum_t x_{it}$ and $\bar{y}_{it} = y_{it} - T^{-1}\sum_t y_{it}$ and define the LSDV estimator for β in (2) as

$$\hat{\beta} = \left[\sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}^2\right]^{-1} \left[\sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}\bar{y}_{it}\right],$$

and consider the following (consistent and inconsistent respectively) estimators of the variance of $\hat{\beta}$

$$\widehat{Var}\left(\hat{\beta}\right) = \frac{1}{T}\frac{\sum_{i=1}^{n}\left[\sum_{t=1}^{T}\bar{x}_{it}\hat{v}_{it}\right]^2}{\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2\right]^2},$$

$$\widehat{Var^*}\left(\hat{\beta}\right) = \frac{1}{nT}\frac{\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{v}_{it}^2}{\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2},$$

where $\hat{v}_{it} = \bar{y}_{it} - \hat{\beta}\bar{x}_{it}$.

Henceforth, we define $\sigma_x^2 = \lim_{n\to\infty} n^{-1}\sum_i \sigma_{x,i}^2$ and $\sigma_u^2 = \lim_{n\to\infty}\left(\lfloor n\lambda\rfloor\right)^{-1}\sum_{i=1}^{\lfloor n\lambda\rfloor}\sigma_{u,i}^2$. The asymptotics for $\hat{\beta}$, $\widehat{Var}\left(\hat{\beta}\right)$ and $\widehat{Var^*}\left(\hat{\beta}\right)$ is given in the following Theorem.

Proposition 1 *Let Assumptions 1-3 hold, and assume that the first $\lfloor n\lambda\rfloor$ units are spurious regressions. Then, as $(n,T)\to\infty$ with $\frac{n}{T}\to 0$ it holds that*

$$\sqrt{n}\left(\hat{\beta}-\beta\right) \xrightarrow{d} \sqrt{\frac{2}{5}\frac{\lambda\sigma_u^2}{\sigma_x^2}+\frac{9}{5}\sigma_\beta^2}\times Z, \tag{4}$$

with $Z\sim N(0,1)$. As $(n,T)\to\infty$

$$nT\left[\widehat{Var}\left(\hat{\beta}\right)\right] \xrightarrow{p} \frac{2}{5}\frac{\lambda\sigma_u^2}{\sigma_x^2}+\frac{9}{5}\sigma_\beta^2, \tag{5}$$

$$nT\left[\widehat{Var^*}\left(\hat{\beta}\right)\right] \xrightarrow{p} \frac{\lambda\sigma_u^2}{\sigma_x^2}+\sigma_\beta^2. \tag{6}$$

Proposition 1 states that $\hat{\beta}$ is estimated consistently at a rate $\sqrt{n}$. This result is typical in panel spurious regression as shown by Kao (1999) and Phillips and Moon (1999).

Equations (5) and (6) provide the probability limits for $\widehat{Var}\left(\hat{\beta}\right)$ and $\widehat{Var^*}\left(\hat{\beta}\right)$: $\widehat{Var}\left(\hat{\beta}\right)$ estimates the asymptotic variance of $\hat{\beta}$ consistently, whilst $\widehat{Var^*}\left(\hat{\beta}\right)$ is an inconsistent estimator as noted by Kao (1999). Similar results are also derived in Trapani (2009).

3 Inference on λ and σ_β^2

This section contains the inferential theory for λ and σ_β^2 which is needed for the purpose of testing for $H_0:\lambda=0$ and $H_0:\sigma_\beta^2=0$ - this will be further developed in Section 4. Further results on the estimation of λ and σ_β^2, which are not directly needed for the purpose of testing, are reported in Section 6.

Define the probability limits in (5) and (6) as ψ_1 and ψ_2 respectively. If ψ_1 and ψ_2 were observable, then one could compute λ and σ_β^2 as

$$\frac{\lambda\sigma_u^2}{\sigma_x^2} = -\frac{5}{7}\psi_1 + \frac{9}{7}\psi_2, \tag{7}$$

$$\sigma_\beta^2 = \frac{5}{7}\psi_1 - \frac{2}{7}\psi_2. \tag{8}$$

Equations (7) and (8) show that (although infeasible) there exists a "direct" estimator for σ_β^2. As far as λ is concerned, this cannot be estimated directly, and estimates of σ_x^2 and of σ_u^2 are required also. From (5) and (6), consistent estimates for ψ_1 and ψ_2 are $\hat{\psi}_1 = n\sum_{i=1}^n \left[\sum_{t=1}^T \bar{x}_{it}\hat{v}_{it}\right]^2 / \left[\sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}^2\right]^2$ and $\hat{\psi}_2 = \sum_{i=1}^n \sum_{t=1}^T \hat{v}_{it}^2 / \sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}^2$ respectively. Thus, one obtains

$$\hat{\sigma}_\beta^2 = \frac{5}{7}\hat{\psi}_1 - \frac{2}{7}\hat{\psi}_2, \tag{9}$$

$$\frac{\widehat{\lambda\sigma_u^2}}{\sigma_x^2} = -\frac{5}{7}\hat{\psi}_1 + \frac{9}{7}\hat{\psi}_2.$$

Since a consistent estimator of σ_x^2 is $\hat{\sigma}_x^2 = \frac{6}{nT^2}\sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}^2$ - see e.g. Kao (1999) - $\lambda\sigma_u^2$ can be estimated by

$$\widehat{\lambda\sigma_u^2} = \left[-\frac{5}{7}\hat{\psi}_1 + \frac{9}{7}\hat{\psi}_2\right]\hat{\sigma}_x^2. \tag{10}$$

Define now $\phi_{nT} = \min\left\{\sqrt{n}, \sqrt{T}\right\}$ and

$$d_0 = \begin{cases} 1 \text{ if } \lambda = 0 \\ 0 \text{ if } \lambda > 0 \end{cases},$$

$$d_\sigma = \begin{cases} 1 \text{ if } \sigma_\beta^2 > 0 \\ 0 \text{ if } \sigma_\beta^2 = 0 \end{cases}.$$

The rates of convergence of $\hat{\sigma}_\beta^2$ and $\widehat{\lambda\sigma_u^2}$ are reported in the following theorem.

Theorem 1 *Let Assumptions 1-3 hold. As* $(n, T) \to \infty$

$$\hat{\sigma}_\beta^2 - \sigma_\beta^2 = (1 - d_0) O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(\frac{1}{\sqrt{T}}\right) + d_\sigma O_p\left(\frac{1}{\sqrt{n}T^{d_0}}\right) + O_p\left(\frac{1}{n}\right) + o_p(1), \tag{11}$$

and

$$\widehat{\lambda\sigma_u^2} - \lambda\sigma_u^2 = d_\sigma O_p\left(\frac{1}{\phi_{nT}}\right) + O_p\left(\frac{1}{\phi_{nT}T^{d_0}}\right) + d_\sigma O_p\left(\frac{1}{n}\right) + o_p(1). \tag{12}$$

Theorem 1 states that $\hat{\sigma}_\beta^2$ and $\widehat{\lambda\sigma_u^2}$ are consistent estimators for σ_β^2 and $\lambda\sigma_u^2$ respectively, as long as $(n, T) \to \infty$; no restrictions on the expansion rate of n and T are required as they pass to infinity. Specifically, $\hat{\sigma}_\beta^2 - \sigma_\beta^2 = O_p(1/\phi_{nT})$ for $\lambda \in (0, 1]$ and $\sigma_\beta^2 > 0$, and $\hat{\sigma}_\beta^2 - \sigma_\beta^2 = O_p\left(1/\min\left\{n, \sqrt{T}\right\}\right)$ for $\lambda = 0$ or $\sigma_\beta^2 = 0$. Thus, a discontinuity is present in the asymptotics of $\hat{\sigma}_\beta^2$ when either λ or σ_β^2 or both are on the boundary. However, contrary to what found in Ng (2008), albeit in a different context, no discontinuities are found in the rate of convergence of $\widehat{\lambda\sigma_u^2}$ when $\lambda = 0$ as long as $\sigma_\beta^2 > 0$.

4 Tests for cointegration and homogeneity

This section provides the next step forward after applying to (2) a test for the null of panel cointegration. As pointed out in the introduction, (2) can be a spurious panel according to a panel cointegration test due to two reasons (not mutually exclusive), namely that $\lambda = 0$ and/or $\sigma_\beta^2 = 0$.

Formally, a test for the *null of panel cointegration* tests for

$$H_0 : \lambda = 0 \text{ and } \sigma_\beta^2 = 0. \tag{13}$$

When H_0 is rejected, then three possible cases can be considered:

1. $\lambda > 0$ and $\sigma_\beta^2 = 0$: the panel is homogeneous and thus pooling is appropriate, but some of the units are spurious regressions. The question would then arise as to the estimation of λ;

2. $\lambda = 0$ and $\sigma_\beta^2 > 0$: the panel is heterogeneous. All the units in the panel are genuine cointegration relationships: the null of panel cointegration is rejected due to pooling, which introduces a nontrivial $I(1)$ component, given by $(\beta_i - \beta)\bar{x}_{it}$, in the error term. In this case, pooling would have to be reconsidered and σ_β^2 estimated;

3. $\lambda > 0$ and $\sigma_\beta^2 > 0$: in this case, the panel is heterogeneous and some of the units are spurious regressions. Theorem 1 and Corollaries 3 and 4 allow to estimate the fraction of spurious regressions in the panel, λ, and the level of heterogeneity, σ_β^2.

Based on the passages in the proof of Theorem 1, two tests are derived as a follow-up for (13).

4.1 Testing for cointegration - $H_0 : \lambda = 0$

This section considers a test for the null hypothesis that *all the units are cointegrated.* Formally

$$\begin{cases} H_0 : \lambda = 0 \text{ and } \sigma_\beta^2 \in (0, +\infty) \\ H_A : \lambda > 0 \text{ and } \sigma_\beta^2 \in [0, +\infty) \end{cases}.$$

Under the alternative that some units are spurious regressions, the possibility that the panel is homogeneous ($\sigma_\beta^2 = 0$) needs to be entertained also. Since, after (12), $\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}} = \frac{\lambda\sigma_u^2}{\sigma_x^2} + O_p\left(T^{-1/2}\right) + O_p\left(n^{-1/2}\right)$, under $H_0 : \lambda = 0$ we have $\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}} = O_p\left(\phi_{nT}^{-1}\right)$. Thus, a natural test statistic for the null of cointegration is (a suitably scaled transformation of) $\phi_{nT} \times \widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}}$. Consider also the class of local alternatives

$$H_A^{(n,T)} : \lambda = \frac{c}{\phi_{nT}} \text{ and } \sigma_\beta^2 \in [0, +\infty),$$

where $c > 0$.

Theorem 2 *Let Assumptions 1-3 hold. As $(n, T) \to \infty$ with $\frac{n}{T} \to 0$*

$$\sqrt{n}\left(\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}}\right) \underset{H_0}{\overset{d}{\to}} \frac{18}{7}\sqrt{\kappa_\beta\delta_\lambda} Z, \tag{14}$$

where $Z \sim N(0,1)$, $\kappa_\beta = E(\beta_i - \beta)^4$, and $\delta_\lambda = E\left[\left(\int \bar{W}^2\right)^2\left(3 - 10\int \bar{W}^2\right)^2\right]$. As $(n, T) \to \infty$ with $\frac{T}{n} \to 0$, $\sqrt{T}\left(\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}}\right) = O_p(1)$. Letting the limiting distribution of $\phi_{nT} \times \left(\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}}\right)$ as $(n, T) \to \infty$ be defined as D, we have that, under $H_A^{(n,T)}$ as $(n, T) \to \infty$, $\phi_{nT} \times \left(\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}}\right) \overset{d}{\to} c\left(\frac{\sigma_u^2}{\sigma_x^2}\right) + D_\lambda$.

In light of Theorem 2, a test statistic for the null that $\lambda = 0$ can be constructed as

$$S_{nT}^{(\lambda)} = \frac{7}{18}\sqrt{\frac{n}{\hat{\kappa}_\beta \times \delta_\lambda}}\left(\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}}\right), \tag{15}$$

where $\hat{\kappa}_\beta$ is a consistent estimate of κ_β; then $S_{nT}^{(\lambda)}$ has a standard normal distribution under H_0 as $(n, T) \to \infty$, as long as $\frac{n}{T} \to 0$. The value of δ_λ is reported in Table 2 in Appendix C. Thus, from the practical viewpoint, the test applies to panels where the time series dimension is larger than the cross sectional one and with $\sigma_\beta^2 > 0$. The null limiting distribution for "local to homogeneous" cases, i.e. for $\sigma_\beta^2 = o(1)$, can in principle be studied using (44). When $\phi_{nT} = \sqrt{T}$, $\sqrt{T}\left(\widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}} - \frac{\lambda\sigma_u^2}{\sigma_x^2}\right) = O_p(1)$; however, this is not a sharp bound and the limiting distribution of this random variable is nonstandard as it depends on the assumptions on the DGP of the $\bar{x}_{it}$s.

The test has power versus local alternatives as $(n, T) \to \infty$ for all combinations of n and T. This is because under $H_A^{(n,T)}$, the test statistic always converges to the null limiting distribution plus a shift.

A consistent estimate of κ_β can be constructed using the micro information as

$$\hat{\kappa}_\beta = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\beta}_i - \widehat{\bar{\beta}}\right)^4, \tag{16}$$

where the $\hat{\beta}_i$s are the OLS estimates from the individual regressions $y_{it} = \beta_i x_{it} + u_{it}$, and $\widehat{\bar{\beta}} = n^{-1}\sum_{i=1}^{n}\hat{\beta}_i$. Using (16) yields an estimator that is consistent under H_0 and $H_A^{(n,T)}$, and bounded in probability under more general alternatives, as shown in Proposition 2 below.

Proposition 2 *Let Assumptions 1-3 hold, and set* $\lambda = \frac{c}{\phi_{nT}}$*, with* $c \in [0, +\infty)$*. Then, as* $(n, T) \to \infty$*,* $\hat{\kappa}_\beta = \kappa_\beta + o_p(1)$ *for* $c \in [0, +\infty)$ *and* $c \to \infty$ *with* $c = o(\phi_{nT})$*. If* $c = O_p(\phi_{nT})$*,* $\hat{\kappa}_\beta = \kappa_\beta + O_p(1)$*.*

Proposition 2 states that $\hat{\kappa}_\beta$ is a consistent estimator for κ_β as $\lambda \to 0$, irrespective of the rate of convergence of λ to zero; thus, it is consistent under local alternatives. The estimator is not consistent when, as $(n, T) \to \infty$, essentially, $\lambda \to \varepsilon > 0$.

4.2 Testing for homogeneity: $H_0 : \sigma_\beta^2 = 0$

In this section, a test for parameter homogeneity is proposed. The null hypothesis is that all the units have the same response to the idiosyncratic covariates $\bar{x}_{it}$, i.e. if $\beta_i = \beta$ for all i. Formally,

$$\begin{cases} H_0 : \sigma_\beta^2 = 0 \text{ and } \lambda \in (0, 1] \\ H_A : \sigma_\beta^2 > 0 \text{ and } \lambda \in [0, 1] \end{cases}.$$

Note that, whilst under the null some units must be spurious regressions ($\lambda > 0$), under the alternative we also entertain the possibility that the panel is genuinely cointegrated so that $\lambda = 0$. We also consider the class of local alternatives

$$H_A^{(n,T)} : \sigma_\beta^2 = \frac{c}{\phi_{nT}} \text{ and } \lambda \in [0,1],$$

with $c > 0$. It holds that

Theorem 3 *Let Assumptions 1-3 hold. Then, as $(n,T) \to \infty$ with $\frac{n}{T} \to 0$*

$$\sqrt{n}\hat{\sigma}_\beta^2 \xrightarrow[H_0]{d} \frac{12}{7}\left(\frac{\lambda\sigma_u^2}{\sigma_x^2}\right)\sqrt{\frac{\delta_{1\sigma}}{\lambda}} \times Z, \tag{17}$$

where $Z \sim N(0,1)$ and $\delta_{1\sigma}$ is defined in Corollary 4. As $(n,T) \to \infty$ with $\frac{T}{n} \to 0$, $\sqrt{T}\hat{\sigma}_\beta^2 = O_p(1)$. Letting the limiting distribution of $\phi_{nT} \times \hat{\sigma}_\beta^2$ as $(n,T) \to \infty$ be defined as D_σ, we have that, under $H_A^{(n,T)}$ as $(n,T) \to \infty$, $\phi_{nT} \times \hat{\sigma}_\beta^2 \xrightarrow{d} c + D_\sigma$.

Theorem 3 suggests the following test statistic for the null that $\sigma_\beta^2 = 0$

$$S_{nT}^{(\sigma)} = \left[\frac{7}{12}\left(\frac{\sigma_x^2}{\lambda\sigma_u^2}\right)\sqrt{\frac{\lambda}{\delta_{1\sigma}}}\right] \times \sqrt{n}\hat{\sigma}_\beta^2; \tag{18}$$

$S_{nT}^{(\sigma)}$ has a standard normal distribution under H_0 as $(n,T) \to \infty$, as long as $\frac{n}{T} \to 0$. The value of $\delta_{1\sigma}$ is reported in Table 2 in Appendix C. The same considerations as for Theorem 2 for the case $\phi_{nT} = \sqrt{T}$ apply here too, and this test too has nontrivial power versus local alternatives. Estimates of λ, $\lambda\sigma_u^2$, σ_x^2, and σ_β^2, which are needed to make the test feasible, are defined in (20), (10), (**??**) and (9) respectively. Similar considerations as for Theorem 2 apply here for the case of "local-to-cointegrated" panel, i.e. for $\lambda = o(1)$.

5 Monte Carlo results

This section reports some evidence from synthetic data on the null rejection frequency and power properties of tests for $H_0 : \lambda = 0$ and $H_0 : \sigma_\beta^2 = 0$. All routines have been written using Gauss 6.0; tables are in Appendix C.

The results shown below are based on 2000 replications. In each experiment, $T + 1000$ data have been created, and the first 1000 observations have been discarded in order to avoid dependence on initial conditions. The DGP for each experiment (the error term u_{it} is modelled as an ARMA process for those units that are cointegrated) is

$$\begin{aligned} y_{it} &= \alpha_i + \beta_i x_{it} + u_{it}, \\ x_{it} &= x_{it-1} + e_{it}^x, \\ u_{it} &= \rho u_{it-1} + e_{it}^u + \theta e_{it-1}^u, \end{aligned}$$

setting $(n,T) = \{(20,100),(20,200),(20,400),(50,200),(50,400),(100,200),(100,400)\}$. The innovations (e_{it}^u, e_{it}^x) are created as i.i.d. Gaussian with unit variance; dynamics in the error term u_{it} is created using $\{\rho,\theta\} = \{0, 0.75\} \times \{-0.75, 0, 0.75\}$. Having set the number of replications equal to 2000, the empirical null rejection frequencies reported in the tables have a 95% confidence interval of width ± 0.01.

Testing for $H_0 : \lambda = 0$

Preliminary results show that the asymptotic version of tests based on (15) are severely undersized unless n is very large; this seems due mainly to the slow convergence of the variance of $\widehat{\lambda\sigma_u^2}$ to its asymptotic value. Although this is merely an indication that the asymptotic test is overly conservative in finite samples, one can construct the following standardised version of the test:

$$\tilde{S}_{nT}^{(\lambda)} = \frac{7}{18}\sqrt{\frac{n}{v_n}} \times \left(\widehat{\lambda\sigma_u^2}\right), \tag{19}$$

where v_n is defined as

$$\begin{aligned} v_n &= \frac{1}{n}\left[\sum_{i=1}^n \left(\hat{\beta}_i - \widehat{\overline{\beta}}\right)^2 \left(\frac{1}{T^2}\sum_{t=1}^T \bar{x}_{it}^2\right)\left(3 - \frac{10}{\hat{\sigma}_x^2 T^4}\sum_{t=1}^T \bar{x}_{it}^2\right)\right]^2 \\ &\quad + \frac{144}{n^3 T^4 \hat{\sigma}_x^4}\left[\sum_{i=1}^n \left(\hat{\beta}_i - \widehat{\overline{\beta}}\right)\left(\sum_{t=1}^T \bar{x}_{it}^2\right)\right]^2 \\ &\quad \times \left[\sum_{i=1}^n \left(\hat{\beta}_i - \widehat{\overline{\beta}}\right)\left(\frac{1}{T^2}\sum_{t=1}^T \bar{x}_{it}^2\right)\left(3 - \frac{10}{\hat{\sigma}_x^2 T^4}\sum_{t=1}^T \bar{x}_{it}^2\right)\right]^2. \end{aligned}$$

This standardisation is based on (44); heuristically, v_n has a "first order" correction (the first term in their expression) and a "second order" one as well - see the proof of Theorem 2.

[Insert Tables 3 and 4a-4b somewhere here]

Simulations were carried out setting $\sigma_\beta^2 \in \{0.1, 0.25, 0.5\}$ - see Trapani and Urga (2009). When analysing the power, two experiments were carried out, setting λ to 0.25 and 0.5 respectively.

As Table 3 illustrates, the test based on (19) has good size properties for $\sigma_\beta^2 = 0.25$ and $\sigma_\beta^2 = 0.5$, reaching the nominal 5% value for n as little as 20 as long as T is sufficiently large. However, the size is affected by the ratio $\frac{n}{T}$; particularly, as $\frac{n}{T}$ departs from zero, the test becomes grossly oversized; this also is in line with the theory, which has been derived under $\frac{n}{T} \to 0$. This can be noticed also in "large" samples, e.g. the case $(n, T) = (100, 200)$ in Table 4. Although not reported, the asymptotic test has, for all the cases in Table 3, size equal to 0.

As far as the power is concerned, Tables 4a and 4b shows that this is affected mainly (as predicted by the theory) by the value of σ_β^2; the ratio $\frac{n}{T}$ does not seem to play a role. The impact of the value of σ_β^2 can be anticipated, albeit indirectly, by looking at the formulation of (15). A test based on $S_{nT}^{(\lambda)}$ rejects the null for large values of the test statistic; since an increase in σ_β^2 is likely to increase κ_β, one could expect that, as σ_β^2 increases, the power should worsen, which is confirmed by the data. The case $\sigma_\beta^2 = 0.5$ sees a pronounced worsening in the performance of $\tilde{S}_{nT}^{(\lambda)}$, and in order to attain a sufficient level of power n has to be decidedly large. Note also the lack of power for large negative MA roots, for both the cases of tests based on $\tilde{S}_{nT}^{(\lambda)}$ and $S_{nT}^{(\lambda)}$. Despite what one could expect in light of the severe undersizement, the asymptotic test seems to have a better power for all cases when $\lambda = 0.5$, and also when $\lambda = 0.25$ as long as n is sufficiently large, and the impact of σ_β^2 is only marginal. Thus, albeit undersized, the asymptotic test seems to have better (and more stable as the model parameters vary) power properties than the one based on $\tilde{S}_{nT}^{(\lambda)}$.

Testing for $H_0 : \sigma_\beta^2 = 0$

The Monte Carlo exercise was carried out setting $\lambda \in \{0.25, 0.5, 0.75\}$, to assess the dependence of size and power on the fraction of spurious regressions. When evaluating power, σ_β^2 was set as $\sigma_\beta^2 \in \{0.1, 0.5\}$; the case $\sigma_\beta^2 = 0.1$ could be regarded as a "local to null" alternative.

[Insert Tables 5 and 6 somewhere here]

Table 5 contains the null rejection frequencies. The test has good size properties for all cases, except the large negative MA root case, when the size is almost equal to zero for every sample size. The test has a tendency to be slightly undersized, although this vanishes as T increases (and, less noticeably, when n increases); the value of λ does not seem to have an impact, although when λ increases, there seems to be a slight improvement in the null rejection frequency which gets closer to its nominal 5% value. This further testifies of the limited impact of n on the test size, as the actual number of units that drives the asymptotics of the test statistic is $\lfloor n\lambda \rfloor$; the remaining units are of a smaller order of magnitude and thus they do not contribute to the asymptotics of the test.

Table 6 shows that the power of the test increases as n and σ^2_β increase, as expected; the test has good power properties for $n = 20$ already, under the alternative $\sigma^2_\beta = 0.5$. For $n = 20$, the impact of λ is in line with the theory, and in line with the comments made earlier for tests based on $S^{(\lambda)}_{nT}$ and $\hat{S}^{(\lambda)}_{nT}$. Since, as one can see from (18), the test statistic $S^{(\sigma)}_{nT}$ decreases with λ, this heuristically means that for large values of the proportion of spurious regressions λ, the power of the test should decrease. This is confirmed by the simulations; however, for larger values of n, this disappears and the power does not seem to be sensitive to λ any more. The test does have some, limited power versus the local alternative $\sigma^2_\beta = 0.1$.

6 Further results

This section contains further results on the estimation of λ and σ^2_β. Whilst not directly necessary for the testing presented in Section 4, the researcher may be interested in *(i)* finding a consistent estimator for λ and *(ii)* constructing confidence intervals for $\hat{\sigma}^2_\beta$. The former is needed in order to make (17) feasible, and it could be of interest to assess how many units are spurious regressions, although the techniques developed here are not able to tell which units are indeed spurious regressions.

6.1 Consistent estimation of λ

In this section, we propose an estimator for λ. To this end, let $\sigma^2_{u,i}$ be the long-run variance of u_{it}, and let $\hat{\sigma}^2_{u,i}$ be an estimator based on the residuals from the unit specific regressions. Define

$$\hat{\psi}_1^* = \frac{n}{\left[\sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}^2\right]^2} \sum_{i=1}^n \left[\sum_{t=1}^T \bar{x}_{it} \frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right]^2,$$

$$\hat{\psi}_2^* = \frac{1}{\sum_{i=1}^n \sum_{t=1}^T \bar{x}_{it}^2} \sum_{i=1}^n \sum_{t=1}^T \left(\frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right)^2,$$

so that a feasible estimator of λ is given by

$$\hat{\lambda} = \left[-\frac{5}{7}\hat{\psi}_1^* + \frac{9}{7}\hat{\psi}_2^*\right] \hat{\sigma}_x^2. \tag{20}$$

Define $\varepsilon_1, \varepsilon_2 \in \left(0, \frac{1}{2}\right)$, and let $\hat{\sigma}_{u,i} = \sigma_{u,i} + O_p\left(T^{-\varepsilon_1}\right)$ for those units i that are spurious regressions, and $\hat{\sigma}_{u,i} = O_p\left(T^{-\varepsilon_2}\right)$ for the other units; let also

$$d_1 = \begin{cases} 1 \text{ if } \lambda = 1 \\ 0 \text{ if } \lambda < 1 \end{cases}.$$

Proposition 3 *Let Assumptions 1-3 hold. As* $(n, T) \to \infty$

$$\begin{aligned} \hat{\lambda} - \lambda &= (1 - d_0) O_p\left(\frac{1}{T^{\varepsilon_1}}\right) + \\ &+ d_\sigma \left[(1 - d_0) O_p(1) + (1 - d_1) O_p\left(T^{\varepsilon_2}\right)\right] O_p\left(\frac{1}{\phi_{nT}}\right) \\ &+ (1 - d_0) O_p\left(\frac{1}{\sqrt{n}\phi_{nT}}\right) + (1 - d_1) O_p\left(\frac{1}{\phi_{nT}^2}\right) + o_p(1). \end{aligned}$$

Consistent estimation of λ is possible for $\lambda \in [0, 1]$ according to Proposition 3, although at a "slow" rate that depends on n and T and also on the rate of convergence of the long-run variance estimators $\hat{\sigma}_{u,i}$. Note that Proposition 3 states that there are discontinuities in the rate of convergence (as found in Ng, 2008) when either λ or σ_β^2 are on the boundary of the parameter space. The proposition is summarized by the table below.

λ \ σ_β^2	0	$\in (0, +\infty)$
0	$O_p\left(\frac{1}{\phi_{nT}}\right)$	$O_p\left(\frac{T^{\varepsilon_2}}{\phi_{nT}}\right)$
$\in (0, 1)$	$O_p\left(\frac{1}{T^{\varepsilon_1}}\right) + O_p\left(\frac{1}{\phi_{nT}}\right)$	$O_p\left(\frac{1}{T^{\varepsilon_1}}\right) + O_p\left(\frac{T^{\varepsilon_2}}{\phi_{nT}}\right)$
1	$O_p\left(\frac{1}{T^{\varepsilon_1}}\right) + O_p\left(\frac{1}{\sqrt{n}\phi_{nT}}\right)$	$O_p\left(\frac{1}{T^{\varepsilon_1}}\right) + O_p\left(\frac{1}{\sqrt{n}}\right)$

Table 1: rates of convergence for $\hat{\lambda} - \lambda$ as $(n, T) \to \infty$.

The rates of convergence for $\hat{\lambda} - \lambda$ depend on the values of λ and σ_β^2. Of course, in some situations, estimation of λ is not strictly necessary. For example, if a test suggests evidence of cointegration in model (2), this implies that $\lambda = \sigma_\beta^2 = 0$, and therefore in this case it would not be required to estimate either λ or σ_β^2. Consider the case $\sigma_\beta^2 = 0$, which is relevant for Theorem 3 when testing for the null of heterogeneity:

$$\hat{\lambda} - \lambda = O_p\left[\frac{1}{T^{\varepsilon_1}}\right] + O_p\left[\frac{1}{n^{\frac{1}{2}(1-d_1)}\phi_{nT}}\right].$$

This result means that the rate of convergence of the long-run variance estimators, ε_1, should be chosen as high as possible in order to maximize the rate of convergence of $\hat{\lambda}$ - see also Theorem 3, where a consistent estimator of λ is needed.

An interesting case is the one where $\lambda \in (0, 1)$ and $\sigma_\beta^2 \in (0, +\infty)$ - presence of some units that are observationally equivalent to spurious regressions and heterogeneity. In this case, the rate of convergence of $\hat{\lambda} - \lambda$ depends on the speed of convergence of the estimators of the long-run variances, ε_1 and ε_2. Assuming, as it could be expected, that $\varepsilon_1 = \varepsilon_2 = \varepsilon$, the optimal rate of convergence ε that maximizes the rate of convergence of $\hat{\lambda}$ can be found as a solution of

$$\min_{\varepsilon}\left[\frac{1}{T^{\varepsilon}} + \frac{T^{\varepsilon}}{\phi_{nT}}\right],$$

which yields $T^{\varepsilon} = \sqrt{\phi_{nT}} = \min\left\{n^{1/4}, T^{1/4}\right\}$. This provides an indication as to the optimal choice of the bandwidth when estimating the $\sigma_{u,i}$s.

Proposition 3 illustrates the consistency of $\hat{\lambda}$. It is not easy to derive the limiting distribution of $\hat{\lambda}$; as shown in greater detail in Appendix B, this depends on the asymptotic bias when estimating the $\sigma_{u,i}$s.

6.2 Inferential theory for $\hat{\sigma}_\beta^2$

As an ancillary result, the following theorem provides the limiting distribution of $\hat{\sigma}_\beta^2$ for $\sigma_\beta^2 \in (0, +\infty)$.

Proposition 4 *Let Assumptions 1-3 hold. As $(n, T) \to \infty$, with $\frac{n}{T} \to 0$ and $\lambda > 0$*

$$\sqrt{n}\left(\hat{\sigma}_\beta^2 - \sigma_\beta^2\right) \xrightarrow{d} \frac{12}{7}\sqrt{\frac{\lambda\sigma_u^2}{\sigma_x^2}}\sqrt{\frac{1}{\lambda}\left(\frac{\lambda\sigma_u^2}{\sigma_x^2}\right)\delta_{1\sigma} + 4\sigma_\beta^2\delta_{2\sigma}} \times Z, \tag{21}$$

where $Z \sim N(0,1)$, $\delta_{1\sigma} = E\left[15\left(\int \bar{W}_1\bar{W}_2\right)^2 - \int \bar{W}_2^2\right]^2$ *and* $\delta_{2\sigma} = E\left[\left(\int \bar{W}_1\bar{W}_2\right)^2\left(15\int \bar{W}_1^2 - 1\right)^2\right]$, *with* $\bar{W}_1$, $\bar{W}_2$ *independent demeaned standard Brownian motions. As* $(n,T) \to \infty$, *with* $\frac{n}{\sqrt{T}} \to 0$ *and* $\lambda = 0$

$$n\left(\hat{\sigma}_\beta^2 - \sigma_\beta^2\right) \xrightarrow{d} \frac{18}{\sqrt{5}}\sigma_\beta^2\sqrt{\delta_{3\sigma}} \times (Z_1Z_2), \tag{22}$$

where $\delta_{3\sigma} = E\left[\left(\int \bar{W}^2\right)^2\left(1 - \frac{60}{7}\int \bar{W}^2\right)^2\right]$, *and* $Z = [Z_1, Z_2]'$ *is two-dimensional, zero mean, normally distributed random variable with*

$$E(ZZ') = \begin{pmatrix} 1 & \rho_\sigma \\ \rho_\sigma & 1 \end{pmatrix},$$

and $\rho_\sigma = 2\sqrt{5}\ E\left[\left(\int \bar{W}^2\right)^2\left(1 - \frac{60}{7}\int \bar{W}^2\right)\right]/\sqrt{\delta_{3\sigma}}$.

Proposition 4 contains all the necessary theory to construct confidence intervals for $\hat{\sigma}_\beta^2$; the quantities $\delta_{1\sigma}$, $\delta_{2\sigma}$, $\delta_{3\sigma}$ and ρ_σ are reported in Table 2 in Appendix C. The values of λ, $\frac{\lambda\sigma_u^2}{\sigma_x^2}$ and σ_β^2 can be replaced by consistent estimators. Similar results could be constructed for $\widehat{\lambda\sigma_u^2}$.

Note the limiting distribution in (22), which is given (up to a rescaling factor) by the product of two zero mean normally distributed random variables. The moments and the distribution function of Z_1Z_2 have been studied by Craig (1936; see also Aroian, 1947; also, Meeker and Escobar, 1994, and MacKinnon *et al.*, 2004, for algorithms to compute moments and critical values). Since $E(Z_1Z_2) = \rho_\sigma < 0$ (see Table 2 in Appendix C), $\hat{\sigma}_\beta^2$ has an asymptotic downward bias. This can be corrected by using

$$\hat{\sigma}_{\beta,BC}^2 = \hat{\sigma}_\beta^2\left(1 - \frac{18\rho_\sigma}{n}\sqrt{\frac{\delta_{3\sigma}}{5}}\right). \tag{23}$$

This small sample correction could be useful upon remembering the restriction $\frac{n}{T} \to 0$, in the light of which n cannot be "too large".

6.3 Multivariate extensions

Most of the results in this paper have been derived and discussed for the case of only one regressor in equation (1). In this case, the algebra greatly simplifies, thereby allowing for a better discussion e.g. of the performance of the test when using synthetic data. This section considers the extension, relevant for empirical applications, where (1) has k regressors, and it presents the details for the implementation of the estimates of the level of heterogeneity and of the fraction of spurious regressions, λ.

Some preliminary notation. In (1), we assume that the k-dimensional vector β_i is i.i.d. across i with $E(\beta_i) = \beta$ and $Var(\beta_i) = \Sigma_\beta$; also, define the long run variance matrix of x_{it} as Ω_{xi} and let $\Omega_x = \lim_{n\to\infty} n^{-1}\sum_{i=1}^n \Omega_{xi}$. We define the pooled OLS estimator for β in $y_{it} = \alpha_i + \beta' x_{it} + u_{it} + (\beta_i - \beta)' x_{it}$ as

$$\begin{aligned}\hat{\beta} &= \left[\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{x}_{it}'\right]^{-1}\left[\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{y}_{it}\right] \\ &= \beta + \left[\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{x}_{it}'\right]^{-1}\left[\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{x}_{it}'(\beta_i - \beta)\right] \\ &\quad + \left[\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{x}_{it}'\right]^{-1}\left[\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right].\end{aligned}$$

Let $Var\left(\hat{\beta}\right)$ denote the asymptotic variance of $\hat{\beta}$. Similar arguments as for Proposition 1 yield

$$\begin{aligned}Var\left(\hat{\beta}\right) &= 36\Omega_x^{-1/2}E\left[\left(\int \bar{W}_1\bar{W}_1'\right)\Omega_x^{1/2}\Sigma_\beta\Omega_x^{1/2}\left(\int \bar{W}_1\bar{W}_1'\right)\right]\Omega_x^{-1/2} \\ &\quad +36\left(\lambda\sigma_u^2\right)\Omega_x^{-1/2}E\left[\left(\int \bar{W}_1\bar{W}_2\right)\left(\int \bar{W}_1\bar{W}_2\right)'\right]\Omega_x^{-1/2},\end{aligned}$$

where $\bar{W}_1$ and $\bar{W}_2$ are two demeaned independent, standard Brownian motions of dimension k and 1 respectively. Consider the two following estimators of $Var\left(\hat{\beta}\right)$

$$\begin{aligned}\hat{\Psi}_1 &= \frac{1}{T}\left[\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{x}_{it}'\right]^{-1}\left\{\sum_{i=1}^n\left[\left(\sum_{t=1}^T \bar{x}_{it}\hat{v}_{it}\right)\left(\sum_{t=1}^T \bar{x}_{it}\hat{v}_{it}\right)'\right]\right\}\left[\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{x}_{it}'\right]^{-1}, \\ \hat{\Psi}_2 &= \frac{1}{nT}\left[\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{x}_{it}'\right]^{-1}\left[\sum_{i=1}^n\sum_{t=1}^T \hat{v}_{it}^2\right].\end{aligned}$$

As $(n,T)\to\infty$, it can be shown that $\hat{\Psi}_1 \xrightarrow{p} Var\left(\hat{\beta}\right)$ and $\hat{\Psi}_2 \xrightarrow{p} 6\Omega_x^{-1}E\left(\int \bar{W}_1'\Omega_x^{1/2}\Sigma_\beta\Omega_x^{1/2}\bar{W}_1\right)+$ $\left(\lambda\sigma_u^2\right)\Omega_x^{-1}$. Note that $E\int \bar{W}_1\bar{W}_1' = I_k/6$, and $E\left[\left(\int \bar{W}_1\bar{W}_2\right)\left(\int \bar{W}_1\bar{W}_2\right)'\right] = I_k/90$;

also, $E\left(\int \bar{W}_1'\Omega_x^{1/2}\Sigma_\beta\Omega_x^{1/2}\bar{W}_1\right) == E\left(\int \bar{W}_1'\Omega_x^{1/2}\otimes \bar{W}_1'\Omega_x^{1/2}\right) vec(\Sigma_\beta)$. It holds that

$$\begin{aligned} & E\left(\int \bar{W}_1'\Omega_x^{1/2}\otimes \bar{W}_1'\Omega_x^{1/2}\right) \\ = \; & E\left[vec\left(\Omega_x^{1/2}\int \bar{W}_1\bar{W}_1'\Omega_x^{1/2}\right)\right]' = \frac{1}{6}\left[vec\left(\Omega_x\right)\right]'. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{36}vec\left(\Omega_x\hat{\Psi}_1\Omega_x\right) &= \Delta_{11}vec\left(\Sigma_\beta\right)+\frac{1}{90}vec\left(\Omega_x\right)\times\lambda\sigma_u^2, \\ \Omega_x\hat{\Psi}_2 &= I_k\times\left[vec\left(\Omega_x\right)\right]'vec\left(\Sigma_\beta\right)+I_k\times\lambda\sigma_u^2, \end{aligned} \tag{24}$$

where $\Delta_{11} = E\left[\left(\Omega_x^{1/2}\int \bar{W}_1\bar{W}_1'\Omega_x^{1/2}\right)\otimes\left(\Omega_x^{1/2}\int \bar{W}_1\bar{W}_1'\Omega_x^{1/2}\right)\right]$. The system has k^2+1 unknowns and $2k^2$ equations; in this respect, the second equation can be rewritten as a scalar one:

$$\frac{1}{k}tr\left(\Omega_x\hat{\Psi}_2\right) = 6\Delta_{22}vec\left(\Sigma_\beta\right)+\lambda\sigma_u^2. \tag{25}$$

Estimation can be made feasible by defining $\hat{\Omega}_x = 6\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{x}_{it}'$ and

$$\hat{\Delta}_{11} = \frac{1}{nT^4}\sum_{i=1}^n\left[\left(\sum_{t=1}^T\bar{x}_{it}\bar{x}_{it}'\right)\otimes\left(\sum_{t=1}^T\bar{x}_{it}\bar{x}_{it}'\right)\right].$$

The solution to system (24)-(25) yields $\widehat{\lambda\sigma_u^2}$ and $\widehat{vec\left(\Sigma_\beta\right)}$. Tests for heterogeneity could be e.g. constructed based on $H_0 : tr\left(\Sigma_\beta\right) = 0$, or on some other matrix norm. It is worth emphasizing that, as k grows large, the choice of the matrix norm, and thus of the test statistics, becomes crucial - see Fan, Fan and Lv (2008).

7 Conclusions

It is well known that, when using a homogeneous nonstationary panel model, spurious regression can be determined by two non mutually exclusive causes: pooling units neglecting the presence of heterogeneity and genuine presence of $I(1)$ errors in some of the units. Thus, when finding evidence of no cointegration, the researcher is left with the question as to the determinants of this. In order to proceed further with the analysis, this paper proposes two tests that would complement a test for panel cointegration applied to (2): one test for the null of homogeneity (and thus presence of spuriousness due to some of the units being genuinely spurious regressions) and one for the null of genuine cointegration in all units of the panel (and thus spuriousness arising only from neglected heterogeneity).

Results are derived by noting that there exist two estimators for the variance of $\hat{\beta}$ in the pooled model (2), one consistent and one inconsistent; note that although the main argument in this paper is based on LSDV estimation, other estimators could be employed. The estimators of the variance have probability limits that are linear combinations of the degree of heterogeneity σ_β^2 and of the proportion of spurious regressions λ (via the transformation $\frac{\lambda\sigma_u^2}{\sigma_x^2}$), and therefore it is possible to estimate σ_β^2 and λ as solutions to a linear system of two equations. The paper studies the properties of the estimated λ and σ_β^2, and tests for $H_0 : \lambda = 0$ (cointegration) and $H_0 : \sigma_\beta^2 = 0$ (homogeneity) are proposed; this problem can be seen as a problem of inference when one parameter is on the boundary of the parameter space, and the proposed tests seem to have good size and power properties. A standardised version of the test for $H_0 : \lambda = 0$ is also proposed, and has better size properties than the asymptotic version which is severely undersized. As an ancillary result, the estimation of λ and σ_β^2 is also discussed. One final note on the presentation of results. Most calculations have been carried out for the univariate case, whereby (1) contains one regressor only; this was done for the sake of the notation, and also because the univariate case, although obviously not always applicable, lends itself to the presentation and discussion of some important stylised facts, such as e.g. the standardised version of tests for $H_0 : \lambda = 0$ or the bias correction for $\hat{\sigma}_\beta^2$. Results concerning the multivariate case are briefly discussed in Section 6.3, where the main purpose of the calculations is to show that the multivariate case. An open and (in the author's view) important question arises as to how to define a test statistic for the null of homogeneity when the number of covariates in (1) is large. This crucially depends on the matrix norm employed, which is likely to affect the choice of the appropriate test statistic, and, therefore, the power properties of the test; this issue is currently under investigation.

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Appendix A: useful Lemmas

Lemma 1 *Let Assumptions 1-3 hold. Then, as $(n, T) \to \infty$, it holds that:*

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2 \overset{a.s.}{=} \frac{\sigma_x^2}{6} + O_p\left(\frac{1}{\sqrt{T}}\right), \tag{26}$$

$$\frac{1}{nT^4}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2 \xrightarrow{p} \frac{\sigma_x^4}{20}, \tag{27}$$

$$\frac{1}{(n\lambda)T^2}\sum_{i=1}^{\lfloor n\lambda\rfloor}\sum_{t=1}^{T}\left[\bar{u}_{it}^{(\lambda)}\right]^2 \overset{a.s.}{=} \frac{\sigma_u^2}{6} + O_p\left(\frac{1}{\sqrt{T}}\right), \tag{28}$$

$$\frac{1}{[n(1-\lambda)]T^2}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\sum_{t=1}^{T}\left[\bar{u}_{it}^{(1-\lambda)}\right]^2 \overset{a.s.}{=} O_p\left(\frac{1}{T}\right), \tag{29}$$

$$\frac{1}{(n\lambda)T^4}\sum_{i=1}^{\lfloor n\lambda\rfloor}\sum_{t=1}^{T}\left[\bar{x}_{it}\bar{u}_{it}^{(\lambda)}\right]^2 \overset{a.s.}{=} \frac{\sigma_x^2\sigma_u^2}{90} + O_p\left(\frac{1}{\sqrt{T}}\right), \tag{30}$$

$$\frac{1}{[n(1-\lambda)]T^4}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\sum_{t=1}^{T}\left[\bar{x}_{it}\bar{u}_{it}^{(1-\lambda)}\right]^2 \overset{a.s.}{=} O_p\left(\frac{1}{T^2}\right), \tag{31}$$

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\bar{u}_{it}^{(f)}\right]^2 \overset{a.s.}{=} O_p\left(\frac{1}{T}\right), \tag{32}$$

$$\frac{1}{nT^4}\sum_{i=1}^{n}\sum_{t=1}^{T}\left[\bar{x}_{it}\bar{u}_{it}^{(f)}\right]^2 \overset{a.s.}{=} O_p\left(\frac{1}{T^2}\right). \tag{33}$$

Proof. Consider (26), and let $W_{1iT}^{(1)} = T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2$. In light of Assumption 1, the $W_{1iT}^{(1)}$s are an MDS with nonzero mean and $E\left|W_{1iT}^{(1)}\right| \leq M_1 E\left(\frac{1}{T^2}\sum_{t=1}^{T}|\bar{x}_{it}|^2\right)$. Assumption 2*(ii)* entails that as $T \to \infty$ $T^{-2}\sum_{t=1}^{T}|\bar{x}_{it}|^2 = O_p(1)$ - see Theorem 5.3 in Park and Phillips (1999). Thus, $E\left|W_{1iT}^{(1)}\right| < \infty$. In light of Assumption 2*(ii)*, as $T \to \infty$ it holds that $W_{1iT}^{(1)} \overset{a.s.}{=} \sigma_x^2 \int \bar{W}^2 + O_p\left(T^{-1/2}\right)$. Therefore, the LLN entails $\frac{1}{n}\sum_{i=1}^{n} W_{1iT}^{(1)} \overset{a.s.}{=} \sigma_x^2 E\left(\int \bar{W}^2\right) + O\left(\frac{1}{\sqrt{T}}\right)$. Kao (1999) proves that $E\left(\int \bar{W}^2\right) = 1/6$. As far as (27) is concerned, define $W_{2iT}^{(1)} = \left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2$; $W_{2iT}^{(1)}$ is an MDS with nonzero mean and

$$E\left|W_{2iT}^{(1)}\right| \leq M_1 E\left|\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}^4\right| \leq M_2 E\left(\frac{1}{T^4}\sum_{t=1}^{T}|\bar{x}_{it}|^4\right). \tag{34}$$

As $T \to \infty$, Assumption 2*(i)* entails $T^{-4}\sum_{t=1}^{T}|\bar{x}_{it}|^4 = O_p(1)$, and therefore $E\left|W_{2iT}^{(1)}\right| < \infty$; the CMT yields $W_{2iT}^{(1)} \overset{a.s.}{=} \sigma_x^4\left(\int \bar{W}^2\right)^2 + O_p\left(T^{-1/2}\right)$. Thus, the

LLN implies that, as $(n,T) \to \infty$ $\frac{1}{n}\sum_{i=1}^{n} W_{2iT}^{(1)} \overset{a.s.}{=} \sigma_x^4 E\left[\left(\int \bar{W}^2\right)^2\right] + O\left(\frac{1}{\sqrt{T}}\right)$, with $E\left[\left(\int \bar{W}^2\right)^2\right] = 1/20$ - see Lemma 2 in Kao (1999). The proof for (28) is very similar to the proof for (26), and thus it is omitted. As far as (29) is concerned, define $W_{3iT}^{(1)} = T^{-1}\sum_{t=1}^{T}\left[\bar{u}_{it}^{(1-\lambda)}\right]^2$; since the $W_{3iT}^{(1)}$s are an MDS, and given that, as $T \to \infty$, $W_{3iT}^{(1)} = O_p(1)$ for all i in light of Assumption 2*(i)*, we have $\frac{1}{nT}\sum_{i=1}^{n} W_{3iT}^{(1)} \overset{a.s.}{=} \frac{1}{T}E\left[W_{3iT}^{(1)}\right] = O_p\left(\frac{1}{T}\right)$. We now turn our attention to (30). Let $W_{4iT}^{(1)} = T^{-4}\left[\sum_{t=1}^{T} \bar{x}_{it}\bar{u}_{it}^{(\lambda)}\right]^2$; $W_{4iT}^{(1)}$ is an MDS with nonzero mean and

$$\begin{aligned} E\left|W_{4iT}^{(1)}\right| &\leq M_1 E\left|\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}^2\left[\bar{u}_{it}^{(\lambda)}\right]^2\right| \\ &\leq M_1 E\left[\left(\frac{1}{T^4}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|^4\right)^{1/2}\left(\frac{1}{T^4}\sum_{t=1}^{T}\left|\bar{u}_{it}^{(\lambda)}\right|^4\right)^{1/2}\right]. \end{aligned}$$

Assumption 2*(i)* implies that, as $T \to \infty$, both $T^{-4}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|^4$ and $T^{-4}\sum_{t=1}^{T}\left|\bar{u}_{it}^{(\lambda)}\right|^4$ are bounded, and therefore $E\left|W_{4iT}^{(1)}\right| < \infty$. As $T \to \infty$, the FCLT yields $W_{4iT}^{(1)} \overset{a.s.}{=} \sigma_x^2\sigma_u^2\left(\int \bar{W}_1\bar{W}_2\right)^2 + O_p\left(T^{-1/2}\right)$. Then $\frac{1}{n}\sum_{i=1}^{n} W_{4iT}^{(1)} \overset{p}{\to} \sigma_x^2\sigma_u^2 E\left[\left(\int \bar{W}_1\bar{W}_2\right)^2\right]$. Baltagi, Kao and Liu (2008) prove that $E\left[\left(\int \bar{W}_1\bar{W}_2\right)^2\right] = 1/90$. Considering now (31), define $W_{5iT}^{(1)} = T^{-2}\left[\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{(1-\lambda)}\right]^2$, the $W_{5iT}^{(1)}$s are an MDS with nonzero mean and with $E\left|W_{5iT}^{(1)}\right| < \infty$, following similar arguments as for (29) and (30). Thus $\frac{1}{nT^2}\sum_{i=1}^{n} W_{5iT}^{(1)} \overset{a.s.}{=} \frac{1}{T^2}E\left[W_{5iT}^{(1)}\right] = O_p\left(\frac{1}{T^2}\right)$.

Last, in order to prove (32) and (33), it is not possible to use the central limit theory employed above, due to the possible presence of cross sectional dependence among units. The theoretical tools that will be employed are a LLN and a CLT for Martingale Difference Sequences (MDS LLN and MDS CLT henceforth), based on achieving cross sectional independence by conditioning upon C - see Assumption 1*(i)*.

Consider first (32), and define $W_{6iT}^{(1)} = T^{-1}\sum_{t=1}^{T}\left[\bar{u}_{it}^{(f)}\right]^2$; $W_{6iT}^{(1)}\Big| C$ is an MDS with nonzero mean, and Assumption 2*(i)* implies $E\left|W_{6iT}^{(1)}\Big| C\right| \leq M_1 E\left(\frac{1}{T^2}\sum_{t=1}^{T}\left|\bar{u}_{it}^{(f)}\Big| C\right|^2\right)$ $< \infty$. Thus, the MDS LLN yields $\frac{1}{nT}\sum_{i=1}^{n} W_{6iT}^{(1)}\Big| C \overset{a.s.}{=} \frac{1}{T}E\left[W_{6iT}^{(1)}\Big| C\right] + o_p(1) = O_p\left(\frac{1}{T}\right)$. Turning now our attention to (33), let $W_{7iT}^{(1)} = T^{-2}\sum_{t=1}^{T}\left[\bar{x}_{it}\bar{u}_{it}^{(f)}\right]^2$;

$W_{7iT}^{(1)} \Big| C$ is an MDS with nonzero mean and

$$\begin{aligned} E\left|W_{7iT}^{(1)}\right| &\leq M_1 E\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\left[\bar{u}_{it}^{(f)}\right]^2\Big| C\right| \\ &\leq M_2 E\left[\left(\frac{1}{T^4}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|^4\right)^{1/2}\left(\frac{1}{T^4}\sum_{t=1}^{T}\left|\bar{u}_{it}^{(\lambda)}\right| C\right|^4\right)^{1/2}\right] < \infty. \end{aligned}$$

Therefore, applying the MDS LLN $\frac{1}{nT^2}\sum_{i=1}^{n} W_{7iT}^{(1)}\Big| C \overset{a.s.}{=} \frac{1}{T^2}E\left(W_{7iT}^{(1)}\Big| C\right) + o_p(1) = O_p\left(\frac{1}{T^2}\right)$. ■

Lemma 2 *Let Assumptions 1-3 hold. Then, as* $(n, T) \to \infty$*, it holds that:*

$$\frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}\left[\left(\beta_i - \beta\right)\bar{x}_{it} + \bar{u}_{it}^{(\lambda)} d_\lambda\right] \overset{a.s.}{=} \left[\frac{\lambda\sigma_u^2\sigma_x^2}{90} + \frac{\sigma_\beta^2\sigma_x^4}{20}\right]^{1/2} Z + O_p\left(\sqrt{\frac{n}{T}}\right), \tag{35}$$

$$\frac{1}{\sqrt{n(1-\lambda)}T^2}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{(1-\lambda)} \overset{a.s.}{=} O_p\left(\frac{1}{T}\sqrt{\frac{n}{T}}\right), \tag{36}$$

$$\frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{(f)} \overset{a.s.}{=} O_p\left(\frac{1}{T}\sqrt{\frac{n}{T}}\right), \tag{37}$$

where $Z \sim N(0, 1)$.

Proof. Consider first (35). Define $W_{1iT}^{(2)} = T^{-2}\sum_{t=1}^{T}\bar{x}_{it}\left[\left(\beta_i - \beta\right)\bar{x}_{it} + \bar{u}_{it}^{(\lambda)} d_\lambda\right]$. The $W_{1iT}^{(2)}$s are an MDS; define $W_{1iT}^{(2)} = \frac{1}{T^2}\left(\beta_i - \beta\right)\sum_{t=1}^{T}\bar{x}_{it}^2 + \frac{1}{T^2}d_\lambda\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{(\lambda)} = W_{1iT,1}^{(2)} + W_{1iT,2}^{(2)}$. Assumption 3 ensures that $E\left[W_{1iT,1}^{(2)}\right] = 0$ for all T; as $T \to \infty$, Assumption 2*(ii)* leads to $E\left[W_{1iT,2}^{(2)}\right] = O_p\left(T^{-1/2}\right)$. Thus, $E\left[W_{1iT}^{(2)}\right] = O_p\left(T^{-1/2}\right)$. Also, a Liapunov condition holds since $E\left|W_{1iT}^{(2)}\right|^{2+\delta} \leq M_1\left[E\left|W_{1iT,1}^{(2)}\right|^{2+\delta} + E\left|W_{1iT,2}^{(2)}\right|^{2+\delta}\right]$, and $E\left|W_{1iT,1}^{(2)}\right|^{2+\delta} \leq \left|\beta_i - \beta\right|^{2+\delta}\frac{1}{T^{4+2\delta}}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|^{4+\delta}$,

$$\begin{aligned} E\left|W_{1iT,2}^{(2)}\right|^{2+\delta} &\leq \frac{1}{T^{4+2\delta}}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|^{2+\delta}\left|\bar{u}_{it}^{(\lambda)}\right|^{2+\delta} \\ &\leq \left(\frac{1}{T^{4+2\delta}}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|^{4+2\delta}\right)^{1/2}\left(\frac{1}{T^{4+2\delta}}\sum_{t=1}^{T}\left|\bar{u}_{it}^{(\lambda)}\right|^{4+2\delta}\right)^{1/2}, \end{aligned}$$

which are both finite in light of Assumptions 2*(i)* and 3. Thus, the CLT yields $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} W_{1iT}^{(2)} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{W_{1iT}^{(2)} - E\left[W_{1iT}^{(2)}\right]\right\} + \frac{1}{\sqrt{n}}\sum_{i=1}^{n} E\left[W_{1iT}^{(2)}\right] = I + II$. As

$(n,T) \to \infty$, $II = O_p\left(\sqrt{n/T}\right)$ and $I \xrightarrow{d} \sqrt{\lim_{(n,T)\to\infty} \frac{1}{n}\sum_{i=1}^{n}\left[W_{1iT}^{(2)}\right]^2} \times Z$, with $Z \sim N(0,1)$. The LLN entails that $\lim_{(n,T)\to\infty} n^{-1}\sum_{i=1}^{n}\left[W_{1iT}^{(2)}\right]^2 \overset{a.s.}{=} \lim_{T\to\infty} E\left[W_{1iT}^{(2)}\right]^2$ with

$$
\begin{aligned}
E\left[W_{1iT}^{(2)}\right]^2 &= E\left[(\beta_i-\beta)^2\right]\left\{\lim_{T\to\infty} E\left[\frac{1}{T^4}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right]\right\} + \lambda \lim_{T\to\infty} E\left\{\frac{1}{T^4}\sum_{t=1}^{T}\left[\bar{x}_{it}^2\bar{u}_{it}^{(\lambda)}\right]^2\right\} \\
&\quad + \lambda E(\beta_i-\beta)\lim_{T\to\infty} E\left\{\frac{1}{T^4}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\sum_{t=1}^{T}\left[\bar{x}_{it}\bar{u}_{it}^{(\lambda)}\right]^2\right\} \\
&= I + II + III.
\end{aligned}
$$

For $T\to\infty$, it holds that $I \overset{a.s.}{=} \sigma_\beta^2\sigma_x^4 E\left[\left(\int \bar{W}_1^2\right)^2\right] = \sigma_\beta^2\sigma_x^4/20$, and $II \overset{a.s.}{=} \lambda\sigma_u^2\sigma_x^2 E\left(\int \bar{W}_1\bar{W}_2\right) = \lambda\sigma_u^2\sigma_x^2/90$. As far as III is concerned, $E(III)=0$ for any T. Thus, as $(n,T)\to\infty$, $\frac{1}{\sqrt{n}}\sum_{i=1}^{n} W_{1iT}^{(2)} \overset{a.s.}{=} \sqrt{\frac{\sigma_\beta^2\sigma_x^4}{20}+\frac{\lambda\sigma_u^2\sigma_x^2}{90}} \times Z + O_p\left(\sqrt{\frac{n}{T}}\right)$. Consider now (36). Define $W_{2iT}^{(2)} = T^{-1}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{(1-\lambda)}$; the $W_{2iT}^{(2)}$s are an MDS and in light of Assumption 2*(ii)*, as $T\to\infty$, $E\left[W_{2iT}^{(2)}\right] = O_p\left(T^{-1/2}\right)$. Also, Assumption 2*(i)* implies that

$$
\begin{aligned}
E\left|W_{2iT}^{(2)}\right|^{2+\delta} &\leq M_1 E\frac{1}{T^{2+\delta}}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|^{2+\delta}\left|\bar{u}_{it}^{(1-\lambda)}\right|^{2+\delta} \\
&= M_1 E\left[\frac{1}{T^{4+2\delta}}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|^{4+2\delta}\right]^{1/2}\left[\frac{1}{T^{4+2\delta}}\sum_{t=1}^{T}\left|\bar{u}_{it}^{(1-\lambda)}\right|^{4+2\delta}\right]^{1/2} = O_p(1).
\end{aligned}
$$

Thus, a CLT holds for $W_{2iT}^{(2)} - E\left[W_{2iT}^{(2)}\right]$ and therefore

$$
\begin{aligned}
& \frac{1}{T}\frac{1}{\sqrt{n(1-\lambda)}}\sum_{i=\lfloor n\lambda\rfloor+1}^{n} W_{2iT}^{(2)} \\
= & \frac{1}{T}\frac{1}{\sqrt{n(1-\lambda)}}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left\{W_{2iT}^{(2)} - E\left[W_{2iT}^{(2)}\right]\right\} + \frac{1}{T}\frac{1}{\sqrt{n(1-\lambda)}}\sum_{i=\lfloor n\lambda\rfloor+1}^{n} E\left[W_{2iT}^{(2)}\right] \\
= & \frac{1}{T}O_p(1) + \frac{1}{T}O_p\left(\sqrt{\frac{n}{T}}\right).
\end{aligned}
$$

As far as (37) is concerned, let $W_{3iT}^{(2)} = T^{-1}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{(f)}$. Conditional on C, $W_{3iT}^{(2)}\Big| C$ is an MDS with mean $E\left[W_{3iT}^{(2)}\Big| C\right] = O_p\left(T^{-1/2}\right)$. Similar passages as for $W_{3iT}^{(2)}$ above yield $E\left|W_{3iT}^{(2)}\Big| C\right|^{2+\delta} < \infty$. Thus, the MDS CLT can be applied and similar passages as above yield $\frac{1}{T}\frac{1}{\sqrt{n}}\sum_{i=1}^{n} W_{10iT}\big| C = \frac{1}{T}O_p(1) + \frac{1}{T}O_p\left(\sqrt{\frac{n}{T}}\right)$. ∎

Appendix B: Proofs and derivations

In order to prove all results, where necessary we shall assume, with no loss of generality, that the first $\lfloor n\lambda \rfloor$ units are spurious regressions, and that the remaining ones are cointegration regressions. Also, conditioning on C will be omitted whenever no ambiguity arises.

Proof of Proposition 1. Let us consider (4), and recall that $\hat{\beta} - \beta = \left[\sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it}^2\right]^{-1} \left\{\sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it} \left[\bar{u}_{it} + (\beta_i - \beta) \bar{x}_{it}\right]\right\}$. From Lemmas 1 and 2 we have $\sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it}^2 = O_p(nT^2)$, and $\sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it} \left[\bar{u}_{it} + (\beta_i - \beta) \bar{x}_{it}\right] = O_p(\sqrt{n}T^2)$; thus, $\hat{\beta} - \beta = O_p\left(n^{-1/2}\right)$. Equation (26) ensures that $(nT^2)^{-1} \sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it}^2 \xrightarrow{p} \sigma_x^2/6$. As far as the numerator of $\hat{\beta} - \beta$ is concerned, we have

$$
\begin{aligned}
& \frac{1}{\sqrt{n}T^2} \sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it} \left[\bar{u}_{it} + (\beta_i - \beta) \bar{x}_{it}\right] \\
= & \frac{1}{\sqrt{n}T^2} \sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it} \left[(\beta_i - \beta) \bar{x}_{it} + \bar{u}_{it}^{(\lambda)} d_\lambda\right] + \frac{1}{\sqrt{n}T^2} \sum_{i=\lfloor n\lambda \rfloor+1}^{n} \sum_{t=1}^{T} \bar{x}_{it} \bar{u}_{it}^{(1-\lambda)} \\
& + \frac{1}{\sqrt{n}T^2} \sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it} \bar{u}_{it}^{(f)} \\
= & \frac{1}{\sqrt{n}T^2} \sum_{i=1}^{n} \sum_{t=1}^{T} \bar{x}_{it} \left[(\beta_i - \beta) \bar{x}_{it} + \bar{u}_{it}^{(\lambda)} d_\lambda\right] + o_p(1),
\end{aligned}
$$

and therefore, using (35) $\sqrt{n}\left(\hat{\beta} - \beta\right) \xrightarrow{d} \sqrt{\frac{2}{5} \frac{\lambda \sigma_u^2}{\sigma_x^2} + \frac{9}{5} \sigma_\beta^2} \times Z$. As far as (5) and (6) are concerned, consider

$$
\begin{aligned}
& \sum_{i=1}^{n} \left(\sum_{t=1}^{T} \bar{x}_{it} \hat{v}_{it}\right)^2 \\
= & \sum_{i=1}^{n} \left(\sum_{t=1}^{T} \bar{x}_{it} \bar{u}_{it}\right)^2 + \sum_{i=1}^{n} \left[(\beta_i - \beta)^2 \left(\sum_{t=1}^{T} \bar{x}_{it}^2\right)^2\right] + \left(\hat{\beta} - \beta\right)^2 \sum_{i=1}^{n} \left(\sum_{t=1}^{T} \bar{x}_{it}^2\right)^2 \\
& + 2 \sum_{i=1}^{n} \left[(\beta_i - \beta) \left(\sum_{t=1}^{T} \bar{x}_{it}^2\right) \left(\sum_{t=1}^{T} \bar{x}_{it} \bar{u}_{it}\right)\right] - 2\left(\hat{\beta} - \beta\right) \sum_{i=1}^{n} \left[(\beta_i - \beta) \left(\sum_{t=1}^{T} \bar{x}_{it}^2\right)^2\right] \\
& - 2\left(\hat{\beta} - \beta\right) \sum_{i=1}^{n} \left[\left(\sum_{t=1}^{T} \bar{x}_{it}^2\right) \left(\sum_{t=1}^{T} \bar{x}_{it} \bar{u}_{it}\right)\right],
\end{aligned}
\tag{38}
$$

and

$$
\begin{aligned}
& \sum_{i=1}^{n}\sum_{t=1}^{T}\hat{v}_{it}^{2} \\
= \; & \sum_{i=1}^{n}\sum_{t=1}^{T}\bar{u}_{it}^{2}+\sum_{i=1}^{n}\left[\left(\beta_{i}-\beta\right)^{2}\left(\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)\right]+\left(\hat{\beta}-\beta\right)^{2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^{2} \\
& +2\sum_{i=1}^{n}\left[\left(\beta_{i}-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right]-2\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it} \\
& -2\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left[\left(\beta_{i}-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)\right]. \qquad (39)
\end{aligned}
$$

Define

$$
\begin{aligned}
\frac{1}{nT^{4}}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\hat{v}_{it}\right)^{2} &= a_{1}+a_{2}+a_{3}+a_{4}+a_{5}+a_{6}, \\
\frac{1}{nT^{2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{v}_{it}^{2} &= b_{1}+b_{2}+b_{3}+b_{4}+b_{5}+b_{6}.
\end{aligned}
$$

It holds that

$$
\begin{aligned}
a_{1} &= \frac{1}{nT^{4}}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left\{\sum_{t=1}^{T}\bar{x}_{it}\left[\bar{u}_{it}^{(\lambda)}+\bar{u}_{it}^{(f)}\right]\right\}^{2}+\frac{1}{nT^{4}}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left\{\sum_{t=1}^{T}\bar{x}_{it}\left[\bar{u}_{it}^{(1-\lambda)}+\bar{u}_{it}^{(f)}\right]\right\}^{2} \\
&= a_{1,1}+a_{1,2},
\end{aligned}
$$

and (30), (31) and (33) entail $a_{1,1}=O_{p}(1)$ and $a_{1,2}=O_{p}\left(T^{-2}\right)$, with $a_{1,1}\xrightarrow{p}\lambda\sigma_{\beta}^{2}\sigma_{x}^{2}/90$. Consider a_{2}; the sequence $\left(\beta_{i}-\beta\right)^{2}\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)^{2}$ is an MDS with nonzero mean and finite expectation in light of Assumption 3 and (34). Thus, using (27), as $(n,T)\rightarrow\infty$, $a_{2}\xrightarrow{p}E\left[\left(\beta_{i}-\beta\right)^{2}\right]E\left[\lim_{T\rightarrow\infty}\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)^{2}\right]=\sigma_{\beta}^{2}\sigma_{x}^{4}/20$. Equations (4) and (27) entail $a_{3}=O_{p}\left(n^{-1}\right)$. As far as a_{4} is concerned, note that $\left(\beta_{i}-\beta\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^{2}\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)$ has mean zero for all T and is an MDS conditional on C. Also, using Assumption 3

$$
\begin{aligned}
& E\left[\left|\beta_i-\beta\right|^{2+\delta}\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right|^{2+\delta}\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right|^{2+\delta}\right] \\
= & E\left(\left|\beta_i-\beta\right|^{2+\delta}\right)E\left[\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right|^{2+\delta}\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right|^{2+\delta}\right] \\
\leq & M_1\left[E\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right|^{4+2\delta}\right]^{1/2}\left[E\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right|^{4+2\delta}\right]^{1/2}, \qquad (40)
\end{aligned}
$$

where

$$
\begin{aligned}
E\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right|^{4+2\delta} & \leq M_1E\left(\frac{1}{T^{8+4\delta}}\sum_{t=1}^{T}\left|\bar{x}_{it}\right|^{8+4\delta}\right), \qquad (41) \\
E\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right|^{4+2\delta} & \leq M_1E\left(\frac{1}{T^{8+4\delta}}\sum_{t=1}^{T}\left|\bar{x}_{it}\bar{u}_{it}\right|^{4+2\delta}\right);
\end{aligned}
$$

as $T\to\infty$, Assumption 2*(i)* implies that both $T^{-(8+4\delta)}\sum_{t=1}^{T}\left|\bar{x}_{it}^2\right|^{8+4\delta}$ and $T^{-(8+4\delta)}$ $\sum_{t=1}^{T}\left|\bar{x}_{it}\bar{u}_{it}\right|^{4+2\delta}$ are bounded - see Theorem 5.3 in Park and Phillips (1999). Thus, the MDS CLT entails $\sum_i\left(\beta_i-\beta\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right) = O_p\left(\sqrt{n}\right)$ and therefore $a_4=O_p\left(n^{-1/2}\right)$. As far as a_5 is concerned, $\left(\beta_i-\beta\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2$ is a zero mean MDS with $E\left[\left|\beta_i-\beta\right|^{2+\delta}\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right|^{4+2\delta}\right]=E\left[\left|\beta_i-\beta\right|^{2+\delta}\right]E\left[\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right|^{4+2\delta}\right]$, which is finite in light of (41). Thus, $\sum_i\left(\beta_i-\beta\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2=O_p\left(\sqrt{n}\right)$, and using (4), $a_5=O_p\left(n^{-1}\right)$. Last, consider a_6. The sequence $\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)$ is an MDS conditional on C with, as $T\to\infty$, $E\left[\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right]=O_p\left(\frac{1}{\sqrt{T}}\right)$. Also, the moment of order $2+\delta$ is finite following similar passages as in (41). The MDS CLT yields $\sum_{i=1}^{n}\left[\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right]=O_p\left(\sqrt{n}\right)+O_p\left(\frac{n}{\sqrt{T}}\right)$, so that, using (4), $a_6=O_p\left(n^{-1}\right)+O_p\left(n^{-1/2}T^{-1/2}\right)$. Thus, as $(n,T)\to\infty$, $\frac{1}{nT^4}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\hat{v}_{it}\right)^2\xrightarrow{p}\frac{\lambda\sigma_\beta^2\sigma_x^2}{90}+\frac{\sigma_\beta^2\sigma_x^4}{20}$; combining this with the denominator we get (5).

As far as (6) is concerned, the proof is fairly similar to that of (5). Let

$$
b_1=\frac{1}{nT^4}\sum_{i=1}^{\lfloor n\lambda\rfloor}\sum_{t=1}^{T}\left[\bar{u}_{it}^{(\lambda)}+\bar{u}_{it}^{(f)}\right]^2+\frac{1}{nT^4}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\sum_{t=1}^{T}\left[\bar{u}_{it}^{(1-\lambda)}+\bar{u}_{it}^{(f)}\right]^2=b_{1,1}+b_{1,2}.
$$

As $(n,T) \to \infty$, (28) and (32) yield $b_{1,1} \xrightarrow{p} \lambda\sigma_u^2/6$, and (29) and (32) entail $b_{1,2} = O_p(T^{-1})$. Similar passages as for a_2 lead to $b_2 \xrightarrow{p} E\left[(\beta_i - \beta)^2\right] \times E\left[\lim_{T\to\infty} T^{-2}\sum_{t=1}^T \bar{x}_{it}^2\right] = \sigma_\beta^2\sigma_x^2/6$. Equations (4) and (26) entail $b_3 = O_p(n^{-1})$. As far as b_4 is concerned, the sequence $(\beta_i - \beta)\left(T^{-2}\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)$ is, conditional on C, a zero mean MDS with finite moment of order $2+\delta$, and therefore the MDS CLT ensures that $b_4 = O_p\left(n^{-1/2}\right)$. Equation (4) and (35)-(37) yield $b_5 = O_p(n^{-1})$. Last, since $(\beta_i - \beta)\left(T^{-2}\sum_{t=1}^T \bar{x}_{it}^2\right)$ is a zero mean MDS with finite moment of order $2+\delta$, $b_6 = O_p(n^{-1})$. Thus, as $(n,T)\to\infty$, $(nT^2)^{-1}\sum_{i=1}^n\sum_{t=1}^T \hat{v}_{it}^2 \xrightarrow{p} \frac{\lambda\sigma_\beta^2}{6} + \frac{\sigma_\beta^2\sigma_x^2}{6}$; combining this with the denominator we get (6). ■

Proof of Theorem 1. Consider (11). Replacing $(nT^2)^{-1}\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}^2$ with its limit from (26) and using (38) and (39), we have

$$
\begin{aligned}
\hat{\sigma}_\beta^2 &= \frac{5}{7}\hat{\psi}_1 - \frac{2}{7}\hat{\psi}_2 \\
&= \frac{12}{7\sigma_x^2}\left[\frac{15}{\sigma_x^2 nT^4}\sum_{i=1}^n\left(\sum_{t=1}^T \bar{x}_{it}\hat{v}_{it}\right)^2 - \frac{1}{nT^2}\sum_{i=1}^n\sum_{t=1}^T \hat{v}_{it}^2\right] \\
&= \frac{12}{7\sigma_x^2}\left\{\frac{15}{\sigma_x^2 nT^4}\sum_{i=1}^n\left(\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)^2 - \frac{1}{nT^2}\sum_{i=1}^n\sum_{t=1}^T \bar{u}_{it}^2\right. \\
&\quad + \frac{1}{nT^2}\sum_{i=1}^n(\beta_i - \beta)^2\left[\frac{15}{\sigma_x^2 T^2}\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2 - \left(\sum_{t=1}^T \bar{x}_{it}^2\right)\right] \\
&\quad + \frac{1}{nT^2}\left(\hat{\beta} - \beta\right)^2\sum_{i=1}^n\left[\frac{15}{\sigma_x^2 T^2}\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2 - \left(\sum_{t=1}^T \bar{x}_{it}^2\right)\right] \\
&\quad + \frac{30}{\sigma_x^2 nT^4}\sum_{i=1}^n\left[(\beta_i - \beta)\left(\sum_{t=1}^T \bar{x}_{it}^2\right)\left(\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)\right] \\
&\quad - \frac{30}{\sigma_x^2 nT^4}\left(\hat{\beta} - \beta\right)\sum_{i=1}^n\left[(\beta_i - \beta)\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2\right] \\
&\quad - \frac{30}{\sigma_x^2 nT^4}\left(\hat{\beta} - \beta\right)\sum_{i=1}^n\left[\left(\sum_{t=1}^T \bar{x}_{it}^2\right)\left(\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)\right] - \frac{2}{nT^2}\sum_{i=1}^n\left[(\beta_i - \beta)\left(\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)\right] \\
&\quad \left. + \frac{2}{nT^2}\left(\hat{\beta} - \beta\right)\left(\sum_{i=1}^n\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right) + \frac{2}{nT^2}\left(\hat{\beta} - \beta\right)\sum_{i=1}^n\left[(\beta_i - \beta)\left(\sum_{t=1}^T \bar{x}_{it}^2\right)\right]\right\} \\
&= \frac{12}{7\sigma_x^2}\left\{A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 + A_9 + A_{10}\right\}. \qquad (42)
\end{aligned}
$$

Consider $A_1 + A_2$:

$$
\begin{aligned}
A_1 + A_2 &= \frac{1}{n}\sum_{i=1}^{\lfloor n\lambda \rfloor}\left[\frac{15}{\sigma_x^2}\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2 - \frac{1}{T^2}\sum_{t=1}^{T}\bar{u}_{it}^2\right] \\
&\quad + \frac{1}{nT^2}\sum_{i=\lfloor n\lambda \rfloor+1}^{n}\left[\frac{15}{\sigma_x^2}\left(\frac{1}{T}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2 - \frac{1}{T}\sum_{t=1}^{T}\bar{u}_{it}^2\right] \\
&= A_{12,1} + A_{12,2}.
\end{aligned}
$$

Consider $A_{12,1}$; as $(n,T) \to \infty$, (30) and (31) yield (conditional on C) $A_{12,1} = 15 \times \frac{\lambda\sigma_u^2}{90} - \frac{1}{6}\lambda\sigma_u^2 + O_p\left(\frac{1}{\sqrt{T}}\right) = O_p\left(\frac{1}{\sqrt{T}}\right)$. Also, conditional on C

$$
\begin{aligned}
& E\left|\frac{15}{\sigma_x^2}\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2 - \frac{1}{T^2}\sum_{t=1}^{T}\bar{u}_{it}^2\right|^{2+\delta} \\
\leq & \; M\left[E\left|\frac{15}{\sigma_x^2}\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2\right|^{2+\delta} + E\left|\frac{1}{T^2}\sum_{t=1}^{T}\bar{u}_{it}^2\right|^{2+\delta}\right],
\end{aligned}
$$

which is finite in light of Assumption 2*(ii)* as $T \to \infty$ - see also the passages in the proof of Lemma 2. Last, since, conditional on C, the sequence $\frac{15}{\sigma_x^2}\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2 - T^{-2}\sum_{t=1}^{T}\bar{u}_{it}^2$ is an MDS, the MDS CLT can be applied to $A_{12,1} - E\left(A_{12,1}\right)$ leading to $A_{12,1} - E\left(A_{12,1}\right) = O_p\left(n^{-1/2}\right)$. Thus, $A_{12,1} = O_p\left(\phi_{nT}^{-1}\right)$. As far as $A_{12,2}$ is concerned, the same considerations apply, thereby yielding $A_{12,2} = T^{-2}O_p\left(\phi_{nT}^{-1}\right)$. As far as A_3 is concerned, define $Z_{1iT} = \left(15/\sigma_x^2\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2 - \left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)$. As $T \to \infty$, from Assumption 2*(ii)* it holds that $E\left(Z_{1iT}\right) = \frac{15}{20}\sigma_x^2 - \frac{1}{6}\sigma_x^2 + O_p\left(\frac{1}{\sqrt{T}}\right) = \frac{7}{12}\sigma_x^2 + O_p\left(\frac{1}{\sqrt{T}}\right)$. The sequence $\left(\beta_i - \beta\right)^2 Z_{1iT}$ is an MDS with mean $E\left[\left(\beta_i - \beta\right)^2 Z_{1iT}\right] = E\left[\left(\beta_i - \beta\right)^2\right]E\left[Z_{1iT}\right] = \frac{7}{12}\sigma_\beta^2\sigma_x^2 + O_p\left(T^{-1/2}\right)$ and with $E\left|\left(\beta_i - \beta\right)^2 Z_{1iT}\right| = \sigma_\beta^2 E\left|Z_{1iT}\right| < \infty$ - see the proof of Lemma 1. Therefore $A_3 \overset{a.s.}{=} \frac{7\sigma_\beta^2\sigma_x^2}{12} + O_p\left(\frac{1}{\sqrt{T}}\right)$. Since $A_4 = \left(\hat{\beta} - \beta\right)^2\left[n^{-1}\sum_{i=1}^{n}Z_{1iT}\right]$, using (4) we get $A_4 = O_p\left(n^{-1}\right)$. As far as A_5 is concerned, we have

$$
\begin{aligned}
\frac{30}{\sigma_x^2 n}\sum_{i=1}^{n}\left(\beta_i - \beta\right)Z_{2iT} &= \frac{30}{\sigma_x^2 n}\sum_{i=1}^{\lfloor n\lambda \rfloor}\left(\beta_i - \beta\right)Z_{2iT}^{(1)} + \frac{30}{\sigma_x^2 n}\sum_{i=\lfloor n\lambda \rfloor+1}^{n}\left(\beta_i - \beta\right)Z_{2iT}^{(2)} \\
&= A_{51} + A_{52},
\end{aligned}
$$

where $Z_{2iT} = \left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)$. Conditional on C the sequence

$\left(\beta_i-\beta\right) Z_{2iT}^{(1)}$ is a zero mean MDS for all T. We have $E\left|\left(\beta_i-\beta\right) Z_{2iT}^{(1)}\right|^{2+\delta}=E\left|\left(\beta_i-\beta\right)\right|^{2+\delta} E\left|Z_{2iT}^{(1)}\right|^{2+\delta}<\infty$ after Assumption 3 and (40). The MDS CLT entails $A_{51}=O_p\left(n^{-1/2}\right)$. As far as A_{52} is concerned, $Z_{2iT}^{(2)}=O_p\left(T^{-1}\right)$ and therefore $A_{52}=O_p\left(\frac{1}{\sqrt{n}T}\right)$. Thus, $A_5=d_\sigma O_p\left(\frac{1}{\sqrt{n}T^{d_0}}\right)$. Consider now A_6; the sequence $\left(\beta_i-\beta\right)\left(T^{-2}\sum_{t=1}^T \bar{x}_{it}^2\right)^2$ is an MDS with zero mean and finite moment of order $2+\delta$ in light of (41). Hence $-A_6=\frac{30}{\sigma_x^2 n}\left(\hat{\beta}-\beta\right)\sum_{i=1}^n\left(\beta_i-\beta\right)\left(\frac{1}{T^2}\sum_{t=1}^T \bar{x}_{it}^2\right)^2$ $=O_p\left(\frac{1}{\sqrt{n}}\right) d_\sigma O_p\left(\frac{1}{\sqrt{n}}\right)=d_\sigma O_p\left(\frac{1}{n}\right)$. As far as A_7 is concerned, define $Z_{3iT}=\left(T^{-2}\sum_{t=1}^T \bar{x}_{it}^2\right)\left(T^{-2}\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}\right)$, so that $-A_7=\frac{30}{\sigma_x^2 n}\left(\hat{\beta}-\beta\right)\sum_{i=1}^{\lfloor n\lambda\rfloor} Z_{3iT}^{(1)}+\left(\hat{\beta}-\beta\right)$ $\frac{30}{\sigma_x^2 n}\sum_{i=\lfloor n\lambda\rfloor+1}^n Z_{3iT}^{(2)}=A_{71}+A_{72}$. Conditional on C, Z_{3iT} is an MDS and, as $T\to\infty$, $E\left[Z_{3iT}^{(1)}\right]=O_p\left(T^{-1/2}\right)$ and $E\left[Z_{3iT}^{(2)}\right]=O_p\left(T^{-3/2}\right)$. From (40), it follows that $E\left|Z_{3iT}\right|^{2+\delta}<\infty$. Thus

$$\begin{aligned}
\sum_{i=1}^{\lfloor n\lambda\rfloor} Z_{3iT}^{(1)} &= \sum_{i=1}^{\lfloor n\lambda\rfloor}\left\{Z_{3iT}^{(1)}-E\left[Z_{3iT}^{(1)}\right]\right\}+\sum_{i=1}^{\lfloor n\lambda\rfloor} E\left[Z_{3iT}^{(1)}\right]=O_p\left(\sqrt{n}\right)+O\left(\frac{n}{\sqrt{T}}\right),\\
\sum_{i=\lfloor n\lambda\rfloor+1}^{n} Z_{3iT}^{(2)} &= \sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left\{Z_{3iT}^{(2)}-E\left[Z_{3iT}^{(2)}\right]\right\}+\sum_{i=\lfloor n\lambda\rfloor+1}^{n} E\left[Z_{3iT}^{(2)}\right]=O_p\left(\frac{\sqrt{n}}{T}\right)+O\left(\frac{n}{T^{3/2}}\right).
\end{aligned}$$

Hence, $A_7=O_p\left(\frac{1}{nT^{d_0}}\right)+O_p\left(\frac{1}{\sqrt{nT}T^{d_0}}\right)$. Consider A_8 and define $Z_{4iT}=T^{-2}\sum_{t=1}^T \bar{x}_{it}\bar{u}_{it}$; we have $-A_8=\frac{2}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\beta_i-\beta\right) Z_{4iT}^{(1)}+\frac{2}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^n\left(\beta_i-\beta\right) Z_{4iT}^{(2)}=A_{81}+A_{82}$. Conditional on C, $\left(\beta_i-\beta\right) Z_{4iT}$ is MDS with zero mean with $E\left|Z_{4iT}\right|^{2+\delta}<\infty$ from (40). Using Assumption 2*(ii)*, we have $A_{81}=O_p\left(n^{-1/2}\right)$ and $A_{82}=O_p\left(\frac{1}{\sqrt{n}T}\right)$, so that $A_8=d_\sigma O_p\left(\frac{1}{\sqrt{n}T^{d_0}}\right)$. As far as A_9 is concerned, note that $A_9=\frac{2}{n}\left(\hat{\beta}-\beta\right)\sum_{i=1}^{\lfloor n\lambda\rfloor} Z_{4iT}^{(1)}+\frac{2}{n}\left(\hat{\beta}-\beta\right)\sum_{i=\lfloor n\lambda\rfloor+1}^n Z_{4iT}^{(2)}=A_{91}+A_{92}$. As $T\to\infty$, Assumption 2*(ii)* implies that $E\left[Z_{4iT}^{(1)}\right]=O_p\left(T^{-1/2}\right)$ and $E\left[Z_{4iT}^{(2)}\right]=O_p\left(T^{-3/2}\right)$. Then $A_{91}=\frac{2}{n}\left(\hat{\beta}-\beta\right)\sum_{i=1}^{\lfloor n\lambda\rfloor}$ $\left[Z_{5iT}-E\left(Z_{5iT}\right)\right]+\frac{2}{n}\left(\hat{\beta}-\beta\right)\sum_{i=1}^{\lfloor n\lambda\rfloor} E\left(Z_{5iT}\right)=O_p\left(\frac{1}{n}\right)+O_p\left(\frac{1}{\sqrt{nT}}\right)$; similar passages yield $A_{92}=O_p\left(\frac{1}{nT}\right)+O_p\left(\frac{1}{\sqrt{nT}T}\right)$. Hence, $A_9=O_p\left(\frac{1}{nT^{d_0}}\right)+O_p\left(\frac{1}{\sqrt{nT}T^{d_0}}\right)$. Last, as far as A_{10} is concerned, the proof for (35) shows that $\sum_{i=1}^n\left(\beta_i-\beta\right)\left(T^{-2}\sum_{t=1}^T \bar{x}_{it}^2\right)$ $=O_p\left(\sqrt{n}\right)$ and therefore $A_{10}=d_\sigma O_p\left(\frac{1}{n}\right)$. Putting all together, we have

$$\hat{\sigma}_\beta^2=\frac{12}{7\sigma_x^2}\times\frac{7\sigma_\beta^2\sigma_x^2}{12}+O_p\left(\frac{1}{\sqrt{T}}\right)+O_p\left(\frac{1}{n}\right)+d_\sigma O_p\left(\frac{1}{\sqrt{n}T^d}\right)+o_p(1). \quad (43)$$

The proof for (12) follows similar passages. We have $\widehat{\lambda\sigma_u^2}=\left[-\frac{5}{7}\hat{\psi}_1+\frac{9}{7}\hat{\psi}_2\right]\hat{\sigma}_x^2$;

replacing $\left(nT^2\right)^{-1}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}^2$ and $\hat{\sigma}_x^2$ with their limit and using (38) and (39)

$$\begin{aligned}
\widehat{\lambda\sigma_u^2} &= 6\left[-\frac{30}{7\sigma_x^2 nT^4}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\hat{v}_{it}\right)^2+\frac{9}{7nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{v}_{it}^2\right] \\
&= \frac{18}{7}\left\{-\frac{10}{\sigma_x^2 nT^4}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2+\frac{3}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{u}_{it}^2\right. \\
&\quad +\frac{1}{nT^2}\sum_{i=1}^{n}\left(\beta_i-\beta\right)^2\left[3\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)-\frac{10}{\sigma_x^2T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right] \\
&\quad +\frac{1}{nT^2}\left(\hat{\beta}-\beta\right)^2\sum_{i=1}^{n}\left[3\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)-\frac{10}{\sigma_x^2T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right] \\
&\quad -\frac{20}{\sigma_x^2 nT^4}\sum_{i=1}^{n}\left[\left(\beta_i-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right] \\
&\quad +\frac{20}{\sigma_x^2 nT^4}\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left[\left(\beta_i-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right] \\
&\quad +\frac{20}{\sigma_x^2 nT^4}\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left[\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right]+\frac{6}{nT^2}\sum_{i=1}^{n}\left[\left(\beta_i-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right] \\
&\quad \left.-\frac{6}{nT^2}\left(\hat{\beta}-\beta\right)\left(\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)-\frac{6}{nT^2}\left(\hat{\beta}-\beta\right)\sum_{i=1}^{n}\left[\left(\beta_i-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\right]\right\} \\
&= \frac{18}{7}\left\{B_1+B_2+B_3+B_4+B_5+B_6+B_7+B_8+B_9+B_{10}\right\}. \qquad (44)
\end{aligned}$$

Assumption 2*(ii)* yields, as $(n,T)\to\infty$, $B_1=-10\times\frac{\lambda\sigma_u^2}{90}+O_p\left(\frac{1}{\sqrt{T}T^{2d_0}}\right)$, and $B_2=3\times\frac{\lambda\sigma_u^2}{6}+O_p\left(\frac{1}{\sqrt{T}T^{d_0}}\right)$; therefore, $B_1+B_2=\frac{7\lambda\sigma_u^2}{18}+O_p\left(\frac{1}{\sqrt{T}T^{d_0}}\right)$. Consider B_3, and define

$$Z_{5iT}=3\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)-\frac{10}{\sigma_x^2}\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2. \qquad (45)$$

As $T\to\infty$, $E\left(Z_{5iT}\right)=3\times\frac{\sigma_x^2}{6}-\frac{10}{\sigma_x^2}\times\frac{\sigma_x^4}{20}+O_p\left(\frac{1}{\sqrt{T}}\right)=O_p\left(\frac{1}{\sqrt{T}}\right)$, and therefore $B_3=\frac{1}{n}\sum_{i=1}^{n}\left(\beta_i-\beta\right)^2\left[Z_{5iT}-E\left(Z_{5iT}\right)\right]+\frac{1}{n}\sum_{i=1}^{n}\left(\beta_i-\beta\right)^2E\left(Z_{5iT}\right)=B_{31}+B_{32}$. Since $\left(\beta_i-\beta\right)^2\left[Z_{5iT}-E\left(Z_{5iT}\right)\right]$ is an MDS with zero mean and finite moment of order $2+\delta$, the CLT entails $B_{31}=O_p\left(n^{-1/2}\right)$. Also, $B_{32}=O_p\left(T^{-1/2}\right)$. Thus

$B_3 = d_\sigma O_p\left[1/\phi_{nT}\right]$. As far as B_4 is concerned, we have

$$
\begin{aligned}
B_4 &= \left(\hat{\beta}-\beta\right)^2 \frac{1}{n}\sum_{i=1}^{n} Z_{5iT} \\
&= \left(\hat{\beta}-\beta\right)^2 \frac{1}{n}\sum_{i=1}^{n} \left[Z_{5iT}-E\left(Z_{5iT}\right)\right] + \left(\hat{\beta}-\beta\right)^2 \frac{1}{n}\sum_{i=1}^{n} E\left(Z_{5iT}\right) \\
&= B_{41}+B_{42}
\end{aligned}
$$

Similar arguments as for B_3, together with $\hat{\beta}-\beta = O_p\left(n^{-1/2}\right)$, lead to $B_4 = O_p\left(\frac{1}{n\phi_{nT}}\right)$. As far as the other terms are concerned, similar passages as in the proof for (11) yield: $B_5 = d_\sigma O_p\left(\frac{1}{\sqrt{nT^{d_0}}}\right)$; $B_6 = d_\sigma O_p\left(n^{-1}\right)$; $B_7 = O_p\left(\frac{1}{nT^{d_0}}\right) + O_p\left(\frac{1}{\sqrt{nT}T^{d_0}}\right)$; $B_8 = d_\sigma O_p\left(\frac{1}{\sqrt{nT^{d_0}}}\right)$; $B_9 = O_p\left(\frac{1}{nT^{d_0}}\right)$ and $B_{10} = d_\sigma O_p\left(n^{-1}\right)$. Putting all together

$$
\begin{aligned}
\widehat{\lambda\sigma_u^2} &= \frac{18}{7}\times\frac{7\lambda\sigma_u^2}{18} + d_\sigma O_p\left(\frac{1}{\phi_{nT}}\right) \\
&\quad + O_p\left(\frac{1}{\sqrt{T}T^{d_0}}\right) + d_\sigma O_p\left(\frac{1}{n}\right) + O_p\left(\frac{1}{\sqrt{n}T^{d_0}}\right) + o_p(1).
\end{aligned}
\tag{46}
$$

■

Proof of Theorem 2. The results follow from (46). Under the null that $\lambda = 0$ and $\sigma_\beta^2 \in (0,+\infty)$, $\widehat{\lambda\sigma_u^2} = O_p\left(\frac{1}{\sqrt{T}}\right) + O_p\left(\frac{1}{\sqrt{n}}\right) + o_p(1)$, and thus $\widehat{\lambda\sigma_u^2} = O_p\left(\phi_{nT}^{-1}\right)$. Consider first the case $\frac{n}{T}\to 0$, which entails $\widehat{\lambda\sigma_u^2} = O_p\left(n^{-1/2}\right)$. From (44), and from the passages thereafter, it follows that the term that dominates in the decomposition is B_3, given by $\frac{18}{7}\times B_3 = \frac{1}{n}\sum_{i=1}^{n}\left(\beta_i-\beta\right)^2 Z_{5iT}$, where Z_{5iT} is defined in (45). Under $\frac{n}{T}\to 0$, $B_3 = \frac{1}{n}\sum_{i=1}^{n}\left(\beta_i-\beta\right)^2\left[Z_{5iT}-E\left(Z_{5iT}\right)\right] + o_p(1)$. The sequence $\left(\beta_i-\beta\right)^2\left[Z_{5iT}-E\left(Z_{5iT}\right)\right]$ is an MDS with mean zero and and finite moment of order $2+\delta$. Thus, the CLT yields

$$
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\beta_i-\beta\right)^2\left[Z_{5iT}-E\left(Z_{5iT}\right)\right] \xrightarrow{d} \sqrt{E\left[\left(\beta_i-\beta\right)^4\right]E\left\{\lim_{T\to\infty}\left[Z_{5iT}-E\left(Z_{5iT}\right)\right]^2\right\}}\times Z,
$$

with $Z\sim N(0,1)$. As $(n,T)\to\infty$, using Assumption 2*(ii)* we have

$$
E\left\{\lim_{T\to\infty}\left[Z_{5iT}-E\left(Z_{5iT}\right)\right]^2\right\} = \sigma_x^4 E\left[\left(\int \bar{W}^2\right)^2\left(3-10\int \bar{W}^2\right)^2\right].
$$

Thus, under H_0 as $(n,T)\to\infty$ with $\frac{n}{T}\to 0$, $\sqrt{n}\times\frac{\widehat{\lambda\sigma_u^2}}{\sigma_x^2} \xrightarrow{d} \frac{18}{7\sigma_x^2}\times\sqrt{\kappa_\beta\sigma_x^4\delta_\lambda}\times Z$. When

$\frac{T}{n} \to 0$, (46) entails $\sqrt{T} \times \widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}} = O_p(1)$. Last, consider $H_A^{(n,T)}$; since $\phi_{nT} \times \widehat{\frac{\lambda\sigma_u^2}{\sigma_x^2}} \overset{a.s.}{=} \phi_{nT} \times \frac{\lambda\sigma_u^2}{\sigma_x^2} + D_\lambda$, the drift term is nonzero as $(n,T) \to \infty$ if $\frac{\lambda\sigma_u^2}{\sigma_x^2} = O\left(\phi_{nT}^{-1}\right)$. ■

Proof of Proposition 2. The individual estimators $\hat{\beta}_i$ are such that

$$\hat{\beta}_i - \beta_i = O_p(1) \text{ for } i = 1, ..., \left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor, \tag{47}$$

$$\hat{\beta}_i - \beta_i = O_p\left(\frac{1}{T}\right) \text{ for } i = \left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor + 1, ..., n. \tag{48}$$

Thus we have

$$\begin{aligned}
\widehat{\bar{\beta}}_i &= \frac{1}{n}\sum_{i=1}^{n}\hat{\beta}_i \\
&= \frac{1}{n}\sum_{i=1}^{n}\beta_i + \frac{1}{n}\sum_{i=1}^{\left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor}\left(\hat{\beta}_i - \beta_i\right) + \frac{1}{n}\sum_{i=\left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor+1}^{n}\left(\hat{\beta}_i - \beta_i\right) \\
&= \beta + o_p(1) + \frac{1}{n}\left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor \max_{i=1,...,\left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor}\left(\hat{\beta}_i - \beta_i\right) + \frac{1}{n}\left(n - \left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor\right)\max_{i=\left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor+1,...,n}\left(\hat{\beta}_i - \beta_i\right) \\
&= \beta + o_p(1) + cO_p\left(\frac{1}{\phi_{nT}}\right) + O_p\left(\frac{1}{T}\right),
\end{aligned}$$

using the LLN for $n^{-1}\sum_{i=1}^{n}\beta_i$. Thus, as $c = o(\phi_{nT})$, $\widehat{\bar{\beta}}_i = \beta + o_p(1)$. Hence for some constants M_j, $j = 1, 2$

$$\begin{aligned}
\hat{\kappa}_\beta &= \frac{1}{n}\sum_{i=1}^{n}\left(\hat{\beta}_i - \widehat{\bar{\beta}}_i\right)^4 \\
&= \frac{1}{n}\sum_{i=1}^{\left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor}\left(\hat{\beta}_i - \widehat{\bar{\beta}}_i\right)^4 + \frac{1}{n}\sum_{i=\left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor+1}^{n}\left(\hat{\beta}_i - \widehat{\bar{\beta}}_i\right)^4 \\
&\leq \frac{1}{n}\sum_{i=1}^{n}\left(\beta_i - \beta\right)^4 + M_1\frac{1}{n}\left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor \max_{i=1,...,\left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor}\left[\left(\hat{\beta}_i - \beta_i\right) - \left(\widehat{\bar{\beta}}_i - \beta\right)\right]^4 \\
&\quad + M_2\frac{1}{n}\left(n - \left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor\right)\max_{i=\left\lfloor \frac{nc}{\phi_{nT}} \right\rfloor+1,...,n}\left[\left(\hat{\beta}_i - \beta_i\right) - \left(\widehat{\bar{\beta}}_i - \beta\right)\right]^4 \\
&= I + II + III.
\end{aligned}$$

The LLN (following from Assumption 3) ensures that $I = \kappa_\beta + o_p(1)$; also, using (47) and (48) respectively, we have $II = cO_p\left(\phi_{nT}^{-1}\right)$, $III = o_p(1)$. Thus, $\hat{\kappa}_\beta = \kappa_\beta + o_p(1) + cO_p\left(\phi_{nT}^{-1}\right)$. ■

Proof of Theorem 3. The results follow from the proof of Corollary 4 below, of which this theorem is a special case. Specifically, since under H_0 with $\lambda > 0$ the term that dominates in (42) is $A_1 + A_2$, (17) follows from (50) setting $\phi_a = 1$ and $\phi_b = 0$. As $(n, T) \to \infty$ with $\frac{T}{n} \to 0$, (43) leads to $\sqrt{T}\hat{\sigma}^2_\beta = O_p(1)$. Last, consider $H_A^{(n,T)}$; since $\phi_{nT} \times \hat{\sigma}^2_\beta \overset{a.s.}{=} \phi_{nT} \times \sigma^2_\beta + D_\sigma$, the drift term is nonzero as $(n, T) \to \infty$ if $\sigma^2_\beta = c\phi^{-1}_{nT}$. ■

Proof of Corollary 3. Recall

$$\begin{aligned}
\hat{\lambda} &= -\frac{5}{7}\frac{n}{\left[\sum_{i=1}^n \sum_{t=1}^T \bar{x}^2_{it}\right]^2}\sum_{i=1}^n\left(\sum_{t=1}^T \bar{x}_{it}\frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right)^2 \times \frac{6}{nT^2}\sum_{i=1}^n\sum_{t=1}^T \bar{x}^2_{it} \\
&\quad +\frac{9}{7}\frac{1}{\sum_{i=1}^n \sum_{t=1}^T \bar{x}^2_{it}}\sum_{i=1}^n\sum_{t=1}^T\left(\frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right)^2 \times \frac{6}{nT^2}\sum_{i=1}^n\sum_{t=1}^T \bar{x}^2_{it} \\
&= \frac{18}{7}\left\{-\frac{10}{T^2\sum_{i=1}^n \sum_{t=1}^T \bar{x}^2_{it}}\sum_{i=1}^n\left(\sum_{t=1}^T \bar{x}_{it}\frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right)^2 + \frac{3}{nT^2}\sum_{i=1}^n\sum_{t=1}^T\left(\frac{\hat{v}_{it}}{\hat{\sigma}_{u,i}}\right)^2\right\} \qquad (49)
\end{aligned}$$

Replacing $(nT^2)^{-1}\sum_{i=1}^n \sum_{t=1}^T \bar{x}^2_{it}$ with its limit (and omitting higher order terms), defining $\hat{v}^+_{it} = \hat{v}_{it}/\hat{\sigma}_{u,i}$, and $\bar{u}^+_{it} = \bar{u}_{it}/\hat{\sigma}_{u,i}$, and recalling that $\hat{v}_{it} = \bar{u}_{it} + (\beta_i - \beta)\,\bar{x}_{it}$

$-\left(\hat{\beta}-\beta\right)\bar{x}_{it}$, it holds that

$$
\begin{aligned}
\frac{7}{18}\hat{\lambda} &= -\frac{10}{nT^4\sigma_x^2}\times\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\hat{v}_{it}^{+}\right)^2+\frac{3}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\hat{v}_{it}^{+2} \\
&= -\frac{10}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)^2-\frac{10}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right] \\
&\quad -\frac{10\left(\hat{\beta}-\beta\right)^2}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left(\frac{1}{\hat{\sigma}_{u,i}}\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2-\frac{20\left(\hat{\beta}-\beta\right)}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2\right] \\
&\quad +\frac{20}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\right] \\
&\quad -\frac{20\left(\hat{\beta}-\beta\right)}{nT^4\sigma_x^2}\sum_{i=1}^{n}\left[\frac{1}{\hat{\sigma}_{u,i}}\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\right]+\frac{3}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{u}_{it}^{+2} \\
&\quad +\frac{3}{nT^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2\sum_{t=1}^{T}\bar{x}_{it}^2\right]+\frac{3\left(\hat{\beta}-\beta\right)^2}{nT^2}\sum_{i=1}^{n}\left[\frac{1}{\hat{\sigma}_{u,i}^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right] \\
&\quad +\frac{6}{nT^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)\right]-\frac{6\left(\hat{\beta}-\beta\right)}{nT^2}\sum_{i=1}^{n}\left[\frac{1}{\hat{\sigma}_{u,i}}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right] \\
&\quad -\frac{6\left(\hat{\beta}-\beta\right)}{nT^2}\sum_{i=1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}^2}\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\right] \\
&= C_1+C_2+C_3+C_4+C_5+C_6+C_7+C_8+C_9+C_{10}+C_{11}+C_{12}.
\end{aligned}
$$

We have $-C_1=\frac{10}{n\sigma_x^2}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)^2+\frac{10}{n\sigma_x^2}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)^2=C_{1,1}+C_{1,2}$. As far as $C_{1,1}$ is concerned, it holds that

$$
\begin{aligned}
C_{1,1} &= \frac{10}{n\sigma_x^2}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\frac{\bar{u}_{it}}{\sigma_{u,i}}\right)^2-\frac{10}{n\sigma_x^2}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{\hat{\sigma}_{u,i}^2-\sigma_{u,i}^2}{\sigma_{u,i}^2}\right)\left(\frac{1}{T^4}\sum_{t=1}^{T}\bar{x}_{it}\frac{\bar{u}_{it}}{\sigma_{u,i}}\right)^2 \\
&= C_{1,1,1}+C_{1,1,2}.
\end{aligned}
$$

Equation (30) yields $C_{1,1,1}=\lambda/9+O_p\left(T^{-1/2}\right)$; as far as $C_{1,1,2}$ is concerned, we have

$$\begin{aligned}
& \frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{\hat{\sigma}_{u,i}^{2}-\sigma_{u,i}^{2}}{\sigma_{u,i}^{2}}\right)\left(\frac{1}{T^{4}}\sum_{t=1}^{T}\bar{x}_{it}\frac{\bar{u}_{it}}{\sigma_{u,i}}\right)^{2} \\
\leq\ & \max_{i}\left|\frac{\hat{\sigma}_{u,i}^{2}-\sigma_{u,i}^{2}}{\sigma_{u,i}^{2}}\right|\times\frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{1}{T^{4}}\sum_{t=1}^{T}\bar{x}_{it}\frac{\bar{u}_{it}}{\sigma_{u,i}}\right)^{2}=O_{p}\left(\frac{1}{T^{\varepsilon_{1}}}\right).
\end{aligned}$$

Consider $C_{1,2}$; using (31) and (33)

$$\begin{aligned}
C_{1,2} &= \frac{10}{n\sigma_{x}^{2}}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\frac{1}{T^{4}}\sum_{t=1}^{T}\bar{x}_{it}\frac{\bar{u}_{it}}{\hat{\sigma}_{u,i}}\right)^{2} \\
&\leq \frac{10}{n\sigma_{x}^{2}}\max_{i}\left|\frac{1}{\hat{\sigma}_{u,i}^{2}}\right|\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\frac{1}{T^{4}}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^{2}=O_{p}\left(T^{\varepsilon_{2}}\right)O_{p}\left(\frac{1}{T^{2}}\right)=O_{p}\left(T^{\varepsilon_{2}-2}\right).
\end{aligned}$$

Thus, $C_1 = -\lambda/9 + (1-d_0)\, O_p\left(T^{-\varepsilon_1}\right) + (1-d_1)\, O_p\left(T^{\varepsilon_2 - 2}\right)$. Consider now $C_2 + C_8$; we have $C_2 + C_8 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2 \left[-\frac{10}{\sigma_x^2 T^4}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2 + \frac{3}{T^2}\sum_{t=1}^{T}\bar{x}_{it}^2\right]$. Let $X_{1iT} = -\left(10/\sigma_x^2\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2 + 3T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2$; then

$$\begin{aligned}
C_2 + C_8 &= \frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2 X_{1iT}^{(1)}\right] + \frac{1}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2 X_{1iT}^{(2)}\right] \\
&= C_{28,1} + C_{28,2}.
\end{aligned}$$

Lemma 1 yields $E\left[X_{1iT}^{(1)}\right] = O_p\left(T^{-1/2}\right)$, and therefore $C_{28,1} = \frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2 \left\{X_{1iT}^{(1)} - E\left[X_{1iT}^{(1)}\right]\right\} + \frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left[\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2 E\left[X_{1iT}^{(1)}\right]\right] = C_{28,1,1} + C_{28,1,2}$.The CLT entails that $C_{28,1,1} = O_p\left(n^{-1/2}\right)$, and $C_{28,1,2} \leq \max_i \left|E\left[X_{1iT}^{(1)}\right]\right| \frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{\beta_i-\beta}{\hat{\sigma}_{u,i}}\right)^2 = O_p\left(\frac{1}{\sqrt{T}}\right)$. As far as $C_{28,2}$ is concerned, we have

$$\begin{aligned}
C_{28,2} &\leq \max_i\left(\frac{1}{\hat{\sigma}_{u,i}^2}\right)\frac{1}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\beta_i-\beta\right)^2\left\{X_{1iT}^{(2)} - E\left[X_{1iT}^{(2)}\right]\right\} \\
&\quad + \max_i\left(\frac{1}{\hat{\sigma}_{u,i}^2}\right)\left\{\frac{1}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\beta_i-\beta\right)^2 E\left[X_{1iT}^{(2)}\right]\right\} \\
&= O_p\left(T^{\varepsilon_2}\right)O_p\left(\frac{1}{\sqrt{n}}\right) + O_p\left(T^{\varepsilon_2}\right)O_p\left(\frac{1}{\sqrt{T}}\right).
\end{aligned}$$

Thus, $C_2+C_8 = d_\sigma\left[(1-d_0) O_p(1) + (1-d_1) O_p(T^{\varepsilon_2})\right] O_p\left(\phi_{nT}^{-1}\right)$. Let us now turn our attention to C_3+C_9; we have

$$\begin{aligned} C_3+C_9 &= \frac{\left(\hat{\beta}-\beta\right)^2}{n}\sum_{i=1}^{n}\left(\frac{1}{\hat{\sigma}_{u,i}}\right)^2 X_{1iT} \\ &= \frac{\left(\hat{\beta}-\beta\right)^2}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\frac{1}{\hat{\sigma}_{u,i}}\right)^2 X_{1iT}^{(1)} + \frac{\left(\hat{\beta}-\beta\right)^2}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left(\frac{1}{\hat{\sigma}_{u,i}}\right)^2 X_{1iT}^{(2)} \\ &= C_{39,1}+C_{39,2}. \end{aligned}$$

In light of (4), similar passages as above lead to $C_{39,1} = O_p\left(n^{-3/2}\right)+O_p\left(n^{-1}T^{-1/2}\right)$, and $C_{39,2} = O_p\left(T^{\varepsilon_2}/n\phi_{nT}\right)$. Thus, $C_3+C_9 = d_\sigma\left[(1-d_0) O_p(1) + (1-d_1) O_p(T^{\varepsilon_2})\right] O_p\left(\frac{1}{n\phi_{nT}}\right)$. Consider C_4, and define $X_{2iT} = (\beta_i-\beta)\left(T^{-4}\sum_{t=1}^{T}\bar{x}_{it}^2\right)^2$; then $-\sigma_x^2\frac{C_4}{20} = \frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left[\frac{1}{\hat{\sigma}_{u,i}}X_{2iT}^{(1)}\right] + \frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left[\frac{1}{\hat{\sigma}_{u,i}}X_{2iT}^{(2)}\right] = C_{4,1}+C_{4,2}$. As far as $C_{4,1}$ is concerned, the CLT and (4) entail $C_{4,1} = O_p\left(n^{-1}\right)$. Also

$$\begin{aligned} C_{4,2} &\leq \frac{\left(\hat{\beta}-\beta\right)}{n}\max_i\left|\frac{1}{\hat{\sigma}_{u,i}}\right|\sum_{i=\lfloor n\lambda\rfloor+1}^{n}X_{2iT}^{(2)} \\ &= \frac{1}{n}O_p\left(\frac{1}{\sqrt{n}}\right)O_p\left(T^{\varepsilon_2}\right)O_p\left(\sqrt{n}\right) = O_p\left(\frac{T^{\varepsilon_2}}{n}\right). \end{aligned}$$

Thus, $C_4 = d_\sigma\left[(1-d_0) O_p(1) + (1-d_1) O_p(T^{\varepsilon_2})\right] O_p\left(n^{-1}\right)$. Turning our attention to C_5 define $X_{3iT} = (\beta_i-\beta)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)$; we have $\sigma_x^2\frac{C_5}{20} = \frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left[\frac{1}{\hat{\sigma}_{u,i}}X_{3iT}^{(1)}\right] + \frac{1}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left[\frac{1}{\hat{\sigma}_{u,i}}X_{3iT}^{(2)}\right] = C_{5,1}+C_{5,2}$. The MDS CLT yields $C_{5,1} = O_p\left(n^{-1/2}\right)$. Since $X_{3iT}^{(2)} = O_p\left(T^{-1}\right)$, $C_{5,2} \leq \max_i\left|\frac{1}{\hat{\sigma}_{u,i}}\right|\frac{1}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}X_{3iT}^{(2)} = O_p\left(T^{\varepsilon_2}\right)O_p\left(\frac{1}{T}\right)O_p\left(\frac{1}{\sqrt{n}}\right)$. Therefore, $C_5 = d_\sigma\left[(1-d_0) O_p(1) + (1-d_1) O_p(T^{\varepsilon_2-1})\right] O_p\left(n^{-1/2}\right)$. As far as C_6 is concerned, let $X_{4iT} = \left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}^{+}\right)\left(T^{-2}\sum_{t=1}^{T}\bar{x}_{it}^2\right)$; then

$$\begin{aligned} \sigma_x^2\frac{C_6}{20} &= \frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left[\frac{1}{\hat{\sigma}_{u,i}}X_{4iT}^{(1)}\right] + \frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\left[\frac{1}{\hat{\sigma}_{u,i}}X_{4iT}^{(2)}\right] \\ &= C_{6,1}+C_{6,2}. \end{aligned}$$

Note that $E\left[X_{4iT}^{(1)}\right]=O_p\left(T^{-1/2}\right)$ and $E\left[X_{4iT}^{(2)}\right]=O_p\left(T^{-3/2}\right)$. Then we have

$$\begin{aligned} C_{6,1} &= \frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\frac{1}{\hat{\sigma}_{u,i}}\left\{X_{4iT}^{(1)}-E\left[X_{4iT}^{(1)}\right]\right\} \\ &\quad +\frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left\{\frac{1}{\hat{\sigma}_{u,i}}E\left[X_{4iT}^{(1)}\right]\right\} \\ &= C_{6,1,1}+C_{6,1,2}. \end{aligned}$$

The CLT and (4) yield $C_{6,1,1}=O_p\left(n^{-1}\right)$ and $C_{6,1,2}\leq\max_i\left|E\left[X_{4iT}^{(2)}\right]\right|\frac{\left(\hat{\beta}-\beta\right)}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}\frac{1}{\hat{\sigma}_{u,i}}=O_p\left(\frac{1}{\sqrt{T}}\right)O_p\left(\frac{1}{\sqrt{n}}\right)$; hence, $C_{6,1,2}=O_p\left(n^{-1/2}T^{-1/2}\right)$, and thus $C_{6,1}=(1-d_0)\left[O_p\left(n^{-1}\right)+O_p\left(n^{-1/2}T^{-1/2}\right)\right]$. As far as $C_{6,2}$ is concerned, similar arguments as for C_5 entail $C_{6,2}=(1-d_1)\,O_p\left(n^{-1/2}T^{\varepsilon_2-1}\right)+(1-d_1)\,O_p\left(T^{\varepsilon_2-3/2}\right)$. Consider now C_7; it holds that $C_7=\frac{3}{nT^2}\sum_{i=1}^{\lfloor n\lambda\rfloor}\sum_{t=1}^{T}\bar{u}_{it}^{+2}+\frac{3}{nT^2}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\sum_{t=1}^{T}\left(\frac{\bar{u}_{it}}{\hat{\sigma}_{u,i}}\right)^2=C_{7,1}+C_{7,2}$. Similar arguments as for $C_{1,1}$ yield $C_{7,1}\overset{a.s.}{=}\lambda/2+O_p\left(T^{-\varepsilon_1}\right)$. Turning our attention to $C_{7,2}$, one can write $C_{7,2}\leq\frac{3}{T}\max_i\left|\frac{1}{\hat{\sigma}_{u,i}^2}\right|\frac{1}{nT}\sum_{i=\lfloor n\lambda\rfloor+1}^{n}\sum_{t=1}^{T}\bar{u}_{it}^2=O_p\left(T^{\varepsilon_2-1}\right)$. Similar passages as above entail that C_{10}, C_{11} and C_{12} have the same asymptotic magnitude as C_5, C_6 and C_4 respectively.

Putting all together, we have

$$\begin{aligned} \frac{18}{7}\times\hat{\lambda} &= -\frac{\lambda}{9}+\frac{\lambda}{2}+(1-d_0)\,O_p\left(\frac{1}{T^{\varepsilon_1}}\right) \\ &\quad +d_\sigma\left[(1-d_0)\,O_p(1)+(1-d_1)\,O_p\left(T^{\varepsilon_2}\right)\right]O_p\left(\frac{1}{\phi_{nT}}\right) \\ &\quad +(1-d_0)\,O_p\left(\frac{1}{\sqrt{n}\phi_{nT}}\right)+(1-d_1)\,O_p\left(\frac{1}{\phi_{nT}^2}\right)+o_p(1). \end{aligned}$$

■

Proof of Corollary 4. Consider first the case $\lambda>0$, which corresponds to (21). As (42) and the passages thereafter show, as $(n,T)\to\infty$ with $\frac{n}{T}\to 0$, the terms that dominate are A_1+A_2, A_5 and A_8; thus, the limiting distribution of

$\sqrt{n}\left(\hat{\sigma}_{\beta}^{2}-\sigma_{\beta}^{2}\right)$ is given by

$$\begin{aligned}
&\frac{12}{7\sigma_x^2}\sqrt{n}\left\{\frac{15}{\sigma_x^2 nT^4}\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2-\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{u}_{it}^2\right.\\
&\left.+\frac{30}{\sigma_x^2 nT^4}\sum_{i=1}^{n}\left[\left(\beta_i-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right]-\frac{2}{nT^2}\sum_{i=1}^{n}\left[\left(\beta_i-\beta\right)\left(\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\right]\right\}\\
=&\ \frac{12}{7\sigma_x^2}\sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{15}{\sigma_x^2 T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2-\frac{1}{T^2}\sum_{t=1}^{T}\bar{u}_{it}^2\right]\right.\\
&\left.\sum_{i=1}^{n}\left(\beta_i-\beta\right)\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\left[\frac{30}{\sigma_x^2 T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)-2\right]\right\},
\end{aligned}$$

and defining

$$\begin{aligned}
Y_{aiT} &= \left(\frac{15}{\sigma_x^2 T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)^2-\frac{1}{T^2}\sum_{t=1}^{T}\bar{u}_{it}^2,\\
Y_{biT} &= \left(\beta_i-\beta\right)\left(\frac{1}{T^2}\sum_{t=1}^{T}\bar{x}_{it}\bar{u}_{it}\right)\left[\frac{30}{\sigma_x^2 T^2}\left(\sum_{t=1}^{T}\bar{x}_{it}^2\right)-2\right],
\end{aligned}$$

we have $\frac{12}{7\sigma_x^2}\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}Y_{aiT}+\frac{1}{n}\sum_{i=1}^{n}Y_{biT}\right)=\frac{12}{7\sigma_x^2}\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}Y_{aiT}+\frac{1}{n}\sum_{i=1}^{\lfloor n\lambda\rfloor}Y_{biT}\right)$ $+O_p\left(\frac{1}{T}\right)$. To prove a CLT for $\sqrt{n}\left(\hat{\sigma}_{\beta}^{2}-\sigma_{\beta}^{2}\right)$, consider the sequence $\frac{1}{\sqrt{n\lambda}}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\phi_a Y_{aiT}+\phi_b Y_{biT}\right)$, for some nonzero ϕ_a and ϕ_b. Conditioning on C, $\left(\phi_a Y_{aiT}+\phi_b Y_{biT}\right)$ is an MDS with, as $T\to\infty$, $E\left(\phi_a Y_{aiT}+\phi_b Y_{biT}\right)=O_p\left(T^{-1/2}\right)$. Also, $E\left|\phi_a Y_{aiT}+\phi_b Y_{biT}\right|^{2+\delta}\leq M\left(\left|\phi_a\right|^{2+\delta}E\left|Y_{aiT}\right|^{2+\delta}+\left|\phi_b\right|^{2+\delta}E\left|Y_{biT}\right|^{2+\delta}\right)$; thus, after Assumption 2*(ii)* and (40), $E\left|\phi_a Y_{aiT}+\phi_b Y_{biT}\right|^{2+\delta}<\infty$. Then as $(n,T)\to\infty$ with $\frac{n}{T}\to 0$, the MDS CLT yields

$$\frac{1}{\sqrt{n\lambda}}\sum_{i=1}^{\lfloor n\lambda\rfloor}\left(\phi_a Y_{aiT}+\phi_b Y_{biT}\right)\stackrel{d}{\to}\sqrt{E\left[\lim_{T\to\infty}\left(\phi_a Y_{aiT}+\phi_b Y_{biT}\right)^2\right]}\times Z, \qquad (50)$$

with $Z\sim N(0,1)$. Last, it holds that, using Assumption 2*(ii)* and the MDS LLN

$$\begin{aligned}
&E\left[\lim_{T\to\infty}\left(\phi_a Y_{aiT}+\phi_b Y_{biT}\right)^2\right]\\
=&\ \phi_a^2 E\left[\lim_{T\to\infty}Y_{aiT}^2\right]+\phi_b^2 E\left[\lim_{T\to\infty}Y_{biT}^2\right]\\
&+2\phi_a\phi_b E\left[\lim_{T\to\infty}Y_{aiT}Y_{biT}\right].
\end{aligned}$$

Lemma 2 yields $E\left[\lim_{T\to\infty} Y_{aiT}^2\right] = \sigma_u^4 E\left[15\left(\int \bar{W}_1\bar{W}_2\right)^2 - \int \bar{W}_2^2\right]^2$, $E\left[\lim_{T\to\infty} Y_{biT}^2\right] = 4\sigma_u^2\sigma_x^2\sigma_\beta^2\times\ E\left[\left(\int \bar{W}_1\bar{W}_2\right)^2\left(15\int \bar{W}_1^2 - 1\right)^2\right]$ respectively. Finally, Assumption 3 entails $E\left(Y_{aiT}Y_{biT}\right) = 0$ for all T. Thus, as $(n,T)\to\infty$, $\frac{1}{\sqrt{n\lambda}}\sum_{i=1}^{\lfloor n\lambda\rfloor} Y_{aiT}$ and $\frac{1}{\sqrt{n\lambda}}\sum_{i=1}^{\lfloor n\lambda\rfloor} Y_{biT}$ are independent. Putting all together, (21) follows.

As far as (22) is concerned, when $\lambda = 0$, (42) and the passages thereafter entail that $\hat{\sigma}_\beta^2 - \sigma_\beta^2 = O_p\left(n^{-1}\right)$ as $(n,T)\to\infty$ with $\frac{n}{\sqrt{T}}\to 0$; the terms that dominate are, in this case, A_4 and A_6. Thus, the asymptotics of $n\left(\hat{\sigma}_\beta^2 - \sigma_\beta^2\right)$ is driven by

$$\frac{12}{7\sigma_x^2}\left\{\frac{1}{nT^2}\left(\hat{\beta}-\beta\right)^2\sum_{i=1}^n\left[\frac{15}{\sigma_x^2T^2}\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2 - \left(\sum_{t=1}^T \bar{x}_{it}^2\right)\right]\right.$$
$$\left.-\frac{30}{\sigma_x^2 nT^4}\left(\hat{\beta}-\beta\right)\sum_{i=1}^n\left[\left(\beta_i-\beta\right)\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2\right]\right\}.$$

Let $Y_{ciT} = T^{-2}\left(\beta_i-\beta\right)\sum_{t=1}^T \bar{x}_{it}^2$ with $E\left(Y_{ciT}\right) = 0$, and recall $\frac{1}{nT^2}\sum_{i=1}^n\left[\frac{15}{\sigma_x^2T^2}\left(\sum_{t=1}^T \bar{x}_{it}^2\right)^2 - \left(\sum_{t=1}^T \bar{x}_{it}^2\right)\right] \overset{a.s.}{=} \frac{7}{12}\sigma_x^2 + o_p(1)$, and $\hat{\beta}-\beta \overset{a.s.}{=} \frac{6}{\sigma_x^2}\frac{1}{n}\sum_{i=1}^n Y_{ciT} + o_p(1)$, for $\lambda = 0$. Then

$$\begin{aligned}
& n\left(\hat{\sigma}_\beta^2 - \sigma_\beta^2\right)\\
&\overset{a.s.}{=} \frac{12}{7\sigma_x^2}\frac{6}{\sigma_x^2}\left[\frac{7}{12}\sigma_x^2\frac{6}{\sigma_x^2}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_{ciT}\right)^2 - \frac{30}{\sigma_x^2}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_{ciT}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_{ciT}d_{iT}\right) + o_p(1)\right]\\
&\overset{a.s.}{=} \frac{72}{7\sigma_x^4}\left[\frac{7}{2}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_{ciT}\right)\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n Y_{ciT}\left(1-\frac{60}{7\sigma_x^2}d_{iT}\right)\right] + o_p(1)\right],
\end{aligned}$$

with $d_{1iT} = T^{-2}\sum_{t=1}^T \bar{x}_{it}^2$. Let (for the sake of the notation) $Y_{diT} = Y_{ciT}\left(1-\frac{60}{7\sigma_x^2}d_{iT}\right)$; it holds that $E\left(Y_{diT}\right) = 0$. Also, conditional on C, the sequences Y_{ciT} and Y_{diT} are an MDS $E\left|Y_{ciT}\right|^{2+\delta} < \infty$, and $E\left|Y_{diT}\right|^{2+\delta} \leq \left[E\left|Y_{ciT}\right|^{4+2\delta}\right]^{1/2}\left[E\left|d_{iT}\right|^{4+2\delta}\right]^{1/2} < \infty$. Consider the sequence $\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(\phi_c Y_{ciT} + \phi_d Y_{diT}\right)$; the MDS CLT yields

$$\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^n\left(\phi_c Y_{ciT} + \phi_d Y_{diT}\right)\\
&\overset{d}{\to}\sqrt{\phi_c^2 E\left(\lim_{T\to\infty} Y_{ciT}^2\right) + \phi_d^2 E\left(\lim_{T\to\infty} Y_{diT}^2\right) + 2\phi_c\phi_d E\left(\lim_{T\to\infty} Y_{ciT}Y_{diT}\right)}\times Z,
\end{aligned}$$

where $Z \sim N(0,1)$ and

$$\begin{aligned}
E\left(\lim_{T\to\infty} Y_{ciT}^2\right) &= E\left[(\beta_i-\beta)^2\right] E\left[\sigma_x^4\left(\int \bar{W}^2\right)^2\right] = \frac{\sigma_\beta^2\sigma_x^4}{20}, \\
E\left(\lim_{T\to\infty} Y_{diT}^2\right) &= E\left[(\beta_i-\beta)^2\right] E\left\{\left[\sigma_x^2\int \bar{W}^2\right]^2\left[1-\frac{60}{7}\int \bar{W}^2\right]^2\right\} \\
&= \sigma_\beta^2\sigma_x^4 E\left[\left(\int \bar{W}^2\right)^2\left(1-\frac{60}{7}\int \bar{W}^2\right)^2\right] \\
&= \sigma_\beta^2\sigma_x^4\delta_{3\sigma}.
\end{aligned}$$

Finally, $E\left(\lim_{T\to\infty} Y_{ciT}Y_{diT}\right) = \sigma_\beta^2\sigma_x^4 E\left[\left(\int \bar{W}^2\right)^2\left(1-\frac{60}{7}\int \bar{W}^2\right)\right]$. This implies that

$$\left[\begin{array}{c} \frac{1}{\sqrt{n}}\sum_{i=1}^n Y_{ciT} \\ \frac{1}{\sqrt{n}}\sum_{i=1}^n Y_{diT} \end{array}\right] \xrightarrow{d} N\left[\left(\begin{array}{c} 0 \\ 0 \end{array}\right), \left(\begin{array}{cc} \sigma_\beta^2\sigma_x^4/20 & \cdot \\ \rho_\sigma\sigma_\beta^2\sigma_x^4\sqrt{\delta_{3\sigma}/20} & \sigma_\beta^2\sigma_x^4\delta_{3\sigma} \end{array}\right)\right].$$

■

Appendix C

$\delta_{1\sigma}$	0.12327
$\delta_{2\sigma}$	0.32067
$\delta_{3\sigma}$	0.84588
ρ_{σ}	-0.7344
δ_{λ}	0.66816

Table 2: simulated values for $\delta_{1\sigma}$, $\delta_{2\sigma}$, $\delta_{3\sigma}$, ρ_{σ} and δ_{λ}. The values have been obtained using 10000 replications. Brownian motions have been simulated over a 2000 points grid.

(ρ,θ)			(n,T)	$(20,100)$	$(20,200)$	$(20,400)$	$(50,200)$	$(50,400)$	$(100,200)$	$(100,400)$
$(0,0)$	σ^2_β	0.10		0.127	0.092	0.084	0.087	0.064	0.097	0.071
		0.25		0.056	0.046	0.033	0.063	0.047	0.066	0.039
		0.50		0.045	0.053	0.039	0.050	0.046	0.057	0.032
$(0.75,0)$	σ^2_β	0.10		0.296	0.107	0.128	0.196	0.089	0.235	0.166
		0.25		0.111	0.067	0.034	0.089	0.085	0.154	0.079
		0.50		0.105	0.044	0.050	0.082	0.053	0.094	0.059
$(0,0.75)$	σ^2_β	0.10		0.142	0.125	0.088	0.165	0.066	0.109	0.094
		0.25		0.094	0.085	0.048	0.093	0.072	0.086	0.070
		0.50		0.091	0.040	0.040	0.055	0.042	0.091	0.055
$(0,-0.75)$	σ^2_β	0.10		0.114	0.060	0.044	0.110	0.065	0.148	0.077
		0.25		0.268	0.058	0.048	0.092	0.071	0.099	0.052
		0.50		0.054	0.040	0.030	0.050	0.044	0.071	0.042

Table 3: size for $H_0 : \lambda = 0$

(ρ,θ)		(n,T)	$(20,100)$	$(20,200)$	$(20,400)$	$(50,200)$	$(50,400)$	$(100,200)$	$(100,400)$
$(0,0)$	σ_β^2	0.10	0.788	0.732	0.735	0.826	0.796	0.944	0.905
			[0.787]	[0.682]	[0.723]	[0.863]	[0.817]	[0.981]	[0.968]
		0.25	0.282	0.272	0.259	0.328	0.345	0.619	0.480
			[0.173]	[0.247]	[0.098]	[0.381]	[0.467]	[0.875]	[0.758]
		0.50	0.211	0.255	0.244	0.281	0.277	0.266	0.230
			[0.053]	[0.156]	[0.162]	[0.281]	[0.245]	[0.296]	[0.157]
$(0.75,0)$	σ_β^2	0.10	0.759	0.688	0.636	0.821	0.772	0.927	0.887
			[0.744]	[0.640]	[0.597]	[0.881]	[0.819]	[0.984]	[0.963]
		0.25	0.749	0.316	0.293	0.464	0.443	0.620	0.511
			[0.713]	[0.200]	[0.262]	[0.556]	[0.512]	[0.913]	[0.797]
		0.50	0.199	0.210	0.254	0.367	0.284	0.305	0.300
			[0.017]	[0.117]	[0.132]	[0.505]	[0.260]	[0.397]	[0.414]
$(0,0.75)$	σ_β^2	0.10	0.822	0.803	0.793	0.896	0.890	0.968	0.959
			[0.760]	[0.709]	[0.674]	[0.895]	[0.873]	[0.988]	[0.985]
		0.25	0.565	0.584	0.646	0.750	0.634	0.844	0.812
			[0.623]	[0.583]	[0.647]	[0.827]	[0.767]	[0.964]	[0.945]
		0.50	0.370	0.381	0.435	0.627	0.491	0.744	0.658
			[0.372]	[0.352]	[0.353]	[0.742]	[0.638]	[0.933]	[0.898]
$(0,-0.75)$	σ_β^2	0.10	0.502	0.238	0.147	0.410	0.240	0.568	0.318
			[0.584]	[0.084]	[0.004]	[0.612]	[0.150]	[0.867]	[0.539]
		0.25	0.490	0.085	0.066	0.140	0.100	0.214	0.115
			[0.404]	[0.000]	[0.000]	[0.005]	[0.000]	[0.176]	[0.001]
		0.50	0.088	0.067	0.079	0.109	0.070	0.087	0.072
			[0.000]	[0.000]	[0.000]	[0.002]	[0.000]	[0.001]	[0.000]

Table 4a: power for $H_0 : \lambda = 0$ (true value $\lambda = 0.25$)

(ρ, θ)		(n, T)	$(20, 100)$	$(20, 200)$	$(20, 400)$	$(50, 200)$	$(50, 400)$	$(100, 200)$	$(100, 400)$
$(0, 0)$	σ_β^2	0.10	0.857	0.874	0.831	0.911	0.909	0.963	0.969
			[0.864]	[0.847]	[0.821]	[0.965]	[0.957]	[0.999]	[0.998]
		0.25	0.492	0.361	0.414	0.593	0.690	0.800	0.665
			[0.522]	[0.325]	[0.718]	[0.745]	[0.828]	[0.989]	[0.958]
		0.50	0.262	0.255	0.318	0.419	0.425	0.542	0.460
			[0.575]	[0.389]	[0.665]	[0.589]	[0.715]	[0.845]	[0.646]
$(0.75, 0)$	σ_β^2	0.10	0.897	0.844	0.818	0.917	0.928	0.969	0.969
			[0.811]	[0.758]	[0.740]	[0.978]	[0.972]	[0.998]	[0.999]
		0.25	0.638	0.629	0.396	0.599	0.583	0.693	0.832
			[0.838]	[0.443]	[0.623]	[0.874]	[0.841]	[0.988]	[0.968]
		0.50	0.240	0.192	0.189	0.330	0.309	0.436	0.410
			[0.175]	[0.332]	[0.145]	[0.802]	[0.860]	[0.741]	[0.807]
$(0, 0.75)$	σ_β^2	0.10	0.867	0.894	0.877	0.965	0.962	0.989	0.986
			[0.859]	[0.842]	[0.872]	[0.976]	[0.981]	[0.997]	[1.000]
		0.25	0.666	0.777	0.847	0.867	0.895	0.962	0.945
			[0.746]	[0.850]	[0.757]	[0.959]	[0.957]	[0.996]	[0.995]
		0.50	0.567	0.666	0.584	0.846	0.860	0.807	0.837
			[0.728]	[0.631]	[0.777]	[0.956]	[0.927]	[0.984]	[0.989]
$(0, -0.75)$	σ_β^2	0.10	0.713	0.547	0.253	0.620	0.285	0.563	0.388
			[0.513]	[0.114]	[0.024]	[0.688]	[0.320]	[0.898]	[0.688]
		0.25	0.149	0.090	0.101	0.129	0.126	0.219	0.141
			[0.354]	[0.000]	[0.000]	[0.014]	[0.001]	[0.239]	[0.003]
		0.50	0.067	0.044	0.040	0.060	0.068	0.104	0.069
			[0.000]	[0.000]	[0.000]	[0.003]	[0.000]	[0.001]	[0.000]

Table 4b: power for $H_0 : \lambda = 0$ (true value of $\lambda = 0.5$)

(ρ,θ)			(n,T)	$(20,100)$	$(20,200)$	$(20,400)$	$(50,200)$	$(50,400)$	$(100,200)$	$(100,400)$
		0.25		0.035	0.041	0.051	0.055	0.065	0.053	0.064
$(0,0)$	λ	0.50		0.040	0.033	0.051	0.057	0.059	0.054	0.055
		0.75		0.031	0.040	0.042	0.050	0.055	0.058	0.055
		0.25		0.015	0.028	0.039	0.029	0.049	0.028	0.046
$(0.75,0)$	λ	0.50		0.029	0.038	0.040	0.046	0.052	0.051	0.061
		0.75		0.034	0.042	0.030	0.053	0.049	0.052	0.054
		0.25		0.043	0.049	0.055	0.067	0.073	0.066	0.069
$(0,0.75)$	λ	0.50		0.042	0.043	0.042	0.058	0.061	0.071	0.061
		0.75		0.042	0.044	0.043	0.061	0.069	0.055	0.064
		0.25		0.000	0.000	0.003	0.000	0.003	0.000	0.000
$(0,-0.75)$	λ	0.50		0.000	0.002	0.006	0.001	0.005	0.001	0.002
		0.75		0.001	0.004	0.008	0.002	0.008	0.000	0.006

Table 5: size for $H_0 : \sigma_\beta^2 = 0$

(ρ, θ)		(n, T)	(20, 100) σ_β^2		(20, 200) σ_β^2		(20, 400) σ_β^2		(50, 200) σ_β^2		(50, 400) σ_β^2		(100, 200) σ_β^2		(100, 400) σ_β^2	
			0.1	0.5	0.1	0.5	0.1	0.5	0.1	0.5	0.1	0.5	0.1	0.5	0.1	0.5
		0.25	0.112	0.718	0.126	0.616	0.109	0.610	0.179	0.904	0.245	0.899	0.466	0.968	0.449	0.954
(0, 0)	λ	0.50	0.092	0.586	0.132	0.487	0.130	0.394	0.184	0.914	0.261	0.924	0.259	0.989	0.318	0.989
		0.75	0.101	0.335	0.083	0.352	0.099	0.625	0.181	0.974	0.221	0.894	0.277	0.994	0.247	0.976
		0.25	0.120	0.255	0.160	0.692	0.180	0.688	0.265	0.925	0.308	0.900	0.381	0.975	0.476	0.958
(0.75, 0)	λ	0.50	0.079	0.686	0.096	0.446	0.094	0.568	0.147	0.907	0.182	0.895	0.343	0.982	0.302	0.983
		0.75	0.106	0.336	0.128	0.481	0.073	0.314	0.256	0.806	0.212	0.820	0.214	0.985	0.272	0.990
		0.25	0.084	0.291	0.080	0.255	0.075	0.242	0.150	0.543	0.117	0.576	0.127	0.956	0.162	0.955
(0, 0.75)	λ	0.50	0.075	0.326	0.078	0.365	0.065	0.249	0.102	0.465	0.104	0.380	0.113	0.705	0.112	0.658
		0.75	0.052	0.293	0.051	0.237	0.068	0.151	0.098	0.405	0.087	0.407	0.110	0.871	0.101	0.698
		0.25	0.304	0.494	0.499	0.829	0.642	0.822	0.842	0.828	0.917	0.800	0.973	0.814	0.946	0.805
(0, −0.75)	λ	0.50	0.386	0.775	0.442	0.859	0.686	0.888	0.766	0.839	0.906	0.796	0.968	0.847	0.978	0.807
		0.75	0.101	0.899	0.534	0.878	0.461	0.866	0.852	0.831	0.913	0.840	0.991	0.852	0.988	0.834

Table 6: power for $H_0 : \sigma_\beta^2 = 0$