Asset Pricing with Bayesian Learning and Signal Distortion

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Abstract: Assuming asymmetric information, heterogeneous beliefs among agents, as well as Bayesian learning from the performance of stock markets, this paper demonstrates that the stock prices would have a sustained deviation from its fundamental values à la the conventional expected present value (EPV) approach of asset pricing. More specifically, the EPV approach would underprice the stocks in a booming market and overprice them in a bearish one. In addition, the deviation is biggest when there is a widespread rumor in the market rather than the announcement of the truth. The intuition is based on the idea that current beliefs of the stock’s quality would affect the expectation of subsequent signals, which is ignored in the traditional asset pricing approach. An additional finding is that speculative manipulations are not profitable at least when all agents have the same accessibility to the signals. When we introduce signal distortions in the model, speculative manipulation is more profitable than the baseline model. As one of the contributions to the mainstream learning or herding literatures on financial markets, this paper is among the first few that endogenizes the price process.

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1 Introduction

Since the seminal paper of Lucas (1978), present value approach has been widely used by securities analysts. With the presence of uncertainty, the fundamental value of a stock is simply the expected present value (EPV) of its future dividend flow. However, this approach fails to explain why the stock prices may have sustained deviation from the related fundamental values, and why there exist speculative manipulation at least in the emerging markets. There are also several empirical puzzles raised later. One of them is the excess volatility of stock prices over that of underlying dividend flows, as pointed out by Shiller (1981) and other following studies. Among the efforts to explain this puzzle, Bulkley and Tonks (1989) suggest that the information content in the latest dividend could lead to higher volatility via its impact on the estimation of dividend growth. Timmermann (1996) illustrates this idea by simulation of an estimation-based asset pricing model. Guidolin and Timmermann (2003) incorporate Bayesian learning into the pricing a stock with a dividend flow from binomial process.

Latest development in social learning and herd behavior provides better tools to study the learning effect in asset pricing with informational frictions.1 As a first effort to introduce a social learning model into an overlapping generation framework, this paper finds that the EPV approach would underprice the stocks in a bullish market and overprice it in a recession which is in line with the observation of stock markets. The intuition is that the current beliefs would affect the expectations of forthcoming signals. If the firm is believed to be good, the subsequent signal is more likely to be encouraging. Hence the investors would expect a higher price in the following period. In other words, the expected capital gain is positive in this case.

Note that the expected capital gain is zero in any version of the traditional asset pricing formulas. It rests on the assumption that current investors makes the inference of the true joint distribution of future dividends based on their estimation, and they anticipate that the investors in the subsequent period would make the same inference. Hence the expected stock price should be the same as the current one. In contrast, this paper regards the observed dividends as signals, which change the beliefs instead of inferences. Since the anticipated distribution of subsequent dividend shocks would depend on current beliefs, the expected capital gain is generically non-zero.

The presence of capital gain in stock pricing formula makes it possible to study the behavior

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1 In the information cascades modeled by Banerjee (1992) and Bikhchandani, Hirshleifer, and Welch (1992), individual learning would be outvoiced by the observation of the history. Avery and Zemsky (1998) showed that unbounded price adjustment could prevent herd behavior. Smith and Sørensen (2000) demonstrated that learning could slow down the convergence rate to the truth. More discussion about different social learning models can be found in Chamley (2003b).
of speculators, who care more about capital gains than dividends. In current settings, speculative manipulation turns out to be non-profitable due to the lack of information privileges and the short period of investments for each agent.

As one of the contributions to the mainstream social learning or herding literatures on financial markets, this paper is among the first few that endogenizes the price process. Currently, most of the learning models assume exogenous payoffs. Some of the latest papers, such as the endogenous timing models in Abreu and Brunnermeier (2003) and Chamley (2003a), try to endogenize the payoff, but they are essentially endogenous shifting between two exogenous price processes. In contrast, this paper derives a completely endogenous price process, which makes it easier to introduce the social learning models into other economic frameworks in near future.

In addition, we also introduce two cases where previous stock buyers have more accurate information about signals than the subsequent investors. The first case introduces the limit-ups in stock prices, while the second one considers animal spirit issues. The distortion in the signals perceived by subsequent investors makes speculative manipulation more profitable.

In the economy depicted in Section 2, we can find that the conventional EPV approach is no long valid generically. In an effort to obtain more insights, Section 3 investigates a simplified case with a special distribution of prior beliefs, and characterizes the endogenous price process. It also demonstrates an example of sustained deviation from fundamental values suggested by the EPV approach. Section 4 discusses the possibility of speculative manipulation. Two examples of signal distortion are demonstrated in Section 5 in an effort to study speculators’ behavior when they have informational advantages. Section 6 concludes. Preliminary works on possible extensions of the model are shown in Appendix A, while the proofs are attributed to Appendix B.

2 Asset Pricing with Bayesian Learning

2.1 General Settings of the Economy

The model is a typical two-period overlapping generation model with two assets. One is a risk-free asset with infinite supply and a constant gross return of \( R \) with \( R \geq 1 \). The investors can also buy common stocks issued by a listed company. Shortselling is forbidden on both markets.

Each generation, indexed by their dates of entrance to the stock market, forms a continuum with a Lebesgue measure of one. Each agent, assumed to be risk neutral, is initially endowed with one unit of capital. At time \( t \) \((t = 0, 1, 2, \ldots)\), the young agents (generation \( t \)) enter the stock market and bid for the common stocks sold by the old ones (generation \( t - 1 \)). If the bidding
price is lower than the market price, he would invest all the money in the risk-free asset. At time $t+1$, they sell all the assets and enjoy their retirement on the beach. Consumption is valued only in the second period of their life, and the utility is a linear function of the amount of capital they earned. These assumptions simplify our model by imposing forced savings on the young investors and inelastic supply of stocks. Even with these simplifications, it is still difficult to find analytic solutions from an integral equation. Nonetheless, it provides interesting and intuitive results.

### 2.2 States and Signals

The nature chooses the quality of the listed firm, $\theta \in \{\theta_H, \theta_L\}$. With an abuse of notation, $\theta$ also indicates the type-$i$ firm’s expected dividend per share paid to shareholders. To ensure the results to be nontrivial, we assume that

$$\theta_L < R - 1 < \theta_H.$$  \hfill (1)

The firm’s realized profit per share for period $(t-1, t]$ is

$$s_t = \theta + \varepsilon_t,$$

where $\{\varepsilon_t\}$ is i.i.d. noise with a probability density function (pdf) of $\phi_\varepsilon(\cdot)$. For simplicity, we assume it to be Gaussian noise with zero mean and constant variance $\sigma_\varepsilon^2$. All the profits generated in that period are paid to the current shareholders as dividend before the market opens. While the true quality of the firm is unknown, the realized profit per share also serves as a signal.

### 2.3 Initial Public Offer (IPO)

After the circulation of its prospectus about the its pre-IPO performance, the firm is listed in the stock market by an initial public offer (IPO) to generation 0 at time 0. The IPO price per share is $P_0 = 1$, while the number of outstanding shares, $S$, depends on the volume of applications. For simplicity, the shares are assumed to be perfectly divisible. After the issuance, the number of shares is fixed but the price, $P_t$, can fluctuate over time.

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2 Since we have a continuum of agents, the agents bidding at the market price is of measure zero.

3 The noises can follow any other distributions. Gaussian distribution is chosen due its simplicity in calculation and easier comparison with other social learning models. Admittedly, this assumption violates limited liability for investments in stock market, but the results of this paper would still hold for other distributions.
2.4 Evolution of Beliefs

All agents first study the prospectus of the listed firm to form a prior belief, $\mu_{-1}$, about the probability of the good state with a cumulative density function (cdf) of $F_{-1}(\cdot)$, which is same for each generation, and assumed to be a common knowledge. Similar to all the models of herding behavior, we assume that the agents can observe the history of prices and the number of shares, $h_t = \{S, P_0, P_1, \ldots, P_{t-1}\}$, but the firm’s past earnings are not recorded.\(^4\) Hence, the young agents have to make an inference from $h_t$ to obtain the estimated signals $\Omega_t = \{\hat{s}_0, \hat{s}_1, \ldots, \hat{s}_t\}$. In contrast, the firm announces its latest performance, $s_t$, to the young investors of generation $t$. Knowing $h_t$ and $s_t$, the young agents update their beliefs to a distribution with a cdf of $F(\cdot; \Omega_t)$.

On the basis of Bayesian inference in Chamley (2003b, section 2.1), a $t$-generation agent with a prior belief of $\mu_{-1}$ would update his beliefs to $\mu_t$ such that

$$\frac{\mu_{\tau}}{1-\mu_{\tau}} = \frac{m(s_{\tau})}{1-\mu_{\tau-1}} \cdot \frac{\mu_{\tau-1}}{1-\mu_{\tau-1}}, \quad \tau = 0, 1, \ldots, t$$

where

$$m(s_t) = \frac{\phi_{\varepsilon}(s_t - \theta_H)}{\phi_{\varepsilon}(s_t - \theta_L)}$$

Recall that the function $\phi_{\varepsilon}(\cdot)$ is the probability density function (pdf) of the noise $\varepsilon_t$. Since it is Gaussian, we can rewrite the updating multiplier as

$$m_t = m(s_t) = \exp \left[ \frac{(\theta_H - \theta_L)(2s_t - \theta_H - \theta_L)}{2\sigma_{\varepsilon}^2} \right].$$

By the belief updating rules of (2), we can derive the cdf $F(\cdot; \Omega_t)$ from $F(\cdot; \Omega_{t-1})$ and $s_t$ as long as we can infer $\hat{s}_\tau$ from $h_{\tau+1}$, for $\tau = 1, 2, \ldots, t-1$. In the following subsection, we describe one of the bidding mechanisms enabling us to do so.

2.5 Bidding Mechanism in the Stock Market

The stock market collects the orders from buyers and sellers in each period, and then generates only one price for each trading date.\(^5\) After comparing the expected payoffs from risk-free asset

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\(^4\)This assumption is not consistent with the reality that the history of a listing firm’s earnings is observable. However, it does not affect the illustration of the price process since we show later that all agents are truth-telling since speculative manipulation provides net losses. The purpose of this assumption is to provide an environment to examine the possibility of speculative manipulations. Ideally, we can come up with a model with earnings recorded but the firm makes some financial manipulations which is only observable to current generation. By comparing the histories of stock prices and earnings, subsequent agents can infer the level of financial manipulation. On the fact that the baseline model is already difficult to solve, the additional complicacy seems unnecessary.

\(^5\)This simplification makes the model more tractable. Actually this assumption, as well as the bidding mechanism we describe here, is in line with the applications in some of over-the-counter (OTC) markets.
and the stocks, young investors would like to hand in limit price order in accordance with their beliefs. If all agents are truthful, agents with higher beliefs will bid for a higher price. On the other hand, the sellers would submit a market price order because they have to sell all the stocks to enjoy their retirement.6

The market collects all the orders and determines the market price $P_t$ such that the number of shares from bidding orders above $P_t$ equals $S$. Suppose the marginal agent has a belief of $\mu_t^*$, then the number of winning buyers is $1 - F(\mu_t^*; \Omega_t)$. Recall that each agent has one unit of capital and the total market value of the company is $SP_t$ at time $t$. Hence we obtain the market price as a function of marginal belief. It also depends on all the inferred and observed signals.

$$P_t = \frac{1 - F(\mu_t^*; \Omega_t)}{S}, \quad (4)$$

where

$$S = SP_0 = 1 - F(\mu_0^*; \Omega_0) \quad (5)$$

The winning buyers have to purchase the stock at a price of $P_t$, while the other young investors would invest in the risk-free asset. Theoretically, we can solve $\mu_t^*$ as a function of $P_t$ from (4)

$$\mu_t^* = \phi(P_t; \Omega_t) = F^{-1}(1 - SP_t; \Omega_t) \quad (6)$$

2.6 The Sequence of Events

The sequence of events within any arbitrary time $t$ can be summarized as follows:

i. the company announces its profit per share $s_t$ to the young generation, and pay the same amount as dividend to the shareholders;

ii. after receiving the dividend, the shareholders of the old generation submit market price orders of their holding stocks;

iii. based on the history $h_t$ and the signal of the company’s performance $s_t$, the young generation forms a belief about the company’s quality, and then bid for the stocks;

iv. the market generates a price $P_t$ for the stock according to the bidding mechanism described above, where the losers, whose bidding price is smaller than the market price, have to invest in the risk-free assets.

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6 Since this market is essentially a limit-order market, there is no bid-ask spread due to the absence of market makers.
2.7 No Arbitrage Condition

For the marginal agent at time $t$, the expected gross rate of return on equity $y_t$ equates the yield from risk-free assets,

$$y_t = \frac{E_t[s_{t+1} + P_{t+1}]}{P_t} = \mu_t^* \theta_H + (1 - \mu_t^*) \theta_L + E_t[P_{t+1} \mid \mu_t^*, \Omega_t] = R.$$  \hspace{1cm} (7)

Note that the number of shares per agent, $S/[1 - F(\mu_t^*; \Omega_t)]$, cancels out from the numerator and denominator. The expectations are taken on both the states of nature $\theta$ and the error term $\varepsilon_t$. In the light of equation (4) and the approach to generate $F(\cdot; \Omega_t)$, we can multiply both sides by $SP_t$, and rewrite equation (7) as

$$R[1 - F(\mu_t^*; \Omega_t)] = S[\mu_t^* \theta_H + (1 - \mu_t^*) \theta_L] + 1 - E[F(\mu_t^*; \Omega_{t+1}) \mid \mu_t^*, \Omega_t]$$  \hspace{1cm} (8)

The left-hand side is the opportunity cost of the capitals invested in the stock market in period $t$, while the right-hand side is the expected return from the dividends and the market value of the stocks in period $t + 1$.

2.8 Comparison with Expected Present Values

Based on the present value approach, we define

$$V_i = \frac{\theta_i}{R - 1}, \quad i = H, L,$$

and rewrite equation (7) as

$$P_t = [\mu_t^* V_H + (1 - \mu_t^*) V_L] + \frac{E_t[P_{t+1} \mid \mu_t^*, \Omega_t] - P_t}{R - 1}.$$  \hspace{1cm} (9)

If we compare it with the expected present value

$$V_t^E = \mu_t^* V_H + (1 - \mu_t^*) V_L$$  \hspace{1cm} (10)

we can find that only when $P_t = E_t[P_{t+1} \mid \mu_t^*, \Omega_t]$, i.e. when there is no expected capital gain, the price is exactly the expectation of the probable present values for the marginal stock investors.

However, we do anticipate capital gains (losses) when the current belief is in favor of good (bad) state, because a good (bad) firm is more likely to generate a positive (negative) signal. Even if the expectation of the capital gain is zero in one period, the random signal in the following period would break the balance. As a result, the EPV approach cannot be a valid solution to this economy with heterogenous beliefs and signals about the firm’s quality, since it ignores the capital gain effect from the expectation of subsequent signals, which is characterized by the last term in equation (9).
3 An Example of Bayesian Learning Pricing

As illustrated above, the EPV approach fails to provide an arbitrage-free pricing function in a stock market with Bayesian learning. The next task is to figure out the effective asset pricing function in this scenario. Theoretically, we can use equation (6) and (8) to substitute out $\mu_t^*$ and obtain an integral equation for $P_t = P(\Omega_t)$. In calculation, the young agents have to obtain an inference about the previous signal from the history of stock prices. Due to belief updating, the signal $s_t$ is incorporated in the transition from $\Omega_{t-1}$ to $\Omega_t$. However, the increasing dimensions in $\Omega_t$ makes it almost impossible to characterize $\Omega_t$ based on the scalar $P_t$. In an effort to reach an analytic solution, we need to relate $\Omega_t$ to some scalar. In this section, we assume a special distribution family, which enables us to overcome this obstacle.

3.1 A Special Distribution of Prior Beliefs

Assume that the likelihood ratio of prior belief, $\mu_{-1}/(1 - \mu_{-1})$, follows a uniform distribution on $[0, b_{-1}]$. As a consequence, the likelihood of the belief of generation $t$, $\mu_t/(1 - \mu_t)$, is also uniformly distributed on the support of $[0, b_t]$. We call this family of distribution as uniform likelihood ratio distribution (ULR). It enables us to employ only one parameter to characterize $F(\cdot; \Omega_t)$

$$F(\mu_t; \Omega_t) = \frac{\mu_t}{b_t(1 - \mu_t)}. \quad (11)$$

The corresponding probability density function (pdf) is

$$f(\mu_t; \Omega_t) = \frac{1}{b_t(1 - \mu_t)^2}. \quad (12)$$

and the upper bound of belief (or maximal belief)

$$\mu_{\text{max}}^t = \frac{b_t}{1 + b_t}. \quad (13)$$

With the help of equation (2) and (3), we know that $b_t$ can be written as a function of estimated signals $\{\hat{s}_\tau\}_{\tau=0}^{t-1}$ and the latest signal $s_t$:

$$b_t \quad \begin{align*} b_t &= b_{-1} m(s_t) \prod_{\tau=0}^{t-1} m(\hat{s}_\tau) \\ &= b_{-1} \exp \left[ \frac{\theta_H - \theta_L}{2\sigma_{\varepsilon}^2} [2s_t + 2 \sum_{\tau=0}^{t-1} \hat{s}_\tau - (t + 1)\theta_H - (t + 1)\theta_L] \right] \end{align*} \quad (14)$$

7 We chose this family of distribution just to simplify the subsequent computation since it is closed under belief updating. Theoretically, other families of distributions with single parameter can also be employed. However, we believe that the qualitative results would not change.
Note that we have one-to-one mapping between two sequences \( \{ \Omega_t \} \) and \( \{ b_t \} \), and we can easily infer the previous signals from the knowledge about the sequence \( \{ b_t \} \).

Given the above cdf, we can equation (4) as
\[
SP_t = 1 - \frac{\mu_t^*}{b_t(1 - \mu_t^*)},
\]
or
\[
\mu_t^* = \frac{b_t(1 - SP_t)}{1 + b_t(1 - SP_t)},
\]
with
\[
\lim_{b_t \to 0} \mu_t^* = 0; \quad \lim_{b_t \to \infty} \mu_t^* = 1.
\]

Note that the no arbitrage condition (7) is essentially
\[
RP_t = [\mu_t^* \theta_H + (1 - \mu_t^*) \theta_L] + \mu_t^* E[P_{t+1}|b_t, \theta_H] + (1 - \mu_t^*) E[P_{t+1}|b_t, \theta_L]
\]
We can write down the integral equation for the endogenous price process \( P_t = P(b_t) \) as
\[
RP(b_t)\{1 + b_t[1 - SP(b_t)]\} - b_t[1 - SP(b_t)]\theta_H - \theta_L
= \int_{-\infty}^{+\infty} P(b_{t+1,L})\phi_\varepsilon(u)du + b_t[1 - SP(b_t)]\int_{-\infty}^{+\infty} P(b_{t+1,H})\phi_\varepsilon(u)du.
\] (15)
where the number of issued shares, \( S \), is given by
\[
R[1 + b_0(1 - S)] - b_0(1 - S)\theta_H - \theta_L
= \int_{-\infty}^{+\infty} P(b_{1,L})\phi_\varepsilon(u)du + b_0(1 - S)\int_{-\infty}^{+\infty} P(b_{1,H})\phi_\varepsilon(u)du.
\]
Observe that the integral equation (15) implies that
\[
\lim_{b_t \to 0} P(b_t) = V_L; \quad \lim_{b_t \to \infty} P(b_t) = V_H.
\]
when \( SV_H < 1 \). It means that, in the absence of uncertainty, equation (15) is identical with the present value approach.

The analytic solution for integral equation (15) is not easy, if not impossible, to obtain. However, the numerical solution can be computed based on the so called brute-force or iteration method. The explanation is quite intuitive. Suppose that the investors expect that the pricing function follows the expected present value approach (EPV) after \( k \) periods. If the sequence of price functions converges as \( k \) tends to infinity, the limit function would be the solution. The subsequent subsection is dedicated to computation of the EPV pricing in a market with heterogeneous prior beliefs.

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This is likely to be the case in reality, where the market value of all stocks is always smaller than capitals available.
3.2 EPV Pricing with Heterogeneous Beliefs

In addition to the bidding equation (4), we assume that the stock price equals to expected present values for the marginal investors. The no-arbitrage condition is no longer effective to calculate this initial function since we have demonstrated that the EPV approach fails to be arbitrage-free.

While the IPO price $P_0 = 1$, we can calculate $\mu_0^*$ from (10) and the EPV assumption $P_0 = V_0^E$,

$$ \mu_0^* = \frac{1 - V_L}{V_H - V_L}. $$

As a result, we can use equations (5), and (11) to infer $b_0$ from the volume of the issued stock

$$ b_0 = \frac{1 - V_L}{(V_H - 1)(1 - S)}. $$

For the subsequent periods, we can equate the prices in (4) and (10) to obtain

$$ \frac{1 - F(\mu_t^*, \Omega_t)}{S} = \mu_t^* V_H + (1 - \mu_t^*)V_L $$

and then derive the relationship between $b_t$ and $\mu_t^*$ with the help of equation (11). Substituting $\mu_t^*$ as a function of $b_t$ into equation (4) yields

$$ S P_t = \frac{1}{2} [1 + SV_H + \frac{1}{b_t} - \sqrt{(1 - \frac{1}{b_t} - SV_H)^2 + \frac{4}{b_t}(1 - SV_L)}] $$

$$ \mu_t^* = 1 - \frac{2}{(b_t + 1 - SV_H b_t) + \sqrt{(b_t + 1 - SV_H b_t)^2 + 4b_t S(V_H - V_L)}} $$

for $t = 1, 2, \ldots$. The proofs in appendix show that both $P_t$ and $\mu_t^*$ are increasing in $b_t$.

3.3 Numerical Solution

With the above initial pricing function, we calculate the numerical results with parameter values of $R = 1.1$, $V_H = 1.8$, $V_L = 0.6$, $b_0 = 1$, $S = 0.5$ and $\sigma_\varepsilon = 0.5(\theta_H - \theta_L)$. In this example, the quality of signal is quite good, since the probability of misleading signals is just 0.1587 if we employ the middle point of $\theta_H$ and $\theta_L$ as the critical point. The iteration method provides a fast convergence rate for the pricing function. For instance, after 10 iterations, the supnorm distance between adjacent pricing functions is only $2.8628 \times 10^{-6}$, while stock prices range from 0.6 to 1.8.

Figures 1 and 2 illustrate the relationship between EPV and Bayesian learning (BL) prices. We can find that the EPV approach would underprice the stock as much as 3.4% of the BL prices when the firm is widely believed to be good and overprices it as much as 1.5% when the most investors are pessimistic. This result agrees with the phenomena of sustained high prices.

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9 The derivation is provided in Appendix.
during the bubble period and seemingly endless low prices in a bearish market. Intuitively, if the marginal investors believe that the quality of the firm is good (bad), the signal in the subsequent period is more likely be encouraging (discouraging). Hence he would expect a capital gain (loss).

Based on this result, we can have some idea about the dynamics of stock prices. Note that $b_t$ will change over time since it is based on the realization of the random signal $s_t$. When the firm is of a good quality, its stock prices would be more likely to stay higher than its EPV until the truth is revealed. The deviation from EPV is biggest when there is a widespread rumor in the market rather than the announcement of the truth.

A noteworthy remark is that the quality of signal is quite important. When the signals are less informative, the investors are less sure about the subsequent signals. Thus the expected stock market return is closer to the risk-free rate. Extremely, when the variance of the signal $\sigma_s^2$ approaches to infinity, we have $E_t[P_{t+1}|\mu_t^*,\Omega_t] = P_t$, and the no arbitrage condition (7) leads to the EPV approach.

4 Speculative Manipulations

Equation (19) essentially provides a pricing function $P(\cdot)$, such that $P_t = P(b_t)$. We have shown that this function is strictly increasing. This provides us an opportunity to study the possibility of speculative manipulation. The $t$-generation agents with a belief just below the cutoff may consider submitting a higher bidding price in an attempt to pretend that the signal is better than it actually is. According to Bayesian learning, the next generation would like to offer a higher price. In short, these investors can sell the stocks at a higher price in the subsequent period, but they have to bid up their purchasing price to disguise the quality of the firm.

Suppose the agents manage to make an illusion that the highest belief is $b_t + \delta$. As a result, the expected yield becomes

$$y_t^{SP} = \frac{\mu_t^*\theta_H + (1 - \mu_t^*)\theta_L}{P(b_t + \delta)} + \frac{E_t\{P[(b_t + \delta)m(s_{t+1})]|\mu_t^*,\Omega_t\}}{P(b_t + \delta)}.$$  \hfill (21)

The first term is a negative dilution effect, since the expected rate of return from dividend is diluted by the higher purchasing price. The second term can be regarded as capital gain effect.

The sign of the capital gain effect is demonstrated in Figure 3, where we define the elasticity of price manipulation as

$$e_t = \frac{dE_t\{P_{t+1}|\mu_t^*\}}{E_t\{P_{t+1}|\mu_t^*\}} \frac{P_t}{dP_t},$$  \hfill (22)

which illustrates how much (in percentage) the expected subsequent price would change if the investors manage to bid up current price by 1%. We can find that $e_t$ is always less than one,
which implies negative capital gain effect.

While both effects are negative, we fail to justify the existence of speculative manipulations. The reason lies in the fact that the later investors’ own signal would reduce the importance of the previous one. If the agents have different accessibility to the signals, it may be possible for the more informative agents to make positive profits. For example, if there is some signal distortion in the inference by the subsequent generation, some investors may be able to benefit from speculative manipulation.

5 Two Examples of Signal Distortion

5.1 Stock Markets with Fluctuation Limits

In several emerging markets, fluctuation limits\textsuperscript{10} are introduced to avoid drastic changes in stock prices. For example, in Chinese Mainland markets, the fluctuation limit is $\pm 10\%$ of the previous closing price, while Taiwan market imposes a range of $\pm 7\%$. As we saw in the baseline model, stock prices completely reveal the signals. In constrast, we only have partial revelation when stock price hits the limit.

For simplicity and illustrative purposes, only limit-ups are considered here.\textsuperscript{11} With limit-ups, stock exchange generates the prices according to the formula

\[
P_t = \min \left\{ \frac{1 - F(\mu_t^*; \Omega_t)}{S}, \lambda P_{t-1} \right\},
\]

where $\lambda$ is one plus the fluctuation limits for price rises. When the stock price hits the upper limit $\lambda P_{t-1}$, each investors can only invest a proportion $\alpha_t$ of their capital endowments, where

\[
\alpha_t = \frac{S\lambda P_{t-1}}{1 - F(\mu_t^*; \Omega_t)}.
\]

The remaining $1 - \alpha_t$ would be invested in the risk-free asset. Assume that $\alpha_t$ is not recorded, which is in line with the practice of most stock markets. Note that the critical belief for reaching limit-up price, $\mu_t^{\text{up}}$, satisfies

\[
\mu_t^{\text{up}} = F^{-1}(1 - S\lambda P_{t-1}; \Omega_t).
\]

\textsuperscript{10} It is also called as price limit or daily trading limit. It is more widely employed in futures markets.

\textsuperscript{11} Limit-ups are referred to the limits for price rises, and limit-downs means the limit for price drops. In a stock market with limit-downs, at least some stock holders fail to sell all of their stocks, and have to sell the remaining in the subsequent trading day. Hence it requires a multi-period (at least three-period) overlapping generation framework, which makes the model more complicated without much contribution.
For simplicity, assume that the likelihood ratio is uniformly distributed. Then

\[ SP_t = \min \left\{ 1 - \frac{\mu^*_t}{b_t(1 - \mu^*_t)}, S\lambda P_{t-1} \right\} \]

and the cutoff belief \( \mu^* \) satisfies both

\[ \mu^*_t \leq \mu^*_t \leq \frac{b_t(1 - S\lambda P_{t-1})}{1 + b_t(1 - S\lambda P_{t-1})}, \]

and the no-arbitrage condition

\[ y = \frac{\mu^*_t \theta_H + (1 - \mu^*_t) \theta_L + \mathbb{E}_t[P_{t+1}|\mu^*_t, \hat{b}_t]}{\lambda P_{t-1}} = R. \]

Here we use \( \hat{b}_t \) instead of \( b_t \), since the limit-up price fails to reveal the true signal. Note that the updating multiplier for subsequent generation is now the ratio of probabilities for stock prices to hit the upper limit in each case

\[ \tilde{m}(\hat{s}_t) = \frac{1 - \Phi_\varepsilon(\hat{s}_t - \theta_H)}{1 - \Phi_\varepsilon(\hat{s}_t - \theta_L)} \]

where \( \Phi \) is the Gaussian cdf. Suppose the price function \( P(b) \) is known, then the agents can infer a signal of \( \hat{s}_t \), along with the cutoff belief \( \mu^*_t \), from the equation system of

\[ \mu^*_t = \frac{m(\hat{s}_t)b_{t-1}(1 - S\lambda P_{t-1})}{1 + m(\hat{s}_t)b_{t-1}(1 - S\lambda P_{t-1})} \]

\[ \lambda P_{t-1}R = \mu^*_t \theta_H + (1 - \mu^*_t) \theta_L + \mathbb{E}_t[P_{t+1}|\mu^*_t, \tilde{m}(\hat{s}_t)b_{t-1}] \]

in the case of limit-up prices.

Observe that there are two different updating multipliers. The signal is known to current generation, hence they employ the traditional belief updating formula, while the signal perceived by the subsequent generation entails the use of the updating multiplier based on Gaussian cdf instead of pdf.

From the model, we can find that stock prices hitting fluctuation limits would distort the revelation of private signals. More specifically, since \( m(\hat{s}_t) < \tilde{m}(\hat{s}_t) \) for Gaussian distribution,\(^{12}\) the perceived signal has an upward bias when the true signal is close to \( \hat{s}_t \). In the case when the true signal is lower but sufficiently close to \( \hat{s}_t \), some of the investors would like to pretend that they have received \( \hat{s}_t \) instead of the true signal in an effort to make use of the upward bias, which makes speculative manipulation more profitable than the baseline model. However the complicacy of the model makes it difficult to obtain even a numerical example.

\(^{12}\)It is based on the fact that the hazard rate, \( \frac{\phi(x)}{1 - \Phi(x)} \), is increasing for Gaussian distribution.
5.2 Exogenous Financial Distress or Animal Spirit

Other disturbances, such as exogenous financial distress and/or animal spirit, would also distort the inference of signals from the stock prices. For example, suppose that, with a probability of $1 - \theta$, each agent in generation $t$ would encounter an exogenous financial distress which restrains him from investing in stock market no matter what signal he receives. The distribution of financial distress is assumed independent of the prior beliefs. The fraction of financially healthy agents, $h_t$, follows a uniform distribution on $[H, 1]$, where $0 < H < 1$. While $h_t$ is known to the $t$-generation only, $H$ is public knowledge.\(^{13}\)

Now the cdf for financially healthy agents becomes

$$F(\mu_t; \Omega_t, h_t) = \frac{F(\mu_t; \Omega_t)}{h_t b_t (1 - \mu_t)}.$$

For simplicity, we employ the uniform likelihood ratio distribution again. Therefore

$$F(\mu_t; \Omega_t, h_t) = \frac{\mu_t}{h_t b_t (1 - \mu_t)}.$$

Stock exchange now generates the stock price by

$$SP_t = 1 - \frac{\mu_t^*}{h_t b_t (1 - \mu_t)}$$

and the no-arbitrage condition turns out to be

$$\frac{\mu_t^* \theta_H + (1 - \mu_t^*) \theta_L + E_t[P_{t+1}|\mu_t^*, \hat{b}_t]}{P_t} = R.$$

Note that the perceived range of signal $[s^L_t, s^H_t]$ is given by

$$m(s^H_t) = \frac{h_t m(s_t)}{H}$$
$$m(s^L_t) = h_t m(s_t)$$

Analogous to the fluctuation limit case, $\hat{b}_t = \tilde{m}(s^H_t, s^L_t) b_{t-1}$, where the perceived updating multiplier is the ratio of the probabilities for the signal falling into the region $[s^L_t, s^H_t]$

$$\tilde{m}(s^H_t, s^L_t) = \frac{\Phi_\varepsilon(s^H_t - \theta_H) - \Phi_\varepsilon(s^L_t - \theta_H)}{\Phi_\varepsilon(s^H_t - \theta_L) - \Phi_\varepsilon(s^L_t - \theta_L)}.$$

\(^{13}\)We can also introduce financial frenzy analogously. For example, we can assume that a fraction of agents would always buy stocks whatever signal he receives. However, it is a bit more complicated since we have to adjust the total shares available to rational agents, while in the financial distress model, $h_t$ only appears in the cumulative distribution function.
This would again lead to a signal distortion. The direction of biases is based on the property of function
\[
\psi(s_t, h_t) = \frac{\phi_x(s_t - \theta_H)}{\Phi_x(s_t^H - \theta_H) - \Phi_x(s_t - \theta_H)}
\]

Similar to the baseline model, we can obtain the integral equation for the price function from equation (24) and (25) by substituting out \( \mu_t^* \). The computational difficulties remains the same.

6 Conclusion

In this paper we establish an overlapping generation model with Bayesian learning of the listed firm’s quality. Although the asset pricing model in this economy is difficult to obtain, we can have some idea from the performance of EPV approach. In a bullish market, the stocks are priced higher than that implied by the present value of subsequent dividend flows. During the recession, we can find sustained existence of underpriced stocks.

In our model, the introduction of endogenous payoffs enables us to investigate the profitability of speculative manipulations. However, the results are not satisfying so far. The reason lies in the fact that every agent has equal access to the signals, and the speculative manipulation fails to disguise the signal by sufficient amount to make it profitable.

The presence of signal distortions would give current buyers more accurate signals than the subsequent investors. For example, when the actual stock price is very close to the limit-up level, the marginal investors would like to bid it up to this upper bound. As a result, the following investors would conceive a much better signal than the true one, so that they are willing to pay a higher price than the baseline model. This would make speculative manipulation more profitable.

There are several other ways to extend the baseline model. The introduction of partial access to the signals would slow down the convergence rate of marginal beliefs toward the true quality. A three-period overlapping generation settings would greatly enrich the analysis of stock market behaviors. In contrast to the passive selling from the old investors in current model, we would have more elastic supply since the mid-aged investors can choose whether to hold or sell. The idea is close to the occupational choice model proposed by Banerjee and Newman (1993). We expect that the increase in the dimension of strategy space provides more room for speculative manipulation. Some of the preliminary works are shown in Appendix A.
Appendix A: Possible Extensions of the Model

In the above model, the marginal beliefs converge to the true quite fast. In addition, the complete access to the signals discourages the possibility of speculative manipulation. Hence, one of the interesting extensions would be the imposition of partial access to the signals. We expect the convergence rate to be much slower, and the speculative manipulation to be more likely.

Another concern of the main model is the forced selling for the old generation. If we extend the model to a three-period overlapping generation framework, it would be more close to reality. A1. Partial Access to the Signals

The major difference from the baseline model is that the young agents have a probability $\pi_t$ to observe the signals, while the uninformed agents would have a belief $\mu^N_t = \hat{\mu}_{t-1}$. As a result, the number of winning buyers is $\pi_t[1 - F(\mu^S_t; \Omega_{t-1}, s_t)] + (1 - \pi_t)[1 - F(\mu^N_t; \Omega_{t-1})]$, where $\mu^N_t$ and $\mu^S_t$ stand for the beliefs of marginal agents in each group respectively. Recall that each agent has one unit of capital and the total market value of the company is $SP_t$ at time $t$. Hence we obtain the market price as a function of marginal belief. It also depends on the $F^N_t$ and $F^S_t$.

$$P_t(\mu^*_t; \sigma_{t+1}) = \frac{\pi_t[1 - F(\mu^S_t; \Omega_{t-1}, s_t)] + (1 - \pi_t)[1 - F(\mu^N_t; \Omega_{t-1})]}{S}, \quad (A1)$$

where

$$S = SP_0 = \pi_0[1 - F(\mu^S_0; s_0)] + (1 - \pi_0)[1 - F(\mu^N_0; \emptyset)] \quad (A2)$$

Now we have no-arbitrage condition for each type of agents

$$y_t = \frac{\mu^S_t \theta_H + (1 - \mu^S_t) \theta_L + E_t[P_{t+1}|\mu^S_t, \Omega_{t-1}, s_t]}{P_t} = R, \quad (A3)$$

and

$$y_t = \frac{\mu^N_t \theta_H + (1 - \mu^N_t) \theta_L + E_t[P_{t+1}|\mu^N_t, \Omega_{t-1}]}{P_t} = R. \quad (A4)$$

If we assume the likelihood ratio to be uniformly distributed, then

$$SP_t = SP(b_t; b_{t-1}) = \pi_t \left[1 - \frac{\mu^S_t}{b_t(1 - \mu^S_t)}\right] + (1 - \pi_t) \left[1 - \frac{\mu^N_t}{b_{t-1}(1 - \mu^N_t)}\right] \quad (A5)$$

We can obtain the relationship between the two marginal beliefs

$$(\mu^S_t - \mu^N_t)(\theta_H - \theta_L)$$

$$= \mu^N_t E_t[P(b_{t+1}; b_t) | b_{t-1}, \theta_H] + (1 - \mu^N_t) E_t[P(b_{t+1}; b_t) | b_{t-1}, \theta_L]$$

$$- \mu^S_t E_t[P(b_{t+1}; b_t) | b_t, \theta_H] - (1 - \mu^S_t) E_t[P(b_{t+1}; b_t) | b_t, \theta_L] \quad (A6)$$
Theoretically, we can solve the $\mu_{t}^{S*}$ and $\mu_{t}^{N*}$ from (A5) and (26) and then put them back to either (A3) or (A4) to obtain the integral equation characterizing the price process. However it is even more difficult to solve this model.

A2. Three-Period Overlapping Generation Framework

Assume that each agent will live for three period, young, adult and old. While they are born with an endowment of one unit of capital and only consume at the end of the third period, they invest when young, modify their investment portfolio when adult, and sell all assets when old.

At period 0, the listed firm issues new shares to the young and adult investors, hence

$$ S = SP_{0} = [1 - F(\mu_{0,0}^{*}; \Omega_{0})] + R[1 - F(\mu_{-1,0}^{*}; \Omega_{0})] $$

$$ = (1 + R) \left[ 1 - \frac{\mu_{0,0}^{*}}{b_{0}(1 - \mu_{0,0}^{*})} \right] $$

where the two marginal beliefs, $\mu_{0,0}^{S*}$ and $\mu_{-1,0}^{S*}$, are equal, since the expected price in the next period are the same. We also impose the assumption of uniform distribution for the likelihood ratio. Here the first subscript stands for the birth date of the agent, while the second one indicates the current period.

A good way to obtain the stock market clearing condition is to imagine that the adults sell all the shares and then make their portfolio decision again. Now the market value of the stocks is given by the young and adult agents’ stock investment. To the adult agents, if the number of investors increases, i.e. $1 - \frac{\mu_{t-1,t}^{s} b_{t-1}(1 - \mu_{t-1,t}^{s})}{b_{t}(1 - \mu_{t-1,t}^{s})} > 1 - \frac{\mu_{t-1,t-1}^{s} b_{t-1}(1 - \mu_{t-1,t-1}^{s})}{b_{t}(1 - \mu_{t-1,t-1}^{s})}$, the previous stock holders would invest all the capital gains and the dividends in stocks, while the previous depositors (risk-free asset holders) would also invest their deposits. This implies

$$ SP_{t} = 1 - \frac{\mu_{t,t}^{s} b_{t}(1 - \mu_{t,t}^{s})}{b_{t-1}(1 - \mu_{t-1,t}^{s})} + \frac{P_{t} + s_{t}}{P_{t-1}} \left[ 1 - \frac{\mu_{t-1,t-1}^{s} b_{t-1}(1 - \mu_{t-1,t-1}^{s})}{b_{t}(1 - \mu_{t-1,t}^{s})} \right] $$

$$ + R \left[ \frac{\mu_{t-1,t-1}^{s} b_{t-1}(1 - \mu_{t-1,t-1}^{s})}{b_{t-1}(1 - \mu_{t-1,t-1}^{s})} - \frac{\mu_{t-1,t}^{s} b_{t}(1 - \mu_{t-1,t}^{s})}{b_{t}(1 - \mu_{t-1,t}^{s})} \right]. $$

Otherwise, we have only a portion of previous stock holders investing in the stock market again, which means

$$ SP_{t} = 1 - \frac{\mu_{t,t}^{s} b_{t}(1 - \mu_{t,t}^{s})}{b_{t-1}(1 - \mu_{t-1,t}^{s})} + \frac{P_{t} + s_{t}}{P_{t-1}} \left[ 1 - \frac{\mu_{t-1,t}^{s} b_{t-1}(1 - \mu_{t-1,t}^{s})}{b_{t}(1 - \mu_{t-1,t}^{s})} \right] $$

Obviously, the model is much more complicated.
Appendix B: Proofs and Derivations

The brute-force (iteration) method to solve integral equation (15):

Consider the mapping \( T \) from bounded continuous function to itself satisfying

\[
(Tg)(b_t) = \frac{\mu^*_t \theta_H + (1 - \mu^*_t) \theta_L + \mu^*_t e_H + (1 - \mu^*_t) e_L}{R}
\]

where

\[
e_i = E[g(b_{t+1})|b_t, \theta_i], \quad i = H, L
\]

\[
\mu^*_t = \frac{b_t[1 - Sg(b_t)]}{1 + b_t[1 - Sg(b_t)]}
\]

We choose \( g \) as the EPV price function (19), and compute \( T^k g \) recursively. While the convergence rate for the sequence \( \{T^k g\} \) is quite fast, the numerical solution for the integral equation (15) is easy to obtain.

The derivation of equation (19):

From equations (18) and (11), we have

\[
1 - \frac{\mu^*_t}{b_t(1 - \mu^*_t)} = S[\mu^*_t V_H + (1 - \mu^*_t) V_L]
\]

For computational convenience, we can define

\[
a^*_t = \frac{\mu^*_t}{1 - \mu^*_t}
\]

and get

\[
1 - \frac{a^*_t}{b_t} = S[\frac{a^*_t V_H}{1 + a^*_t V_H} + \frac{1}{1 + a^*_t V_L}]
\]

or

\[
0 = S[a^*_t V_H + V_L] - (1 - \frac{a^*_t}{b_t})(1 + a^*_t)
\]

\[
= \frac{(a^*_t)^2}{b_t} - (1 - \frac{1}{b_t} - SV_H)a^*_t + SV_L - 1
\]

Note that \( S < 1 \), and \( V_L < 1 \), the quadratic equation has two real roots with different signs. While only the positive root is meaningful here, we have

\[
a^*_t = \frac{1}{2}(b_t - 1 - SV_H b_t) + \frac{1}{2} \sqrt{(b_t - 1 - SV_H b_t)^2 + 4b_t(1 - SV_L)}
\]

Putting it into (4) yields

\[
SP_t = 1 - \frac{a^*_t}{b_t} = \frac{1}{2} \left[ 1 + SV_H + \frac{1}{b_t} - \sqrt{(1 - \frac{1}{b_t} - SV_H)^2 + \frac{1}{b_t}(1 - SV_L)} \right]
\]

\[
= \frac{1}{2} \left[ 1 + SV_H + \frac{1}{b_t} - \sqrt{(1 + \frac{1}{b_t} - SV_H)^2 + \frac{4S}{b_t}(V_H - V_L)} \right]
\]
with

\[ \lim_{b_t \to \infty} SP_t = \min\{1, SV_H\}. \]

Note that we can rewrite the expression as

\[
SP_t = \frac{1}{2} \left[ (1 + SV_H + \frac{1}{b_t})^2 - (1 + \frac{1}{b_t} - SV_H)^2 - \frac{4S}{b_t} (V_H - V_L) \right]
\]

\[ = \frac{2SV_H + \frac{2S}{b_t} V_L}{1 + SV_H + \frac{1}{b_t} + \sqrt{(1 + \frac{1}{b_t} - SV_H)^2 + \frac{4S}{b_t} (V_H - V_L)}} \]

we can obtain

\[ \lim_{b_t \to 0} SP_t = SV_L. \]

Defining

\[ x_t = \frac{1}{b_t}, \]

we have

\[
\frac{d(SP_t)}{dx_t} = 1 - \frac{2(x - 1 + SV_H) + 4(1 - SV_L)}{2\sqrt{(1 - x - SV_H)^2 + 4x(1 - SV_L)}} < 0
\]

because

\[
(1 - x - SV_H)^2 + 4x(1 - SV_L) - [(x - 1 + SV_H) + 2(1 - SV_L)]^2
\]

\[ = 4x(1 - SV_L) - 4(x - 1 + SV_H)(1 - SV_L) - 4(1 - SV_L)^2
\]

\[ = 4x(1 - SV_L)(SV_L - SV_H) < 0. \]

Hence \( P_t \) is strictly increasing in \( b_t \). When \( b_t \) tends to infinity, \( SP_t \) goes to \( \min\{1, SV_H\} \); when \( b_t \) tends to zero, \( SP_t \) goes to \( SV_L \).

In addition, we can compute the belief of marginal investor

\[
\mu_t^* = \frac{a_t^*}{1 + a_t^*} = 1 - \frac{1}{1 + a_t^*}
\]

\[ = 1 - \frac{2}{(b_t + 1 - SV_H b_t) + \sqrt{(b_t - 1 - SV_H b_t)^2 + 4b_t(1 - SV_L)}} \]

\[ = 1 - \frac{2}{(b_t + 1 - SV_H b_t) + \sqrt{(b_t + 1 - SV_H b_t)^2 + 4b_t(SV_H - V_L)}} \]

\[ = 1 - \frac{\sqrt{(b_t + 1 - SV_H b_t)^2 + 4b_t(SV_H - V_L)} - (b_t + 1 - SV_H b_t)}{2b_t(SV_H - V_L)} \]
with
\[
\lim_{b_t \to 0} \mu^*_t = 0; \quad \lim_{b_t \to \infty} \mu^*_t = \begin{cases} 
1 & \text{if } SV_H < 1 \\
1 - SV_L & \text{if } SV_H \geq 1 
\end{cases}
\]
and \(\mu^*_t\) strictly increasing in \(b_t\).

A numerical example

Let
\[
P^L_{t+1} = \int_{-\infty}^{+\infty} P(b_{t+1,L})\phi_\varepsilon(u)du
\]
\[
P^H_{t+1} = \int_{-\infty}^{+\infty} P(b_{t+1,H})\phi_\varepsilon(u)du
\]
we have
\[
RP(b_t)\{1 + b_t[1 - SP(b_t)]\} - b_t[1 - SP(b_t)]\theta_H - \theta_L = P^L_{t+1} + b_t[1 - SP(b_t)]P^H_{t+1}.
\]

Then
\[
Rb_tSP_t^2 - P_t(R + Rb_t + b_t\theta_H + b_tSP^H_{t+1}) + \theta_L + P^L_{t+1} + b_tP^H_{t+1} + b_t\theta_H = 0
\]
When \(P_t = 0\), the left hand side is positive. When \(P_t = 1/S\), the left hand side is
\[
Rb_t - (R + Rb_t + b_t\theta_H S + b_tP^H_{t+1}) + S\theta_L + SP^L_{t+1} + b_tSP^H_{t+1} + b_tS\theta_H = -R + S\theta_L + SP^L_{t+1} < -R + \theta_L + 1 < 0
\]
since \(S < 1\), \(SP^L_{t+1} < 1\), and \(\theta_L < R - 1\). Hence there is one root in \((0, 1/S)\) and another in \((1/S, \infty)\). Obviously only the smaller one is reasonable, since the other leads to \(SP_t > 1\).

To figure out the number of shares, \(S\), just set \(P(b_0) = 1\), which results in
\[
R\{1 + b_0[1 - S]\} - b_0[1 - S]\theta_H - \theta_L = P^L_1 + b_0[1 - S]P^H_1
\]
\[
S = 1 - \frac{R - \theta_L - P^L_1}{b_0(\theta_H + P^H_1 - R)}.
\]
References


Figure 1: Stock prices and Cutoff Beliefs

- **EPV**
- **Bayesian Learning Prices**
Figure 2: Price ratio and Cutoff Beliefs

Ratio between Bayesian Learning prices and EPV vs. Cutoff Beliefs
Figure 3: Elasticity of Expected Price on Current Price