Value at Risk and Expected Shortfall under Regime Switching

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November 8, 2003

Abstract

This paper models the joint distribution of stock and bond returns as a multivariate Markov switching process. We find evidence that four states are needed to capture the joint distribution of returns on these asset classes. This gives rise to rich patterns in the term structure of risk measures such as the Value at Risk and expected shortfall as a function of the state probabilities and investment horizon. Compared to a Gaussian IID and a multivariate GARCH specification, in general the regime switching model suggests higher tail losses and expected shortfall. We also study real-time measures of risk and out-of-sample forecasts generated by the proposed econometric specification.

1 Introduction

Quantitative models are now routinely used in risk management and have been the subject of extensive academic interest as witnessed by the many survey papers and monographs on the topic, see, e.g., Duffie and Pan (1997), Manganelli and Engle (2001) and Christoffersen (2003). The early literature was mainly built around volatility models, emanating from the study by Engle (1982) on autoregressive conditional heteroskedasticity. Subsequently, the literature has progressed to study measures such as Value at Risk (VaR) or expected
shortfall. Value at Risk is used to monitor risk exposure but also to gauge the risk-adjusted performance of portfolio managers and for regulatory purposes under the Basel agreement.

Though most studies have concentrated on modeling risk at relatively short horizons using high frequency data, as argued by Christoffersen, Diebold and Schuerman (1998), the relevant horizon can be very long and depends on the economic problem at hand. Our interest in this paper lies in studying risk in relation to strategic asset allocation, i.e. the allocation to broad asset classes such as T-bills (cash), bonds and stocks. To many mutual funds and pension funds the strategic asset allocation is the single most important determinant of the risk and return characteristics of their portfolio and their investment horizon tends to be very long. The decision on how much to allocate to such asset classes is typically based not on their predictive return distribution over the next few days or weeks, but rather over the next months or even years.

Regime switching models are now routinely used to capture the dynamics of financial returns and we study portfolio risk in the context of multivariate mixtures of normal distributions where the mixing weights - or state probabilities - are driven by a first-order Markov process. The proposed multivariate Markov switching specification is very flexible and can capture skews, kurtosis and other well-documented deviations from the Gaussian benchmark. Furthermore, a latent state variable allows for rich dynamics in the return distribution which gives rise to interesting term-structures in the VaR or expected shortfall even at very long horizons. This is an important insight. Using GARCH models on daily stock returns data, Christoffersen and Diebold (2000) found that volatility is not generally predictable more than a few weeks ahead in time, but similar results have not been established for other measures such as VaR or expected shortfall. In particular, their result does not preclude that other moments or even quantiles of the return distribution could be predictable further ahead in time. When modeling returns data at a lower frequency (monthly), it is possible that new predictive patterns emerge, e.g. related to higher volatility around recession periods, c.f. Hamilton and Lin (1996).

Our econometric analysis models the joint distribution of returns on US stocks and bonds using monthly post-war data. We find evidence that a four-state model is required to ad-
equately characterize time-variations in their joint distribution. Mean returns, volatilities and correlations are found to vary significantly across these states in a way that gives rise to important variations in real-time estimates of the risk measures.

The contribution of the paper is four-fold. First, we provide econometric estimates for a four-state multivariate regime switching model for the joint distribution of monthly stock and bond returns. Second, we derive implications of economic regimes for the ‘term structure’ of moments of portfolio returns and of risk measures such as VaR and expected shortfall. Despite the popularity of regime switching models, to our knowledge such results have not previously been derived. Third, we provide a comparison of the regime switching model against two benchmarks, namely a Gaussian IID model and a GARCH(1,1) specification. Fourth, we demonstrate the out-of-sample predictive performance of the regime switching model in terms of the accuracy of its VaR predictions.

The structure of the paper is as follows. Section 2 introduces the regime switching specification and provides empirical estimates. Section 3 provides theoretical results on the moment generating function of portfolios whose returns are driven by a multivariate regime switching process and characterizes the term structure of VaR and expected shortfall under regime switching. It also considers two benchmark models, namely a model with independently and identically distributed (iid) Gaussian increments and a multivariate GARCH model. Section 4 provides real-time estimates of risk measures and also reports results from an out-of-sample forecasting experiment using model-free diagnostic tests. Section 5 concludes with a discussion of our results.

2 Estimation Results

A number of studies have found evidence of nonlinearities in the dynamics of financial return series related to the presence of regime switching. This is well documented for returns on currencies (Engel and Hamilton (1990)), bonds (Driiffill and Sola (1994) Gray (1996), Ang and Bekaert (2002a)) and stocks (Ang and Bekaert (2002b), Perez-Quiros and Timmermann (2001), Turner, Startz and Nelson (1989), Veronesi (1999) and Whitelaw (2001)).
2.1 Econometric Specification

We build on this evidence in constructing a model for the joint distribution of asset returns. This is the natural object of interest when studying risk in relation to strategic asset allocation. To capture the possibility of regimes in asset returns, consider an $n \times 1$ vector of asset returns in excess of the T-bill rate, $r_t = (r_{1t}, ..., r_{nt})'$.\footnote{Since the risk-free rate is either known in advance or has a much lower variability than stock and bond returns it is common practice to subtract the risk-free rate and model excess returns.} Suppose that the mean, covariance and possibly also (auto and cross-) serial correlations in returns are driven by a common state variable, $S_t$, that takes integer values between 1 and $k$:

$$r_t = \mu_{s_t} + \sum_{j=1}^{p} A_{j,s_t} r_{t-j} + \Sigma_{s_t} \epsilon_t. \quad (1)$$

Here $\mu_{s_t} = (\mu_{1s_t}, ..., \mu_{ns_t})'$ is an $n \times 1$ vector of mean returns in state $s_t$, $A_{j,s_t}$ is the $n \times n$ matrix of autoregressive coefficients associated with lag $j \geq 1$ in state $s_t$, and $\epsilon_t = (\epsilon_{1t}, ..., \epsilon_{nt})' \sim N(0, I_n)$ follows a multivariate normal distribution with zero mean and identity covariance matrix so that return innovations have covariance

$$E \left[ \left( r_t - \mu_{s_t} - \sum_{j=1}^{p} A_{j,s_t} r_{t-j} \right) \left( r_t - \mu_{s_t} - \sum_{j=1}^{p} A_{j,s_t} r_{t-j} \right)' | s_t \right] = \Sigma_{s_t} \Sigma_{s_t}' = \Omega_{s_t}. \quad (2)$$

Regime switches in the state variable, $S_t$, are assumed to be governed by the transition probability matrix, $P$, with elements

$$\Pr(s_t = i | s_{t-1} = j) = p_{ji}, \quad i, j = 1, ..., k. \quad (3)$$

Each regime is thus the realization of a first-order Markov chain with constant transition probabilities. We refer to $\theta$ as the column vector that collects all the unknown model parameters, i.e. $\theta = \{\mu_s, \{A_{j,s}\}_{j=1}^{p}, \Omega_s, P\}$ where $s = 1, 2, ..., k$.

While simple, this model is quite general and allows asset returns to have different means, variances and correlations in different states. This means that risk measures are likely to vary considerably across states. For example, knowing that the current state is a persistent high return, low volatility state will make risky assets more attractive.
The model (1) - (3) is of course heavily parameterized. We justify using the model on three grounds. First, many parameters have an economic interpretation that are of separate interest. This makes it easier to characterize risk both at a given point in time and also as it evolves over time and parameter estimates get recursively updated. Second, under the assumption that the maintained parametric model provides a reasonably adequate representation of the data generating process we can compute risk measures at several time horizons and thus characterize the entire term structure of risk. Third, although our parameter estimates will be biased in case the dynamics of the first and second moments is misspecified, the advantage of working with a parametric model is that standard errors tend to be smaller than if the parametric assumptions were dispensed with. As we shall see, in fact the empirical results suggest that most parameters are quite precisely estimated. Furthermore, as pointed out by Marron and Wandt (1992), Gaussian mixtures have the ability to flexibly approximate a range of distributions.

2.2 Data

We are interested in studying allocation to three major US asset classes, namely stocks, bonds and T-bills. Strategic asset allocation decisions have recently been the subject of considerable interest (e.g. Brennan, Schwarz and Lagnado (1997) and Campbell and Viceira (2001)). The joint dynamics of stock and bond returns underlying strategic asset allocation decisions have been studied far less than the returns on individual asset classes, however. Yet, for many pension and mutual fund managers, the strategic asset allocation decision accounts for close to 90% of the variability in asset returns and is far more important than decisions on how to select securities within individual asset classes, c.f. Blake, Lehmann and Timmermann (1999).

Our analysis uses monthly returns on a value-weighted portfolio of common stocks listed on the NYSE. We also consider the return on a portfolio of 10-year government bonds. Returns are calculated applying the standard continuous compounding formula, \( \tilde{r}_t \equiv \ln S_t - \ln S_{t-1} \), where \( S_t \) is the asset price, inclusive of any cash distributions (dividends, coupons) between time \( t - 1 \) and \( t \). To obtain excess returns \( r_t \), we subtract the 30-day T-bill rate.
from these returns, \( r_t \equiv \tilde{r}_t - r^f_t \). Our sample is January 1954 - December 1999, a total of 552 observations. All data is obtained from the Center for Research in Security Prices. Hence the empirical analysis focuses on a bivariate model of stock and bond excess returns, taking the risk-free rate as given (non-stochastic).

### 2.3 Determining the Number of States

The number of states, \( k \), in (1) is a key determinant of the proposed model. If \( k = 1 \), we are back to the standard linear/Gaussian VAR(\( p \)) model used in much of the existing literature. As \( k \) rises, it becomes increasingly easy to fit complicated dynamics and deviations from the normal distribution in asset returns. However, this comes at the cost of having to estimate more parameters which can lead to deteriorating out-of-sample model performance.

To determine the appropriate number of regimes in the bivariate model (2), we studied a range of bivariate specifications with different numbers of states, \( k \), and different lag orders, \( p \). Table 1 presents the results. We report the outcome of likelihood ratio tests for linearity using the Davies (1977) upper bounds for the critical values to account for the presence of unidentified parameters under the null hypothesis. Linearity is strongly rejected by all models with two or more states. To select a model we further report values of the Hannan-Quinn information criterion which supports a four-state specification without lagged returns (\( k = 4, p = 0 \)).

The lower part of the table takes a model with \( k \geq 4 \) regimes and \( p = 0 \) as a benchmark and experiments with \( k = 5, 6, \) and 9 states. These models were based on evidence of three states in the univariate bond and stock return series. Nine regimes represents the case in which the underlying univariate states are completely independent. Expanding the number of states means increasing the number of parameters to be estimated and the information criterion heavily penalizes the models with five or more regimes. We conclude that four states are sufficient to fit the joint process for monthly bond and stock returns.
2.4 Interpreting the regimes

Table 2 reports parameter estimates from the four-state specification for the joint distribution of stock and bond (excess) returns. For comparison we also show results for a single-state model where excess returns follow a Gaussian IID process. Stock returns in excess of the T-bill rate had a sample mean of 8% per annum and an annualized volatility of 15%. Bond (excess) returns had a marginally positive (and insignificant) sample mean of about 1% per annum with half the volatility of stock returns.

These sample averages do not reveal what appears to be significant time-variations in the first and second moments of the joint distribution of stock and bond returns across the four regimes identified in Table 2. Mean returns on stocks vary from 1.3% per month - almost double its unconditional mean - in the third state to minus 8.5% per month in the first state. Interestingly, all four estimates of mean stock returns are statistically significant at conventional levels. Far less variation is observed in the mean returns of long-term bonds and none of the mean estimates is significant for bonds returns.

Turning to the volatility and correlation parameters, Table 2 shows that stock return volatility varies between 10% and almost 19% per annum, with state 1 displaying the highest value. Bond return volatility varies even more across states, going from a very low value of 1.1% per annum in state three to 11.6% per annum in state four. Correlations between stock and bond returns go from -0.85 in state 1 to 0.44 in state four. In state 3 stock and bond returns are essentially uncorrelated. These important differences in correlations across states are crucial for long-run asset allocation and risk management purposes and strongly support using a multi-state model. Transition probability estimates show that state 1 is a transitory state lasting on average around two months, while the three remaining states are highly persistent, lasting on average between 14 and 40 periods.

Taken together, this suggests that state 1 is a temporary bear state that picks up large

\[ \sigma \equiv \sigma_m \times \sqrt{12} \]

Volatilities are annualized by applying the standard (but incorrect) formula $\sigma \equiv \sigma_m \times \sqrt{12}$, where $\sigma_m$ is the monthly volatility. As pointed out by Christoffersen, Diebold, and Schuermann (1998) in the presence of time-varying and mean-reverting volatility, such a scaling may be grossly incorrect. We apply the scaling only for comparison with estimates reported in the literature.
negative and highly volatile stock returns that are negatively correlated with bond returns. States 2-4 are more similar in terms of mean returns on stocks but vary considerably in terms of stock and bond market volatility - with state 3 representing calmer markets and state 4 being more volatile - and in terms of stock and bond return correlations that are close to zero in state 3 and much higher in state 4. The volatility of bond returns is also much higher in regime 4 than in regimes 2 and 3.

A plot of the smoothed state probabilities from the bivariate model is provided in Figure 1. State 1 is transitory with relatively isolated spikes capturing high volatility episodes that quickly die out and only occur in 3% of the sample. Some of these periods are associated with important events in the recent history of financial markets, like the recessions caused by the two oil price shocks, the crash of October 1987 and the ‘Asian flu’ in the Summer of 1998. State 3 identifies a single historical episode from 1962-1966, although its ergodic probability is 7%. States 2 and 4, on the other hand, are visited frequently and for long periods of time, capturing 90% of the total sample (67% and 23%, respectively). The fact that their overall ergodic probabilities are only 10% does not mean that states 1 and 3 are unimportant for risk management and modeling purposes, however. Clearly the left tail of the distribution of stock and bond returns is significantly affected particularly by state 1 and the possibility of shifting to this state even if starting from another state.

3 Term Structure of Risk

The four-state model fitted in the previous section offers a rich econometric specification. Despite its popularity as an econometric model and the considerable evidence of regimes in the asset return distribution, no previous study seems to have explored the implications of regime switching for the moments of multi-period return distributions and risk measures such as Value at Risk or expected shortfall. This is the object of the analysis in this section.

This finding is based on full-sample, smoothed state probabilities. However, the real-time (filtered) state-3 probabilities were clearly non-zero on a number of other occasions, so that this state matters to real-time risk estimates.
3.1 Moments of Multi-period Portfolio Returns

To gain a better understanding of risk measures under regime switching, we first characterize the moments of the multi-period predictive density of portfolio returns, $R_{t+h} = \omega' r_{t+h}$, where $\omega$ is a vector of portfolio weights that sum to unity, i.e. $\omega' \ell = 1$. Conditional on being in state $s_t$ at time $t$, the $h$-step ahead distribution of portfolio returns, $\mathcal{P}(R_{t+h}|S_t = s_t)$ is not a simple normal density $N(\omega' \mu_{t+h}, \omega' \Omega_{t+h} \omega)$, but is instead a mixture of $k$ normal densities. For example, the unconditional predictive distribution of $R_{t+h}$ is a mixture of $k$ normal densities with mixture weights given by the $h$-step predicted probabilities of each of the $k$ regimes, $\bar{\pi}_{s_{t+h}, s_t} = P^h \bar{\pi}$, where $\bar{\pi}$ is the vector of ergodic state probabilities. Portfolio returns one period ahead can thus be written as

$$R_{t+1} = \omega' r_{t+1} = \omega' \mu_{s_{t+1}} + \omega' \Sigma_{s_{t+1}} \epsilon_{t+1}. $$

It follows that (unconditionally)

$$E[\omega' r_{t+1}] = \pi' P \mu_M \omega = \pi'_{t+1,t} \mu_M \omega = \bar{\mu}_{t+1}. \quad (4)$$

where

$$\mu_M = [\mu_{is_t}] = \begin{pmatrix} \mu_{11} & \mu_{21} & \cdots & \mu_{n1} \\ \mu_{12} & \mu_{22} & \cdots & \mu_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{1k} & \mu_{2k} & \cdots & \mu_{nk} \end{pmatrix}. $$

$\pi_{t+1,t} = \pi' P$, is the vector of one-step-ahead probabilities that accounts for possible transitions between $t$ and $t+1$. The expected value of a linear combination of regime switching processes is a probability-weighted linear combination of regime-specific means, with weights provided by the vector $\omega$. The probability weights are provided by the elements of $\pi_{t+1,t}$ which depend on the transition probabilities and the state probabilities at time $t$. Using that $\Pr(s_{t+h}|s_t) = P^h$ and $\pi_{s_{t+h}} = \sum_{s_{t+1}=1}^k \pi_{s_t} P^h_{s_t, s_{t+h}}$, the extension to the multi-period conditional mean of the aggregate process is straightforward:

$$E[\omega' r_{t+h}] = \pi' P^h \mu_M \omega = \pi'_{t+h,t} \mu_M \omega = \bar{\mu}_{t+h}. \quad (5)$$
To derive the one-period variance of portfolio returns, define \( \bar{\mu}_{jt+1} = \pi'_{t+1}\mu_M e_j \). It is easily shown that:

\[
E\left[ (\omega' r_{t+1} - E[\omega' r_{t+1}])^2 \right] = \sum_{s_{t+1}=1}^{k} \pi_{s_{t+1}} (\mu'_M - p_{t+1} \otimes \iota'_k)(\mu_M - p'_{t+1} \otimes \iota_k)\omega + \\
+ \sum_{s_{t+1}=1}^{k} \pi_{s_{t+1}} \omega'\Sigma_{s_{t+1}} \Sigma'_{s_{t+1}} \omega,
\]

where \( \pi_{s_{t+1}} = \Pr(s_{t+1} = j|\Omega_t) = \pi' P_{s_{t+1}} \) and \( \mu'_M = \mu'_M P' \pi \). Again this expression is easily generalized to the multi-period case:

\[
E\left[ (\omega' r_{t+h} - E[\omega' r_{t+h}])^2 \right] = \sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}} (\mu'_M - p_{t+h} \otimes \iota'_k)(\mu_M - p'_{t+h} \otimes \iota_k)\omega + \\
+ \sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}} \omega'\Sigma_{s_{t+h}} \Sigma'_{s_{t+h}} \omega.
\]

Notice that the variance of the portfolio return is not merely a linear combination of the unconditional covariance matrices under the \( k \) regimes:

\[
\sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}} \Sigma_{s_{t+h}} \Sigma'_{s_{t+h}} = \sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}} \Omega_{s_{t+h}}.
\]

Instead a probability-weighted sum of matrices collecting the cross-products of deviations of state-specific means from their means, \( (\mu'_M - p_{t+h} \otimes \iota'_k)(\mu_M - p'_{t+h} \otimes \iota_k) \), enters the expression for the portfolio variance. This is relevant to understanding tail risks under regime switching. The parameter estimates in Table 2 showed large differences in mean returns particularly for stocks and equation (7) shows that this will generate a higher spread than if mean returns were similar across states.

To understand the tail probabilities underlying VaR, it is also useful to consider the portfolio skew and kurtosis. Since

\[
E\left[ (\omega' r_{t+1} - E[\omega' r_{t+1}])^3 \right] = \sum_{s_{t+1}=1}^{k} \pi_{s_{t+1}} \left\{ (\omega' \mu_{s_{t+1}} - \pi'_{t+1}\mu_M \omega)^3 + 3 (\omega' \mu_{s_{t+1}} - \pi'_{t+1}\mu_M \omega) (\omega' \Omega_{s_{t+1}} \omega)^2 \right\},
\]

where \( \pi_{s_{t+1}} = \Pr(s_{t+1} = j|\Omega_t) = \pi' P_{s_{t+1}} \) and \( \mu'_M = \mu'_M P' \pi \). Again this expression is easily generalized to the multi-period case:
it follows that the $h$-step ahead coefficient of skewness is given by

$$
\text{skew}_{t+h} = \frac{\sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}} \left\{ \left( \omega' \mu_{s_{t+h}} - \pi'_{t+h} \mu_M \omega \right)^3 + 3 \left( \omega' \mu_{s_{t+h}} - \pi'_{t+h} \mu_M \omega \right) \left( \omega' \Omega_{s_{t+h}} \omega \right) \right\}}{\left[ \sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}} \omega' \left( \mu'_{M} - \pi'_{t+h} \otimes \iota'_{k} \right) \left( \mu_{M} - \pi'_{t+h} \otimes \iota_{k} \right) \omega + \sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}} \omega' \Omega_{s_{t+h}} \omega \right]^2}.
$$

(8)

In the absence of regime switching in the conditional means, portfolio returns have zero skew. However, in situations such as here where the mean return can be large and negative while simultaneously the variance is high (as in state 1), a negative skew is induced, thereby leading to a higher risk of a large loss when compared to a symmetric return distribution.

The fourth moment of portfolio returns can be written as follows:

$$
E[(\omega' r_{t+1} - E[\omega' r_{t+1}])^4] = \sum_{s_{t+1}=1}^{k} \pi_{s_{t+1}} \left\{ \left( \omega' \mu_{s_{t+1}} - \pi'_{t+1} \mu_M \omega \right)^4 + 6 \left( \omega' \mu_{s_{t+1}} - \pi'_{t+1} \mu_M \omega \right)^2 \times \right.
\times \left( \omega' \Omega_{s_{t+1}} \omega \right)^2 + 3 \left( \omega' \Omega_{s_{t+1}} \omega \right)^4 \right\}.
$$

Hence the $h$-step ahead coefficient of kurtosis is given by:

$$
\kappa_{t+h} = \frac{\sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}} \left\{ \left( \omega' \mu_{s_{t+h}} - \pi'_{t+h} \mu_M \omega \right)^4 + 6 \left( \omega' \mu_{s_{t+h}} - \pi'_{t+h} \mu_M \omega \right)^2 \left( \omega' \Omega_{s_{t+h}} \omega \right)^2 + 3 \left( \omega' \Omega_{s_{t+h}} \omega \right)^4 \right\}}{\left[ \sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}} \omega' \left( \mu'_{M} - \pi'_{t+h} \otimes \iota'_{k} \right) \left( \mu_{M} - \pi'_{t+h} \otimes \iota_{k} \right) \omega + \sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}} \omega' \Omega_{s_{t+h}} \omega \right]^4}.
$$

(9)

The skew and kurtosis reflect in a complicated manner the mean returns, variances and covariances in the individual states as well as their differences. They clearly also depend on the transition probability matrix through the period $t + h$ state probabilities, $\pi_{s_{t+h}}$. Kurtosis will generally be higher, however, when the variance parameters vary across states as we found they did in Table 2 for both stocks and bonds.

These expressions can readily be generalized to obtain the moment generating function.
of portfolio returns in period $t+h$:

$$E[(\omega' r_{t+h} - E[\omega' r_{t+h}])^2] = \sum_{s_{t+h}=1}^{k} E[(\omega' r_{s_{t+h}} - E[\omega' r_{s_{t+h}}])^2 | s_{t}] \Pr(s_{t})$$

$$= \sum_{s_{t+h}=1}^{k} \left( (\omega' \mu_{s_{t+h}} - \pi'_{t+h} \mu_{M} \omega) + \omega' \Omega_{s_{t+h}} \omega \varepsilon_{t+h} \right)^{r} \pi_{s_{t+h}}$$

$$= \sum_{s_{t+h}=1}^{k} \pi_{s_{t+h}} \sum_{j=0}^{r} rC_{j} \kappa_{j} \left( \omega' \mu_{s_{t+h}} - \pi'_{t+h} \mu_{M} \omega \right)^{r-j} (\omega' \Omega_{s_{t+h}} \omega)^{j},$$

(10)

where $rC_{j} = r!/(r-j)!j!$ and

$$\kappa_{j} = \prod_{h=1}^{j/2} (2h-1), \text{ if } j \text{ is even and }$$

$$\kappa_{j} = 0, \text{ otherwise.}$$

These results extend to the multivariate case many of the results in Timmermann (2000).

### 3.2 Multi-period VaR

Value at Risk has been extensively studied in finance, c.f., e.g. Britten-Jones and Schaeffer (1999), Berkowitz and O’Brien (2002) and Christoffersen (2003). Following conventional practice we report Value-at Risk at the 100\(\alpha\)% level for \(\alpha = 0.01\). Let the weight on stocks and bonds be \(\omega = (\omega^{stock}, \omega^{bond})'\). Then the cumulated \(h\)-period return on the portfolio comprising stocks, bonds and T-bills from period \(t\) to period \(t+h\) is

$$R_{t:t+h} = (1 - \omega' \nu_{2}) \exp \left( hr^{f} \right) + \omega' \exp \left( R_{t+h} + hr^{f} \nu_{2} \right),$$

(11)

where \(r^{f}\) is the risk-free rate and \(\nu_{2}\) is a 2 \times 1 vector of ones. Value at Risk at the \(h\)-period horizon is simply the \(\alpha\) quantile of the conditional probability distribution of \(R_{t:t+h}\):

$$\Pr(R_{t:t+h} \leq VaR_{t:t+h}^{\alpha} | \mathcal{F}_{t}) = \alpha,$$

(12)

where \(\mathcal{F}_{t}\) is the period-\(t\) information set. For a given asset allocation, \(\omega\), The \(VaR\) estimate depends on the horizon, \(h\), the significance level, \(\alpha\), the information set, \(\mathcal{F}_{t}\), and on the
econometric model at hand. As portfolio weights are changed, the VaR estimate will also change. We therefore study five generic portfolios representing different levels of aggressiveness in the strategic asset allocation. These portfolios comprise (i) 50% stocks and 50% bonds, (ii) 50% bonds and 50% T-bills, (iii) 100% bonds, (iv) 50% stocks and 50% T-bills; and (v) 100% stocks.

3.3 Calculation of VaR under Regime Switching

Because of the nonlinearity introduced by regime switching, $VaR_{t:t+h}$ is best computed by Monte Carlo simulation. For example, when $r_t$ is generated by a $k$-state regime switching model, for a given set of portfolio weights, $\omega$, the c.d.f. of the single-period portfolio return, $F(R_{t:t+1}(\omega))$ becomes

$$F(R_{t:t+1}(\omega) \leq \kappa) = \int_{-\infty}^{\kappa} \sum_{s_{t+1}=1}^{k} \pi_{s_{t+1}} \phi \left( \frac{x - \mu'_{s_{t+1}} - \omega' \Sigma_{s_{t+1}} \omega + \omega' (\mu'_{s_{t+1}} + \mu'_{t+k} \otimes t_k) (\mu'_{t+k} - \mu'_{s_{t+1}} \otimes t_k) \omega}{\omega' \Sigma_{s_{t+1}} \omega + \omega' (\mu'_{s_{t+1}} + \mu'_{t+k} \otimes t_k) (\mu'_{t+k} - \mu'_{s_{t+1}} \otimes t_k) \omega} \right) dx$$

$$= \sum_{s_{t+1}=1}^{k} \pi_{s_{t+1}} \int_{-\infty}^{\kappa} \phi \left( \frac{x - \mu'_{s_{t+1}} - \omega' \Sigma_{s_{t+1}} \omega + \omega' (\mu'_{s_{t+1}} + \mu'_{t+k} \otimes t_k) (\mu'_{t+k} - \mu'_{s_{t+1}} \otimes t_k) \omega}{\omega' \Sigma_{s_{t+1}} \omega + \omega' (\mu'_{s_{t+1}} + \mu'_{t+k} \otimes t_k) (\mu'_{t+k} - \mu'_{s_{t+1}} \otimes t_k) \omega} \right) dx$$

where $\Phi(\cdot; \mu_{s_{t+1}}, \Sigma_{s_{t+1}}, \omega)$ is the standard normal c.d.f., and $\phi(\cdot)$ is the corresponding p.d.f.

The conditional c.d.f. for portfolio returns given the current information $\mathcal{F}_t$, $\hat{F}(R_{t:t+h}(\omega) \leq \kappa|\mathcal{F}_t)$, can therefore be computed as follows:

$$\hat{F}(R_{t:t+1}(\omega) \leq \kappa|\mathcal{F}_t) = \int_{-\infty}^{\kappa} \sum_{s_{t+1}=1}^{k} \pi_{s_{t+1}} \phi \left( \frac{x - \mu'_{s_{t+1}} - \omega' \Sigma_{s_{t+1}} \omega + \omega' (\mu'_{s_{t+1}} + \mu'_{t+k} \otimes t_k) (\mu'_{t+k} - \mu'_{s_{t+1}} \otimes t_k) \omega}{\omega' \Sigma_{s_{t+1}} \omega + \omega' (\mu'_{s_{t+1}} + \mu'_{t+k} \otimes t_k) (\mu'_{t+k} - \mu'_{s_{t+1}} \otimes t_k) \omega} \right) dx$$

$$= \sum_{s_{t+1}=1}^{k} \pi_{s_{t+1}} \int_{-\infty}^{\kappa} \phi \left( \frac{x - \mu'_{s_{t+1}} - \omega' \Sigma_{s_{t+1}} \omega + \omega' (\mu'_{s_{t+1}} + \mu'_{t+k} \otimes t_k) (\mu'_{t+k} - \mu'_{s_{t+1}} \otimes t_k) \omega}{\omega' \Sigma_{s_{t+1}} \omega + \omega' (\mu'_{s_{t+1}} + \mu'_{t+k} \otimes t_k) (\mu'_{t+k} - \mu'_{s_{t+1}} \otimes t_k) \omega} \right) dx$$

$$= \sum_{s_{t+1}=1}^{k} \pi_{s_{t+1}} \phi(\kappa; \mu_{s_{t+1}}, \Sigma_{s_{t+1}}, \omega). \quad (13)$$

The (row) vector $\pi_{s_{t+1},s_t} \equiv [\Pr(s_{t+1} = 1|\mathcal{F}_t) \Pr(s_{t+1} = 2|\mathcal{F}_t) \cdots \Pr(s_{t+1} = k|\mathcal{F}_t)]$ is a random variable and Hamilton (1994, p. 679) shows that it can be expressed as a first-order vector
autoregression,

\[ \pi_{t+1} = \pi_t P + v_{t+1} \]  (14)

where \( v_{t+1} \equiv \pi_{t+1} - E[\pi_{t+1}|F_t] \) is a martingale difference sequence and \( E[\pi_{t+1}|F_t] = 0 \).

Using that \( \pi_{st+h,s_t} \) is a random variable with transitions driven by (3) means that the VaR can be computed as follows:

1. Draw a large number \( Q \) of excess returns \( \{r^q_i\}_{q=1}^Q (i = 1, 2, ..., h) \) from the bivariate four-state regime switching model. The draws are generated consistent with the stochastic process for the regime switches (3) using the transition probabilities, \( \hat{P} \).

2. Temporally aggregate excess returns to get multiperiod returns \( \tilde{r}^q_{t,t+h} \equiv \sum_{i=1}^{h} r^q_i + hr^f \).

3. Form \( h \)-period portfolio returns using the portfolio weights \( \omega \), \( R^q_{t,t+h}(\omega) \equiv (1 - \omega' t_2) \exp(hr^f) + \omega' \exp(\tilde{r}^q_{t,t+h}) \).

4. Calculate the \( \alpha \) percentile of the resulting simulated distribution for multiperiod portfolio returns, \( \{R^q_{t,t+h}(\omega)\}_{q=1}^Q \).

Since some regimes occur relatively infrequently (mainly regimes 1 and 3), we set the number of Monte Carlo simulations to a relatively large number, \( Q = 50,000 \). We report results for the five portfolios listed above at different horizons \( h \).

Results are provided in Figure 2 which shows VaR term structures at the 1% level for horizons going from a single month to two years (24 months). We use the convention of reporting \( VaR \) as a positive number, so that a large positive number represents a large loss. We show outcomes initiating from each of the four regimes as well as a “high uncertainty” scenario which puts equal weights on the four states \( \pi = \iota_4/4 \). Several interesting patterns emerge. In the case of the equal-weighted stock-bond portfolio it is clear that there are nonmonotonic term structures even at horizons shorter than two years. Furthermore, VaR patterns differ significantly across the four states. Starting from state 3, the VaR estimate declines as a function of the horizon, \( h \), indicating that portfolio risk is always reduced the longer the horizon when starting from this state. In contrast, starting from regime 1, the
VaR initially rises from about 10% at the 1-month horizon to around 25% at the 6-month horizon before declining as a result of the high likelihood of switching to a state with higher mean return and lower volatility. Similar non-monotonic patterns are observed when starting from states two and four. For the pure stock portfolio the 1% VaR estimate is as high as 60% when starting from the first state. In contrast, for the pure bond or 50% bond and 50% T-bill portfolio, state 4 is associated with the highest VaR levels.

At even longer horizons, Figure 3 shows stronger non-monotonicities in the VaR term structures. At sufficiently long horizons the VaR estimates always become negative, indicating that the risk-return trade-off is such that although the spread of the return distribution widens for larger values of \( h \), the probability distribution simultaneously shifts to the right due to the positive mean portfolio return. At short horizons the first effect dominates while at very long horizons beyond three or five years the second effect dominates.

### 3.4 Benchmark Models

To better understand and interpret the effect of regime switching on VaR, for comparison we also report VaR estimates based on two Gaussian vector autoregressions (VARs). The first assumes that innovations are drawn from a homoskedastic distribution with constant variance. Although this model ignores deviations from the Gaussian distribution that are clearly present in the stock returns data, it is less misspecified at the multi-month horizons we are considering here than at the shorter (daily) horizons often under consideration.

To see how one would expect VaR to evolve as a function of \( h \) under this model where \( R_t \sim INN(\mu, \sigma^2) \), notice that \( R_{t:t+h} \sim N(\mu h, \sigma^2 h) \). Hence

\[
VaR_{t:t+h} = \mu h + \sigma \sqrt{h} \Phi^{-1}(\alpha),
\]

Differentiating with respect to \( h \), we get the first order condition

\[
\mu + \frac{\sigma \Phi^{-1}(\alpha)}{2 \sqrt{h}} = 0.
\]

The horizon where VaR is minimized (assuming a positive mean return, \( \mu \)) is therefore

\[
h^* = \frac{\sigma^2 (\Phi^{-1}(\alpha))^2}{4 \mu}.
\]
The second order condition is easily verified to be satisfied so that \( h^+ \) gives the value of the largest VaR. For positive \( \mu \) and for small values of \( h \), VaR may initially decline, but for longer horizons it will eventually be dominated by the mean term, \( \mu h \), which grows linearly in \( h \).

We calibrate the parameters of this model to match the first two moments of the four-state regime switching model when \( h = 1 \):

\[
\begin{align*}
\mu &= \pi_{t+1,t}^i \mu_M, \\
\sigma^2 &= \sum_{s_{t+1}=1}^{k} \pi_{s_{t+1}}^i \omega' (\mu_M' - \bar{m}_{t+1} \otimes \bar{l}_k^i) (\mu_M' - \bar{m}_{t+1} \otimes \bar{l}_k^i) \omega + \sum_{s_{t+1}=1}^{k} \pi_{s_{t+1}}^i \omega' \Omega_{s_{t+1}},
\end{align*}
\]

where \( \pi_{s_{t+1}} = \sum_{s_{t+1}=1}^{k} \pi_{s} P_{s,s_{t+1}} \). Equation (15) is then used to compute the Value at Risk.

Our second benchmark is a multivariate GARCH(1,1) model with time-varying conditional variance of the form:

\[
\begin{align*}
\mathbf{r}_{t+1} &= \mathbf{\mu} + \mathbf{\eta}_{t+1}, \\
\sigma_{i,i,t}^2 &= \alpha_{ii} + \beta_{i0} \eta_{it}^2 + \beta_{ii} \sigma_{ii,t-1}, \\
\sigma_{i,j,t}^2 &= \psi_{ij} \sigma_{ii,t} \sigma_{jj,t}, \quad i, j = 1, 2
\end{align*}
\]

where \( \mathbf{\eta}_{t+1} = (\eta_{1t+1}, \eta_{2t+1})' \) are heteroskedastic return innovations defined as \( \eta_{it+1} = \sigma_{ii,t} \varepsilon_{it+1} \), and \( \mathbf{\varepsilon}_{t+1} = (\varepsilon_{1t+1}, \varepsilon_{2t+1})' \sim IN(\mathbf{0}, \mathbf{I}_2) \) so that \( \mathbf{\eta}_{t+1} \sim N(\mathbf{0}, \Omega_t) \). \( \Sigma_t = [\sigma_{ij,t}] \) is the conditional covariance matrix and \( \psi_{ij} \) is the conditional correlation coefficient which is assumed to be constant. This model was first proposed by Bollerslev (1990). We found no evidence of ARCH effects in the univariate bond return series and therefore imposed that \( \beta_{i0} = \beta_{i1} = 0 \) for the bond return variance equation (\( i = 2 \)).

The parameters of the model (17) were estimated by maximum likelihood to obtain the following estimates (standard errors are in parenthesis):

\[
\begin{align*}
\mathbf{r}_{t+1} &= \begin{bmatrix} 0.00676 \\ 0.00088 \end{bmatrix} + \mathbf{\eta}_{t+1}, \quad \mathbf{\eta}_{t+1} \sim N(\mathbf{0}, \Omega_t) \\
\Omega_t &= \begin{bmatrix} 0.00013 & 0.0683 \eta_{1t}^2 + 0.8617 \sigma_{11,t-1}^2 \\ 0.5427 \times \sqrt{0.00047} \times \sqrt{0.00013 + 0.0683 \eta_{1t}^2 + 0.8617 \sigma_{11,t-1}^2} & 0.00047 \end{bmatrix} \\
&= \begin{bmatrix} 0.00013 & 0.0683 \eta_{1t}^2 + 0.8617 \sigma_{11,t-1}^2 \\ 0.5427 \times \sqrt{0.00047} \times \sqrt{0.00013 + 0.0683 \eta_{1t}^2 + 0.8617 \sigma_{11,t-1}^2} & 0.00047 \end{bmatrix}
\end{align*}
\]
The estimated mean parameters are very similar to those found in Table 2 for the IID Gaussian case. Typical estimates are obtained for the GARCH(1,1) model fitted to stock returns, although the implied persistence is modest ($\hat{\beta}_{10} + \hat{\beta}_{11} = 0.93$). For this model we compute VaR using a simulation approach similar to that adopted for the regime switching process. Again we set the number of Monte Carlo trials to $Q = 50,000$.

Figure 4 compares the VaR under the three models. Obviously, the largest difference arises when comparing the regime switching model vs. the IID Gaussian model where differences are large for both stocks and bonds and grow as a function of the horizon, $h$. For instance, the 1% VaR of a pure stock portfolio computed under the regime switching model is in excess of 10% for all investment horizons and peaks at roughly 35% for $h$ close to two years. However, it never reaches 10% under the IID Gaussian model. Using a simple IID Gaussian model that ignores the strong evidence of regimes therefore leads to substantial underestimation of the stock market risk. This claim seems not to depend on $\alpha$ as the deviations of the two sets of VaR estimates remain large when $\alpha = .05$.

Differences between the VaR estimates implied by the GARCH and regime switching models are less pronounced and depend on the portfolio composition. Differences are largest for the pure equity and stock and cash portfolios where the VaR estimate under regime switching is 5-10% higher than under GARCH at the longer horizons. For instance the 1% VaR for a 100% equity portfolio is at most 25% under the GARCH model, but peaks at 35% under regime switching. In contrast, the GARCH model implies a higher VaR estimate for bonds than the regime switching model with a difference of 300-400 basis points emerging at the longest horizon.

### 3.5 Expected shortfall

Expected shortfall is another commonly reported measure of risk. It is defined as the expected loss conditional on the (cumulated) loss exceeding the VaR. At the $h$-period horizon, the conditional expected shortfall is hence given by

$$ES_{t:t+h}^\alpha = E[(R_{t:t+h}|R_{t:t+h} \leq VaR_{t:t+h})|\mathcal{F}_t].$$
When single-period returns are $IIN(\mu, \sigma^2)$, the $h$–period expected shortfall is given by

$$ES^\alpha_{t:t+h} = \mu h - \sigma \sqrt{h} \frac{\phi((VaR^\alpha_{t:t+h} - \mu h)/\sigma \sqrt{h})}{\Phi((VaR^\alpha_{t:t+h} - \mu h)/\sigma \sqrt{h})},$$

where $\phi(x) = \exp(-x^2/2)/\sqrt{2\pi}$. Using that $VaR_{t:t+h} = \mu h + \sigma \sqrt{h\Phi^{-1}(\alpha)}$, we have

$$ES^\alpha_{t:t+h} = \mu h - \sigma \sqrt{h} \frac{\phi(\Phi^{-1}(\alpha))}{\Phi^{-1}(\alpha)}. \quad (19)$$

This reaches a minimum at

$$h^{**} = \frac{\sigma^2 \phi(\Phi^{-1}(\alpha))^2}{4\mu^2 \alpha^2}. \quad (20)$$

Again the expected shortfall grows linearly in $h$ at long horizons.

Figure 5 reports the expected shortfall for the same configurations of initial state probabilities that were considered in Figures 2 and 3. Again we follow the convention of reporting a negative return as a positive loss. Variations in the expected shortfall are very large across the four regimes, giving further credence to the importance of accounting for regime switching. Unsurprisingly, the expected shortfall is highest for portfolios invested in stocks when starting from the negative return state (state 1) where it is close to twice as high as its value when starting from any of the other states. At short horizons the sensitivity of the expected shortfall with respect to $h$ also tends to be higher when starting from state 1 than from any other state. In contrast, starting from state 4 leads to the highest VaR estimate for the bond portfolios. Interestingly, for horizons between one and 24 months there is weaker evidence of non-monotonicities in the expected shortfall than in the VaR estimates shown in Figure 2.

Figure 6 compares the expected shortfall under the regime switching model to its values under the IID Gaussian and GARCH(1,1) benchmarks. Once again, the regime switching model indicates a systematically higher expected shortfall for stocks than under either of the benchmarks. However, the expected shortfall is now also highest under regime switching for the bond portfolios. Introducing ARCH in asset returns does increase the expected shortfall when compared to the simple Gaussian IID model but not by the full distance to the regime switching specification.
4 Real-time Risk Estimates

To assess the real time performance of our model we first graphed the sequence of term structures of VaR estimates for the portfolios under consideration. For each month, \( t \), running from 1980:1 to 1999:12, we thus estimated the parameters of the regime switching model recursively by MLE to obtain estimates, \( \hat{\theta}_t \). We then plot the real-time VaR estimates conditional on these estimates, \( \text{VaR}^{\alpha}_{t,t+h}(\hat{\theta}_t) \).

Results from this pseudo real-time analysis are shown in Figure 7. For simplicity we only consider VaR estimates. There are very strong variations over time in the VaR term structures. Most of the time, VaR term structures are upward sloping for values of \( h \) shorter than 24 months. However, the steepness of the VaR curves varies significantly over time going from very steep in the early 1980s to somewhat flatter in the early-to-mid nineties. VaR levels also vary significantly and are far higher in the eighties than in the nineties, reflecting the bull market in the latter period which led to higher estimated mean returns and a decline in volatility levels.

Once again, non-monotonicities in VaR term structures emerge from the real-time estimates, particularly towards the end of the sample. Interestingly, although changes to the conditional term-structure patterns mostly evolve gradually over time, on some occasions they can also change quite quickly over the course of a few months.

4.1 Econometric Tests of out-of-sample Predictions

Econometric specifications used in risk management are best judged by their out-of-sample performance. This provides an appropriate method to control for overfitting, which could be a concern for the four-state regime switching model which requires the estimation of 32 parameters.

To assess the out-of-sample performance of the forecasting model we undertook a range of econometric tests, using 1980-1999 as our out-of-sample period. The method for evaluation of interval forecasts proposed by Christoffersen (1998) is ideal for evaluating the VaR forecasts.
The test is based on an indicator variable, $I_{t:t+h}$, defined as follows:

$$I_{t:t+h} = \begin{cases} 
1 & \text{if } R_{t:t+h} < VaR_{t:t+h}^α(\hat{\theta}_t) \\
0 & \text{otherwise} 
\end{cases}$$

Following Christoffersen, the sequence of VaR forecasts $\{VaR_{t:t+h}^α(\hat{\theta}_t)\}^T_{t=1}$ are said to be efficient with respect to information set $\mathcal{F}_t$ if

$$E[I_{t:t+h}|\mathcal{F}_t] = α \text{ for all } t \text{ and } h.$$ 

For the one-period horizon ($h = 1$) Christoffersen shows that testing $E[I_{t:t+h}|\mathcal{F}_t] = E[I_{t:t+h}|I_{t-h,t}, I_{t-h-1,t-1}, ...]$ is equivalent to testing that the sequence $\{I_{t:t+h}\}$ is identically and independently distributed Bernoulli with parameter $α$, $\{I_{t:t+h}\} \overset{iid}{\sim} Bern(α)$. This result readily extends to $h > 1$ provided that non-overlapping $h$-period return data is used.

The sequence of VaR forecasts $\{VaR_{t:t+h}^α(\hat{\theta}_t)\}^T_{t=1}$ therefore has correct conditional coverage if $\{I_{t:t+h}\} \overset{iid}{\sim} Bern(α)$ for all $t$. Under the null hypothesis, the likelihood function of the sequence of indicator variables is therefore

$$L(α; I_1, ..., I_T) = (1 − α)^{n_0} α^{n_1},$$

where $n_0$ is the number of cases where $I_{t:t+h} = 0$, and $n_1 = n − n_0$. Under an alternative, $Π_1$, of first-order Markovian dependence in the indicator function, defining $n_{ij}$ as the number of observations where value $i$ was followed by value $j$, the likelihood function is

$$L(Π_1; I_1, ..., I_T) = (1 − π_{01})^{n_{00}} π_{01}^{n_{01}} (1 − π_{11})^{n_{10}} π_{11}^{n_{11}}.$$ 

This requires estimating two parameters (since the probabilities sum to one across columns).

The likelihood ratio test of correct coverage and independence is

$$LR = -2 \log \left[ L(α; I_1, ..., I_T)/L(Π_1; I_1, ..., I_T) \right] \overset{asy}{\sim} χ^2_2.$$ 

(22)

To test for misspecification in the expected shortfall, we investigate whether the difference between the realized and expected shortfall is itself predictable. Conditional on the VaR threshold being exceeded we thus test whether the following holds:

$$E[(R_{t:t+h} - ES_{t:t+h}|R_{t:t+h} < VaR_{t:t+h})|\mathcal{F}_t] = 0.$$
We test for serial correlation in the innovation to the shortfall.

We also consider how far ahead in time VaR and expected shortfall are predictable by reporting p-values for runs tests, using the non-parametric methodology proposed in Christoffersen and Diebold (2000). The chief advantage of this method is that it is not dependent on the model specification at hand. Define a run as a string of consecutive zeros or ones in the sequence $\{I_{t:t+h}\}_{t=1}^{T-h}$. Let $r \geq 1$ be the total number of runs. It can be shown that under the null that $\{I_{t:t+h}\}_{t=1}^{T-h}$ is a random sequence the (discrete) exact distribution of $r$, given $n_1$ and $n_0$ is:

\[
Pr(r|n_0, n_1) = \frac{f_r}{n} \quad r = 1, 2, ..., R
\]

\[
f_{r=2s+j} = \begin{cases} 
2\binom{n_0-1}{s-1}\binom{n_1-1}{s-1} & \text{if } j = 0 \\
R_{r=2s-1}^{n-2s} & \text{if } j = 0 \\
2\min\{n_0, n_1\} & \text{if } n_0 = n_1 \\
2\min\{n_0, n_1\} + 1 & \text{otherwise}
\end{cases}
\]

\[
R = \begin{cases} 
2\min\{n_0, n_1\} & \text{if } n_0 = n_1 \\
2\min\{n_0, n_1\} + 1 & \text{otherwise}
\end{cases}
\]

Empirical results for the tests are reported in Table 3, in all cases based on non-overlapping data. They show that the unconditional coverage for the regime switching model is not too far from the 1% level in most cases. In general it hovers around 0-2%, rising only to 8% for bonds and T-bills at the 24 month horizon. Using the $S$-statistic suggested by Christoffersen and Diebold (1998) on the non-overlapping data, there is generally only weak evidence against the regime switching model at the longer horizons, although the evidence is stronger at the shortest horizons.

For the IID Gaussian model the results in shown in Table 4 reveal that the nominal coverage is generally quite good at short horizons, $h \leq 6$. However, as the horizon grows beyond six months, the unconditional coverage probability grows far above its theoretical value of 1%, leading to rejections of this model based on a simple t-test for the intercept in a regression that has $I_t - \alpha$ as the dependent variable. This is consistent with the plot in Figure 4 which showed that the difference between the 1% VaR of the four-state model and that of the Gaussian IID model grows as the horizon increases and it shows that the

---

4 For instance the sequence \{0,0,1,0,1,1,0,0,1,0\} contains 7 runs.
Gaussian IID model systematically underestimates portfolio risk at long horizons.

5 Conclusion

We considered the effect of state probabilities and investment horizon on standard measures of risk such as Value at Risk and expected shortfall using a four-state mixture model driven by a first-order homogenous Markov chain. Some of the states of this model - notably the first state - clearly identify outliers in the asset return distribution so this model is well suited to capture tail behavior in returns. Indeed, when compared to standard parametric models such as the Gaussian and GARCH(1,1) model, the regime switching model generally implied higher VaR estimates and expected shortfall.

Interestingly, our out-of-sample forecasting results suggested that the relative performance of the regime switching model is better at long horizons between 6 months and two years, while the nominal coverage probability of the simpler IID Gaussian model is not too far off at shorter horizons. This suggests the importance of accounting for regime switching dynamics in calculations of risk for strategic asset allocation problems that often assume a long investment horizon.

Our analysis concentrated on three relatively simple and popular models for return distributions. This had the advantage that it allowed us to contrast the four-state regime switching model to two commonly used benchmarks. It is of course possible to extend these models by considering other distributional assumptions. For example, the regime switching model could be extended by assuming mixtures of Gaussian and fat-tailed distributions such as student-t distributions, c.f. Perez-Quiros and Timmermann (2001). Although this would make the model even more flexible, it would introduce further parameters to be estimated so it is not certain that this will improve the model. It seems more important to have a model such as the one considered here that allows for outliers and lets the outlier probability vary over time.
References


Selection of Regime Switching Model for Excess Stock and (Long-Term) Bond Returns

The table reports estimates for the multivariate Markov switching conditionally heteroskedastic VAR model:

\[ r_t = \mu_S + \sum_{j=1}^{p} A_{j,S} r_{t-j} + \varepsilon_t \]

where \( r_t = [r_t^v, r_t^b]' \) is the vector collecting continuously compounded, monthly returns on the value-weighted NYSE index and on a CRSP portfolio of long-term government bonds. \( \varepsilon_t = [\varepsilon_{t1}, \varepsilon_{t2}]' \sim N(0, \Omega) \). The unobserved state variable \( S_t \) is governed by a first-order Markov chain that can assume \( k \) distinct values. \( p \) autoregressive terms are considered. Excess returns are the difference between portfolio returns on 30-days T-bills. The sample period is 1954:01 – 1999:12. MSIAH\((k,p)\) stands for Markov Switching Intercept Autoregressive Heteroskedasticity model with \( k \) states and \( p \) autoregressive lags. Linearity tests are performed using Davies’ (1977) upper bounds for the critical values from standard likelihood ratio tests, to account for the presence of parameters unidentified under the null hypothesis.

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<th><strong>Model</strong> ((k,p))</th>
<th><strong>Number of parameters</strong></th>
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<td>-8.562</td>
</tr>
<tr>
<td>MSIAH(9,0)</td>
<td>117</td>
<td>2494.42</td>
<td>367.9865 ((0.000))</td>
<td>-8.257</td>
</tr>
</tbody>
</table>
Table 2

Estimates of a Four-State Regime Switching Model for Excess Stock and (Long-Term) Bond Returns

The table shows estimation results for the four-state vector regime switching model:

\[ r_t = \mu_{s_t} + \varepsilon_t \]

where \( r_t = [r^s_t, r^b_t]' \) is the vector collecting continuously compounded, monthly returns on the value-weighted NYSE index and on a CRSP portfolio of long-term government bonds. \( \mu_{s_t} \) is the intercept vector in state \( s_t \), \( A_{p_t} \) is the matrix of autoregressive coefficients associated with lag \( j \geq 1 \) in state \( s_t \) and \( \varepsilon_t = [\varepsilon_{1t}, \varepsilon_{2t}]' \sim N(0, \Omega_{s_t}) \). The unobserved state variable \( S_t \) is governed by a first-order Markov chain that can assume \( k \) distinct values. \( p \) autoregressive terms are considered. The data are monthly and obtained from the CRSP tapes. Excess returns are the difference between portfolio returns and yields on 30-days T-bills. The sample period is 1954:01 – 1999:12.

<table>
<thead>
<tr>
<th>Panel A – Single State Model</th>
<th>Value-weighted stocks</th>
<th>Long-term bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Mean excess return</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value-weighted stocks</td>
<td>0.007 (0.002)</td>
<td>0.001 (0.001)</td>
</tr>
<tr>
<td>Long-term bonds</td>
<td>0.147***</td>
<td>0.075***</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B – Two State Model</th>
<th>Value-weighted stocks</th>
<th>Long-term bonds</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Mean excess return</td>
<td></td>
<td></td>
</tr>
<tr>
<td>State 1</td>
<td>-0.085 (0.020)</td>
<td>-0.001 (0.008)</td>
</tr>
<tr>
<td>State 2</td>
<td>0.009 (0.002)</td>
<td>-0.000 (0.001)</td>
</tr>
<tr>
<td>State 3</td>
<td>0.013 (0.005)</td>
<td>0.000 (0.001)</td>
</tr>
<tr>
<td>State 4</td>
<td>0.010 (0.005)</td>
<td>0.004 (0.003)</td>
</tr>
<tr>
<td>2. Correlations/Volatilities</td>
<td></td>
<td></td>
</tr>
<tr>
<td>State 1:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value-weighted stocks</td>
<td>0.187***</td>
<td></td>
</tr>
<tr>
<td>Long-term bonds</td>
<td>-0.851***</td>
<td>0.084***</td>
</tr>
<tr>
<td>State 2:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value-weighted stocks</td>
<td>0.124***</td>
<td></td>
</tr>
<tr>
<td>Long-term bonds</td>
<td>0.200**</td>
<td>0.057***</td>
</tr>
<tr>
<td>State 3:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value-weighted stocks</td>
<td>0.100***</td>
<td></td>
</tr>
<tr>
<td>Long-term bonds</td>
<td>-0.029</td>
<td>0.011**</td>
</tr>
<tr>
<td>State 4:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Value-weighted stocks</td>
<td>0.166***</td>
<td></td>
</tr>
<tr>
<td>Long-term bonds</td>
<td>0.443*</td>
<td>0.116***</td>
</tr>
<tr>
<td>3. Transition probabilities</td>
<td></td>
<td></td>
</tr>
<tr>
<td>State 1</td>
<td>0.494**</td>
<td>0.022</td>
</tr>
<tr>
<td>State 2</td>
<td>0.018*</td>
<td>0.977***</td>
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<tr>
<td>State 3</td>
<td>0.000</td>
<td>0.027</td>
</tr>
<tr>
<td>State 4</td>
<td>0.015</td>
<td>0.056**</td>
</tr>
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</table>

* denotes 10% significance, ** significance at 5%, *** significance at 1%.
Table 3
Out-of-Sample Tests of Predictive Accuracy for Value at Risk under Regime Switching

The table reports the results of out-of-sample tests concerning the 1% Value at Risk for multi-period portfolio returns calculated from the four-state vector regime switching model:

\[ r_t = \mu_s + \varepsilon_t \]

where \( r_t = [r_t^1, r_t^2] \) collects monthly returns on the value-weighted NYSE index and on a CRSP portfolio of long-term government bonds, and \( \varepsilon_t = [\varepsilon_t^1, \varepsilon_t^2] \sim N(0, \Omega_s) \). The tests/predictive accuracy measures consist of: the unconditional coverage probability; Christoffersen and Diebold’s (2000) \( \psi \) test capturing deviations from iid-ness (persistence) of an indicator variable \( I_t \) that records portfolio returns below the VaR; nonparametric runs tests that assess the effective degree of randomness of \( I_t \); regression tests in which \( I_t \) is regressed on four lags. Tests are performed for a variety of portfolios and horizons. We perform calculations using non-overlapping portfolio returns.

<table>
<thead>
<tr>
<th></th>
<th>h=1</th>
<th>h=2</th>
<th>h=4</th>
<th>h=6</th>
<th>h=12</th>
<th>h=18</th>
<th>h=24</th>
</tr>
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<tbody>
<tr>
<td><strong>50% Bonds + 50% T-bills</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Unconditional coverage prob.</td>
<td>0.017</td>
<td>0.021</td>
<td>0.013</td>
<td>0.021</td>
<td>0.024</td>
<td>0.036</td>
<td>0.079</td>
</tr>
<tr>
<td>( \psi ) statistic test – p-value</td>
<td>0.582</td>
<td>0.339</td>
<td>NA†</td>
<td>NA†</td>
<td>0.374</td>
<td>NA†</td>
<td>0.163</td>
</tr>
<tr>
<td>Nonparametric runs test – runs</td>
<td>9</td>
<td>7</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Nonparametric runs test – p-value</td>
<td>0.000</td>
<td>0.008</td>
<td>NA†</td>
<td>0.001</td>
<td>0.050</td>
<td>NA†</td>
<td>0.097</td>
</tr>
<tr>
<td>Regression F-test on ( I_t ) – p-value</td>
<td>0.000</td>
<td>0.019</td>
<td>NA†</td>
<td>1.000</td>
<td>0.995</td>
<td>NA†</td>
<td>0.685</td>
</tr>
<tr>
<td>Regression const. t-test – p-value</td>
<td>0.305</td>
<td>0.209</td>
<td>NA†</td>
<td>1.000</td>
<td>0.803</td>
<td>NA†</td>
<td>0.183</td>
</tr>
<tr>
<td><strong>100% Bonds</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
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<td>NA†</td>
<td>0.163</td>
</tr>
<tr>
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<td>7</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Nonparametric runs test – p-value</td>
<td>0.000</td>
<td>0.008</td>
<td>NA†</td>
<td>0.001</td>
<td>0.050</td>
<td>NA†</td>
<td>0.097</td>
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<td>NA†</td>
<td>1.000</td>
<td>0.995</td>
<td>NA†</td>
<td>0.685</td>
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<td>0.305</td>
<td>0.209</td>
<td>NA†</td>
<td>1.000</td>
<td>0.803</td>
<td>NA†</td>
<td>0.183</td>
</tr>
<tr>
<td><strong>50% Stocks + 50% T-bills</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Unconditional coverage prob.</td>
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<td>0.008</td>
<td>0.017</td>
<td>0.004</td>
<td>0.000</td>
<td>0.009</td>
<td>0.000</td>
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<tr>
<td>( \psi ) statistic – p-value</td>
<td>0.884</td>
<td>0.967</td>
<td>0.857</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
</tr>
<tr>
<td>Nonparametric runs test – runs</td>
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<td>3</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Nonparametric runs test – p-value</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
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<tr>
<td>Regression F-test on ( I_t ) – p-value</td>
<td>0.000</td>
<td>1.000</td>
<td>1.000</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
</tr>
<tr>
<td>Regression const. t-test – p-value</td>
<td>0.303</td>
<td>0.561</td>
<td>1.000</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
</tr>
<tr>
<td><strong>50% Stocks + 50% Bonds</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
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<td>0.013</td>
<td>0.017</td>
<td>0.013</td>
<td>0.004</td>
<td>0.013</td>
<td>0.018</td>
<td>0.023</td>
</tr>
<tr>
<td>( \psi ) statistic – p-value</td>
<td>0.884</td>
<td>0.754</td>
<td>0.857</td>
<td>NA†</td>
<td>NA†</td>
<td>0.253</td>
<td>NA†</td>
</tr>
<tr>
<td>Nonparametric runs test – runs</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Nonparametric runs test – p-value</td>
<td>0.000</td>
<td>0.008</td>
<td>0.017</td>
<td>0.001</td>
<td>NA†</td>
<td>0.071</td>
<td>NA†</td>
</tr>
<tr>
<td>Regression F-test on ( I_t ) – p-value</td>
<td>0.998</td>
<td>0.997</td>
<td>1.000</td>
<td>1.000</td>
<td>NA†</td>
<td>1.000</td>
<td>NA†</td>
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<tr>
<td>Regression const. t-test – p-value</td>
<td>0.356</td>
<td>0.264</td>
<td>1.000</td>
<td>1.000</td>
<td>NA†</td>
<td>1.000</td>
<td>NA†</td>
</tr>
<tr>
<td><strong>100% Stocks</strong></td>
<td></td>
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<tr>
<td>Unconditional coverage prob.</td>
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<td>0.013</td>
<td>0.017</td>
<td>0.004</td>
<td>0.013</td>
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<td>( \psi ) statistic – p-value</td>
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<td>0.754</td>
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</tr>
<tr>
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<td>5</td>
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<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
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<tr>
<td>Regression F-test on ( I_t ) – p-value</td>
<td>0.000</td>
<td>0.997</td>
<td>1.000</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
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<tr>
<td>Regression const. t-test – p-value</td>
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<td>0.264</td>
<td>1.000</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
</tr>
</tbody>
</table>

† = No multiperiod portfolio returns are below the VaR measure in the non-overlapping sample.
Table 4

Out-of-Sample Tests of Predictive Accuracy for Value at Risk under IID Gaussian Returns

The table reports the results of out-of-sample tests concerning the 1% Value at Risk for multi-period portfolio returns calculated from the single-state vector model:

\[ r_i = \mu + \varepsilon_i \]

where \( r_i = [r_i^T, r_i^b]^T \) collects monthly returns on the value-weighted NYSE index and on a CRSP portfolio of long-term government bonds, and \( \varepsilon_i = [\varepsilon_i^T, \varepsilon_i^b]^T \sim N(0, \Omega) \). The tests/predictive accuracy measures consist of: the unconditional coverage probability; Christoffersen and Diebold’s (2000) S test capturing deviations from iid-ness (persistence) of an indicator variable \( I_t \) that records portfolio returns below the VaR; nonparametric runs tests that assess the effective degree of randomness of \( I_t \); regression tests in which \( I_t \) is regressed on four lags. Tests are performed for a variety of portfolios and horizons. We perform calculations using non-overlapping portfolio returns.

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<tbody>
<tr>
<td><strong>50% Bonds + 50% T-bills</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Unconditional coverage prob.</td>
<td>0.021</td>
<td>0.021</td>
<td>0.021</td>
<td>0.021</td>
<td>0.070</td>
<td>0.081</td>
<td>0.190</td>
</tr>
<tr>
<td>$S$ statistic test – p-value</td>
<td>0.294</td>
<td>0.754</td>
<td>0.857</td>
<td>0.667</td>
<td>0.034</td>
<td>0.025</td>
<td>0.004</td>
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<tr>
<td>Nonparametric runs test – runs</td>
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<td>5</td>
<td>3</td>
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<td></td>
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<tr>
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<td>0.017</td>
<td>0.025</td>
<td>0.041</td>
<td>0.071</td>
<td>0.060</td>
</tr>
<tr>
<td>Regression F-test on $I_t$ – p-value</td>
<td>0.002</td>
<td>0.997</td>
<td>0.999</td>
<td>0.998</td>
<td>0.812</td>
<td>0.989</td>
<td>0.685</td>
</tr>
<tr>
<td>Regression const. t-test – p-value</td>
<td>0.202</td>
<td>0.264</td>
<td>0.322</td>
<td>0.253</td>
<td>0.049</td>
<td>0.085</td>
<td>0.018</td>
</tr>
<tr>
<td><strong>100% Bonds</strong></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>0.021</td>
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<td></td>
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</tr>
<tr>
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<td>0.025</td>
<td>0.021</td>
<td>0.026</td>
<td>0.022</td>
<td>0.026</td>
<td>0.033</td>
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<td>0.109</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>0.025</td>
<td>NA†</td>
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<td>3</td>
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<tr>
<td>Nonparametric runs test – p-value</td>
<td>0.000</td>
<td>0.001</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>0.071</td>
<td>NA†</td>
</tr>
<tr>
<td>Regression F-test on $I_t$ – p-value</td>
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<td>0.967</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>1.000</td>
<td>NA†</td>
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<td>Regression const. t-test – p-value</td>
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<td>0.064</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>0.084</td>
<td>NA†</td>
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<tr>
<td><strong>50% Stocks + 50% Bonds</strong></td>
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<td></td>
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<td>0.025</td>
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<td>0.048</td>
<td>0.059</td>
<td>0.064</td>
</tr>
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<td>0.339</td>
<td>0.857</td>
<td>NA†</td>
<td>0.681</td>
<td>0.755</td>
<td>0.713</td>
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<td>Nonparametric runs test – runs</td>
<td>5</td>
<td>7</td>
<td>3</td>
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<tr>
<td>Nonparametric runs test – p-value</td>
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<td>0.008</td>
<td>0.017</td>
<td>NA†</td>
<td>0.050</td>
<td>0.071</td>
<td>0.097</td>
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<tr>
<td>Regression F-test on $I_t$ – p-value</td>
<td>1.000</td>
<td>0.987</td>
<td>0.999</td>
<td>NA†</td>
<td>0.995</td>
<td>1.000</td>
<td>0.685</td>
</tr>
<tr>
<td>Regression const. t-test – p-value</td>
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<td>0.125</td>
<td>0.322</td>
<td>NA†</td>
<td>0.803</td>
<td>0.000</td>
<td>0.018</td>
</tr>
<tr>
<td><strong>100% Stocks</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Unconditional coverage prob.</td>
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<td>0.025</td>
<td>0.021</td>
<td>0.026</td>
<td>0.022</td>
<td>0.026</td>
<td>0.033</td>
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<td>$S$ statistic – p-value</td>
<td>0.942</td>
<td>0.109</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>0.025</td>
<td>NA†</td>
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<tr>
<td>Nonparametric runs test – runs</td>
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<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>Nonparametric runs test – p-value</td>
<td>0.000</td>
<td>0.001</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>0.071</td>
<td>NA†</td>
</tr>
<tr>
<td>Regression F-test on $I_t$ – p-value</td>
<td>1.000</td>
<td>0.967</td>
<td>NA†</td>
<td>NA†</td>
<td>NA†</td>
<td>1.000</td>
<td>NA†</td>
</tr>
<tr>
<td>Regression const. t-test – p-value</td>
<td>0.606</td>
<td>0.125</td>
<td>0.322</td>
<td>NA†</td>
<td>0.803</td>
<td>0.000</td>
<td>0.018</td>
</tr>
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</table>

† = No multiperiod portfolio returns are below the VaR measure in the non-overlapping sample.
Figure 1

Smoothed State Probabilities: Four-State Model for Excess Stock and (Long-Term) Bond Returns

The graphs plot the smoothed probabilities of regimes 1-4 for the multivariate Markov Switching model describing excess returns on the continuously compounded, monthly returns on the value-weighted NYSE index and on a CRSP portfolio of long-term government bonds.
Figure 2

1% VaR under Regime Switching – Effects of the Horizon and of State Beliefs
The graphs plot the (negative of the) first percentile of the distribution of returns for various portfolios comprising
stocks, bonds, and 1-month T-bills as a function of the investment horizon and of probability beliefs concerning the
current state.
Figure 3

1% VaR under Regime Switching – Long-Horizon Effects

The graphs plot the (negative of the) first percentile of various portfolios comprising stocks, bonds, and 1-month T-bills as a function of the investment horizon and of probability beliefs concerning the current state.
Figure 4

1% VaR – Comparing Regime Switching and Benchmarks

The graphs plot the (negative of the) first percentile of various portfolios comprising stocks, bonds, and 1-month T-bills as a function of the investment horizon. The regime-switching VaR is calculated under the assumption the current vector of state probabilities correspond to the vector of ergodic (long-run) probabilities ([0.031 0.667 0.07 0.232]'), while the Gaussian VaR is calculated under the assumption that:

$$r_t = \mu^* + \epsilon_t^*$$

where $\epsilon_t^* = [\epsilon_{1t}^* \epsilon_{2t}^*] \sim N(0, \Omega^*)$, with mean vector matching the ergodic means from the four state model ($\mu^* = \sum_{j=1}^{4} \pi_j^{\text{erg}} \mu_j$), and covariance matrix matching the ergodic covariance under regime switching (see equation in the text).
Figure 5

1% Expected Shortfall – Effects of the Horizon and of State Beliefs

The graphs plot the 1% shortfall for various portfolios comprising stocks, bonds, and 1-month T-bills as a function of the investment horizon and of probability beliefs concerning the current state.
1% Expected Shortfall – Comparing Regime Switching and Benchmarks

The graphs plot the 1% shortfall of various portfolios comprising stocks, bonds, and 1-month T-bills as a function of the investment horizon. The regime-switching VaR is calculated under the assumption the current vector of state probabilities correspond to the vector of ergodic (long-run) probabilities \((0.031 \ 0.667 \ 0.07 \ 0.232)\), while the Gaussian VaR is calculated under the assumption that:

\[
\begin{align*}
    r_t &= \mu^* + \varepsilon_t^* \\
    \varepsilon_t^* &= [\varepsilon_{1t}^* \varepsilon_{2t}^*]^{\text{IID} \ N(0, \Omega^*)},
\end{align*}
\]

where \(\varepsilon_t^* = [\varepsilon_{1t}^* \varepsilon_{2t}^*]^{\text{IID} \ N(0, \Omega^*)}\), with mean vector matching the ergodic means from the four state model \((\mu^* = \sum_{j=1}^{4} \pi^e_j \mu_j)\), and covariance matrix matching the ergodic covariance under regime switching (see equation in the text).
Figure 7

Real Time 1% VaR Under Regime Switching

The graphs show the 1% VaR for 5 different portfolios calculated under the assumption that an investor recursively estimates a four state regime switching models and employs the resulting parameter estimates and smoothed probabilities to forecast the density of cumulative portfolio returns.