The Critical Kurtosis Value and Skewness Correction.

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The Critical Kurtosis Value and Skewness Correction.

In the empirical option pricing literature, it is generally agreed that the pronounced volatility smirks are signs of a strongly negative risk neutral skewness. Given that actual skewness does not seem to be negative enough, a theory that explains such a skewness differential is needed. The paper provides conditions under which skewness in a Lévy process is corrected. For small values of risk aversion, jump induced skewness is corrected only when excess kurtosis is greater than a critical value that depends on squared skewness. Interestingly, keeping kurtosis fixed, the maximum skewness correction occurs for symmetric processes. Some example processes that are extensively used in the literature are discussed separately.
Introduction

A Lévy process extends the Brownian motion by incorporating jumps that arrive at some, potentially infinite, rate. Central examples of infinite jump activity models are the normal inverse Gaussian by Barndorff-Nielsen (1998), the generalized hyperbolic by Eberlein, Keller and Prause (1998), the pure gamma (Heston, 1993), and the Variance Gamma by Madan and Seneta (1990).

Broadly speaking, the accelerated pace of introduction of Lévy models in asset pricing is motivated for statistical, economic, and portfolio management reasons. Statistically, there is now evidence against using a geometric Brownian motion to model prices, especially the deviations of out of the money option prices from the Black-Scholes-Merton predictions.

Economically, since Clark (1973), Lévy processes are motivated by the need to model a stochastic rate of economic activity that does not coincide with the deterministic rate at which time evolves. Ané and Geman (2000) show that such a rate of economic activity can be proxied by transaction volume. Geman, Madan and Yor (2001) argue that if the time change is not locally deterministic, then market prices must be purely discontinuous. Geman, Madan and Yor (2002) address the recovery issue, i.e. estimating the latent rate of economic activity by observing prices. Huang and Wu (2004) empirically analyze option pricing models based on time changed Lévy specifications. Carr and Wu (2003b) argue that time changed Lévy processes are fundamental for modern asset pricing since they can capture stochastic volatility and leverage effects simultaneously.
On the portfolio allocation front, it is hard to justify the existing plethora of diverse financial instruments when such instruments are redundant. Lévy processes can facilitate the study of incomplete markets, where options are not replicable, in a tractable way. In such markets, derivative securities are important for asset allocation as is demonstrated in Carr, Jin and Madan (2001). Essentially, jumps in the price process force agents to face "large" risks and can also play a role in explaining the need for risk management. Some even suggest that disentangling the pure jump from the diffusive component may be at the core of risk management, since the diffusive risks are hedgeable. The $R^2$ of the pure jump component is then a measure of how "hedgeable" the risk is, or equivalently how "redundant" derivative securities are. Ait-Sahalia (2004) finds that in a mixture of Brownian motion with a pure jump Cauchy process maximum likelihood estimators can still asymptotically distinguish the two sources of risk. Carr and Wu (2003c) propose a method to distinguish among continuous and discontinuous components in the price process by studying the speed at which ATM option prices tend to zero as they expire.

In the empirical option pricing literature, it is generally believed that the pronounced volatility smirks are signs of a strongly negative risk neutral skewness. Since empirical return skewness is not always high enough, it is then argued that the risk neutral skewness is the result of risk correction. But so far no theory that explains how skewness is generated due to risk correction for a broad universe of return generating processes and preferences has been formalized. The paper closes some of this gap by providing a first order approx-
imation to the skewness correction for a homogeneous Lévy process by agents who exhibit a constant degree of relative risk aversion, \( \gamma \).

It is informally believed, and some approximate results in the literature have suggested, that the source of the excessively negative risk neutral skewness is excess kurtosis. In the main theorem of the paper, I show that leftward correction for market skewness\(^{1}\), \( \text{SKEW}^* - \text{SKEW} \), is actually driven only by the excess kurtosis above a \textit{critical value} that for all Lévy processes equals one-and-a-half times the squared skewness of the process, \( \frac{3}{2} \text{SKEW}^2 \),

\[
\text{SKEW}^* - \text{SKEW} \approx -\gamma \left[ (\text{KURT} - 3) - \frac{3}{2} \text{SKEW}^2 \right] \text{STD}
\]

where the approximation is correct up to a first order of the agent’s risk aversion.

Thus, a fat-tailed return distribution generates an increasingly negative skewness, only to the extent that excess kurtosis, \( \text{KURT} - 3 \), exceeds the critical value. In the special case of symmetric distributions, the critical value is zero, and the entire excess kurtosis generates skewness correction. For skewed processes, as is the general case of jump processes, the critical kurtosis can be quite high, and skewness correction small, or even zero. This points to a counterintuitive result; for a given kurtosis, skewness is more heavily corrected for the more symmetric processes.

For example, the highly skewed pure jump gamma process is directly scaled under the change of measure, and thus excess kurtosis does not generate any

\(^{1}\)A star superscript denotes a risk neutral quantity.
skewness correction. This implies that, in the gamma process, kurtosis is exactly equal to the critical value. In another interesting example, when a gamma risk is mixed with a diffusion, I show that the existence of the Brownian component raises excess kurtosis beyond its critical value and results in a leftward skewness correction.

The gamma is more commonly used as a subordinator that changes time in a Brownian motion. A diffusion with its time changed by a gamma is called a Variance Gamma (VG) process. Madan, Seneta (1990), Madan, Milne (1991), developed the symmetric VG process, while Madan, Carr and Chang (1998) employed a more general asymmetric version of the VG process to successfully match option smiles. A VG process is equal to the difference of two gamma processes. I show that for a VG process skewness is always corrected to the left. Intuitively, the positive gamma component is scaled down, while the negative gamma is inflated, and thus, under Q, the VG process will have more negative skewness.

The paper is organized as follows: In the first section the general corrections for Lévy cumulants are discussed. In the next section, it is shown that gamma risks are scaled. This implies that skewness for gamma risks is not corrected. In the third section, the mixture of a gamma with a Brownian motion is studied, and it is shown that in this case skewness is actually corrected. In the final section, the general skewness correction formula for all Lévy processes is developed, and some examples are presented.
1 Discounting under Lévy type uncertainty

I assume the log-price of a stock, $X_t = \log S_t$, follows a Lévy process,

$$X_t = dt + \sigma_1 W_t^1 + \int_0^t \int_{-\infty}^\infty xN(ds, dx)$$

(1)

where $W_t^1$ is a diffusion, and $N(dt, dx)$ is the jump counter. Further, assume that moments and cumulants are well defined.\textsuperscript{2} In the moment generating function, $\mathcal{M}(s)$, of a Lévy processes, time gets factored out,

$$\mathcal{M}(s) = \mathcal{E}e^{sX_t} = e^{tK(s)}$$

(2)

where $K(s)$ is the cumulant generating function of the Lévy process\textsuperscript{3}.

The stock drift equals

$$\mu = d + \frac{1}{2} \sigma_1^2 + \int_{-\infty}^{\infty} (e^x - 1)\Pi(dx)$$

(3)

where $\Pi(dx)$ is the Lévy measure.

\textsuperscript{2}The analyticity of the characteristic function at zero implies the existence of the moment and cumulant generating function (see Lukacs (1970) ch. 7). Recently, Carr and Wu (2003a) motivate the use of a log-stable model for option pricing by observing that the implied volatility smirk does not flatten out, and even slightly steepens as maturity grows. Such processes do not have analytic characteristic functions.

\textsuperscript{3}In the mathematical literature of Lévy processes, the Laplace exponent, $\Phi(\lambda)$, defined through $Ee^{-\lambda X_t} = e^{-t\Phi(\lambda)}$, is more often used. Here the cumulant generating function is more appropriate since it generates cumulants which are central to the main theorems. The two functions are related through $K(s) = -\Phi(-s)$. 
1.1 Risk neutral cumulants.

In this section, we are seeking to extend the basic diffusion risk correction for the case of Lévy processes. For CRRA agents, the risk neutral process is an exponentially tilted version of the original process $X_t$. The negative risk aversion "tilts" the $P$ measure

$$\left( \frac{dQ}{dP} \right)_t = e^{-\gamma X_t - tK(-\gamma)}$$

(4)

This measure change is also known as an Esscher transform (see Kallsen and Shiryaev, 2002). Given (4), the risk neutral cumulant function of $X_t$ is a first difference of the actual $K(s)$,

$$K^s(s) = \frac{1}{t} \ln \mathcal{E}^s e^{sX_t} = K(s - \gamma) - K(-\gamma)$$

(5)

The cumulants of a Lévy process are horizon-scaled derivatives at zero of its cumulant function. From (5), risk-neutral cumulants are recovered by differentiating at $s = -\gamma$. Risk neutral cumulants are thus functions of risk aversion,

$$c_n^* = c_n^*(\gamma)$$

(6)

When risk neutral cumulants are explicitly written as functions of $\gamma$, it is useful to think of actual cumulants as risk corrected cumulants for risk neutral agents,

$$c_n = c_n^*(0)$$

(7)

In Merton’s case, $X_t = dt + \sigma_1 W^1_t$, the cumulant function equals

$$K(s) = ds + \frac{1}{2} \sigma^2_1 s^2$$

(8)
Applying (5), we directly get

$$K^*(s) = (d - \sigma_1^2 \gamma)s + \frac{1}{2}\sigma_1^2 s^2$$

(9)

which corresponds to a diffusion with the same volatility

$$X_t = (d - \sigma_1^2 \gamma)t + \sigma_1 W_t^{1*}$$

(10)

where $W_t^{1*}$ is a standard Brownian motion under $Q$. In this case, the risk neutral drift is lowered by

$$\mu^* = \mu - \sigma_1^2 \gamma$$

(11)

which implies that

$$\left[ \frac{\partial \mu^*}{\partial \gamma} \right]_{\gamma=0} = -\sigma_1^2$$

(12)

that is, the drift correction is given by the variance. The following theorem generalizes the above result for cumulants of higher order.

**Theorem 1.** For Lévy processes the derivative at zero of the $n^{th}$ risk neutral cumulant equals the negative $(n+1)^{th}$ cumulant,

$$\left[ \frac{\partial c_n^*}{\partial \gamma} \right]_{\gamma=0} = -c_{n+1}$$

(13)

**Proof:** From (5), the $n^{th}$ risk neutral cumulant equals the $n^{th}$ derivative of the cumulant function $K(s)$ at $-\gamma$,

$$c_n^*(\gamma) = \frac{\partial^n K(-\gamma)}{\partial s^n}$$

(14)
It is then clear that

\[ \frac{\partial c_n^*(\gamma)}{\partial \gamma} = - \frac{\partial^{n+1} K(-\gamma)}{\partial s^{n+1}} = -c_{n+1}^*(\gamma) \]  

Equation (15)

Since \( c_{n+1}^*(0) = c_{n+1} \), we have

\[ \frac{\partial c_n^*(0)}{\partial \gamma} = -c_{n+1} \]  

Equation (16)

\[ \text{1.1.1 The term structure of Lévy cumulants.} \]

In diffusions it helps to factor the term out and express equations in terms of rate of variance, \( \sigma^2 \), rather than variance

\[ \text{VAR}(h) = \sigma^2 h \]  

Equation (17)

The linear dependence of Lévy cumulants on the term, \( c_n(h) = c_n h \), introduces specific term structures on skewness and excess kurtosis (Konikov and Madan, 2002). The annual quantities\(^4\) for skewness and excess kurtosis, relate to their horizon dependent counterparts through

\[ \text{SKEW}(h) = \frac{c_3 h}{(c_2 h)^{3/2}} = \text{SKEW}_h^{-1/2} \]  

Equation (18)

where

\[ \text{SKEW} = \frac{c_3}{\sigma^3} \]  

Equation (19)

is the annual skewness, and

\[ \text{KURT}(h) - 3 = \frac{c_4 h}{(c_2 h)^2} = (\text{KURT} - 3)h^{-1} \]  

Equation (20)

\(^4\)Although, these are not rates.
where

\[ \text{KURT} - 3 = \frac{c_t}{\sigma^4} \]  

is the annual excess kurtosis.

2 An economy where excess kurtosis does not generate risk neutral skewness.

A broadly used example of a pure jump Lévy process is the gamma process. Even though the gamma process has been used alone as the return generating process (as in Heston (1993) in the context of option pricing), it is more appropriately either mixed with a Brownian motion, or, because of its one-sided jump activity, used as a subordinator that captures time changes in the construction of the symmetric Variance Gamma, in Madan and Seneta (1990), Madan and Milne (1991), and the asymmetric Variance Gamma, in Madan, Carr and Chang (1998).

For \( l \) and \( v > 0 \), the pure jump gamma process, \( \gamma_t(l, v) \) is distributed as a gamma variate with \( vt \) degrees of freedom

\[ f(\gamma_t = x|\gamma_0 = 0) = \frac{e^{-x/l}x^{vt-1}}{l^{vt}\Gamma(vt)} \]  

The Lévy character of the process is due to the fact that the degrees of freedom of independent gamma variates are additive. Its drift equals

\[ \int_{-\infty}^{\infty} x\Pi(dx) = lv \]  

10
and its variance rate is given by

\[ \sigma^2 = l^2 v \]  

(24)

The gamma process is pure jump, with an infinite arrival rate of small jumps. For \( l > 0 \), the \( \text{Lévy measure of the process,} \)

\[ \Pi(dx) = \frac{v}{x} e^{-x/l} dx \quad \text{for} \ x > 0 \]  

implies a concentration of jumps around zero, in the sense that the arrival rate of jumps away from zero by any \( \epsilon > 0 \) is finite. For

\[ X_t = dt + \gamma_t(l, v) \]  

(26)

the cumulant function equals

\[ \mathcal{K}(s) = ds + \int_{-\infty}^{\infty} (e^{sx} - 1) \Pi(dx) = ds - v \log(1 - ls) \]  

(27)

When manipulating the parameters of the gamma, in order to calibrate higher moments, attention is needed in keeping the drift fixed so that the phenomena captured are not due to a changing drift; from (23), unlike a Brownian motion, a pure gamma always has a non-zero drift.

When \( l > 0 \) the jumps of \( X_t \) are always positive. In financial applications sometimes we need negative jumps. Since a positive gamma process always jumps up, a negative gamma process \( (l < 0) \) will always jump down.

2.1 A higher moment constraint for the gamma process.

Successive derivatives of \( \mathcal{K}(s) \) at zero recover the \( n^{th} \) cumulant of the gamma

\[ c_n(h) = c_n = (n - 1)! l^n v h \]  

(28)
Normalized squared skewness equals

\[ \text{SKEW}(h)^2 = \frac{(c_3h)^2}{(c_2h)^3} = \frac{4}{vh} \]  

(29)

Excess kurtosis equals

\[ \text{KURT}(h) - 3 = \frac{c_4h}{(c_2h)^2} = \frac{6l^4vh}{(l^2vh)^2} = \frac{6}{vh} \]  

(30)

The two previous relations imply that, for all horizons,

\[ \text{KURT} - 3 = \frac{3}{2}\text{SKEW}^2 > 0 \]  

(31)

This relation will be shown to have implications as to how skewness is corrected for the gamma.

2.2 For the gamma process skewness is invariant.

It is informally believed that excess kurtosis is related to the risk neutral skewness implicit in option smiles. Furthermore, a mistake in the proof of Theorem 2 in Bakshi, Kapadia and Madan (2003), leads them to argue that excess kurtosis always generates skewness correction. I show that the gamma process is a counter-example since its excess kurtosis does not lower risk neutral skewness.

From (5) the risk neutral cumulant function equals

\[ \mathcal{K}^*(s) = d(s - \gamma) - v \log(1 - l(s - \gamma)) + d \gamma + v \log(1 + l\gamma) \]

\[ = ds - v \log(1 - l^*s) \]  

(32)

where \( l^* = \varphi l \) with

\[ \varphi = (1 + l\gamma)^{-1} \]  

(33)
The $K^*(s)$ in (32) implies that

$$X_t = dt + \gamma_t^*(l^*, v)$$

where $\gamma_t^*$ is a gamma process under $Q$.

Comparing (10) and (34), a major difference with respect to risk correction is that, while Brownian volatility remains the same, the cumulants of the gamma process are scaled by $\varphi$,

$$c_n^* = \varphi^n c_n$$

Thus, risk neutral skewness remains invariant,

$$\text{SKEW}^* = \frac{c_3^*}{(c_2^*)^{3/2}} = \text{SKEW}$$

Excess kurtosis is also invariant,

$$\text{KURT}^* - 3 = \frac{c_4^*}{(c_2^*)^2} = \text{KURT} - 3$$

### 2.3 Gamma scaling.

From (33), arrival rates for positive jumps are scaled down, while for negative jumps arrivals are scaled up.

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3 Critical Kurtosis and Risk Neutral Skewness.

Having studied the pure gamma, a more realistic example Lévy process is one that contains both continuous path and pure gamma components, as in (1).

3.1 Limitations of the pure gamma process.

A reason against using a pure gamma process as in (26), is that skewness and excess-kurtosis are constrained through (31). A very first analysis, using index moment data from Table 1.1 (p. 21) in Campbell, Lo and MacKinlay (1997), suggests that for value- and equal-weighted index returns as well as individual stocks the above constraint is too restrictive (see Table 2). Essentially, the pure gamma process generates too much skewness relative to kurtosis. Furthermore, because of this constraint, in the pure gamma economy it is impossible to isolate the effects of skewness and kurtosis on asset prices.

Another undesirable feature of the pure gamma model is that, due to its one-sided jump activity, returns are always bounded from below or above depending on the sign of $l$. When dealing with log-returns such a bound in their support is unnecessary and hard to justify on economic grounds. Furthermore, if the model is misspecified, such hard bounds in the support tend to produce maximum likelihood estimates that are sensitive to outliers.
3.2 The mixed process.

I thus study a process that combines both a gamma risk $X_t^\gamma$, and a diffusive risk,

$$X_t = dt + X_t^\gamma + \sigma_1 W_t^1$$

with $X_t^\gamma = \gamma(t, \nu)$. Obviously, such a process does not suffer from a bounded return support. But more importantly, in this economy the undesirable equality (31) is relaxed in the right direction as the next lemma, which is proved in the appendix, shows.

**Lemma 1.** In (38), excess kurtosis can attain any value that satisfies the inequality

$$\text{KURT} - 3 > \frac{3}{2} \text{SKEW}^2$$

In the appendix, it is actually shown that

$$\text{KURT} - 3 = \frac{1}{R_o^2} \frac{3}{2} \text{SKEW}^2$$

where $R_o^2$ of the total variance is explained by the gamma component in (38).

As can be seen from $R_o^2 = \frac{\text{SKEW}^2}{\text{KURT} - 3}$, keeping other things equal, a larger Brownian component will tend to increase excess kurtosis faster than skewness squared and will lead to a point further inside the parabola (dotted curve in figure 2).
3.3 The critical kurtosis value.

From (40), the relation of $\text{KURT}_t$ to the value $\frac{3}{2}\text{SKEW}^2$ is of direct significance in determining the relevant magnitude of the gamma component in (38). This relation is actually proven later to be of significance for all Lévy processes. In anticipation of the general result, I define

Definition 1. For any Lévy process the critical excess kurtosis value equals $\frac{3}{2}\text{SKEW}^2$.

Since, the $R_o^2$ of the gamma risk is a measure of the non-hedgeability of the underlying uncertainty, the proximity of actual excess kurtosis to its critical value becomes a measure of how far the economy is from the purely diffusive Black-Scholes-Merton paradigm.

Calibration of the mixed model (38) depends on whether return moments satisfy $\text{KURT}_t - 3 \geq \frac{3}{2}\text{SKEW}^2$.

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3.4 Risk neutral and actual skewness for the mixed gamma.

In the appendix, I prove that the introduction of the diffusion in (38) implies a leftward correction for risk neutral skewness.

Lemma 2. By mixing the gamma process with a diffusion (38), risk neutral skewness is corrected to the left.
4 Skewness Correction for Lévy processes

In explaining the excessive negative risk neutral market skewness implied from option smirks, it has been argued in Theorem 2 (page 109) in Bakshi, Kapadia and Madan (2003) that excess kurtosis always leads to skewness correction. As we have already shown, the gamma represents a counter-example.

The problem is that recursive equations (49), (50) and (51) in BKM should be stated in terms of cumulants, $c_n$, rather than central moments, $m_n$. More specifically, equation (51), in BKM p.138,

$$m_3 = m_3 - \gamma m_4 + o(\gamma)$$

implies a relation analogous to (13) for moments.

Actually, using equation (13) the third moment is corrected as follows,

$$\left[ \frac{\partial m_3}{\partial \gamma} \right]_{\gamma=0} = \left[ \frac{\partial c_3}{\partial \gamma} \right]_{\gamma=0} = -c_4 = -m_4 + 3\sigma^4$$

5BKM state their relations (49), (50) and (51) in terms of simple and not central moments, but this does not make a difference since they treat a zero mean shock.

6Here the little $o$ notation is used, $\lim_{\gamma \to 0} \frac{o(\gamma)}{\gamma} = 0$
So the correct relation becomes

\[ m_4^* = m_3 - \gamma m_4 + 3\sigma^4 \gamma + o(\gamma) \quad \text{not in Eq. (41)} \]  \hspace{1cm} (43)

It will be shown here that, up to a first order approximation, the results of the previous section apply to all Lévy processes. More specifically, for any Lévy process, skewness correction is generated only when excess kurtosis is beyond the critical kurtosis level. In the special case of symmetric distributions, the critical value is zero, and the correction implied by the entire excess kurtosis still applies. This points to a seemingly paradoxical fact; keeping kurtosis constant, the more symmetric a process is the more its skewness is corrected.

The general formula for any Lévy process is based on (13), and is proven in the appendix.

**Theorem 2.** *For all exponentially tilted Lévy processes, skewness is corrected to the left only when excess kurtosis, \( \text{KURT}_3 - 3 \), is higher than the critical kurtosis value, \( \frac{3}{2} \text{SKEW}^2 \), and the magnitude of correction is given by*

\[ \text{SKEW}^* - \text{SKEW} = -\gamma \left[ (\text{KURT} - 3) - \frac{3}{2} \text{SKEW}^2 \right] \text{STD} + o(\gamma) \]  \hspace{1cm} (44)

### 4.1 Symmetric processes

In this case, critical kurtosis is zero and thus the entire excess kurtosis generates skewness correction.

**Corollary 1.** *When the underlying process is symmetric, excess kurtosis generates risk-neutral skewness via*
$SKEW^* = -\gamma(KURT - 3)STD + o(\gamma)$ \hspace{1cm} (45)

4.2 Pure Gamma

It has been shown in (36) that for the pure gamma case (26) no skewness correction takes place. This is also suggested from the general formula (44), since as is shown in (31) in this case excess kurtosis exactly equals the critical value.

4.3 Mixed gamma

From equation (39), when mixing a gamma with a Brownian motion the excess kurtosis is always greater than the critical kurtosis value of the process thus generating a leftward skewness correction. The amount of correction depends on the gamma $R_g^2$ in the mixture.

4.4 Time changed Brownian motion.

The Variance Gamma (VG) process is a time changed Brownian motion

$$X_t = \theta h_t + \sigma o W^o(h_t)$$ \hspace{1cm} (46)

where a pure gamma process

$$h_t = \gamma_t(l, v)$$ \hspace{1cm} (47)

is used to measure the passage from real $t$ to trading time. It has been shown (e.g. Madan, Carr and Chang, 1998) that the VG is equal to the difference of
two pure gamma processes,\(^7\)

\[ X_t = \gamma^u_t(l_u, v) - \gamma^d_t(l_d, v) \quad \text{where} \quad l_u, l_d > 0 \quad (48) \]

The following lemma is proved in the appendix,

**Lemma 3.** For a VG process excess kurtosis always exceeds the critical kurtosis and thus risk neutral skewness is corrected to the left.

The scaling factor (33) provides some intuition on the fine jump structure of the VG under the risk neutral measure. The positive component is *deflated*,

\[ \varphi_u = \frac{1}{1 + l_u \gamma} < 1 \quad (49) \]

while the negative is *inflated*,

\[ \varphi_d = \frac{1}{1 - l_d \gamma} > 1 \quad (50) \]

From

\[ l_u^* = \varphi_u l_u < l_u \quad (51) \]

we see that the risk neutral Lévy measure of the positive gamma

\[ \Pi_u^*(dx) = \frac{v}{x} e^{-x/l_u^*} dx \quad \text{for} \quad x > 0 \quad (52) \]

implies that, under the risk neutral measure, positive jumps always arrive at a lower rate

\[ \Pi_u^*(dx) < \Pi_u(dx) \quad (53) \]

The opposite happens to risk neutral negative jumps that arrive at a higher than actual rate. This explains why risk neutral skewness is more negative than actual.

\(^7\)The parameters are related by, \( l_u - l_d = \theta l \), and \( l_u l_d = \frac{1}{2} \sigma^2 \).
4.5 Symmetric Variance Gamma

In the symmetric VG case (Madan, Seneta (1990)) the drift of the Brownian motion is zero, $\theta = 0$. This means that the two Lévy measures are exact opposites, $l_u = l_d$, and thus skewness is zero. This time SKEW* is approximated by (45).
5 Conclusion

In the empirical literature, the pronounced option smirks have been generally attributed to a strongly negative risk neutral skewness, but so far no general theory of skewness correction in the context of all Lévy processes has been presented. The paper extends the literature by developing approximate conditions that lead to skewness correction for such processes.

It is shown that skewness is not necessarily corrected for fat-tails. Instead it is shown that, for any Lévy process, skewness is, for small values of the risk aversion, corrected by the excess kurtosis beyond a critical value. For symmetric processes, critical kurtosis is zero and thus the entire excess kurtosis corrects skewness.

The paper leads to some results that can be empirically tested. Another open question is what happens for other types of agents. Finally, it is not clear what values of risk aversion would be small enough for the approximation to be good.
A Appendix

Proof of lemma 1. Let $R_o^2$ of the total variation be explained by the gamma component in (38). The diffusive component does not enter in higher order cumulants, which are entirely generated by the gamma,

$$\text{SKEW}(h) = \frac{c_3 h}{(\sigma^2 h)^{3/2}} = \frac{c_3^\o h}{\sigma_\o^3 h^{3/2} / R_o^3} = \text{SKEW}_\o(h) R_o^3$$ (54)

and,

$$\text{KURT}(h) - 3 = \frac{c_4 h}{(\sigma^2 h)^2} = \frac{c_4^\o h^2}{\sigma_\o^4 h^2 / R_o^4} = (\text{KURT}_\o(h) - 3) R_o^4$$ (55)

Substituting (54), and (55), back in the constraint (31) shows that

$$\text{KURT}(h) - 3 = \frac{3}{2} \text{SKEW}_\o(h)^2 R_o^4 = \frac{1}{R_o^2} \left( \frac{3}{2} \text{SKEW}(h)^2 \right)$$

Furthermore, any skewness to kurtosis combination that satisfies the inequality can be attained by varying the gamma $R^2$ in the mixture.

Proof of lemma 2. The case of a negative gamma risk is treated first. In this case it has to be shown that when actual skewness is negative ($l < 0$), risk neutral skewness is lower than actual skewness, $\text{SKEW}^* < \text{SKEW} < 0$. The key observation is that, while the exponential tilting of a gamma results in a $\varphi$-scaled gamma process, the exponentially tilted diffusive component will remain a Brownian motion with the same volatility. Furthermore, the orthogonality of continuous and pure jump components allows the exponential tilting to be applied independently. Under the risk neutral measure, the total risk will again be a mixed gamma with a diffusion, and thus all the formulas of the previous
section still apply. Applying (54), the total risk neutral skewness equals

\[ \text{SKEW}^* = \text{SKEW}_o R_o^3 = \text{SKEW}_o R_o^3 \]  \hspace{1cm} (56)

since, skewness of the pure gamma component is invariant under tilting.

Skewness correction then depends on the magnitude of \( R_o^3 \) relative to \( R_3^3 \). A negative \( X_t^o \) is inflated, \( \varphi > 1 \). Thus, while the Brownian component retains the same variance under both measures, the non-diffusive component is inflated in the risk neutral measure. Thus \( R_o^3 > R_3^3 \) and,

\[ \text{SKEW}^* < \text{SKEW}_o R_o^3 = \text{SKEW} < 0 \]  \hspace{1cm} (57)

Equation (56) is still valid in the positive skewness case. But when \( X_t^o \) is positive, \( \varphi < 1 \). Thus, \( R_o^3 < R_3^3 \) and,

\[ 0 < \text{SKEW}^* < \text{SKEW}_o R_o^3 = \text{SKEW} \]  \hspace{1cm} (58)

**Proof of theorem 2.** Using (13), the derivative of the corrected skewness with respect to risk aversion at zero equals

\[
\begin{align*}
\frac{\partial \text{SKEW}^*}{\partial \gamma} &\bigg|_{\gamma=0} \\
&= \frac{\partial}{\partial \gamma} \left[ \frac{c_3}{c_5^{3/2}} \right]_{\gamma=0} \\
&= \left[ \frac{\partial c_3}{\partial \gamma} c_5^{3/2} - \frac{3}{2} c_3^a c_5^{5/2} \frac{\partial c_3^a}{\partial \gamma} \right]_{\gamma=0} \\
&= - \left( c_4 \sigma^{-3} - \frac{3}{2} c_3^a \sigma^{-5} \right) \\
&= - \left[ (\text{KURT} - 3) - \frac{3}{2} \text{SKEW}^2 \right] \text{STD} \hspace{1cm} (59)
\end{align*}
\]
Finally expanding $SKEW^\gamma$ for $\gamma \neq 0$ around zero and using the above value for $\frac{\partial SKEW^\gamma}{\partial \gamma}|_{\gamma=0}$ we recover (44).

**Proof of lemma 3.** From (28) we can recover the cumulants of the (possibly symmetric) VG process

\begin{align*}
c_2 &= (l_u^2 + l_d^2)v \\
c_3 &= 2(l_u^3 - l_d^3)v \\
c_4 &= 6(l_u^4 + l_d^4)v
\end{align*}

Excess kurtosis is given by

$$KURT - 3 = \frac{6}{v} \frac{(l_u^4 + l_d^4)}{(l_u^2 + l_d^2)^2}$$

and the critical kurtosis for the VG process equals

$$\frac{3}{2} SKEW^2 = \frac{6}{v} \frac{(l_u^3 - l_d^3)^2}{(l_u^2 + l_d^2)^3}$$

Then excess beyond critical kurtosis is always positive

$$KURT - 3 - \frac{3}{2} SKEW^2 = \frac{6l_u^2 l_d^2 (l_u + l_d)^2}{v(l_u^2 + l_d^2)^3} > 0$$
<table>
<thead>
<tr>
<th></th>
<th>Standard deviation</th>
<th>Excess Skewness</th>
<th>Critical Kurtosis</th>
<th>Gamma Risk</th>
<th>Brownian Risk</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>STD</td>
<td>SKEW</td>
<td>KURT-3</td>
<td>SKEW²</td>
<td>1-R²</td>
</tr>
</tbody>
</table>

### Panel A: Daily Returns

<table>
<thead>
<tr>
<th>Stock</th>
<th>STD</th>
<th>SKEW</th>
<th>KURT-3</th>
<th>SKEW²</th>
<th>1-R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value-Weighted Index</td>
<td>.82</td>
<td>-1.33</td>
<td>34.92</td>
<td>2.65</td>
<td>7.6%</td>
</tr>
<tr>
<td>Equal-Weighted Index</td>
<td>.76</td>
<td>-0.93</td>
<td>26.03</td>
<td>1.30</td>
<td>5.0%</td>
</tr>
<tr>
<td>IBM</td>
<td>1.42</td>
<td>-0.18</td>
<td>12.48</td>
<td>0.05</td>
<td>0.4%</td>
</tr>
<tr>
<td>General Signal Corp.</td>
<td>1.66</td>
<td>0.01</td>
<td>3.35</td>
<td>≈ 0</td>
<td>≈ 0</td>
</tr>
<tr>
<td>Wrigley Co.</td>
<td>1.45</td>
<td>-0.00</td>
<td>11.03</td>
<td>≈ 0</td>
<td>≈ 0</td>
</tr>
<tr>
<td>Interlake Corp.</td>
<td>2.16</td>
<td>0.72</td>
<td>12.35</td>
<td>0.78</td>
<td>6.3%</td>
</tr>
<tr>
<td>Raytech Corp.</td>
<td>3.39</td>
<td>2.25</td>
<td>59.40</td>
<td>7.60</td>
<td>12.8%</td>
</tr>
<tr>
<td>Ampco-Pittsburgh Corp.</td>
<td>2.41</td>
<td>0.66</td>
<td>5.02</td>
<td>0.65</td>
<td>13%</td>
</tr>
<tr>
<td>Energen Corp.</td>
<td>1.41</td>
<td>0.27</td>
<td>5.91</td>
<td>0.11</td>
<td>1.85%</td>
</tr>
<tr>
<td>General Host Corp.</td>
<td>2.79</td>
<td>0.74</td>
<td>6.18</td>
<td>0.82</td>
<td>13.3%</td>
</tr>
<tr>
<td>Garan Inc.</td>
<td>2.35</td>
<td>0.72</td>
<td>7.13</td>
<td>0.78</td>
<td>11%</td>
</tr>
<tr>
<td>Continental Materials Corp.</td>
<td>5.24</td>
<td>0.93</td>
<td>6.49</td>
<td>1.30</td>
<td>20%</td>
</tr>
</tbody>
</table>

### Panel B: Monthly Returns

<table>
<thead>
<tr>
<th>Stock</th>
<th>STD</th>
<th>SKEW</th>
<th>KURT-3</th>
<th>SKEW²</th>
<th>1-R²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value-Weighted Index</td>
<td>4.33</td>
<td>-0.29</td>
<td>2.42</td>
<td>0.126</td>
<td>5.2%</td>
</tr>
<tr>
<td>Equal-Weighted Index</td>
<td>5.77</td>
<td>0.07</td>
<td>4.14</td>
<td>0.0073</td>
<td>0.2%</td>
</tr>
<tr>
<td>IBM</td>
<td>6.18</td>
<td>-0.14</td>
<td>0.83</td>
<td>0.029</td>
<td>3.5%</td>
</tr>
<tr>
<td>General Signal Corp.</td>
<td>8.19</td>
<td>-0.02</td>
<td>1.87</td>
<td>0.0006</td>
<td>0.03%</td>
</tr>
<tr>
<td>Wrigley Co.</td>
<td>6.68</td>
<td>0.30</td>
<td>1.31</td>
<td>0.135</td>
<td>10.3%</td>
</tr>
<tr>
<td>Interlake Corp.</td>
<td>9.38</td>
<td>0.67</td>
<td>4.09</td>
<td>0.67</td>
<td>16.5%</td>
</tr>
<tr>
<td>Raytech Corp.</td>
<td>14.88</td>
<td>2.73</td>
<td>22.70</td>
<td>11.18</td>
<td>49.25%</td>
</tr>
<tr>
<td>Ampco-Pittsburgh Corp.</td>
<td>10.64</td>
<td>0.77</td>
<td>2.04</td>
<td>0.89</td>
<td>43.6%</td>
</tr>
<tr>
<td>Energen Corp.</td>
<td>5.75</td>
<td>1.47</td>
<td>12.47</td>
<td>3.24</td>
<td>26%</td>
</tr>
<tr>
<td>General Host Corp.</td>
<td>11.67</td>
<td>0.35</td>
<td>1.11</td>
<td>0.18</td>
<td>16.5%</td>
</tr>
<tr>
<td>Garan Inc.</td>
<td>11.30</td>
<td>0.76</td>
<td>2.30</td>
<td>0.87</td>
<td>37.6%</td>
</tr>
<tr>
<td>Continental Materials Corp.</td>
<td>17.76</td>
<td>1.13</td>
<td>3.33</td>
<td>1.915</td>
<td>57.5%</td>
</tr>
</tbody>
</table>
Table 1. Calibration of the mixed model.

Moments are from Campbell, Lo and MacKinlay (1997) Table 1.1 (p. 21). Campbell et.al. calculate standard deviation, skewness and excess kurtosis, using CRSP returns from 1962 to 1994, for value- and equal-weighted NYSE and AMEX indices, as well as, for ten individual stocks. Individual stocks are chosen so that each stock, based on its 1979 end-of-year market cap, represents a different capitalization decile. In the last three columns, the gamma-diffusion mixed model (38) is calibrated. The mixed model can match both skewness and kurtosis. The pure gamma would plainly fail since most risk (highest $R^2$) comes from the diffusion component.
\[ R^2_o = \frac{\varphi^2 \sigma_o^2}{\varphi^2 \sigma_o^2 + R^2_o} \]

<table>
<thead>
<tr>
<th>( R^2_o )</th>
<th>Total</th>
<th>Gamma</th>
<th>Diffusive</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q</td>
<td>( \sigma^2 = \varphi^2 \sigma_o^2 + \sigma_1^2 )</td>
<td>( \sigma^2 = \varphi^2 \sigma_o^2 )</td>
<td>( \sigma_1^2 = \sigma_1^2 )</td>
</tr>
<tr>
<td></td>
<td>SKEW* = ( \frac{\sigma_3^3}{R^2_o} )</td>
<td>SKEW* = ( \sigma_o )</td>
<td>SKEW* = ( \sigma_1 )</td>
</tr>
<tr>
<td></td>
<td>KURT* = 3 = (KURT - 3) ( \frac{R^4_o}{R^2_o} )</td>
<td>KURT* = KURT*</td>
<td>KURT* = 3</td>
</tr>
<tr>
<td>P</td>
<td>( \sigma^2 = \sigma_o^2 + \sigma_1^2 )</td>
<td>( \sigma^2 = R^2_o \sigma^2 )</td>
<td>( \sigma_1^2 = R^2_1 \sigma^2 )</td>
</tr>
<tr>
<td></td>
<td>SKEW = ( \text{SKEW}_o R^3_o )</td>
<td>SKEW = ( \text{SKEW}_o )</td>
<td>SKEW = ( \text{SKEW}_1 )</td>
</tr>
<tr>
<td></td>
<td>KURT = 3 = ( \frac{1}{R^2_o} \frac{1}{2} \text{SKEW}^2 )</td>
<td>KURT = 3 = ( \frac{1}{R^2_o} \frac{1}{2} \text{SKEW}^2 )</td>
<td>KURT = 3</td>
</tr>
</tbody>
</table>

**Table 2. Risk correction in the mixed economy.** The relation between risk neutral and actual moments depends on the relative \( R^2_o \) of the gamma process. For negative gamma, \( \varphi \) is larger than one, and variance is inflated, \( \sigma_o^2 = \varphi^2 \sigma_o^2 > \sigma_o^2 \), while diffusive volatility remains unchanged. Thus, \( R^2_o \) is higher than the actual \( R^2_o \). When the gamma is positive, \( \varphi < 1 \), and the gamma component is deflated leading to a smaller \( R^2_o \).
References


[21] Dilip Madan, Peter Carr, and E. Chang. The variance gamma process and

[22] Dilip Madan and Frank Milne. Option Pricing with VG Martingale Com-

[23] Dilip Madan and Eugene Seneta. The Variance Gamma (V.G.) Model for
Figure 1. Gamma scaling

From (33), as a function of skewness \( \varphi = (1 + \frac{1}{2}\sigma \text{ SKEW } \gamma)^{-1} \). \( \varphi \) is greater than one for negative gamma, and less than one for positive. Also, \( \frac{\partial \varphi}{\partial \gamma} < 0 \), and \( \frac{\partial^2 \varphi}{\partial \gamma^2} > 0 \). Here \( \sigma = 20\% \).
Figure 2. The gamma $R_2^2$. Points inside the parabola are generated by (38). Here, SKEW = 2, and KURT−3 = 10 > $\frac{3}{2}$SKEW$^2 = 6$. Thus $R_2^2 = 60\%$ for the gamma risk.
Figure 3. SKEW* as a function of KURT−3. When kurtosis is raised beyond the critical value, the diffusive component gains weight and risk neutral skewness declines. $\sigma = 20\%$ and $\gamma = 2$. 

\[ SKEW = \frac{3 \cdot SKEW^2}{2} \]

\[ KURT = 3 \cdot SKEW^2 / 2 \]

\[ KURT > 3 \cdot SKEW^2 / 2 \]