Indeterminacy in the free-trade world

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Abstract

We show that indeterminacy arises in a discrete-time competitive two-country dynamic model of international trade in which externalities, imperfect competition, public goods, and government intervention are assumed away. The present model is a standard dynamic trade model in the sense that there is neither an international credit market nor international factor mobility, and these intrinsic features are a source of indeterminacy. Indeterminacy is implied by the condition for the existence of a steady state.

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1 Introduction

Recently, researchers majoring in economic dynamics have shown a particular interest in indeterminacy which is defined as the existence of a continuum of dynamic equilibrium paths starting from historically given initial conditions of state variables and converging to a common steady state: It is indeterminate in a decentralized market economy as to which equilibrium path is realized. Kazuo Nishimura is a leading academic economist who has made considerable contributions to this topic: joint research with Jess Benhabib and Qinglai Meng (Benhabib and Nishimura (1998) and Benhabib, Meng and Nishimura (2000))
have made ground-breaking contributions to indeterminacy in multi-sector dynamic general equilibrium frameworks of exogenous and endogenous growth in which returns to scale are socially constant but privately decreasing.\(^2\)

His other notable contribution to this topic is that while almost all contributions to the literature on indeterminacy have assumed a closed economy, his joint work with Shimomura (Nishimura and Shimomura (2002b)) is the first to study indeterminacy in the standard two-country dynamic Heckscher-Ohlin model in which, using a standard trade-theory term, \textit{factor-generated external economies of scale} are assumed. As Nishimura and Shimomura (2002b) discuss, if indeterminacy takes place, we cannot explain the pattern of international trade based on the initial international distribution of capital stocks. That is, indeterminacy may violate the long-run trade pattern predicted by the dynamic Heckscher-Ohlin theorem (Chen (1992)); a dynamic version of a fundamental theorem in trade theory.

Nishimura and Shimomura (2002b) followed the preceding literature on indeterminacy in assuming factor-generated externalities. Later, Nishimura and Shimomura (2005) found that, even if not only such externalities but also other market-distortional factors like imperfect competition, public goods and trade policies are assumed away, indeterminacy is still possible in a dynamic continuous-time, free-trade, and two-country model based on the standard assumptions in trade theory such that (i) two commodities, a pure consumption good and a consumable capital, are produced by using capital and labor under constant-returns-to-scale technologies, (ii) while commodities are freely traded among countries, factors of production are internationally immobile and (iii) there is no international credit market.\(^3\) Shimomura (2004) also obtains an indeterminacy result in a continuous-time dynamic two-country model in which there are two tradable goods, durable and non-durable consumption goods, both of which are produced by using constant primary factors of production.

These two papers share a problem that needs to be resolved; to guarantee the indeterminacy result, the sum of the rate of capital depreciation and the rate of time preference must be \textit{exactly} equal between two countries; moreover, the domestic rate of time preference must be internationally different. However small the international difference in the sum may be, the indeterminacy result is not generally guaranteed. Such a restrictive condition is apparently undesirable.

In this paper, we construct a simple dynamic version of the Ricardian model of international trade and derive an indeterminacy result under some mild conditions on the model parameters. Moreover, we show that under the conditions the existence of a steady state is sufficient for indeterminacy. That is, if there exists a steady state in the dynamic trade model, there must be multiple steady states such that there is a continuum of equilibrium paths starting from the neighborhood of one of the steady states; the conditions for the existence of

\(^2\)See also Nishimura, Shimomura and Wang (2006) and Mino Nishimura, Shimomura and Wang (2005).

\(^3\)There is a huge body of literature in international macroeconomics in which there is an international credit market. On the other hand, it is usually assumed away in a dynamic trade model which focuses on trade patterns.
the steady state are less restrictive than the condition imposed in the above
two papers in the sense that the former conditions are no longer of “knife-edge”
type.

This paper is organized as follows. Section 2 sets up the model. Section 3
discusses under what conditions a steady state exists and shows that if a
steady state exists, two non-degenerate steady states exist. Section 4 shows
that there is a continuum of equilibrium paths starting from a stock of the
durable consumption good which is sufficiently close to one of the two steady
states. Section 5 makes some concluding remarks.

2 The Model

The two-country model in this paper is a specific version of the dynamic trade
model developed in Shimomura (1993). The trading world consists of two coun-
tries called India and the US, and two goods are traded between the countries.
Both goods are consumption goods, but one of them, say wheat, is a purely non-
durable good while the other, say computer, is a durable good with a constant
rate of depreciation; both goods are produced by using only labor with constant
labor coefficients, \((a_1, a_2)\) in the US and \((a_1^*, a_2^*)\) in India.

Let us begin with an explanation of optimal consumption plans of US and
Indian households.

2.1 Households

Each US household supplies \(L\) units of labor inelastically, while each Indian
household supplies \(L^*\) units of labor. The population of each country is nor-
malized to be unity.

Concerning preferences, we assume, for simplicity, that the representative US
household derives utility from consuming wheat \(c\) and operating computers \(B\),
while the representative household in India derives utility only from consuming
wheat, say \(c^*\).

2.1.1 The US household

We assume that the US felicity function, \(u_t = U(c_t, B_t)\), is increasing and con-
cave in \(c_t\) and \(B_t\), and satisfies the Inada conditions. The objective of the US
household is to maximize the discounted sum of utility:

\[
\max_{u_t, S_t} \sum_{t=0}^{\infty} u_t \rho^t \quad \text{subject to} \quad \sum_{t=0}^{\infty} u_t \rho^t
\]

\[
S_{t+1} = (1 + r_t)S_t + w_t L - E(p_t, R_t, u_t), \quad S_0; \text{ given,}
\]
where $S_t$, the asset owned by a household at period $t$, consists of the stock of computers and the net credit $F_t$ owned by the US household; the labor endowment of the household, $L$, is positive and time-invariant; $r_t$ the interest rate, $R_t$ the rental rate of computers, $w_t$ the wage rate, and $p_t$ the price of wheat in terms of computers; $\rho$ the discount factor, which is positive, time-invariant, and in between 0 and 1. We assume that one unit of computer generates one unit of service flow during each period. So, $B_t$ is equal to the stock of computers: $S_t = B_t + F_t$. Computers serve as the numeraire.

$E(p_t, R_t, u_t)$ is called the period-$t$ expenditure function; we define it as

$$E(p_t, R_t, u_t) \equiv \min_{c_t, B_t} p_t c_t + R_t B_t$$

subject to $u_t \leq U(c_t, B_t)$.

The envelope properties of the expenditure function ensure that the partial derivatives of $E(p_t, R_t, u_t)$ with respect to $p_t$ and $R_t$, denoted as $E_{p_t}$ and $E_{R_t}$ respectively, are equal to the optimal solution to this minimization problem.

Let us solve the US household’s optimization problem formulated by (1) and (2). Associated with this problem is the Lagrangian

$$\Upsilon = \sum_{t=0}^{\infty} [u_t \rho^t + \lambda_t ((1 + r_t) S_t + w_t L - E(p_t, R_t, u_t) - S_{t+1})]$$

The first-order conditions are

$$\frac{\partial \Upsilon}{\partial u_t} = \rho^t - \lambda_t E_u(p_t, R_t, u_t) = 0 \quad (3)$$

$$\frac{\partial \Upsilon}{\partial S_{t+1}} = -\lambda_t + \lambda_{t+1} (1 + r_{t+1}) = 0, \quad (4)$$

where $E_u(p_t, R_t, u_t)$ denotes the partial derivative of $E(p_t, R_t, u_t)$ with respect to $u_t$. Since we assume a concave felicity function, the second-partial derivative of $E(.)$ with respect to $u$, $E_{uu}(.)$, is nonnegative. Thus, the second-order conditions are weakly satisfied.\(^4\)

Defining $\theta_t \equiv \lambda_t / \rho^t$, we can rewrite (3) and (4) as

$$0 = 1 - \theta_t E_u(p_t, R_t, u_t) \quad (5)$$

$$0 = -\theta_{t-1} + \rho \theta_t (1 + r_t) \quad (6)$$

For given time profiles $\{p_t\}_{t=0}^{\infty}$, $\{R_t\}_{t=0}^{\infty}$, $\{w_t\}_{t=0}^{\infty}$ and $\{r_t\}_{t=0}^{\infty}$, and initial condition $S_0$, if a value of $\theta_{-1}$ is chosen, the dynamical system which consists of (2), (5) and (6) determines the pair of time profiles $\{S_t\}_{t=0}^{\infty}$ and $\{u_t\}_{t=0}^{\infty}$.

\(^4\)As we shall see later, we specify the felicity function such that it is strictly concave in a relevant region. So, the second-order condition clearly holds.
The following lemma states under what conditions the pair is optimal from the viewpoint of the US household.

**LEMMA 1:** If the value $\theta_{-1}$ is chosen in such a way that the transversality condition
\[
\lim_{t \to \infty} S_{t+1} \theta_1 \rho^t = 0,
\]
holds for given $\{p_t\}_{t=0}^{\infty}$, $\{R_t\}_{t=0}^{\infty}$, and $S_0$, then the pair $\{\{S_t\}_{t=0}^{\infty}, \{u_t\}_{t=0}^{\infty}\}$ which is derived from the dynamical system (2), (5) and (6) is an optimal consumption plan of the US household.

**Proof:** The proof is available from the authors on request. (QED)

### 2.1.2 The Indian household

Let us turn to the representative household. He solves the following problem
\[
\max_{c^*} \sum_{t=0}^{\infty} U^*(c^*_t) \rho^*_t
\]
subject to
\[
F^*_{t+1} = (1 + r^*_t)F^*_t + w^*_tL^* - p_tC^*_t, \quad F^*_0 \text{ given,}
\]
where $r^*_t$ is the Indian interest rate. The discount factor $\rho^*_t$ is time-invariant and in between 0 and 1. His felicity depends only on the consumption of wheat, say $U^*(c^*_t)$. Therefore, his optimization problem is
\[
\max_{F^*_t} \sum_{t=0}^{\infty} U^*((1 + r^*_t)F^*_t + w^*_tL^* - F^*_{t+1})/p_t)\rho^*_t,
\]
where $L^*$ is the labor endowment of the Indian household and $w^*_t$ is the wage rate in India at period $t$. Assuming that the felicity function is increasing and strictly concave, we derive the first-order condition
\[
U'_t((1 + r^*_t)F^*_t + w^*_tL^* - F^*_{t+1})/p_t)/(1 + r^*_t) \rho^*_t/p_t = U'_t\left((1 + r^*_{t+1})F^*_{t+1} + w^*_tL^* - F^*_{t+2} \right)/p_t/(1 + r^*_{t+1}) \rho^*_t/p_{t+1} \quad (8)
\]
Like Lemma 1, we can prove that for given time profiles $\{p_t\}_{t=0}^{\infty}$, $\{w_t\}_{t=0}^{\infty}$ and $\{r_t\}_{t=0}^{\infty}$ and initial $F^*_0$, the pair of time profile $\{F^*_t\}_{t=0}^{\infty}$, $\{c^*_t\}_{t=0}^{\infty}$ satisfying the budget condition and (8) is an optimal plan for the Indian household, if the transversality condition
\[
\lim_{t \to \infty} F^*_{t+1} U'_t((1 + r^*_t)F^*_t + w^*_tL^* - F^*_{t+1})/p_t)\rho^*_t/p_t = 0, \quad (9)
\]
is satisfied.
2.2 Production side

We shall focus on the case in which labor coefficients satisfy the following inequality

\[
\frac{a_1}{a_2} < \frac{a_1^*}{a_2^*},
\]  

which means that the US has a comparative advantage in the production of wheat; under free trade the US exports wheat and India exports computers. (10) itself does not imply that both countries are completely specialized to the production of each of the two goods; either the US or India can produce both goods. In what follows, however, we assume that US produces only wheat, while India is completely specialized to the production of computers. This assumption is justified if \( p \) is determined in between \( \frac{a_1}{a_2} \) and \( \frac{a_1^*}{a_2^*} \).

These restrictive Ricardian assumptions, complete specialization in both countries and only one primary factor of production, are not necessary for indeterminacy to occur. As we outlined in Section 1, Nishimura and Shimomura (2005) show that there is an equilibrium characterized by indeterminacy in a two-country standard dynamic trade model in which there are two factors of production, physical capital and labor, and production is incompletely specialized in each country. Shimomura (2004) assumes multiple primary factors of production and incomplete specialization in one of the trading countries.

As we already stated in the introductory section, Nishimura and Shimomura (2005) made another restrictive assumption such that the sum of the constant rates of time preference and capital depreciation is the same between the two countries. On the other hand, our more recent works (Chen, Nishimura, and Shimomura 2005), Kikuchi and Shimomura (2006a, 2006b) introduce endogenous time preferences into a multi-country dynamic general equilibrium model to study how international differences in preferences, technologies and initial factor endowments may affect the pattern of international trade in the steady state where at least one of the trading countries is incompletely specialized. Tradable goods are produced by using a primary and time-invariant factor (labor) and a reproducible factor (capital). Thus, it is our research agenda to investigate under what conditions indeterminacy takes place in the new dynamic trade model.

Let us return to our main business in this paper. Under complete specialization of the production of each good, the world outputs of computers and wheat are written as \( Y^* \equiv L^*/a_2^* \) and \( Y \equiv L/a_1 \), respectively. We also have the two (price) = (average cost) conditions for each period, one for computer and the

\(^5\)As will be made clear in the subsequent sections, the steady-state \( p \) is independent of labor coefficients \( a_1 \) and \( a_2^* \). Hence, these coefficients can be chosen in such a way that \( p \) is in between the two ratios.
other for wheat,

\[ 1 = w_t^* a_2^* \]  
\[ p_t = w_t a_1 \]  

(11)  

(12)

It follows that the factor incomes of the US and India are

\[ w_t L = (p_t/a_1) L = p_t Y \]  
\[ w_t^* L^* = (1/a_2^*) L^* = Y^* \]  

(13)  

(14)

2.3 Market-clearing conditions

2.3.1 Credit and rental markets

We assume that there is no international credit market, while each country has its own domestic credit market. The market-clearing condition of the US credit market is \( F_t = 0 \), which means that

\[ S_t = B_t \]  

(15)

On the other hand, the market-clearing condition of the Indian credit market is \( F_t^* = 0 \). It follows from the Indian budget constraint and (11) that the Indian demand function for wheat is written as

\[ c_t^* = w_t^* L^* \]  
\[ p_t = L^* a_2^* \]  

(16)

and the Euler condition (8) becomes

\[ \frac{U_t^*(L^*/a_2^* p_t)}{U_t^*(L^*/a_2^* p_{t+1}) p_t} = \frac{(1 + r_t^*) p_t}{p_{t+1}} \]  

(17)

which determines the time profile of the equilibrium interest rate in India \( r_t^* \), given the time profiles of the price of wheat. (17) means that the marginal rate of substitution between the period-\( t \) wheat and the period-\( (t+1) \) wheat is equal to their present-value price ratio. Note that this equality implies that the interest rate \( r_{t+1}^* \) equals the rate of time preference in the steady state where all prices are kept constant over time. Since \( F_t^* = 0 \) for each period \( t \), the transversality condition becomes

\[ \lim_{t \to \infty} 0 \times U_t^*(1 + r_t^*) \times 0 + \frac{L^*}{a_2^* p_t} p_t = 0, \]

which holds as long as \( p_t \) converges to a positive constant.
Next, let us consider the computer rental market. The stock of computers at period $t$ is written as $B_t$. On the other hand, the demand for computer services in the rental market is $E_R(p_t, R_t, u_t)$. Thus, the market-clearing condition at period $t$ is

$$0 = E_R(p_t, R_t, u_t) - B_t \quad (18)$$

Moreover, we assume instantaneous arbitration between the stock of computers and net credit, i.e.,

$$1 + r_t = R_t + 1 - \delta \quad (19)$$

holds at each period $t$.

### 2.3.2 The international wheat market

Finally, let us consider the wheat market. The supply of wheat is $Y$. On the other hand, the US demand for wheat is $E_p(p_t, R_t, u_t)$. Since the Indian demand for wheat is, from (16),

$$c^*_t = Y^*/p_t,$$

the international market-clearing condition is

$$0 = Y^*/p_t + E_p(p_t, R_t, u_t) - Y \quad (20)$$

### 2.4 The dynamic two-country model of international trade

Putting together the US and Indian households’ consumption behavior and market-clearing conditions, we have the dynamic two-country model of international trade as follows.

\[
\begin{align*}
B_{t+1} &= (R_t + 1 - \delta)B_t + p_t Y - E(p_t, R_t, u_t) \\
\theta_t &= \frac{\theta_{t-1}}{\rho(R_t + 1 - \delta)} \quad (21) \\
0 &= 1 - \theta_t E_u(p_t, R_t, u_t) \quad (22) \\
0 &= E_R(p_t, R_t, u_t) - B_t \quad (23) \\
0 &= Y^*/p_t + E_p(p_t, R_t, u_t) - Y \quad (24)
\end{align*}
\]

The substitution of (22) into (23) yields

$$0 = 1 - \frac{\theta_{t-1} E_u(p_t, R_t, u_t)}{\rho(R_t + 1 - \delta)} \quad (26)$$

For a historically given $B_0$ and a chosen $\theta_{-1}$, (24), (25) and (26) determine $p_0, R_0$ and $u_0$. Substituting $B_0, \theta_{-1}, p_0, R_0$ and $u_0$ into (21) and (22), we derive $B_1$ and $\theta_0$. Repeating a parallel argument, we obtain a time profile $(B_t, \theta_{t-1}, p_t, R_t, u_t), t = 0, 1, 2, 3, ..., \text{which depends on what value is chosen for } \theta_{-1}$. Based on Lemma 1, we have the first proposition.
Proposition 1: If $\theta_{-1}$ is chosen in such a way that the time profile starting from $(B_0, \theta_{-1})$ converges to a steady state where all variables are time-invariant and make sense from an economic viewpoint, then the time profile is an equilibrium path.

3 The existence of a steady state

The steady state of the dynamical system (21)-(25) is a solution to the system of equations

\begin{align}
0 &= (R - \delta)B + pY - E(p, R, u) \\
1 &= \rho(R + 1 - \delta) \\
0 &= 1 - \theta E_u(p, R, u) \\
0 &= E_R(p, R, u) - B \\
0 &= Y^* - p[Y - E_p(p, R, u)]
\end{align}

The steady-state rental rate is uniquely determined as $\frac{1}{\rho} - 1 + \delta$. Combining (27) and (30), we obtain

$$0 = (R - \delta)E_R(p, R, u) + pY - E(p, R, u)$$

Since the expenditure function is linearly homogeneous in $p$ and $R$, the following identity holds

$$E(p, R, u) = pE_p(p, R, u) + RE_R(p, R, u)$$

It follows from (31) that (32) can be rewritten as

$$\delta E_R(p, R, u) = p[Y - E_p(p, R, u)] = Y^*$$

Let $z \equiv \bar{R}/p$, where $\bar{R} \equiv \frac{1}{\rho} - 1 + \delta$, the steady-state rental rate. Since $E(p, R, u)$ is linearly homogeneous in $p$ and $R$, its partial derivatives, $E_R(p, R, u)$ and $E_p(p, R, u)$, are homogeneous of degree zero. Therefore,

$$E_R(p, \bar{R}, u) = E_R(1, z, u) \quad \text{and} \quad E_p(p, \bar{R}, u) = E_p(1, z, u)$$

Hence, the following lemma is established from (33).

**Lemma 2:** Consider the system of equations

\begin{align}
E_R(1, z, u) &= Y^*/\delta \\
E_p(1, z, u) &= Y - (Y^*/\bar{R})z
\end{align}

If this system has a solution $(\bar{z}, \bar{u})$, the rest of steady-state variables is uniquely determined as

\begin{align}
p &= \bar{R}/\bar{z} \\
\theta &= 1/E_u(\bar{R}/\bar{z}, \bar{R}, \bar{u}) \\
B &= E_R(1, \bar{z}, \bar{u})
\end{align}
3.1 A mapping from the price ratio \((z)\) into itself

To derive conditions for the existence of a pair \((z, u)\) that satisfies (34) and (35), we construct a mapping as follows. See Figure 1. For example, take a price ratio \(z(= \bar{R}/p)\), say \(z_0\), in the interval \([0, \frac{Y^*}{\bar{R}}]\). \(HQJ\) is the graph of the line (35). Then, we derive the intersection of the horizontal line \(QMF\) and the vertical line \(ER = Y^*/\delta\). Denote the intersection by \(M\). The absolute value of the slope of the indifference curve crossing \(M\), \(sM's\) is equal to the price ratio \(\Theta(z_0)\). By repeating the same procedure for any other \(z\) in the interval \([0, \frac{Y^*}{\bar{R}}]\), we derive a continuous mapping from the price ratio into itself. We denote it by \(\Theta(z)\). As is clear from (34) and (35), the steady-state ratio \(\bar{R}/p\) is a fixed point of \(\Theta(z)\) in the interval \([0, \frac{Y^*}{\bar{R}}]\).

The existence of a fixed point depends on how we specify the felicity function \(u = U(c, B)\). For example, if it is strictly quasi-concave and homothetic, the absolute value of the slope of indifference curves monotonically decreases as we move down to the horizontal axis of coordinates along the vertical line \(B = Y^*/\delta\). In this case \(\Theta(z)\) is monotonically decreasing. If the slope is zero at the intersection of the vertical line \(ER = Y^*/\delta\) and the horizontal axis of coordinates, then \(0 = \Theta(\bar{R}Y/Y^*)\) and \(0 < \Theta(0)\) : the fixed point uniquely exists.

3.2 Multiple steady states

In this paper, we specify the felicity function as follows.

\[
U(c, B) = \begin{cases} 
\alpha \ln c + \beta \ln B - \gamma cB & \text{for } cB \leq \beta/\gamma, \ c \geq 0 \text{ and } B \geq 0 \\
(\alpha - \beta) \ln c + \beta[\ln(\beta/\gamma) - 1] & \text{for } cB > \beta/\gamma, \ c \geq 0 \text{ and } B \geq 0,
\end{cases}
\]  

(36)

where we assume \(\alpha > \beta > 0\). The felicity function is always increasing and strictly concave in \(c\) and \(B\) satisfying \(cB \leq \beta/\gamma\), and non-decreasing and concave in any positive \(c\) and \(B\). Since concavity implies quasi-concavity, the function satisfies the standard properties as a felicity function.\(^6\)

In contrast to the case of homothetic felicity functions, the mapping \(\Theta(z)\) constructed from this felicity function is not monotone. Let us explicitly derive the mapping from the felicity function. Along the vertical line \(B = Y^*/\delta\) the absolute value of the slope of each indifference curve is

\[
\frac{U_B}{U_c} = \begin{cases} 
\frac{|\beta - \gamma Y^*/\delta|c}{|\alpha - \gamma Y^*/\delta|Y^*/\delta}, & \text{if } c \leq \frac{\beta\delta}{\alpha\gamma} \\
0, & \text{if } c > \frac{\beta\delta}{\alpha\gamma}
\end{cases}
\]

\(^6\)See Doi, Iwasa, and Shimomura (2006), where the properties of the felicity function was studied in the context of the standard static consumer theory.
Substituting \( c = Y - (Y^*/\bar{R})z \), we explicitly derive the mapping as follows.

\[
\Theta(z) \equiv \begin{cases} 
\frac{-\frac{\alpha \delta}{Y^* \gamma} \{z - \frac{\beta \delta}{Y^* \gamma} \}}{\frac{\alpha \delta}{Y^* \gamma} \{z - \frac{\beta \delta}{Y^* \gamma} \}}, & \text{if } \frac{\bar{R}Y}{Y^*} (Y - \frac{\beta \delta}{Y^* \gamma}) < z \leq \frac{\bar{R}Y}{Y^*} \\
0, & \text{if } 0 \leq z \leq \frac{\bar{R}Y}{Y^*} (Y - \frac{\beta \delta}{Y^* \gamma})
\end{cases}
\]

where we assume

\[
\frac{\alpha \delta}{Y^* \gamma} > Y \geq \frac{\beta \delta}{Y^* \gamma} \tag{37}
\]

This assumption means that the graph of the mapping can be depicted as shown in Figure 2, where

\[
z_\alpha \equiv \frac{\bar{R}Y}{Y^*} (Y - \frac{\alpha \delta}{Y^* \gamma}) < 0, \quad z_\beta \equiv \frac{\bar{R}Y}{Y^*} (Y - \frac{\beta \delta}{Y^* \gamma}) \geq 0, \quad \bar{z} \equiv \frac{\bar{R}Y}{Y^*} > 0
\]

It is bell-shaped with

\[
0 = \Theta(z_\beta) = \Theta(\bar{z})
\]

Let us examine under what conditions the graph of \( \Theta(z) \) and the 45°-line have intersections, i.e., fixed points, like \( z_1(\varepsilon) \) and \( z_2(\varepsilon) \). First, suppose that \( Y \) is equal to \( \beta \delta \frac{Y^*}{Y^* \gamma} \), which is equivalent to \( z_\beta = 0 \). Then

\[
\Theta(z)|_{z_\beta=0} = -\frac{\frac{R}{\beta \delta} \{z - \frac{\beta \delta}{Y^* \gamma} \}}{\frac{\alpha \delta}{Y^* \gamma} \{z - \frac{\beta \delta}{Y^* \gamma} \}},
\]

the graph of which is depicted by the broken curve. Differentiating \( \Theta(z)|_{Y = \frac{\beta \delta}{Y^* \gamma}} \) with respect to \( z \) at \( z = 0 \), we have

\[
\frac{d}{dz} \Theta(z)|_{z_\beta=0} \bigg|_{z=0} = \frac{\rho \delta \beta}{\{1 - \rho (1 - \delta)\}(\alpha - \beta)}
\]

where we use the definition \( \bar{R} \equiv \frac{1}{\rho} - 1 + \delta \). Inspection of the graph of \( \Theta(z)|_{z_\beta=0} \) in Figure 2 ensures that if

\[
\frac{\rho \delta \beta}{\{1 - \rho (1 - \delta)\}(\alpha - \beta)} > 1, \tag{38}
\]

which is equivalent to

\[
\frac{1 - \rho + 2 \rho \delta}{1 - \rho + \rho \delta} > \frac{\alpha}{\beta}, \tag{39}
\]

then \( \Theta(z)|_{z_\beta=0} \) has two fixed points, 0 and \( z_0 > 0 \). It follows from the continuity of the roots of the algebraic equation with respect to parameters that at least as long as \( Y \) is greater than but sufficiently close to \( \frac{\beta \delta}{Y^* \gamma} \), then \( \Theta(z) \) has two positive fixed points, \( z_1(\varepsilon) \) and \( z_2(\varepsilon) \), both of which are in the interval \( (z_\beta, \bar{z}) \). We now obtain the lemma as follows.
Lemma 3: If (i) \( \alpha > \beta \), (ii) (39) holds and (iii) \( \varepsilon \equiv Y - \frac{\beta \delta}{\pi} \) is positive but sufficiently close to zero, then \( \Theta(z) = z \) has two positive real roots, say \( z_1(\varepsilon) \) and \( z_2(\varepsilon) \), such that
\[
z_{\beta} < z_1(\varepsilon) < z_2(\varepsilon) < \bar{z}
\]
and
\[
\lim_{\varepsilon \to 0} z_1(\varepsilon) = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} z_2(\varepsilon) > 0
\]

Once a steady-state \( z \) is obtained, \( c \) and \( u \) are uniquely determined by
\[
c = Y - \frac{\beta \delta}{\pi} \quad \text{and} \quad u = U(Y - \frac{\beta \delta}{\pi}, \frac{\beta \delta}{\pi})
\]
respectively. We arrive at the multiple steady-state results.

Proposition 2: Under the conditions on parameters (i), (ii) and (iii) stated in Lemma 2, there are two steady states, \( i = 1, 2 \).

\[
R_i = \bar{R}, \quad u_i = U(Y - \frac{\beta \delta}{\pi}, \frac{\beta \delta}{\pi}), \quad p_i = \frac{\bar{R}}{z_i(\varepsilon)}
\]
\[
\pi_i = \frac{1}{E R} (R_i / z_i(\varepsilon), \bar{R}, u_i), \quad B_i = E R (1, z_i(\varepsilon), u_i)
\]

4 Indeterminacy

Let us linearize the dynamical system (21), (22), (24), (25), and (26) around a steady state to check the number of characteristic roots which are in between 0 and 1. The characteristic equation is
\[
\Omega(x) \equiv \begin{vmatrix}
\hat{R} + 1 - \delta - x & 0 & -E_u & 0 & Y - E_p \\
0 & 1 - x & 0 & -\rho \theta & 0 \\
0 & -E_u & -\theta E_u & 0 & -\theta E_{up} \\
-1 & 0 & E_{Ru} & 0 & 0 \\
0 & 0 & pE_{pu} & pE_{pR} & pE_{pp} + E_p - Y
\end{vmatrix} = 0
\]
(40)

As a result of tedious calculations, we get the lemma.

Lemma 4: The characteristic equation is
\[
\Omega(x) = \frac{(x - (1 - \delta)) \theta}{p} \Gamma(x) = 0
\]

where
\[
\Gamma(x) \equiv (x - 1)\{\delta E_R (E_{uu}E_{RR} - E_{uR}^2) + E_{uR}^2 E_{RR} \} - x \rho E_u (E_u E_{RR} R - \delta E_{Ru} E_R)
\]
(41)

Proof: The proof is available from the authors on request. (QED)

According to Lemma 4, one characteristic root is \( 1 - \delta \), which is positive and smaller than one and the other root is the solution to \( \Gamma(x) = 0 \). Inspecting
the definition of $\Gamma(x)$, $E_{uu} > 0$, and $E_{RR} < 0$ imply that $\Gamma(0) > 0$. Thus, the other root is in between 0 and 1, if either $\Gamma(-1) < 0$ or $\Gamma(1) < 0$. However, we can prove that the felicity function (36) implies that $\Gamma(-1) > 0$.\footnote{The proof of $\Gamma(-1) > 0$ is available from the authors on request.} Therefore, we have the following lemma.

**Lemma 5:** The other characteristic root is in between $-1$ and 1, if and only if
\[
1 - \frac{\delta E_R E_{Ru}}{RE_u E_{RR}} < 0. \tag{42}
\]

A necessary condition for this inequality to hold is $E_{Ru} < 0$.

**Proof:** Substituting $z = 1$ into (43),
\[
\Gamma(1) = -\rho E_u (E_{RR} \bar{R} - \delta E_{Ru} E_R ) = -\rho E_u^2 E_{RR} \bar{R} \left[ 1 - \frac{\delta E_R E_{Ru}}{RE_u E_{RR}} \right]
\]
Since $E_{RR} < 0$, $\Gamma(1) < 0$ if and only if (42) holds. (QED)

Lemma 5 tells us that if the durable consumption good is a “normal good”, i.e., $E_{Ru} > 0$, then (42) does not hold. It follows that while one root is $(1 - \delta)$ which is in between $-1$ and 1 due to Lemma 4, the other root is outside the closed interval $[-1, 1]$. That is, the steady state is saddle-point stable.

Now let us examine under what conditions the inequality (42), i.e., an indeterminacy result, is established. Specifically, we shall show that the inequality (42) holds at the steady state which corresponds to the lower fixed point $z_1(\varepsilon)$ defined in Lemma 3. As is clear from Figure 2, the mapping $\Theta(z)$ cuts the $45^0$-line from below at the lower fixed point $z_1(\varepsilon)$ and from above at the larger fixed point $z_2(\varepsilon)$, which means that
\[
\frac{d\Theta(z)}{dz} \bigg|_{z=z_1(\varepsilon)} > 1 \text{ and } \frac{d\Theta(z)}{dz} \bigg|_{z=z_2(\varepsilon)} < 1
\]
Let us calculate the derivative $\frac{d\Theta(z)}{dz}$. Totally differentiating
\[
E_R(1, \Theta(z), u) = \frac{Y^*}{\delta}
\]
\[
E_p(1, \Theta(z), u) = Y - \frac{Y^*}{R} z
\]
with respect to $z$, $u$ and $\Theta(z)$, we see that
\[
E_{RR} d\Theta(z) + E_{Ru} du = 0
\]
\[
E_{pR} d\Theta(z) + E_{pu} du = -\frac{\delta}{R} E_R dz,
\]
from which we derive
\[
\frac{d\Theta(z)}{dz} = \frac{\delta E_{Ru} E_R}{R(E_{pu} E_{RR} - E_{pR} E_{Ru})}
\] (43)

Because of the linear homogeneity of \(E(p, R, u)\) and \(E_u(p, R, u)\) with respect to \(p\) and \(R\),
\[
0 = E_{pR}(1, z, u) + z E_{RR}(1, z, u)
\]
\[
E_u(1, z, u) = E_{up}(1, z, u) + z E_{uR}(1, z, u)
\]
has to hold. Taking these equalities into account, we can continue from (43) as follows.
\[
(43) = \frac{\delta E_{Ru} E_R}{R(E_{pu} E_{RR} + \Theta(z) E_{RR} E_{Ru})}
\]
\[
= \frac{\delta E_{Ru} E_R}{R E_{RR} E_u},
\]
which implies that
\[
\text{sign}[\frac{d\Theta(z)}{dz} - 1] = -\text{sign}[1 - \frac{\delta E_{Ru} E_R}{R E_{RR} E_u}]
\] (44)

It follows that at the lower fixed point \(z_1(\varepsilon)\) where \(\frac{d\Theta(z)}{dz} > 1\), the term \(1 - \frac{\delta E_{Ru} E_R}{R E_{RR} E_u}\) must be negative. It follows from Lemma 5 that \(\Gamma(1) < 0\); the two characteristic roots at the steady state are both in between 0 and 1.

Based on the foregoing analysis, we now derive the main result of this paper.

**Theorem:** Suppose that the parameters of the model satisfies the following inequalities
\[
1 - \rho + 2\rho \delta > 0 \quad \alpha \beta > 1 \quad \text{and} \quad \varepsilon \equiv Y - \frac{\beta \delta}{Y^*} > 0
\]
If \(\varepsilon\) is sufficiently small, then the dynamical system (21)-(25) has two steady states. One of them is a saddle, while the other is a sink. Thus, if the initial stock \(S_0(=B_0)\) is in a neighborhood of the latter steady state, then there is a continuum of equilibrium paths which converges to it and which equilibrium path is realized is indeterminate.

**Remark 1:** Note that the above restrictions imposed on parameters are just a sufficient condition for indeterminacy to hold. Under any condition that guarantees a bell-shaped \(\Theta(z)\) for \(z > 0\), the existence of a steady state implies multiple steady states and local indeterminacy around one of the steady states. In a word, the slope of indifference curves \(\Theta(z)\) is non-monotone along the vertical line \(B = Y^*/\delta\); if the slope is greater than one at the fixed point \(z = \Theta(z)\), then local indeterminacy takes place at that point.

One may think that the non-monotonicity is a severe restriction. Part of our future research agenda is to study to what extent we can relax it. However, it
can be noted that Nishimura and Shimomura (2005) derive local indeterminacy
under the following utility function

$$U(c, B) = \alpha(c + B) - \frac{\beta}{2}(c^2 + B^2) - \gamma cB, \quad \alpha > 0, \beta > \gamma > 0,$$

where the commodity space is restricted to the area in which both marginal
utilities are positive. We can check that the relationship between $c$ and the
slope of indifference curves is monotonic for this utility function. Thus, the
non-monotonicity is not a crucial condition for local indeterminacy.

Remark 2: The Jacobian matrices evaluated at the steady states are non-

singular: causality is guaranteed at least in a neighborhood of each steady state.

5 Concluding Remarks

Let us remark briefly on the implications of our results for international trade
timey. It is assumed in the basic trade models like the Heckscher-Ohlin model
and the specific factor model that while commodities are freely traded, fac-
tors of production are internationally immobile. Unless market distortions like
externalities, public goods, government interventions are incorporated into the
model, a static trading equilibrium is Pareto-optimal. However, a dynamic equi-
librium path which is generated from a multi-country dynamic general equilib-
rium model of international trade is generally Pareto-suboptimal. This because
the rate of marginal substitution between present consumption and future con-
sumption can be different between home and foreign households along the path,
if there is no international factor mobility.

This Pareto-suboptimality can generate new theoretical issues in interna-
tional economics. The indeterminacy discussed in this paper is an example
of them. Part of our future research agenda is to study other implications of
Pareto-suboptimality in international economics.

If international factor mobility is allowed and/or an international credit mar-

kets exist, there is no room for indeterminacy without any other source of market
distortion; a dynamic general equilibrium path is Pareto-optimal and coincide
with the optimal solution of a planner's problem. Thus, from the viewpoint
of international economics, indeterminacy is an interesting and important issue
because international economics has its raison d'être in investigating the intrin-
sic properties of the world economy which is at an intermediate stage between
a segmented world consisting of a set of autarkic economies and a perfectly
integrated world.

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Figure 1: the mapping $\Theta(Z)$
Figure 2: the graph of $\Theta(z)$

$\Theta(z)$

The 45° line

$z_\alpha$

$z_\beta$

$z_0$

$z_2(\varepsilon)$

$z_1(\varepsilon)$

$\overline{z}$

$z$