Mathematical Miscalculations and Monopoly Pricing Strategies\textsuperscript{1}

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Abstract

Much of the literature in microeconomics focuses attention on single-unit pricing where the announcement is a price for single units. This linear pricing strategy allows a consumer, whom desires more than one good, to pay the constant per-unit price for any quantity. In practice, firms often use more complicated linear pricing strategies where multiple units are assigned a cost for the bundle and a single unit can be purchased at the per-unit price. I provide an explanation for the use of both linear pricing strategies.

The mechanism I employ is the experimentally common phenomenon of mathematical miscalculations. When determining the per-unit price a computational error may lead to a consumer purchasing a good mistakenly or may lead to a consumer not purchasing a unit when it is in his best interest to do so. It is these two features that distinguish the two pricing strategies. Furthermore, I show that multi-unit pricing can, in certain environments, outperform nonlinear pricing strategies.
1 Introduction

When studying a market attention is paid to the price at which goods are traded. Typically, the price of a good is the cost that must be paid by a consumer for one unit of the good. I refer to this as single-unit pricing. If a consumer wants to buy more than one unit he simply pays the constant per-unit price for each good. Thus, single unit pricing is a linear pricing strategy. Nonlinear pricing strategies have been shown to be a useful to a firm with market power. By selecting a per-unit price that varies with the number of units purchased a firm is able to acquire more information and surplus.

Besides nonlinear pricing and single-unit pricing another type of linear pricing strategy is commonly used in practice. Oftentimes, a firm announces a price for multiple units of a good. For example, two units of a good may be purchased for $X$. A consumer may buy the two units for $X$ or buy one unit at a price of $\frac{X}{2}$. I refer to such a linear pricing strategy as multi-unit pricing. One might argue that single-unit and multi-unit are equivalent strategies. In this paper I analyze and provide an explanation for the use of both single-unit and multi-unit linear pricing strategies as well as nonlinear pricing strategies.

The mechanism I employ is the experimentally relevant phenomenon that individuals frequently make mathematical miscalculations. That is, when presented with a price a consumer may need to engage in a mathematical problem to determine the cost if a different quantity of the good is desired. I show that, given the occurrence of occasional miscalculations, there exists environments where each of the three pricing strategies is optimal.

To illustrate this I consider a monopolist pricing to sell a good to a popu-
lation of consumers whom differ in the utility received from various quantities of the good. Each consumer knows the utility he gains from each quantity of goods and must calculate the cost of each bundle to determine his payoff. If the cost for a particular quantity is not explicitly stated the consumer must do the mathematical problem to determine the cost. Hence, I model miscalculations as a small, exogenous probability that a mistake is made when doing this computation.

Mathematical miscalculations are shown to have two effects. First, they can induce a purchase when it is not optimal for the consumer to make one. Thus, multi-unit pricing has the advantage of providing additional sales to the monopolist. Also, miscalculations may cause a consumer to not purchase a good when it is in his best interest to buy the good. As a result, single-unit and nonlinear pricing have the advantage of eliminating mistakes. It is these opposing features of mathematical miscalculations that distinguishes the strategies.

The paper is organized as follows. The standard model is presented in Section 2 while Section 3 introduces the extension incorporating mathematical miscalculations. Section 4 studies miscalculations when the monopolist selects linear prices while Section 5 considers nonlinear pricing. Section 6 concludes.

2 The Model

There are two types of consumers, denoted $A$ and $B$, who may buy a particular good sold by a monopolist. Normalize the size of the population to unity and let $\alpha \in (0, 1)$ be the fraction of the population that is of type
A. Let $u_i^q$ denote the utility a consumer of type $i$ receives from buying a quantity $q$ of the good. The good is indivisible so consumers buy quantities that are nonnegative integers. Furthermore, utility is zero with no purchases and the functions exhibit diminishing marginal utility. Label the consumers so that $u_A^1 \geq u_B^1$. The total payoff received by a consumer of type $i$ is $u_i^q - P$ where $P$ is the total amount spent on goods.

A monopolist, unable to distinguish between the consumers, prices to sell the product. I assume that the cost of production is zero.\footnote{This is done to simplify the analysis. Alternatively, one can think of the utility as gain net of production costs.} For simplicity assume that the each consumer’s utility function is strictly increasing over the first two units and each is sufficiently concave so that the monopolist never finds it optimal to sell three or more units to any consumer but it is profitable to sell two units to either consumer.\footnote{With no production cost this assumption requires that $u_i^2 > u_i^1 > 0 > u_i^q \forall i \in \{A, B\}$ and $\forall q > 2$. If the solution is generalized with a cost of $c^q$ then it is simply required that $u_i^2 > c^q \forall i \in \{A, B\}$ and $\forall q \leq 2$; along with $u_i^q \leq c^q \forall i \in \{A, B\}$ and $\forall q > 2$.} I therefore focus on pricing to sell at most two units to each consumer.

Let $m_i$ denote the increase in utility to a consumer of type $i$ when buying a second unit. Hence, $m_i = u_i^2 - u_i^1$ and, from the assumption of diminishing marginal utility, $u_i^1 > m_i$. This implies that $u_A^1 \geq u_B^1 > m_B$. The utility functions can take one of three general cases. For Case I, $A$ consumers, who receive a higher utility when consuming one unit, continue to receive a higher utility when consuming two units and the gap between the utilities increases. Figure 1 illustrates this case.

Hence, Case I is defined so that $u_A^1 > m_A > u_B^1 > m_B$. For Case II, $A$ consumers still receive a higher utility than $B$ consumers when buying two

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units but the gap shrinks so that $u^1_A \geq u^1_B > m_A > m_B$. This case is depicted in Figure 2.

Finally, for Case III, $B$ consumer’s utility can catch up and pass consumers of type $A$ so that $u^1_A \geq u^1_B > m_B > m_A$. This final case is depicted in Figure 3. As will be shown, the pricing decisions of the monopolist differ in each of the cases. Furthermore, mathematical miscalculations impact the outcome differently in each.

A monopolist has three general pricing strategies available. It can sell the good by announcing a single-unit price where a price is quoted for each unit desired. Also, the monopolist can announce a multi-unit price. For example, the monopolist may announce that two units of the good can be purchased for a price $X$. A consumer may buy two units at the price $X$ or one unit at $\frac{X}{2}$. Both of these pricing strategies, the single-unit and multi-unit, are linear pricing strategies. Linear pricing strategies are those where the per-unit price
Figure 2: Utility Functions in Case II

Figure 3: Utility Functions in Case III
is unaffected by changes in the number of units purchased. Alternatively, a third pricing strategy the monopolist can use is to sell two units together and charge a price for the bundle along with selling single units. This is referred to as nonlinear pricing since the price of the bundle of two need not be twice the price of the one single unit.

3 Mathematical Miscalculations

Mathematical miscalculations are a frequent problem that individuals face. They are unavoidable and common. Experimental evidence supports this as a problem. Rubinstein, Meyer, and Evans (2001) conduct experiments where subjects are given simple math problems to be solved. Their goal is to study the mental process of task switching. They show that the frequency of mistakes ranges from only 2.3% to 5.6% when subjects perform addition and subtraction problems the rate of miscalculations jumps to 10.7% to 13.0% when they must complete simple division problems. It is this experimentally relevant phenomenon, mathematical miscalculations, that I incorporate into consumer purchasing behavior to study a monopolist selecting among various pricing strategies.

To incorporate mathematical miscalculations I use the frequency of error that arises when individuals must do straightforward division problems. When the monopolist is engaging in a multi-unit pricing strategy it is up to the consumer to do the mathematical exercise of determining the per-unit price. While there is no mistaking what is the price of two units there is a chance that a consumer makes a miscalculation determining whether or not

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3Complete details of the experiments can be found in their paper.
to buy one unit. As a consequence, a mathematical miscalculation may either induce a sale when it is in the consumer’s best interest not to purchase a unit of the good or cause a consumer not to purchase a unit when he should do so. To model such mistakes, let $\lambda$ be the probability that a consumer makes a mathematical miscalculation. Since the size of the population is normalized to unity $\lambda$ also represents the proportion of the population that will make such an error. Thus, if the monopolist announces a price "two units for $X$" a consumer of type $i$ knows, with certainty, that the payoff from buying two units is $u^2_i - X$. But, if the consumer prefers to buy only one unit, with probability $1 - \lambda$ he buys one unit and with probability $\lambda$ he makes a miscalculation and believes it is best not to buy the good. Furthermore, if it is best to purchase zero units of the good a consumer does so with probability $1 - \lambda$. With probability $\lambda$ he buys a unit. Three things should be pointed out regarding the mechanism for which consumers make mistakes. First, at extremely high prices (those with a per-unit above $u^1_A$) consumers do not mistakenly purchase a good. This eliminates the opportunity for the monopolist to charge a price beyond the level that any consumer would be willing to pay for a unit of the good and make a positive amount of sales. Secondly, consumers only make miscalculations when doing division problems. Even though errors are frequently made in all types of mathematical problems, miscalculations in division problems are much more common. Finally, a mistake is made when implementing a best response not when calculating which is the optimal purchase for a consumer. Thus, a mathematical miscalculation is an exogenous probability that when a consumer’s best response is to purchase zero units

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4Rubinstein, Meyer, and Evans (2001) illustrate that the error rate for addition problems never exceeds 5.1% while division problems never have error rates below 10.7%.
one is bought and the probability that when a consumer’s best response is to buy one unit of the good none are purchased.

The modeling of mistakes here is similar to that used in McCannon (2004) studying the effect of consumer mistakes in price competition games. In his paper the mechanism generating the mistaken purchase is unmodeled. The environment has consumers desiring only one unit and multiple firms competing to explain price dispersion and positive profits with homogeneous goods. Here, I study mathematical miscalculation’s effect on the pricing of a monopolist.

Similar setups occur in other literatures. For example, in the refinement literature, Young (1993) uses mistakes as a way to generate the use of conventions. Specifically, player $j$ makes a mistake with probability $\alpha_j\epsilon$ where $\alpha_j$ is player specific and $\epsilon$ is common. Kandori, Mailath and Rob (1993) study long run equilibrium in 2x2 games with mutations. They assume that a player chooses his best response with a probability $1 - 2\epsilon$ and randomizes over the two strategies playing each with probability $\epsilon$. Therefore, the model presented here can be thought of as a simplification of these models where mistakes occur only by consumers, occur with the same frequency by all consumers, and are generated by mistaken division problems when a multi-unit pricing strategy is used. Similar assumptions are made in evolutionary games. Hehenkamp (2002) studies an evolutionary model of price competition with consumers who sluggishly learn prices. With a positive probability they randomly select a firm to buy from. Hehenkamp models random learning in a similar manner that miscalculations occur in this model. The literature on public goods giving acknowledges the impor-
tance of mistakes. Palfrey and Prisbey (1996 & 1997) attempt to extract the random "noise" of mistakes from individual’s feeling of "warm-glow". They remark, "the anomalies [in experimental data] might be cause for a serious reexamination of the theory, as the signal trouble for current economic models". (p.829) This paper is a modest attempt to do so.

4 Linear Pricing and Miscalculations

As stated, the monopolist, in this framework, has three general pricing strategies from which to choose. I first focus on the two linear pricing strategies, single-unit and multi-unit pricing, to illustrate the effect mathematical miscalculations have on a market.

4.1 Case I

Consider Case I where A consumers value one unit more than B consumers and the gap in utility increases when two units are purchased. Consider the multi-unit pricing strategy of "two units for a price". Each consumer must decide whether to buy zero, one, or two units. He prefers one unit to none if the payoff is greater, or rather, \( u^1_i \geq p \) where \( p \) is the per-unit price of the good sold by the monopolist. A consumer prefers to buy two units rather than one unit when the payoff is greater, or rather, \( u^2_i - 2p \geq u^1_i - p \). This simplifies to requiring that \( p \leq m_i \). Thus, when selecting a multi-unit price, there exists intervals divided by \( m_i \) and \( u^1_i \) of the per-unit prices at which consumer behavior is unchanged throughout. For example, every per-unit price selected by the monopolist between \( m_B \) and \( u^1_B \) results in B consumers preferring to buy one unit and A consumers preferring to buy two units.
(since $m_A$ is greater than $u_B^1$ in Case I). It follows that if a monopolist is to select a price in a particular interval the best price to choose is the upper bound of the interval. Therefore, the following table describes the per-unit price intervals, consumer behavior, and resulting profit in the four possible scenarios.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Price Range</th>
<th>Units to $A$</th>
<th>Units to $B$</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$p \leq m_B$</td>
<td>2</td>
<td>2</td>
<td>$2m_B$</td>
</tr>
<tr>
<td>(b)</td>
<td>$p \in (m_B, u_B^1]$</td>
<td>2</td>
<td>$1 - \lambda$</td>
<td>$[2\alpha + (1 - \alpha)(1 - \lambda)]u_B^1$</td>
</tr>
<tr>
<td>(c)</td>
<td>$p \in (u_B^1, m_A]$</td>
<td>2</td>
<td>$\lambda$</td>
<td>$[2\alpha + (1 - \alpha)\lambda]m_A$</td>
</tr>
<tr>
<td>(d)</td>
<td>$p \in (m_A, u_A^1]$</td>
<td>$1 - \lambda$</td>
<td>$\lambda$</td>
<td>$[\alpha (1 - \lambda) + \lambda (1 - \alpha)]u_A^1$</td>
</tr>
</tbody>
</table>

Throughout the paper, (a) refers to pricing to sell two units to both types of consumers where (b) and (c) refer to selling two to one type and one and zero respectively to the other type of consumer. Finally, (d) refers to pricing to sell one unit to $A$ consumers. Figure 4 illustrates the areas for values of $\alpha$ and $\lambda$ for which each scenario is optimal. Each line in the figure represents an indifference curve between two scenarios. The derivation of the figure is given in the appendix.\(^5\)

Consider situations where there is a low probability of a mathematical miscalculation and most of the population is of type $B$, or rather, the area where (a) is best. In such a case, a monopolist, using a multi-unit pricing strategy, finds it optimal to charge a low price so that both types of consumers prefer to buy two units of the good. With so much of the population

\(^5\)Figure 4 represents Case I when $2m_A \geq u_A^1$, $4m_B \geq u_A^1 \geq 2m_B \geq \frac{u_A^1 u_B^1}{u_A^1 + u_B^1}$, and $2m_B \leq \frac{u_B^1 (m_A + m_B)}{m_A}$. The implication of relaxing these assumptions is discussed in the appendix.
of type $B$, it must charge a price that eliminates their mistakes. Since, in Case I, $B$ consumers value two units less than do $A$ consumers this implies that $A$ consumers also buy two units. In the area where $(b)$ is best more of the consumers are of type $A$ and miscalculations are still quite infrequent. Since there are more $A$ consumers and less $B$ consumers the monopolist raises the price to generate more revenue from the $A$ consumers and, as a consequence, it is now favorable for $B$ consumers to buy only one unit. Similarly, for the area where $(c)$ is preferred, most of the consumers are of type $A$ and miscalculations are more frequent. The monopolist raises the price even further to extract more profit from $A$ consumers selling only to mistaken $B$ consumers. Finally, consider the region where $(d)$ is the monopolist’s preferred scenario. Notice that this is best when there are mostly $B$ consumers and there is a high likelihood of miscalculations. In this scenario the monopolist chooses a
high price to sell only one unit to $A$ consumers. Mathematical miscalculations cause some $A$ consumers to not purchase when it is in their best interest to do so but also causes some $B$ consumers to purchase when it is not best for them to make the purchase.

Figure 4 illustrates the best price to charge when the monopolist selects a multi-unit pricing strategy. Single-unit pricing, which eliminates the need for a division computation, avoids mathematical miscalculations. Thus, the payoff to the monopolist using single-unit pricing is Table 1 with the requirement that $\lambda = 0$. Thus, one simply needs to consider the $\alpha$-axis in Figure 4 to determine the best single-unit pricing strategy at the different distributions of consumer types in the population. As a result, in Case I, if most are of type $B$, a price of $m_B$, which induces two units to be purchased by both, is best. For large values of $\alpha$ a price of $m_A$, which extracts more surplus from the $A$ consumers, is best. At intermediate values where the population is more evenly divided a price of $u_B^1$, selling two units to $A$ consumers, is the most preferred single-unit pricing strategy for the monopolist.6

If the monopolist has the choice between engaging in single-unit or multi-unit pricing which would it prefer? Notice that the four scenarios in Case I represent four different ways in which miscalculations affect the monopolist’s payoff. In (a) the payoff does not change with the frequency of miscalculations and the monopolist is indifferent between a single-unit or multi-unit pricing strategy that results in a per-unit price of $m_B$. In (b) miscalculations decrease the profit. Thus, a single-unit price is always better. In the area

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6This requires that the $ab$ – line and $bc$ – line intersect the $\alpha$-axis at values between zero and one. From the appendix, the $\alpha$-intercepts are $\frac{2m_B - u_B^1}{u_B^1}$ and $\frac{u_B^1}{2m_A - u_B^1}$ respectively. Thus, this requires that $2m_B \geq u_B^1$ and $2m_Am_B \leq u_B^1 (m_A + m_B)$. 

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where \((c)\) is best miscalculations increase the monopolist’s payoff making this a desirable strategy. Finally, for the area where \((d)\) is best notice that for every fixed value of \(\alpha\) if \(\lambda\) decreases to zero another scenario is better. Since single-unit pricing is represented as the \(\alpha\)-axis in Figure 4 there always exists a single-unit price that dominates. The impact mathematical miscalculations have on the pricing of a monopolist is stated in the following proposition.

**Proposition 1** In Case I there exists environments where the best linear pricing strategy is a multi-unit price that profits from mistakes of \(B\) consumers. Furthermore, there exists environments where the best linear price is a single-unit price that eliminates lost sales due to miscalculations.

**Proof.** To prove the first statement I need to show that there exists environments where multi-unit pricing is best. In \((c)\) and \((d)\) (when \(\alpha \leq \frac{1}{2}\)) miscalculations increase profit so that multi-unit pricing is better than the corresponding single-unit price. Consider, first, \((c)\). In Case I, \((c)\) is the best multi-unit pricing strategy when \(\alpha = \lambda = 1\). Single-unit pricing can be represented in Table 1 when \(\lambda = 0\). If \(\alpha = 1\) and \(\lambda = 0\) the payoff in \((c)\) is greater than \((a)\) and \((b)\). Also, if \(2m_A \geq u^1_A\) then \((c)\) is preferred to \((d)\) when \(\lambda = 0\). Furthermore, since mistakes increase the monopolist’s profit in \((c)\), the multi-unit price two for \(2m_A\) results in a greater profit than the single-unit price of one for \(m_A\). Therefore, any region where a price of one for \(m_A\) is best the price two for \(2m_A\) is better. The final consideration is the single-unit price of \(u^1_A\) ((d) when \(\lambda = 0\)) when \(u^1_A \geq 2m_A\). From Table 1, the multi-unit price in \((c)\) is better than the single-unit price in \((d)\) when \([2\alpha + (1 - \alpha)\lambda] m_A \geq \alpha u^1_A\). Simplifying, this holds when \(\lambda \geq \frac{\alpha(u^1_A - 2m_A)}{(1 - \alpha)m_A}\).
When this inequality is satisfied there exists a range of values for $\alpha$ and $\lambda$ for which the price of $2m_A$ for two units is the best linear pricing strategy.

Now consider $(d)$. When $\alpha \leq \frac{1}{2}$ miscalculations increase the payoff, which is required for this to be the best linear pricing strategy. For there to exist such a region it must be that if $\lambda$ decreases to zero $(d)$ remains the best scenario or else single-unit pricing would dominate. From Table 1, when $\lambda = 0$, $(d)$ generates a greater profit than $(a)$ if $\alpha \geq \frac{2m_B}{u_B}$, than $(b)$ if $\alpha \geq \frac{u_B}{u_A - u_B}$, and a greater profit that $(c)$ if $u_A^1 \geq 2m_A$. The first two expression must also be less than $\frac{1}{2}$. Thus, if $u_A^1 \geq \max \{2m_A, 3u_B^1, 4m_B\}$ then there exist values of $\alpha$ for which single-unit pricing in $(d)$ is the best single-unit price. Since the payoff in $(d)$ increases with miscalculations this also implies that there exists a region where the announcement two units for $2u_A^1$ is the best linear pricing strategy.

To prove the second statement I need to show that there exists an area where a single-unit price dominates the best multi-unit price. First, notice that the monopolist’s payoff in $(b)$ is strictly decreasing in $\lambda$. Thus, a price of $m_A$ for one dominates two for $2m_A$. Also, in $(d)$, miscalculations induce sales from $B$ consumers but lose $A$ consumers. Thus, when $\alpha \geq \frac{1}{2}$ the announcement of one for $u_A^1$ dominates two for $2u_A^1$. Therefore, areas where $(b)$ is the best multi-unit pricing strategy and ranges with $\alpha \geq \frac{1}{2}$ where $(d)$ is the best multi-unit pricing strategy suffice to illustrate that single-unit pricing is the monopolist’s optimal linear pricing strategy.

Consider, first, $(d)$. When $\alpha \geq \frac{1}{2}$ the payoff in $(d)$ is decreasing with $\lambda$. Therefore, the point where $(d)$ gives the highest payoff is where $\alpha = 1$ and $\lambda = 0$. At this point $(a)$, $(b)$ and $(c)$ give $2m_B$, $2u_B^1$ and $2m_A$ respectively.
Since \( m_A > u^1_B > m_B \) in Case I, (c) is the greatest of the three. Therefore, there exists a range with \( \alpha \geq \frac{1}{2} \) where (d) is the optimal multi-unit pricing strategy if \( u^1_A \geq 2m_A \). Now consider (b). The payoff is decreasing in \( \lambda \) so consider multi-unit pricing when \( \lambda = 0 \). From Table 1, (b) is preferred to (a) when \( \lambda = 0 \) if \( \alpha \geq \frac{2m_B-u^1_B}{u^1_B} \) which is less than one and (b) is preferred to (c) when \( \lambda = 0 \) if \( \alpha \geq \frac{u^1_B}{2m_A-u^1_B} \) which is greater than zero. Since, at \( \lambda = 0 \), (c) is preferred to (d) when \( 2m_A \geq u^1_A \) (and it was already shown for \( u^1_A \geq 2m_A \)) if \( \alpha \) takes a value in the interval \( \left( \frac{2m_B-u^1_B}{u^1_B}, \frac{u^1_B}{2m_A-u^1_B} \right) \) then there exists a range where the price of two for \( 2u^1_B \) is the best multi-unit pricing strategy. ■

Therefore, what effect does mathematical miscalculations have in Case I? When most of the population is of type \( A \) the monopolist prefers to announce a price of two units for \( 2m_A \). This price extracts all surplus from \( A \) consumers buying two units. Some type \( B \) consumers will mistakenly purchase one unit of the good. For intermediate values of \( \alpha \) the monopolist must announce the single-unit price of one for \( u^1_B \). At this price \( A \) consumers prefer to buy two and \( B \) consumers prefer to buy one. The use of single-unit pricing eliminates mistakes made by \( B \) consumers miscomputing their cost. Only for small values of \( \alpha \) is the monopolist indifferent between pricing two for \( 2m_B \) and one for \( m_B \) since both types buy two units.

### 4.2 Case II

Now consider Case II where \( A \) consumers, who value one unit more than \( B \) consumers, continue to value two units more but the gap in utility diminishes. Again, consider the strategy of multi-unit pricing. The following table describes the per-unit price intervals, consumer behavior, and resulting
profit in the four possible scenarios of prices.

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</tr>
<tr>
<td>(b)</td>
<td>$p \in (m_B, m_A]$</td>
<td>2</td>
<td>$1 - \lambda$</td>
<td>$[2\alpha + (1 - \alpha) (1 - \lambda)] m_A$</td>
</tr>
<tr>
<td>(c)</td>
<td>$p \in (m_A, u_B^1]$</td>
<td>$1 - \lambda$</td>
<td>$1 - \lambda$</td>
<td>$(1 - \lambda) u_B^1$</td>
</tr>
<tr>
<td>(d)</td>
<td>$p \in (u_B^1, u_A^1]$</td>
<td>$1 - \lambda$</td>
<td>$\lambda$</td>
<td>$[\alpha (1 - \lambda) + \lambda (1 - \alpha)] u_A^1$</td>
</tr>
</tbody>
</table>

Figure 5 illustrates the areas for values of $\alpha$ and $\lambda$ for which each scenario is the optimal multi-unit price. Again, each line in the figure represents the indifference curve between two scenarios. The derivation of the figure is given in the appendix.\(^7\)

\(^7\)Figure 5 represents Case II when $2m_A (u_A^1 + m_A) \geq u_A^1 u_A^1$, $u_B^1 \geq 2m_B$, and $4m_B \geq u_A^1 \geq 2m_B$. The implication of relaxing these assumptions is discussed in the appendix.
Figure 5 illustrates the best price to charge when the monopolist selects a multi-unit pricing strategy. Single-unit pricing, which eliminates the need for a division computation, eliminates mathematical miscalculations. As in the previous section the payoff to the monopolist is Table 2 with the requirement that \( \lambda = 0 \). One simply needs to consider the \( \alpha \)-axis in Figure 5 to determine the best single-unit pricing strategy at the different distributions of consumer types in the population.

Again, the question can be asked as to if the monopolist has the choice between engaging in single-unit or multi-unit pricing which would it prefer? Notice that miscalculations decrease the monopolist’s profit in both (b) and (c). Therefore, single-unit pricing dominates either of these. Also, as depicted in Figure 5, for every point that has (a) as the best multi-unit pricing strategy has the feature that if the frequency of error dropped to zero another strategy would be better. As is illustrated in the appendix, this occurs when \( u^1_B \geq 2m_B \). Instead, if this additional requirement did not hold, then the monopolist is indifferent between announcing one for \( m_B \) and two for \( 2m_B \). Finally, consider (d). Notice that mathematical miscalculations increase the monopolist’s profit by causing some \( B \) consumers to mistakenly purchase a unit but lowers its profit by losing some sales due to type \( A \) consumers being in error. For this to be the best linear pricing strategy there must be more \( B \) consumers, \( \alpha \leq \frac{1}{2} \). As illustrated in Figure 5 and shown in the appendix, when \( \alpha \leq \frac{1}{2} \) the region in which the scenario (d) is the best multi-unit pricing strategy requires \( \lambda > 0 \). As a consequence, there exists a single-unit price that results in a greater payoff. Therefore, unlike Case I, there is no environment where a multi-unit pricing strategy is best. The following proposition
states the results in Case II.

**Proposition 2** In Case II a single-unit pricing strategy weakly dominates all multi-unit pricing strategies.

**Proof.** As illustrated in Table 2 mathematical miscalculations reduce the monopolist’s payoff in (b) and (c). Thus, a single-unit pricing strategy dominates. In (a) the monopolist is indifferent between announcing two units for $2m_B$ and one for $m_B$ since both types of consumers buy two units requiring no divisional computation and thus allowing no mistakes. Thus, single-unit pricing weakly dominates a multi-unit pricing strategy. Finally, consider (d). The announcements two for $2u^1_A$ and one for $u^1_A$ result in payoffs of $[\alpha (1 - \lambda) + (1 - \alpha) \lambda] u^1_A$ and $\alpha u^1_A$ respectively. Setting the former greater than the latter and simplifying the multi-unit price is better when $\alpha \leq \frac{1}{2}$. Consider the line of indifference between (a) and (d) from Table 2, $\lambda^{ad} = \frac{2m_B - \alpha}{1 - 2\alpha}$. Notice that $\lambda^{ad}$ is discontinuous at $\alpha = \frac{1}{2}$ and intercepts the $\lambda$-axis at $\frac{2m_B}{u^1_A} > 0$. At $\alpha = \lambda = 0$ (a) is preferred by the monopolist to (d). Therefore, when $\alpha \leq \frac{1}{2}$ the region in which (d) is the best multi-unit pricing strategy, if it exists, must be for values of $\lambda$ strictly greater than zero. But, since single-unit pricing can be represented by $\lambda = 0$, there must be a single-unit price that dominates the multi-unit pricing strategy in (d). Since all four possible scenarios have been shown, each multi-unit pricing strategy is weakly dominated by some single-unit pricing strategy. ■

What effect does mathematical miscalculations have on the pricing behavior of a monopolist in Case II? If the monopolist selects a linear pricing strategy it announces a price for single units. The lost sales due to mathe-
matical miscalculations force the monopolist to select a price that eliminates the potential for error.

4.3 Case III

Finally, consider the case where $B$ consumers, who value the first unit relatively less, put more weight on a second unit than do $A$ consumers. As before, the following table describes the per-unit price intervals, consumer behavior, and resulting profit from the four possible scenarios of multi-unit pricing.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Price Range</th>
<th>Units to $A$</th>
<th>Units to $B$</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>$p \leq m_A$</td>
<td>2</td>
<td>2</td>
<td>$2m_A$</td>
</tr>
<tr>
<td>(b)</td>
<td>$p \in (m_A, m_B]$</td>
<td>$1 - \lambda$</td>
<td>2</td>
<td>$[\alpha (1 - \lambda) + 2 (1 - \alpha)] m_B$</td>
</tr>
<tr>
<td>(c)</td>
<td>$p \in (m_B, u_B^1]$</td>
<td>$1 - \lambda$</td>
<td>$1 - \lambda$</td>
<td>$(1 - \lambda) u_B$</td>
</tr>
<tr>
<td>(d)</td>
<td>$p \in (u_B^1, u_A^1]$</td>
<td>$1 - \lambda$</td>
<td>$\lambda$</td>
<td>$[\alpha (1 - \lambda) + \lambda (1 - \alpha)] u_A^1$</td>
</tr>
</tbody>
</table>

Figure 6 illustrates the areas for values of $\alpha$ and $\lambda$ for which each scenario is optimal. As before, the derivation of the figure is given in the appendix.\(^8\)

As in Case II, miscalculations reduce the monopolist’s payoff in (b) and (c) so that single-unit pricing is preferred. For every value of $\alpha$ at which (a) is the best multi-unit pricing strategy another dominates if $\lambda$ is reduced. Thus, single-unit pricing is better. Finally, as before in (d), the multi-unit price of two for $2u_A^1$ is better than one for $u_A^1$ when $\alpha \leq \frac{1}{2}$.

\(^8\)Figure 6 represents Case III when $u_B^1 \geq 4m_A$ and $m_B = \frac{u_A^1 u_B}{2u_A^1 - u_B}$. The implication of relaxing these assumptions is discussed in the appendix.
Proposition 3 In Case III there exists environments where the best linear pricing strategy is a multi-unit price that profits from mistakes of type B consumers. Furthermore, there exists environments where the best linear price is a single-unit price that eliminates lost sales due to miscalculations.

Proof. To prove the first statement it is required that there exists an environment with $\alpha \leq \frac{1}{2}$ where the multi-unit price of two for $2u_A^1$, (d), is the most preferred linear pricing strategy. Single-unit pricing, as stated, is represented by Table 3 when $\lambda = 0$. Thus, for there to exist a region where the multi-unit price of two for $2u_A^1$ is the best linear pricing strategy it must be that this price is the best multi-unit pricing strategy when $\alpha$ is fixed and $\lambda$ drops to zero. Therefore, consider Table 3 when $\lambda = 0$. The payoff from (d) is greater than (a) if $\alpha \geq \frac{2m_A}{u_A^1}$, greater than (b) if $\alpha \geq \frac{2m_B}{u_A^1 + m_B}$, and greater than (c) if $\alpha \geq \frac{u_B^1}{u_A^1}$. Also, as stated, the value of $\alpha$ must fall below $\frac{1}{2}$. Therefore,
each of this points must take values less than $\frac{1}{2}$ as well. Setting each less than one-half the existence of a region where multi-unit pricing is the best linear pricing strategy requires that $u^1_A \geq \max \{2u^1_B, 3m_B, 4m_A\}$. Therefore, if this inequality holds then there exists a range of values of $\alpha$ for which one unit for $u^1_A$ is the best single-unit price. Finally, since $(d)$ is the only scenario that is increasing in $\lambda$, the existence of this range of values of $\alpha$ guarantees that the multi-unit price of two for $2u^1_A$ is the best linear pricing strategy.

To prove the second statement I need to show that there exists an area where the best multi-unit pricing strategy is either $(b)$ or $(c)$, since both are decreasing in $\lambda$ and are therefore dominated by a single-unit pricing strategy. Consider the point where $\alpha = \lambda = 0$. At this point, from Table 3, a gives the monopolist a payoff of $2m_A$ whereas $(b)$ generates $2m_B$ in profit. Since, in Case III, $m_B > m_A$ ($a$) is not the best announcement. Also, $(d)$ results in a zero profit. Thus, the best multi-unit price for values of $\alpha$ and $\lambda$ near zero is either $(b)$ or $(c)$, as was to be shown.

What effect does mathematical miscalculations have? As in the previous cases there exists environments where mistakes reduce sales and the monopolist must select a strategy that eliminates them. Also, as in Case I, there exists environments where the monopolist prefers a multi-unit price. Here, this occurs only when most of the consumers are type $B$ so that pricing to sell one unit to $A$ consumers and none to $B$ consumers generate many mistaken sales at a high price. Unlike Case I, this is the only way in which the miscalculations can be used to the monopolist’s advantage.
5 Nonlinear Pricing and Miscalculations

A monopolist, unable to differentiate between consumers, may instead engage in price discrimination. Since, in the model presented here, consumers buy zero, one or two units of the good the monopolist may select to sell two units to one type of consumer and one unit to the other. To successfully engage in nonlinear pricing it must be that both types of consumers want to purchase - individual rationality - and that neither wants to make a different purchase - incentive capability.

Suppose first that the monopolist prices to sell two units to A consumers and one unit to B consumers. Let $p_A$ and $p_B$ denote the price for two and one unit respectively. The individual rationality constraints are

\[ IR_A : \quad u^2_A - p_A \geq 0 \]  
\[ IR_B : \quad u^1_B - p_B \geq 0 . \]  

Each consumer can deviate by buying the others intended bundle or buying two single units. As a result, there are two incentive compatibility constraints for each type of consumer

\[ IC^1_A : \quad u^2_A - p_A \geq u^1_A - p_B \]  
\[ IC^2_A : \quad u^2_A - p_A \geq u^2_A - 2p_B \]  

for A consumers and

\[ IC^1_B : \quad u^1_B - p_B \geq u^2_B - p_A \]  
\[ IC^2_B : \quad u^1_B - p_B \geq u^2_B - 2p_B \]
for $B$ consumers. Notice that if $m_B > m_A$ then there are no prices that satisfy $IC_A^1$ and $IC_B^1$. If $B$ consumers gain more utility purchasing a second unit than $A$ consumers then there are no prices that induce $A$ consumers to buy a second unit where $B$ consumers does not have the incentive to do so as well.\footnote{The constraint $IC_A^1$ is satisfied if and only if $p_A \leq p_B + m_A$. But $IC_B^1$ is satisfied if and only if $p_A \geq p_B + m_B$. Therefore, as stated, if $m_B > m_A$, which is Case III, there is no set of prices $p_A$ and $p_B$ that satisfy both constraints. As a consequence, the monopolist cannot engage in a nonlinear pricing strategy selling two units to $A$ consumers and one to $B$ consumers in Case III.} Since Case III is defined where $m_B > m_A$ we can restrict attention to Cases I and II. The following proposition states the nonlinear pricing strategy when the monopolist is selling two units to $A$ consumers and one unit to $B$ consumers.

**Proposition 4** In Cases I and II the monopolist, engaging in nonlinear pricing, sells one unit to $B$ consumers extracting all of their surplus and sells two units to $A$ consumers charging the highest price that keeps them preferring two units.

**Proof.** The monopolist must select prices so that each consumer’s individual rationality and incentive compatibility constraints are satisfied. Consider first $A$ consumers. It follows from (1) that $p_A \leq u_A^2$. From (3) it must be that $p_A \leq p_B + m_A$ and $p_A \leq 2p_B$. Since the monopolist is selecting the prices that maximize its payoff it follows that $p_A = \min \{u_A^2, p_B + m_A, 2p_B\}$.

Now consider type $B$ consumers. From (2), $p_B \leq u_B^1$. From (4), $p_B \leq p_A - m_B$ and $p_B \geq m_B$. Thus, $p_B = \min \{p_A - m_B, u_B^1\}$. Since the price to $B$ consumers must be less than $u_B^1$ the monopolist never prices $p_A = u_A^2$.

This occurs because even at $p_B = u_B^1$ the price $u_B^1 + m_A$ (which equals
\(u^2_A - (u^1_A - u^1_B))\) is less than \(u^2_A\). Consider, first, Case I. Since \(m_A > u^1_B\), a price of \(p_B + m_A\) must always be more than \(2p_B\). Therefore, \(p_A = 2p_B\). Suppose that \(p_B = p_A - m_B\). As a result, \(p_B = m_B\) and \(p_A = 2m_B\). Instead, if \(p_B = u^1_B\) then \(p_A = 2u^1_B\), which gives a greater payoff. Therefore, in Case I, the monopolist, selling two units to A consumers and one to B consumers, charges \(u^1_B\) for one unit and extracts all surplus from B consumers. The monopolist charges \(2u^1_B\) for two units where A consumers are indifferent between buying the bundle and buying two single units.

Now consider Case II. First, suppose that \(p_B = p_A - m_B\). If it selects \(p_A = p_B + m_A\) then this implies that \(p_B = p_B + m_A - m_B\), which cannot occur. If, \(p_A = 2p_B\) then this implies that \(p_B = m_B\) and \(p_A = 2m_B\). Instead, suppose that \(p_B = u^1_B\). This implies that \(p_B + m_A\) is less than \(2p_B\) so that it selects \(p_A = u^1_B + m_A\). Notice that the prices charged to both A and B consumers are greater. Thus, the best pricing strategy for the monopolist selling two units to A consumers and one unit to B consumers in Case II is to charge \(u^1_B + m_A\) for two units and \(u^1_B\) for one unit where all of B consumer’s surplus is taken and A consumers are indifferent between buying two units and one. ■

In both Case I and Case II the monopolist is able to extract all of the surplus from B consumers to whom it is only selling one unit. The monopolist sells two units to A consumers but is unable to extract all of their surplus. Instead, it prices up to the point at which A consumers are indifferent between purchasing the two units for the stated price and making another purchasing decision. In Case I it is the point where A consumers would rather buy two of the monopolist’s single units than the bundle of two. In Case II it is
the point where $A$ consumers would prefer to buy only one unit rather than two. Furthermore, notice that in Case I the pricing strategy is linear. Thus, price discrimination is ineffective and the optimal pricing strategy is as is previously described with linear pricing.

Instead, suppose that the monopolist prices to sell one unit to $A$ consumers and two units to $B$ consumers. Now, let $p_A$ and $p_B$ denote the price for one and two units respectively. The individual rationality constraints are

\begin{align*}
IR_A & : \ u_A^1 - p_A \geq 0 \\
IR_B & : \ u_B^2 - p_B \geq 0 .
\end{align*}

Again, each consumer can deviate by buying the other types intended bundle or by buying two single units. Thus,

\begin{align*}
IC_A^1 & : \ u_A^1 - p_A \geq u_A^2 - p_B \\
IC_A^2 & : \ u_A^1 - p_A \geq u_A^2 - 2p_A
\end{align*}

for $A$ consumers and

\begin{align*}
IC_B^1 & : \ u_B^2 - p_B \geq u_B^1 - p_A \\
IC_B^2 & : \ u_B^2 - p_B \geq u_B^2 - 2p_A
\end{align*}

for $B$ consumers. Notice now that if $m_A > m_B$ then there are no prices that satisfy $IC_A^1$ and $IC_B^1$.\footnote{The constraint $IC_A^1$ is satisfied if and only if $p_A \leq p_B - m_A$. But $IC_B^1$ is satisfied if and only if $p_A \geq p_B - m_B$. Therefore, as stated, if $m_A > m_B$, which occurs in both Case I and II, there is no set of prices $p_A$ and $p_B$ that satisfy both constraints. As a consequence, the monopolist can engage in a nonlinear pricing strategy selling two units to $A$ consumers and one to $B$ consumers in Case III.}

Thus, attention can be restricted to Case III. The following proposition illustrates the monopolist’s optimal nonlinear pricing strategy.
Proposition 5 In Case III the monopolist, engaging in nonlinear pricing, sells one unit to A consumers and two units to B consumers extracting all the surplus from each.

Proof. The monopolist selects prices that satisfy each type’s individual rationality and incentive compatibility constraints. Consider first A consumers. It follows from (5) that \( p_A \leq u_A^1 \). From (7), \( p_A \leq p_B - m_A \) and \( p_A \geq m_A \). Since the monopolist is maximizing its payoff \( p_A = \min \{u_A^1, p_B - m_A\} \).

Now consider B consumers. From (6), \( p_B \leq u_B^2 \). From (8), \( p_B \leq p_A + m_B \) and \( p_B \leq 2p_A \). Thus, \( p_B = \min \{u_B^2, p_A + m_B, 2p_A\} \). Suppose that \( p_A = u_A^1 \).

Since \( u_A^1 > m_B \), \( 2p_A > p_A + m_B \). Also, since \( p_A + m_B = u_B^2 + u_A^1 - u_B^1 \) is greater than \( u_B^2 \), \( p_B = u_B^2 \). Therefore, the monopolist selects a price of \( u_A^1 \) for one unit and \( u_B^2 \) for two units. These prices extract all of the surplus from both A consumers buying one unit and B consumers buying two units.

In Case III the monopolist is able to extract all surplus from both types of consumers. This is possible because A consumers, who place much weight on consuming one unit, put little weight on a second unit whereas B consumers, whom place little weight on the first unit, have a relatively greater gain when purchasing a second unit. Thus, the monopolist is able to sell one to A consumers and two to B consumers, extract all of the surplus from each, and not drive either to change their buying behavior.

If the monopolist has the choice between single-unit, multi-unit and nonlinear pricing which does it choose? Consider, first, Case I. Proposition 4 shows that if the monopolist sells two units to one type and one unit to another the best announcement is a linear price. Therefore, nonlinear pricing has no advantage and the optimal pricing is as previously described in
Proposition 1.

From Case II Proposition 4 illustrates that the nonlinear price selection is to price one unit for \( u_B^1 \) and two for \( u_B^1 + m_A \). This results in a profit of \( u_B^1 + \alpha m_A \). As shown in Proposition 2 the best linear pricing strategy is to choose a price for single units. From Table 2, single-unit pricing in (b) and (c) are dominated by nonlinear prices. Pricing to sell two units to both types, (a), is better than nonlinear pricing when the payoff is greater, or rather, when \( \alpha \leq \frac{2m_B - u_B^1}{m_A} \). For a population distribution that satisfies this inequality it is best to offer single units for \( m_B \) and sell two units to all consumers. Finally, pricing to sell a unit to only A consumers, (d), is better than nonlinear pricing when the payoff is greater. Setting \( \alpha u_A^1 \) greater than \( u_B^1 + \alpha m_A \) and simplifying the price of one unit for \( u_A^1 \) is better when \( \alpha \geq \frac{u_B^1}{u_A^1 - m_A} \). In these environments single-unit pricing is the best strategy for the monopolist.

Finally, in Case III, Proposition 5 illustrates that the monopolist, engaging in nonlinear pricing, sells one unit to A consumers and two units to B consumers extracting all surplus from both. The profit the monopolist earns if engaging in nonlinear pricing dominates all other pricing strategies. The one exception is the case where most of the population is of type A. Here, if the revenue generated from two units is greater than selling one unit to A consumers then it is better to offer each consumer a per-unit cost of \( m_A \) to increase the total number of sales.\(^{11}\) Like Case II, when most of the

\(^{11}\)The verification of this is straightforward. The payoff to the monopolist when selecting a nonlinear price is \( \alpha u_A^1 + (1 - \alpha) u_B^2 \). The payoff from the only viable multi-unit pricing strategy is \( |\alpha (1 - \lambda) + (1 - \alpha) \lambda| u_A^1 \). Since \( u_B^2 \geq u_A^1 \) in Case III nonlinear pricing dominates. The payoff from the four single-unit pricing strategies follows from Table 3 with \( \lambda = 0 \). It follows immediately that (b), (c), and (d) give payoffs that are less than the
consumers are of type $A$ it is best to choose a price that extracts all of the surplus from sales to $A$ consumers (either the one unit in Case II or two units in Case III). Unlike Case II, nonlinear pricing is the optimal pricing strategy both for more evenly distributed populations as well as populations that are mostly $B$ consumers since, in Case III, the monopolist is able to extract all of the surplus from them.

6 Conclusion

In practice many types of linear pricing strategies are used. Economic theory tends to focus on prices that take the form of a cost for each single unit purchased. More complicated pricing strategies that announce a cost for more than one unit, which I refer to as multi-unit pricing, has not been mentioned in the literature. In this paper I provide a reason for the existence of both single-unit and multi-unit pricing strategies. The mechanism that differentiates the two linear pricing strategies is the occurrence of mathematical miscalculations illustrated in experimental psychology. In an environment where a monopolist is selecting a pricing strategy to sell units of a good to a unobservably, heterogeneous population of consumer whom make mathematical miscalculations there exists situations where the monopolist prefers single-unit pricing to multi-unit pricing and other situations where multi-unit pricing is preferred to single-unit pricing to take advantage of the miscalculations.

nonlinear pricing for any value of $\alpha$. Therefore consider the price of one for $m_A$ generating a payoff of $2m_A$. Simplifying, this payoff is greater than when selecting the nonlinear price when $\alpha \geq \frac{u_B^2 - 2m_A}{u_B - u_A^1}$ which is greater than zero in Case III. As a result, if $2m_A \geq u_A^1$ there exists values of $\alpha$ for which nonlinear pricing is not optimal.
Mathematical miscalculations have two effects on consumers. First, they may induce a consumer to buy a unit of the good when it is actually not in his best interest to do so. In such a situation, a monopolist may prefer to select a cost for multiple units of the good so that consumers, when determining whether to buy zero or one unit of the good, must engage in a division problem where they might make an error. Secondly, a mathematical miscalculation may cause a consumer to not purchase a unit of the good when it is best for him to buy it. In this case, a monopolist may want to announce a price for every single unit of the good to remove the division problem and eliminate the possibility of losing sales due to mathematical miscalculations. The opposing effects of the mechanism distinguish the two linear pricing strategies of single-unit pricing and multi-unit pricing.

Furthermore, when faced with a heterogeneous population of consumers a monopolist has the opportunity to engage in second degree price discrimination selling one unit for more than one-half of the price of two units. This nonlinear pricing strategy is useful to extract more surplus from the consumers than simple single-unit pricing. It is shown that there exists environments where multi-unit pricing is preferential even when nonlinear pricing strategies are allowed. This occurs in Case I where \( A \) consumers value both the first and second unit more than \( B \) consumers value any amount. In this case, nonlinear pricing fails to adequately separate the two groups of consumers allowing the additional sales from mistaken consumers when multi-unit pricing is used to dominate. Future work could expand on the occurrence of mathematical miscalculations in more complicated environments that the one presented here. For example, if multiple firms were competing
for the sales the reduced market power might cause the firms to act differently. Also, the miscalculations only occur when a division problem is required. An extension to include addition problems would distinguish single-unit pricing from linear pricing schedules. Furthermore, mathematical miscalculations are one example of a mistake that can be made by a consumer that firms react to for their advantage. Future work could incorporate other psychological phenomenon expressed in the experimental literature to market activity. Finally, the experiments cited here are conducted in laboratories. Empirical evidence of miscalculations from actual market activity could provide valuable insight into the form that such mistakes take in practice and what responses firms have to these errors.

7 References


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8 Appendix

The goal of the appendix is to derive the best multi-unit pricing strategy for each given value of $\alpha$ and $\lambda$. I look at multi-unit pricing in each of the three cases independently.

8.1 Case I

Consider the point where $\alpha = \lambda = 1$. At this point (c) gives the monopolist a greater payoff than any other scenario. From Table 1, consider the lines of indifference between (c) and the other three scenarios.

- **ac - line**: $\lambda^{ac} = \frac{2m_B - 2\alpha m_A}{(1-\alpha)m_A}$
- **bc - line**: $\lambda^{bc} = \frac{(1+\alpha)u_B^1 - 2\alpha m_A}{(1-\alpha)(m_A + u_B^1)}$
- **cd - line**: $\lambda^{cd} = \frac{\alpha(2m_A - u_A^1)}{(1-2\alpha)u_A^1 - (1-\alpha)m_A}$

Notice that the *ac - line* and *bc - line* are downward sloping, concave, and approach $-\infty$ as $\alpha$ moves closer to one. The *cd - line* is discontinuous at $\alpha = \frac{u_A^1 - m_A}{2u_A^1 - m_A} \in (0, 1)$ and if $2m_A \geq u_A^1$ it is increasing and convex. Therefore, the area above these three indifference curves is the region where the monopolist prefers (c) as in Figure 4.
Now consider (a). From Table 1, consider the indifference curves between (a) and the scenarios (b) and (d).

\[ \lambda_{ab} = \frac{(1+\alpha)u_B^1 - 2m_B}{(1-\alpha)u_B^1} \]
\[ \lambda_{ad} = \frac{2m_B - \alpha u_A^1}{(1-2\alpha)u_A^1} \]

The \( ab - line \) is increasing, convex, and as \( \alpha \) approaches 1 its value goes to \( \infty \). The \( ad - line \) is discontinuous at \( \alpha = \frac{1}{2} \) and if \( 4m_B \geq u_A^1 \) it is increasing. Furthermore, the \( ad - line \) crosses through the feasible region when \( u_A^1 \geq 2m_B \). Thus, Figure 4 illustrates Case I when \( 4m_B > u_A^1 > 2m_B \).

There exists an area where (a) is preferred if the \( ad - line \) lies above the \( ab - line \). At \( \alpha = 0 \) the \( ad - line \) takes the value \( \frac{2m_B - u_A^1}{u_A^1} \) and the \( ab - line \) takes the value \( \frac{u_B^1 - 2m_B}{u_B^1} \). Thus, if \( 2m_B \geq \frac{u_A^1 u_B^1}{u_A^1 + u_B^1} \), then there exists a range where (a) is best.

From transitivity, the \( ab - line \), \( ac - line \), and the \( bc - line \) must all intersect at the same point as do the \( ad - line \), \( ac - line \), and \( cd - line \). The area below the \( ab - line \) and the \( bc - line \) are those values of \( \alpha \) and \( \lambda \) for which the monopolist prefers (b). Such values exist when the \( ab - line \) intersects the \( \alpha \)-axis at a point less than the \( bc - line \) does. This requires that \( \frac{2m_B - u_B^1}{u_B^1} \leq \frac{u_B^1}{2m_A - u_B^1} \), or rather, \( 2m_B \leq \frac{u_B^1(m_A + m_B)}{m_A} \). The values of \( \alpha \) and \( \lambda \) above the \( ad - line \) and \( cd - line \) is where (d) is preferred. From Table 1, when \( \alpha = 0 \) and \( \lambda = 1 \) (d) generates a greater payoff than (b) or (c) and is better than (a) when \( u_A^1 \geq 2m_B \). Hence, when this inequality holds a range where (d) is the preferred multi-unit strategy exists.

As a consequence, multi-unit pricing in Case I takes the form as depicted in Figure 4 when the assumptions (1) \( 2m_A \geq u_A^1 \), (2) \( 4m_B \geq u_A^1 \geq 2m_B \geq \frac{u_A^1 u_B^1}{u_A^1 + u_B^1} \), and (3) \( 2m_B \leq \frac{u_B^1(m_A + m_B)}{m_A} \). What implication does relaxing these as-
sumptions have? If (1) fails to hold then the \( cd - line \) is decreasing and convex. Since it passes through the point \( \alpha = \lambda = 0 \), is discontinuous at a value of \( \alpha \) between zero and one, and (c) is preferred at \( \alpha = \lambda = 1 \) relaxing the first assumption results in the only region where (c) is optimal being that where both \( \alpha \) and \( \lambda \) take large values. The first inequality in (2) results in the \( ad - line \) being downward sloping. Instead, if \( u_A^1 \geq 4m_B \) then the \( ad - line \) is increasing and convex, but the existence of a range where (d) is the best multi-unit pricing strategy is now guaranteed. If the second inequality in (2) does not hold then there is no region where (d) is the best scenario. If the last inequality in (2) is violated then there is no region where (a) is best. Finally, if (3) is violated then there is no area where (b) is the preferred scenario.

8.2 Case II

Consider the point where \( \alpha = \lambda = 1 \). At this point (c) and (d) give a zero payoff. Since, in Case II, \( m_A > m_B \), (b) has the greatest payoff. From Table 2, consider the lines of indifference between (b) and the other three scenarios.

\[
ab - line : \quad \lambda^{ab} = \frac{(1+\alpha)m_A-2m_B}{(1-\alpha)m_A} \\
bc - line : \quad \lambda^{bc} = \frac{(1+\alpha)m_A-u_B^1}{(1-\alpha)m_A-u_B^1} \\
bd - line : \quad \lambda^{bd} = \frac{m_A-\alpha(u_A^1-m_A)}{u_A^1+m_A-\alpha(2u_A^1+m_A)}
\]

Notice that the \( ab - line \) is increasing, convex and goes to \( \infty \) as \( \alpha \) approaches one. The \( bc - line \) is decreasing and convex from the point \( \alpha = 0 \) and \( \lambda = 1 \). The \( bd - line \) is discontinuous at the point \( \alpha = \frac{u_A^1+m_A}{2u_A^1+m_A} \in (0,1) \) and is increasing, as depicted in Figure 5, when \( 2m_A(u_A^1+m_A) \geq u_A^1u_A^1 \). The region outlined in Figure 5 is the values of \( \alpha \) and \( \lambda \) where (b) is most
preferred by the monopolist.

Now consider (c). From Table 2, consider the lines of indifference between this scenario and (a) and (d).

\[ ac - line : \quad \lambda^{ac} = \frac{u_B^1 - 2m_B}{u_B^1} \]
\[ cd - line : \quad \lambda^{cd} = \frac{u_B^1 - \alpha u_A^1}{u_B^1 + (1-2\alpha)u_A^1} \]

Notice that \( ac - line \) is a straight line and crosses through the feasible region when \( u_B^1 \geq 2m_B \). Furthermore, by transitivity, the \( ab - line, ac - line, \) and the \( bc - line \) must all intersect at the same point. Thus, the \( cd - line \) is as is shown in Figure 5. Similarly, transitivity implies that the \( bc - line, bd - line, \) and the \( cd - line \) must all intersect at the same point. Since the \( cd - line \) is decreasing it must be as Figure 5 illustrates. Therefore, the values of \( \alpha \) and \( \lambda \) within the \( ac - line, bc - line, \) and the \( cd - line \) are those where the monopolist prefers (c).

Now consider (a). The monopolist prefers (a) to (b) and (c) when \( \alpha \) and \( \lambda \) take values that lie above the \( ab - line \) and \( ac - line \). Thus, consider the indifference curve between (a) and (d). From Table 2,

\[ ad - line : \quad \lambda^{ad} = \frac{2m_B - \alpha}{1 - 2\alpha} \cdot \]

Notice, that the \( ad - line \) is increasing if \( 4m_B \geq u_A^1 \) and a region of feasible values of \( \alpha \) and \( \lambda \) exist where (d) is the optimal multi-unit pricing strategy so long as (d) is preferred at \( \alpha = 0 \) and \( \lambda = 1 \), or rather, if \( u_A^1 \geq 2m_B \).

As a consequence, multi-unit pricing in Case II takes the form as depicted in Figure 5 when the assumptions (1) \( 2m_A (u_A^1 + m_A) \geq u_A^1 u_A^1 \), (2) \( u_B^1 \geq 2m_B \), and (3) \( 4m_B \geq u_A^1 \geq 2m_B \). What implication does relaxing these assumptions have? If the first assumption the \( bd - line \) is downward sloping
so that \((b)\) is preferred in the region near \(\alpha = \lambda = 0\) and the area near \(\alpha = \lambda = 1\). If (2) is violated then there is no region where \((c)\) is the preferred scenario and the monopolist selects either \((a)\) or \((d)\). If the first inequality in (3) fails to hold the \(ad – line\) is downward sloping. Finally, if the second inequality in (3) is violated then there is no area where \((d)\) is the preferred scenario.

8.3 Case III

At the point where \(\alpha = \lambda = 1\), \((a)\) is the preferred multi-unit pricing strategy for the monopolist. At this point the other three scenarios generate a zero profit. Furthermore, notice that when \(\alpha = 1\) and \(\lambda \neq 1\) \((d)\) is always more profitable than \((b)\) and \((c)\). Therefore, consider the line of indifference between \((a)\) and \((d)\). From Table 3, setting the profit of the two equal it follows that

\[
ad - line: \quad \lambda^{ad} = \frac{2m_A - \alpha}{1 - 2\alpha}.
\]

If \(u_1^A \geq 4m_A\) then this is a decreasing and convex function as depicted in Figure 6. Next, consider the indifference curve between \((c)\) and \((d)\). Setting the payoffs from Table 3

\[
\text{cd} - line: \quad \lambda^{cd} = \frac{u_1B - \alpha u_1A}{u_1A + u_1B - 2\alpha u_1A}.
\]

Notice that it is decreasing, concave, and intersects the \(\lambda\)-axis and \(\alpha\)-axis at points strictly between zero and one. Therefore, in the area above the \(ad – line\) \((a)\) is preferred to \((d)\) which is preferred to \((c)\). In the area below the \(cd – line\) \((c)\) is preferred to \((d)\) which is preferred to \((a)\). Finally, I must
consider (b). From Table 3 the indifference curve between (b) and (c) is

\[ bc - line : \lambda^{bc} = \frac{u_1^b - 2m_B + \alpha m_B}{u_1^B - \alpha m_B}. \]

which is decreasing and concave. Points under the \( bc - line \) are cases where the monopolist prefers (c) to (b). Therefore, to guarantee that there exists a range for which (b) is most preferred and there exists an area where (c) is most preferred it must be that the \( bc - line \) intersects the \( \lambda \)-axis and the \( \alpha \)-axis at points less than where the \( cd - line \) intersects the axes. The \( bc - line \) and \( cd - line \) intersect the \( \lambda \)-axis at \( \frac{u_1^b - 2m_B}{u_1^B - \alpha m_B} \) and \( \frac{u_1^b}{u_1^A + u_1^B} \) respectively. Thus, the later is greater when \( m_B \geq \frac{u_1^A u_1^b}{2u_1^A - u_1^B} \). The \( bc - line \) and \( cd - line \) intersect the \( \alpha \)-axis at \( \frac{2m_B - u_1^b}{m_B} \) and \( \frac{u_1^b}{u_1^A} \) respectively. Thus, the later is greater when \( m_B \leq \frac{u_1^A u_1^b}{2u_1^A - u_1^B} \). Therefore, there exists regions for which (b) and (c) are optimal if and only if \( m_B = \frac{u_1^A u_1^b}{2u_1^A - u_1^B} \); otherwise only one region exists. Finally, the indifference curve between (b) and (d) must, by transitivity, go through the point where the \( cd - line \) and \( bc - line \) intersect, as illustrated. Therefore, Figure 6 illustrates the case when \( u_1^A \geq 4m_A \) and \( m_B = \frac{u_1^A u_1^b}{2u_1^A - u_1^B} \).

Suppose the parameters did not satisfy the previous two conditions. If \( m_B \) exceeds this value then there is no range of parameters for which (b) is the one preferred by the monopolist. If \( m_B \) falls short of this value there is no parameter values for which (c) is most preferred. Finally, suppose that \( u_1^A < 4m_A \). This implies that the \( ad - line \) is increasing and convex for values of \( \alpha \) less than \( \frac{1}{2} \) and increasing and concave for values of \( \alpha \) greater than \( \frac{1}{2} \). Also, consider the indifference curve between (a) and (c). It is a horizontal line \( \lambda^{ac} = \frac{u_1^b - 2m_A}{u_1^B} \). The \( ac - line \), \( ad - line \), and \( cd - line \) must all intersect at the same point. It follows that this point cannot occur for values of \( \alpha \) and \( \lambda \) between zero and one. Suppose, by way of contradiction, that the intersection
point occurs in the feasible area. Then there must be an area where \((d)\) is preferred to \((a)\) and \((a)\) is preferred to \((c)\). By transitivity, this area must have the monopolist preferring \((d)\) to \((c)\). Since the \(ad\) -- line is increasing, the \(ac\) -- line is horizontal, and the \(cd\) -- line is decreasing the \(cd\) -- line goes through this area so that there exists points where \((c)\) is preferred by the monopolist to \((a)\). Since this violates transitivity the intersection between the three lines does not occur for values of \(\alpha\) and \(\lambda\) between zero and one.