Chain Differentiation Rule for Autocorrelated Random Sequences; Alternative to Ito lemma

Vadim Mezrin*

March 30, 2004

ABSTRACT

A majority of market processes include an autocorrelated component due to various factors, for example information shifts or psychological tendencies, the security valuation models would have account for this autocorrelation. Current models mostly focus on autocorrelation in either interest rates or volatility of returns. The model proposed in this paper takes a different approach; it considers an autocorrelation of returns. This approach necessitates a new chain differentiation rule because Ito lemma that is traditionally used for this purpose can only be applied for independent identically distributed processes.

This paper introduces a general framework for complex stochastic processes with autocorrelated increments. A new general analytical expression for a chain differentiation rule is developed based for underlying process with serially correlated increments.

The chain differentiation rule proposed in this paper serves to replace the Ito lemma in the presence of autocorrelation in increments. For the special case of no autocorrelation, it converges to the Ito lemma.

To verify the analytical results a Monte Carlo procedure, based on the new chain differentiation rule, is created. The validity of the simulation procedure is verified by observing the absence of the mean value drift in the simulation results. When the same criterion is applied to the simulation based on Ito lemma, there is a significant drift in the mean value. Finally, the analytical results obtained in this paper are verified using the new Monte Carlo procedure. The results obtained in this comparison are identical to within the rounding error, thus validating our findings. The formulas obtained here will have application in various areas of finance including but not limited to interest rate derivatives, option and security valuation, real option analysis of projects.

*Rutgers Business School at the Rutgers the State University of New Jersey.
A traditional approach to working with an analytical function of a stochastic process, like lognormally distributed sequence, involves using an Ito lemma for chain differentiation. This lemma can only be applied for independent identically distributed processes. However, there is a wide array of papers documenting a presence of autocorrelation in asset returns. Conrad and Kaul (1988) examine the autocorrelation of returns for size grouped portfolios and find first order autocorrelation of weekly returns. For longer periods Fama and French (1988) find autocorrelation of returns on diversified portfolios. Similarly Peterba and Summers (1988) find evidence of serial correlation. Since a majority of market processes include an autocorrelated component due to various factors, for example information shifts or psychological tendencies, the security valuation models would have account for this autocorrelation.

This paper develops a general analytical expression for a chain differentiation rule based on the assumption that the state increments of the underlying process are serially correlated. The rule should be used instead of Ito lemma when the process is governed by autocorrelated increments.

The expressions derived in this paper presents a general rule to be used in differentiating functions of stochastic process. For a special case of autocorrelation coefficient equal to zero, our equations converge to Ito lemma.

We will develop a framework of random, normally distributed, process $x$, with autocorrelated increments $\zeta$ that have volatility $\sigma^2$ and autocorrelation coefficient $\rho$. Both parameters can be estimated using historical data, Hull (1999), Andrews (1993). Autocorrelated returns $\zeta$ will be constructed using independent identically distributed (iid) normal random variables $\epsilon$ with arbitrary volatility $\sigma^2_{\epsilon}$. This framework is then used to calculate the values of a stochastic integral, providing the general chain differentiation rule applicable to an arbitrary stochastic process.

For demonstration purposes, we apply the new rule in the case of lognormally distributed variables. This process is most commonly used to approximate stock prices in the capital markets. All analytical expressions obtained are verified using Monte Carlo simulations. The
results obtained in this comparison are identical to within the rounding error, thus validating our findings. The formulas obtained here will have application in various areas of finance including but not limited to interest rate derivatives, option and security valuation, real option analysis of projects.

The rest of this article of organized as follows: Section I develops the framework for a process with autocorrelated increments. Section II decomposes the autocorrelated random sequence by using sums of independent random variables. Section III starts to derive a new chain differentiation rule based on the results of previous sections and lists some of the difficulties in attempting to do so. Section IV introduces a new stochastic process that allows rigorous calculation of stochastic integral and completes the derivation of the chain differentiation rule. Section V applies the new chain differentiation rule to the lognormally distributed processes. Section VI develops numerical Monte Carlo procedure to simulate stochastic processes with autocorrelation in increments, additionally the results of this simulation are compared with the analytical results obtained in Section V we confirm the accuracy of the new analytical approach. Concluding remarks and avenues for further research are offered in Section VII. Items of additional interest are provided in Appendix.

I. Serially Correlated Stochastic Variable Model

Any stochastic process under investigation is usually looked at as a sequence of states observed over some time interval $\Delta t$, even though the true process itself might be based on significantly smaller time intervals or in the limit might be continuous. For practical purposes, the process is usually identified by its distribution function and a number of parameters. The most commonly used parameters include mean, volatility, and autocorrelation. In the modern financial literature, particular attention is being paid to the additive process with stochastic increments (stock returns, market shocks, news announcements, etc.). For the analytical work to follow we would like to construct a framework of general random processes $S = f(x)$, such
that $x$ is a normally distributed variable having autocorrelated increments $\zeta$ with volatility $\sigma^2$ and autocorrelation coefficient $\rho$ measured over a fixed interval $\Delta t$. Function $f(x)$ can be an analytical expression of a stochastic normally distributed variable. Both parameters $\sigma^2$, and $\rho$ can be estimated using historical data on some time interval $\Delta t$ (daily, weekly, or monthly observations).

Let $x$ be defined by the following stochastic equation (applicable in the environment with interest rates)

$$\Delta x_n = r\Delta t + \sqrt{\Delta t} \cdot \zeta_n(\sigma^2, \rho).$$ \hspace{1cm} (1)

with

$$\mu_\zeta = \text{mean}(\zeta_n) = 0; \quad \sigma^2 = \sum_{n=1}^{N} \zeta_n^2 / N; \quad \rho = \rho_1 = \sum_{n=1}^{N} \zeta_n \cdot \zeta_{n+1} / N; \quad N \to \infty.$$ \hspace{1cm} (2)

Increments $\zeta_n$ are sometimes referred to as $d\zeta_n$, we will use both of them interchangeable through later sections. This stochastic equation is similar to random work process, however we do not require the stochastic term to be iid, it can be a random serially correlated process. It must be pointed out that introduction of interest rate in the form $r\Delta t$ creates a positive autocorrelation in the process $x_n$ even if $\zeta_n$ would be a random normally distributed variable.

It can be shown that for small values of $r$, this additional term for autocorrelation is equal to $\frac{r^2}{2\sigma^2}\Delta t$. This term is relatively insignificant because it is proportional to $\Delta t$ and the square of $r$ both of which are quite small. As usual, it will be neglected in the calculations to follow. However, the calculations can be easily extended to take it into consideration.

We would like to consider a generic stochastic differential equation of the form

$$\Delta v_n = K_1(x)\Delta t + \sqrt{K_2(x)} \cdot \zeta_n(\sigma^2, \rho).$$ \hspace{1cm} (3)
Values of $x_n$ follow a random process with normally distributed autocorrelated increments $\zeta_n$ that have volatility $\sigma^2$ and autocorrelation coefficient $\rho = \rho_1$ measured over time period $\Delta t$ ($\rho_i$ is a correlation coefficient measured over time period $i \cdot \Delta t$, where $i=1,2,...$).

In the past, the complexities of the autocorrelated sequences have prevented extensive research in this area. In this paper, we propose to deconstruct the increments $\zeta_n$ to their basic form, an infinite sum of iid normal variables $\epsilon_n$. This decomposition allows us to create analytical expressions for a chain differentiation rule applicable in the case of a stochastic process with autocorrelated increments.

II. Decomposition of serially correlated sequences using iid normal variables.

One of the possible avenues to decomposing autocorrelated sequences is by means of using an AR(1) process. This is the approach that will be explored in this paper.

A stable AR(1) process of the form

$$\zeta_n(\sigma^2, \rho) = \alpha \cdot \zeta_{n-1}(\sigma^2, \rho) + \epsilon_n(\sigma^2_\epsilon); \text{ where } \zeta_{-k} = 0; \ k >> 1. \tag{4}$$

will be used to describe the autocorrelated increments $\zeta_n$, where $\epsilon_n(\sigma^2_\epsilon)$ are normal and iid. This process starts at time $-k\Delta t$ ($k >> 0$) much before the time $t = 0$, time when process $x$ is started, because the sequence $\zeta$ needs some time to become autocorrelated with volatility $\sigma^2_\epsilon$, and autocorrelation coefficient $\rho_1 = \rho$. Volatility and autocorrelation coefficient of a random sequence converge to a constant value only after a certain number of iteration has been completed. The start time $t_{-k}$ for process $\zeta$ have to satisfy condition $|t_{-k}| >> \tau_{corr}$ ($\tau_{corr}$ is correlation time as given by (A1)).
Equation (4) has two unknown parameters: $\alpha$ and $\sigma^2_\varepsilon$. Both of them can be found by enforcing the mean, volatility, and correlation conditions, i.e. the volatility must equal $\sigma^2$, $\mu_\varepsilon = 0$, and $\rho_1 = \rho = \sum_{n=1}^N \varepsilon_n \cdot \varepsilon_{n+1}/N$, when $N \to \infty$. Solving the resulting equations and substituting their values back into original equation (4) gives

$$\varepsilon_n(\sigma^2, \rho) = \rho \varepsilon_{n-1}(\sigma^2, \rho) + \varepsilon_n(\sigma^2), \quad \sigma^2_\varepsilon = (1 - \rho^2)\sigma^2. \quad (5)$$

Therefore, if stochastic increments are to have $\varepsilon$ to have a volatility of $\sigma^2$, the stochastic term $\varepsilon(\sigma^2_\varepsilon)$ must be normally distributed with volatility $\sigma^2_\varepsilon = (1 - \rho^2)\sigma^2$. As can be seen, in order to obtain an autocorrelated sequence with volatility $\sigma^2$, the iid random variables used to construct it have to have a different volatility $\sigma^2_\varepsilon$ determined by two independent parameters: desired $\sigma^2$ and $\rho$. For zero autocorrelation, $\sigma^2_\varepsilon$ becomes equal to $\sigma^2$.

Correlation coefficient between elements $\varepsilon_t$ and $\varepsilon_{t+n}$ (time difference $n \cdot \Delta t$) for AR(1) model is given by a well known relationship

$$\rho_n = \rho_1^n \equiv \rho^n. \quad (6)$$

To fully apply these findings, a distribution function $F(x)$ of the process $x$ has to be determined. It will be a Normal distribution function because random variable $x_n$ at time $t_n$ is an infinite sum of iid normally distributed variables $\varepsilon$. To find a weak form solution of equation (5) a volatility of this distribution at time $t_n$ has to be found. After recursively applying equation (4) and allowing $k \to \infty$ in the final expression, the autocorrelated sequence is replaced by an infinite sum of iid normal variables, therefore yielding that

$$\sigma^2_{t_n} = \sigma^2_{eff} \cdot [t_n - 2\rho \frac{1-\rho^n}{1-\rho^2} \cdot \Delta t], \quad where \quad \sigma^2_{eff} = \frac{1+\rho}{1-\rho} \sigma^2. \quad (7)$$

From the above equation the volatility has two components, the component proportional to $t_n$ and an additional term with a specific time dependency due to $\rho^n$ element. The impact of the
second component diminishes as time increases, i.e. for $n \gg 1$ volatility for process $x$ can be written as

$$\sigma^2_{n} \approx \sigma^2_{eff} \cdot t_n, \text{ where } t_n = n\Delta t.$$  \hspace{1cm} (8)

It follow from the above equations that the distribution function of the process $x$ has volatility proportional $\sigma^2_{eff}$ rather then $\sigma^2$ as it was for iid increments. This new effective volatility $\sigma^2_{eff}$ is governed by correlation coefficient and can significantly differ from $\sigma^2$.

### III. Chain differentiation rule for autocorrelated processes.

Our intension is to calculate stochastic process $S = f(x)$ that is an arbitrary function of $x$. An Ito calculus, rather then Stratonovich calculus will be used for further calculations. As is well known, for Ito calculus, a chain differential rule has to be calculated because classical chain differential rule is no longer applicable. An Ito lemma cannot be used for this case because it was derived for independent random increments $\zeta$.

Therefore the goal of this paper is to derive a chain differential rule for the process $f(x)$. Starting with Taylor expansion for function $f(x)$ with respect to both $x$ and $t$, we obtain

$$df[x(t), t] = \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} dt^2 + \ldots + \frac{\partial f}{\partial x} dx(t) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dx(t))^2 + \ldots$$  \hspace{1cm} (9)

take into consideration stochastic equation (3) yields

$$df[x(t)] = f'[x(t)]\left\{ K_1(x) dt + \sqrt{K_2(x)} d\zeta(t) \right\} + \frac{1}{2!} f''[x(t)] K_2(x) d\zeta^2(t) + O(dt^2).$$  \hspace{1cm} (10)

This differential equation is a symbolic form of integral equation

$$\Delta f(x) = \int_t^{t+\Delta t} \left\{ f'[x(t)]\left\{ K_1(x) dt + \sqrt{K_2(x)} d\zeta(t) \right\} + \frac{1}{2!} f''[x(t)] K_2(x) d\zeta^2(t) \right\}.$$  \hspace{1cm} (11)
the most difficult step in solving these equation is calculating a stochastic integral

\[ I = \frac{1}{2} \int_{t}^{t+\Delta t} \{ f''[x(t)]K_2(x) d\xi^2(t) \}. \]  

(12)

A stochastic process \( x_n \) is defined on time-net with interval \( \Delta t \). In order to calculate integral in equation (12), the intervals have to be further subdivided allowing the number of intervals to approach infinity in the limit. However, the constraints placed on the process \( x \), by its definition and requirements to maintain specific volatility and autocorrelation coefficient, do not leave any freedom to obtain intermediate values for variable \( x \), unlike the case of iid variables. We cannot let time interval \( dt \to 0 \) and calculate stochastic integral on some relatively large time interval \( \Delta t >> dt \), as it is usually done for iid process. This is a major obstacle rigorous calculation of stochastic integral (12).

IV. Serially Correlated Stochastic Process with time lag approaching zero

In this paper we want to introduce an approach that will allows us to elevate the problem posed at the end of previous section. A new autocorrelated stochastic process has to be created. Let \( y_i \) be a process on a finer time-net with time intervals \( \Delta \tau = \Delta t / m \) (where \( m \) is odd number \( m = 1, 3, 5, \ldots \)). Additionally, a subset of \( \{ y_i \} \) such that \( i = m \cdot n, n = 0, 1, 2, \ldots, N \) has to have all properties of the process \( x_n \) on time-net with intervals \( \Delta t \). If we take every \( m' \)th element from set \( \{ y_i \} \) and calculate volatility and correlation coefficient based in this subset the results must equal to \( \sigma^2 \) and \( \rho \) respectively. See, for example Fig (1). At the same time, entire sequence \( \{ y_i \} \) is determined by two "macro" parameters: \( \sigma^2 \) and \( \rho \) calculated on the time interval \( \Delta t \), where \( \Delta t >> \Delta \tau \).
Figure 1. Graphical representation of time-net for process $x_n$, process $y_i$ as well as regular time axis.

Stochastic integral can be calculated based on fine time-net allowing $\Delta \tau \rightarrow 0$ (it is equivalent to $m \rightarrow \infty$). The major parameters for this new autocorrelated process are:

$$\text{subset} \{y_i\} = \{x_n\}, \ i = m \cdot n; \ n = 0, 1, 2, ..., N;$$

(13)

In this equation $m$ is a parameter that can be very large, in the limit $m \rightarrow \infty$ as was discussed earlier. From this condition follow $\mu_y = \text{mean}(y) = 0$ and

$$\sigma^2 = \sum_{n=0}^{N} y_{m,n}^2 / N; \ \rho = \sum_{n=0}^{N-1} y_{m,n} \cdot y_{m,(n+1)}/N,$$

(14)

where $m = 1, 3, 5, ...$ and $N \rightarrow \infty$ (see equation (2)).

A new stochastic equation for $\{y_i\}$ similar to (3) can be written as

$$\Delta y_i = K_1(y) \Delta t + \sqrt{K_2(y)} \cdot \xi_i(\sigma^2_{z_i}; \gamma).$$

(15)
this equation is based on a new stochastic autocorrelated sequence $\xi_i$ with mean $\mu_\xi = 0$, volatility $\sigma_\xi^2$, and correlation coefficient $\gamma$. Similarly to previous section, an autocorrelated random increments $\xi_i$ are decomposed to normally distributed iid variables utilizing the AR(1) process. The equation similar to (4) can be written as

$$
\xi_i(\sigma_\xi^2, \gamma) = \gamma \xi_{i-1}(\sigma_\xi^2, \gamma) + \delta_i(\sigma_\delta^2).
$$

(16)

where $\delta_i$ is iid random variable with $\mu_\delta = 0$ and a volatility $\sigma_\delta^2$. This leaves three parameters: $\gamma$, $\sigma_\xi^2$, and $\sigma_\delta^2$ that have to be determined. However only two of this parameters $\gamma$, and $\sigma_\delta^2$ have to be calculated because $\sigma_\xi$ will be uniquely determined by the other two. Variables $\gamma$, and $\sigma_\delta^2$ can be found from equations (14) after taking into account condition (6) as:

$$
\gamma = \rho^{1/m}; \quad \sigma^2 = \sum_{n=0}^{N} \frac{y_{m,n}^2}{N},
$$

(17)

where $m = 1, 3, 5, \ldots$ and $N \rightarrow \infty$.

Equations (15), (16), and (17) fully describe new $\{y_i\}$ stochastic process with autocorrelated increments. The stochastic process $y_i$ will have a subset $\{y_{mn}\}, n = 0, 1, 2, \ldots, N$ with correlation coefficient and volatility for increments equal $\rho$ and $\sigma^2$ even $\Delta\tau \rightarrow 0$ (or $m \rightarrow \infty$).

For the new process $\{y_i\}$ equations equivalent to (10), (11), and (12) are written as

$$
df[y(t)] = f'[y(t)] \{K_1(y)dt + \sqrt{K_2(y)}d\xi(t)\} + \frac{1}{2!}f''[y(t)]K_2(y)d\xi^2(t) + O(dt^2). \tag{18}
$$

$$
\Delta f(y) = \int_{t}^{t+\Delta t} \{f'[y(t)] \{K_1(y)dt + \sqrt{K_2(y)}d\xi(t)\} + \frac{1}{2!}f''[y(t)]K_2(y)d\xi^2(t)\}. \tag{19}
$$

$$
I = \frac{1}{2} \int_{t}^{t+\Delta t} \{f''[y(t)]K_2(y)d\xi^2(t)\}. \tag{20}
$$

In this expressions $\Delta\tau$ and $d\xi$ can be very small and allow for rigorous calculation of stochastic integral (20) on time interval $\Delta t$ (in this case $m$ is a number of sub-intervals on time interval
Equations (18) - (20) give a chain differentiation rule for a function of a stochastic process (15) with arbitrary functions $K_1(y)$ and $K_2(y)$.

V. Lognormally Distributed Process with Serially Correlated Increments

This section will apply the procedure developed in the previous section to a case of a widely applicable lognormally distributed random process. For a lognormally distributed stochastic process $S(x) \equiv f(x) = \ln(x)$, and $K_1(x) = x \cdot r$ and $K_2(x) = x^2 \Delta t$. Equations (3), (10), and (12) will transform to

$$\Delta x_i = x \cdot r \Delta t + x \sqrt{\Delta t} \cdot \xi_i(\sigma_\xi^2, \rho).$$

$$\Delta S \equiv \Delta f = r\Delta t + \sqrt{\Delta t} \xi(t) - \frac{1}{2} \Delta t \xi^2(t).$$

$$I = -\frac{1}{2} \Delta t \int_t^{t+\Delta t} d\xi^2(t), \text{ where } \Delta t = t_{n+1} - t_n,$$

where $\xi_i(\sigma_\xi^2, \rho)$ is given by equation (16).

Let's calculate stochastic integral on time interval $\Delta t$ based on fine time-net with intervals $\Delta \tau$. Equation (17) can be transformed to

$$\sigma^2 = \sum_{n=0}^{N-1} \left( \sum_{i=1}^{m} \xi_{m,n+i}(\sigma_\xi^2) \right)^2 / N,$$

in this equation, using the properties of AR(1) process, the inner sum can be converted to the following

$$\sum_{i=1}^{m} \xi_{m,n+i} = \delta_{m,(n+1)} + (1 + \gamma) \delta_{m,(n+1)-1} + (1 + \gamma + \gamma^2) \delta_{m,(n+1)-2} + \ldots + (1 + \gamma + \ldots + \gamma^{m-1}) \delta_{m,n+1} +$$

$$+ \gamma(1 + \gamma + \ldots + \gamma^{m-1}) (\delta_{m,n} + \gamma \delta_{m,n-1} + \ldots + \gamma^{n+mk} \delta_{-mk}).$$

10
\( \delta \equiv \delta(\sigma_0^2) \) is iid with volatility \( \sigma_0^2 \), \( k \to \infty \) as was discussed earlier, and \( |\rho|^m \to 0 \) as \( m \to \infty \) because \( |\rho| < 1 \). After substituting this expression into (21) and allowing \( N \to \infty, (k \to \infty) \), obtain

\[
\sigma_\delta^2 = \frac{(1 - \rho^{1/m})^2}{m} \sigma_{eff}^2, \quad \text{where} \quad \sigma_{eff}^2 = \frac{1 + \rho}{1 - \rho} \sigma^2, \quad \gamma = \rho^{1/m}. \quad (26)
\]

Distribution function \( F(y) \) of the process \( y \) is also a normal distribution function because random variable \( y_i \) in any time \( t_i = i \cdot \tau \) is an infinite sum of iid normal distributed variables \( \delta_i \). Distribution function on time points \( t_n = m \cdot n\Delta\tau = n\Delta t \) has volatility given by

\[
\sigma_{t_n}^2 = \sigma_{eff}^2 \left[ n - 2\rho^{1/m} \frac{1 - \rho^n}{m(1 - \rho^{2/m})} \right] \Delta t. \quad (27)
\]

This expression include parameter \( m \) and as was discussed earlier, we are looking for a limit when \( \Delta \tau \to 0 \) (or \( m \to \infty \)). Taking into account formula: \( a^g \approx 1 + g \cdot \ln(a) \) when \( 0 < a < 1 \) and \( g \to 0 \) and \( m \to \infty \) (i.e. \( \Delta \tau \to 0 \)), equation (27) for \( \sigma_{t_n}^2 \) becomes

\[
\sigma_{t_n}^2 = \sigma_{eff}^2 \left[ t_n - \text{sign}(\rho) \frac{1 - \rho^n}{\ln|1/\rho|} \Delta t \right], \quad \text{where} \quad t_n = n \cdot \Delta t. \quad (28)
\]

and \( |1/\rho| \) is a module of \((1/\rho)\). As can be seen from equation (28), the second term decays rapidly as \( \sim 1/n \) when \( n \) increase and volatility becomes equal to

\[
\sigma_{t_n}^2 \approx \sigma_{eff}^2 t_n \equiv \sigma_{eff}^2 \cdot n \cdot \Delta t. \quad (29)
\]

this expression equals to (8). The expression (28) is similar to (7) in structure, but differs slightly in the second term. As was mentioned earlier, the second term becomes negligibly small for large values of \( t \). Therefore the two sequences \( \{x_n\} \) and \( \{y_i\} \) converge after a few initial elements, with \( \{y_i\} \) being ’smoother’ than simple AR(1) process given by (1) because of an increased number of elements.
Stochastic integral (23) can be calculated with similar approach described earlier. Alternatively, it can be calculated using the definition of the variance and equation (28). After simplifying the resulting expression becomes

\[ I = -\frac{1}{2} \sigma_{eff}^2 \left[ 1 - \text{sign}(\rho) \cdot \rho^n \frac{1 - \rho}{\ln|1/\rho|} \right] \Delta t. \]  

(30)

This is a basis for a chain differential rule for lognormal distribution with autocorrelated increments.

As can be seen from (30) the second term decays rapidly when \( n \) increases due to term \( \rho^n \), therefore for large values of \( n \) stochastic integral (30) becomes equal

\[ I \approx -\frac{1}{2} \sigma_{eff}^2 \Delta t. \]  

(31)

Let us compare two set of autocorrelated random variables: \( \{x_n\} \) and subset \( \{y_{m,n}\} \) on the same time net \( \Delta t \). It should be noted that both sets have the same autocorrelation coefficient \( \rho \) and the only difference is in the volatility term. For \( t > \tau_{corr} \) both of them have volatility equal \( \sigma_{eff} \cdot t \) but for the \( t < \tau_{corr} \) they differ slightly, according expressions (7) and (28). Fig. 2 presents both curves. The difference is diminishing, therefore subset \( \{y_{m,n}\} \) is a good representation of a set \( \{x_n\} \).

To summarize, the differential form of stochastic equation for \( S \) which represents a chain differentiation rule for a lognormal stochastic process with autocorrelated increments is:

\[ \Delta S = (r - \frac{1}{2} \sigma_{eff}^2 \left[ 1 - \text{sign}(\rho) \cdot \rho^n \frac{1 - \rho}{\ln|1/\rho|} \right]) \Delta t + \sqrt{\Delta t} d\xi(t). \]  

(32)

where

\[ \xi_i(\sigma_2^2, \gamma) = \gamma \xi_{i-1}(\sigma_2^2, \gamma) + \delta_i(\sigma_2^2). \]  

(33)

and

\[ \sigma_2^2 = \frac{(1 - \rho^{1/m})^2}{m} \sigma_{eff}^2, \quad \text{where} \quad \sigma_{eff}^2 = \frac{1 + \rho}{1 - \rho} \sigma^2, \quad \gamma = \rho^{1/m}. \]  

(34)
Figure 2. Volatility ratios $\frac{\sigma^2}{\sigma_{eff}^2}$ for processes $x$ (dashed line) and $y$ (solid line). Correlation coefficient $\rho = 0.2$. 
VI. Monte Carlo Simulation of Autocorrelated Process with Serially Correlated increments

We would like to formulate a Monte Carlo simulation technique for autocorrelated process. To simulate process $S$ a differential form of stochastic equation in form (32) has to be used. However, this is not very useful for Monte Carlo simulation purposes. It has to re-written in an integral form, which can be done by utilizing chain differential rule in form (30) which yields

$$S_{n+1} = S_n \cdot \exp \left[ r \cdot \Delta t + \sqrt{\Delta t} \cdot \zeta_n (\sigma^2, \rho) - \frac{1}{2} \sigma_{\text{eff}}^2 (\rho) \left[ 1 - \text{sign}(\rho) \cdot \rho^n \cdot \frac{1-\rho}{ln|1/\rho|} \right] \Delta t \right]. \quad (35)$$

where $n = 0, 1, 2, \ldots, N$, $N = \frac{T}{\Delta t}$. The initial value for (35) is $S_0$ and sequence of $\zeta_n$’s is given by

$$\zeta_n (\sigma^2, \rho) = \rho \cdot \zeta_{n-1} (\sigma^2, \rho) + \epsilon_n ((1 - \rho)^2 \sigma^2), \quad (36)$$

where $n = -k, -k+1, \ldots, 0, 1, 2, \ldots, N$, $k \approx 20 - 30$. Value of $k$ can chosen to be $k \approx 20 - 30$ for any reasonable correlation coefficient, therefore, sequence $\zeta_n$ starts $k$ points earlier than $S_0$. Utilizing equation (35), (36) it is easy to simulate lognormal autocorrelation process $S$ with reasonable large interval $\Delta t$ using a chain differential rule introduced earlier in this paper as described in Mezrin (2002).

This Monte Carlo technique will be compared with the analytical expression to verify its validity. There are two ways that we can take, first a the procedure can be used to simulate the autocorrelation sequences and compute the variance of the process. This variance should be equal to the value obtained using analytical expressions given by equation (28). Second method involves finding the mean of the sequence using simulation and comparing it with the analytical value to verify the validity of the chain differentiation rule.

A stochastic process $S$ will be simulated utilizing equation (35), (36). Volatility values of $S$ are calculated at different time $t_n$. Result of this calculation are presented on Fig. 3 for $r = 0$. 


Figure 3. Comparison of volatility ratios $\frac{\sigma_n^2}{\sigma_{eff}^2}$ for process $y_n$. Theoretical curve given by equation (28) (solid line) and Monte Carlo simulation (solid dots). Correlation coefficient $\rho = 0.2$. 10 million trajectories.

As can be seen, the result of this simulation clearly show that the expression for volatility is correct.

Let’s discuss in details why a specific chain differentiation rule is needed when working with a function of a stochastic process. In the absence of an additional term, when the mean value $\mu$ of the function $S$ is calculated, a drift will be clearly evident in its value. For example, when working with a lognormally distributed process $S$, where $ln(S)$ is normal iid process, in the absence of an additional term equal to $-\frac{1}{2}\sigma^2 t$ (Ito lemma, see Hull, 1999), a drift equal to $exp(\frac{1}{2}\sigma^2 t)$ (even for $r = 0$) will be present. Therefore it can be concluded that this additional term is necessary to ‘stabilize’ the mean of the process over time.
When using a Monte Carlo approach to verify the behaviors of the autocorrelated process, we can use three different approaches: ignore the chain differentiation rule altogether, use a traditional Ito Lemma approach, or apply the chain differentiation rule developed in the previous sections. However, in this case, an additional term $-\frac{1}{2}\sigma^2 t$ will be insufficient to stabilize the mean. As illustrated in Fig 4, when this term is added, the mean of the process will drift over time, even if not as much as in the case of no additional terms being added. In this figure a stochastic process $S$ is simulated utilizing equation (32). The means are mean of $S$ is calculated at different time $t_n$. As can be clearly seen, only the application of the chain differentiation rule developed in the earlier sections enforces the stability of the mean over time that confirms our findings. This serves to verify the validity of the new chain differentiation rule.
VII. Conclusions and Further Research.

We have developed an analytical expression defining a chain differentiation rule for the case of autocorrelated process increments. As a special case for no autocorrelation, our rule reduces to a well known Ito lemma. However, in the presence of autocorrelated increments, our rule is free of bias.

The procedure described in the paper can have an extensive array of applications. See Mezrin (2003) for application to option pricing. It can be further extended to areas of stock derivative pricing, interest rate derivates and real options. This chain differentiation rule provides a proper procedure to be used if there is a possibility that the underlying process has serially correlated increments. Furthermore, the procedure developed here can also be applied to Monte Carlo simulations.

References


Appendix A. Correlation Decay Time

It is convenient for illustration purposes to introduce a correlation decay time $\tau_{corr}$ as

$$|\rho(t)| \sim \exp\left(-t/\tau_{corr}\right). \quad (A1)$$

In this case the expression for the autocorrelation coefficient can be written as

$$|\rho| = |\rho(t_1)| \sim \exp\left(-\Delta t/\tau_{corr}\right); \quad |\rho^n| = |\rho(t_n)| \sim \exp\left(-t/\tau_{corr}\right); \quad \text{where} \quad t \equiv t_n. \quad (A2)$$

Otherwise the correlation decay time can be looked at as a time interval required for the absolute value of the correlation coefficient to decrease by the factor of $e \approx 2.73$. Expression (A1) can alternatively be written as

$$\tau_{corr} = \frac{\Delta t}{\ln(1/|\rho|)} \geq 0. \quad (A3)$$

where $\Delta t$ represents the time increments used to measure the correlation coefficient $\rho$ and volatility $\sigma^2$. 