Optimal Partnership Contracts: Foundation and Duality

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Abstract

We use the duality in linear programming to solve the problem of optimal contracts with moral hazards. We show the importance of allowing the partners to throw away outputs under some contingencies. A two-step procedure is used to find the optimal contracts. The first step minimizes the loss from undistributed outputs, and in the second step, a second best solution is found. A characterization of the optimal contracts in 2-by-2-by-2 partnership games is offered. Such contracts implement an optimal strategy profile which either has no incentive cost to implement or is near a pure strategy profile.

1 Introduction

We study the problem of designing optimal contracts when there are moral hazard problems among the players due to unobservable actions. There is a huge literature on the design of optimal contracts under asymmetric information, encompassing many areas of study such as the mechanism design, the theory of auctions, the theory of regulation and procurements, and the incentive theory of public goods. By comparison, the problem of designing optimal contracts while providing incentives to the players are relatively underdeveloped. Part of the reason may be due to complexity with moral hazard problems. As noted in Laffont and Mortimort (2002), "As in the case of adverse selection, asymmetric information also plays a role in the design of the optimal incentive contract

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under moral hazard. However instead of being an exogenous uncertainty for the principal, uncertainty is now endogenous". This endogeneity is a source of the complexity.

We study the optimal contract problem in a partnership game with the joint production of a single output subject to production uncertainty. We want to find the optimal\(^1\) incentive compatible sharing contracts. We characterize the optimal contracts with two players, two pure strategies, and two public signals (referred to as two-by-two-by-two games). The method can be applied to the case with many players, pure strategies and signals. A general analysis is however beyond the scope of this paper. Because of the complexity, it is necessary that the solution be studied in the important but simpler two-by-two-by-two games.

A weaker form of the budget constraints is used in the paper. The weaker budget constraint means that in designing the sharing contracts, partners do not fully distribute all the produced output under all contingencies. The better decisions supported by such incentive schemes can more than compensate the loss from the undistributed outputs for the partners. This can occur quite easily, and an example will be given. Although the possibility of using such sharing rules are mentioned in Alchian and H. Demsetz (1972), and Holmstrom (1982), they did not pursue its implications. With the weaker budget constraints, we offer a characterization of the optimal contracts. Such contracts implement an optimal strategy profile which either has no incentive cost to implement or is near a pure strategy profile. This characterization makes it a relatively simple matter to find an optimal strategy profile and a linear programming solution gives us an optimal contract for its implementation.

Most of the literature on optimal contract design take the so-called "differential" approach (Laffont and Maskin (1979)). Our approach in this paper is the "linear inequality methods" surveyed in d’Aspremont and Gérard-Varet (1998).

Section 2 starts with justification of the weaker budget constraints. Second 3 defines the concept of optimal contracts under the weak budget constraints, and establishes an essential incentive compatibility result. Second 4 offers an important duality result for our analysis. The duality result is then applied in section 5 to analyze the solution of the step one optimization problem in general 2-by-2-by-2 games. In section 6, we characterize the optimal contracts.

\(^1\)It is well-known (see Legros and Matthews (1993)) that optimal contracts may not exist, and the best we can hope for is the asymptotically optimal contracts. We will omit the word "asymptotically" for the rest of the paper.
2 Weaker Budget Constraints

Let the partners be denoted by $i = 1, 2, \ldots, n$. There are only finitely many output levels $y_k \in Y, k = 1, 2, \ldots, K$. The total output (or revenue) produced by the partners depends on their joint actions or efforts as well as the stochastic factors. There are finitely many actions (pure strategies) for each partner. Let $A_i$ be the set of all pure strategies of partner $i$. The actions of the partners are not observable, but the total revenue is. However, the total revenue $y$ is an imperfect signal of the action profile $a = (a_1, a_2, \ldots, a_n), a_i \in A_i$. The distribution of revenue on $Y$ depends on $a$. Let $\pi_k(a)$ be the probability of revenue $y_k$ when the action profile $a$ is chosen. The expected total revenue is denoted by $E(a) = \sum_{k=1}^{K} \pi_k(a)y_k$.

Let $s_{ik}$ be the share of revenues by partner $i$ when the realized revenue is $y_k$. We refer to $s_{ik}$ as the share contracts. We allow $s_{ik}$ to be positive or negative so both rewards and punishments may be used in the incentive design. Thus partner $i$ gets the expected revenue $E_i(a) = \sum_{k=1}^{K} \pi_k(a)s_{ik}$ when the joint action is $a$. The pure strategies of partner $i$ will be indexed by $a_i^j \in A_i$. A mixed strategies $\alpha_i$ of partner $i$ is the strategy with probability $\alpha_i(a_i^j)$ of using the pure strategy $a_i^j$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be the mixed strategy profile. We shall let $\pi(\alpha)$ be the probability distribution on $Y$ with probability $\pi_k(\alpha_1)$ of revenue $y_k$. The expected total revenue will be denoted by $E(\alpha) = \sum_{k=1}^{K} \pi_k(a)y_k$.

The utility function of partner $i$ is given by $u_i(w_i, a_i) = w_i - v_i(a_i)$, where $w_i$ is the revenue received by the partner. Let $v_i(\alpha_i) = \sum_j \alpha_i(a_i^j)v_i(a_i^j)$ be the expected cost of effort of the mixed strategy $\alpha_i$. Given the strategy profile $\alpha_{-i}$ of other partners, each partner will maximize his or her net payoff $u_i(\alpha) = E_i(\alpha_i, \alpha_{-i}) - v_i(\alpha_i)$ from the chosen strategy $\alpha_i$. Given the quasi-linear utility functions of the partners, it is natural to consider the efficient strategy profiles $\alpha$ which maximizes the total net benefit $W(\alpha) = E(\alpha) - \sum_{i=1}^{n} v_i(\alpha_i)$. This is the concept of efficiency we will use for the rest of the paper. The partnership incentive problem is to find the revenue dependent share contracts $s_{ik}$ so that the partners have the incentive to choose the profile $\alpha$ with the highest possible $W(\alpha)$.

The budget constraints commonly used in the literature is

\[ \sum_i s_{ik} = y_k \] (1)

In this paper, we shall use the following weaker version of the budget constraints

\[ \sum_i s_{ik} \leq y_k \] (2)
The following example shows that the partners can achieve better results if they don’t distribute all the revenues in all contingencies.

**Example 1** There are two possible outputs \( y_1 = 24, y_2 = 12 \). When both partners work, the probability of high output is \( \frac{2}{3} \). If only one works, the probability is \( \frac{1}{6} \), and if neither works, it is 0. The disutility (cost) of work is 3.5 for both, and shirking has 0 cost.

\[
E(w, w) - 3.5 = 20 - 7 = 13
\]

\[
E(w, s) - 3.5 = E(s, w) - 3.5 = 10.5
\]

\[
E(s, s) - 0 = 12
\]

Hence it should be clear that \((work, work)\) is the efficient strategy profile of the game. We will first show that this is not an incentive compatible equilibrium for any choice of the contracts \( s_{ik} \) satisfying (1). The payoff matrix of the game is

\[
\begin{array}{c|cc}
\text{work} & \text{shirk} \\
\hline
\text{work} & (-3.5+\frac{2}{3}s_{11}+\frac{1}{3}s_{12}, & (-3.5+\frac{1}{6}s_{11}+\frac{5}{6}s_{12} , \frac{1}{6}s_{21}+\frac{5}{6}s_{22}) \\
& -3.5+\frac{2}{3}s_{21}+\frac{1}{3}s_{22}) & (s_{12}, s_{22}) \\
\text{shirk} & (\frac{1}{6}s_{21}+\frac{5}{6}s_{22}, & -3.5+\frac{1}{6}s_{11}+\frac{5}{6}s_{12}) \\
& & \\
\end{array}
\]

The incentive property for player one requires that

\[-3.5+\frac{2}{3}s_{11}+\frac{1}{3}s_{12} \geq \frac{1}{6}s_{11}+\frac{5}{6}s_{12}\]

which yields

\[s_{11} - s_{12} \geq 7\]

Similarly, the equilibrium property for player two implies that

\[s_{21} - s_{22} \geq 7\]

Summing the two requirements together, and use (1), we get

\[y_1 - y_2 \geq 14\]

which is contradictory, as \( y_1 = 24, y_2 = 12 \). We now show that \((work, work)\) is incentive compatible with the weaker budget constraint (2). Let

\[s_{11} = s_{21} = 12, s_{12} = s_{22} = 5\]  (3)
When the output is 12, only 10 units are distributed to the players, and 2 units are disposed of. With such contracts in place, it is easily verified that (work, work) is incentive compatible. Note that even though the efficient strategy (work, work) is now incentive compatible, it does not mean that the contract is efficient. The total amount of 2 units of output is wasted when the outcome is bad. This loss of efficiency can be computed as
\[ \frac{1}{3} \times 2 = \frac{2}{3} \]

By including this measure of inefficiency into our consideration, the efficiency measure of the contract is given by the expected total output minus the sum of the expected costs and the expected waste from disposal. Therefore in such an equilibrium, the efficiency measure is given by
\[ \frac{2}{3} \times 24 + \frac{1}{3} \times 12 - \frac{2}{3} = 12 \frac{1}{3} \]

It can be shown (see Cheng (2004)) that with the exact budget constraint, we cannot do better than (shirk, shirk) with the efficiency measure 12. Allowing for the weaker budget constraints (thus throwing away resources occasionally) achieves a more efficient outcome.

3 Defining Optimal Contracts

With the use of the weaker budget constraint, we need to define the concept of optimal share contract and find a solution of this optimal contract problem. For this purpose, an important concept is needed: admissible strategy profiles.

The concept of admissible strategies is first introduced in Cheng (2000). Here we extend it to the case of many partners. Let \( \alpha \) be any mixed strategy profile of the partners. We say that the deviation \( \alpha'_i \) by partner \( i \) is non-detectable if
\[ \pi(\alpha'_i, \alpha_{-i}) = \pi(\alpha) \]

We say that the strategy profile \( \alpha \) is inadmissible if for some partner \( i \), there exists a non-detectable deviation strategy \( \alpha'_i \) which has a lower cost for partner \( i \)
\[ v_i(\alpha'_i) < v_i(\alpha_i) \]

If \( \alpha \) is not inadmissible, then we say that it is admissible.

If \( \alpha \) is inadmissible, then there is a partner who finds it profitable to deviate from \( \alpha \). Such deviation cannot be prevented by a change of the share contracts. As a result, \( \alpha \) cannot be a Nash equilibrium for
the partners with any share contracts $s_{ik}$. We have proved therefore the following result.

**Proposition 2** If $\alpha$ is not admissible, then it cannot be an incentive compatible equilibrium for the partners through the design of the output-contingent share contracts.

We now show that admissibility is sufficient as well as necessary for the incentive compatibility in the design of share contracts with the weaker budget constraint.

**Theorem 3** The strategy profile $\alpha$ is admissible if and only if it is an incentive compatible equilibrium for the partners through the design of the output-contingent share contracts satisfying the weaker budget constraint (2).

**Proof.** To find an incentive compatible contract, we look for the solution of the following system of linear inequalities in $s_{ik}$

$$\sum_k \left[ \pi_k(\alpha) - \pi_k(a^j, \alpha_{-i}) \right] s_{ik} \geq v_i(\alpha_i) - v_i(a^j_i) \text{ for all } a^j_i$$

Let the set $A_i$ be indexed by $a^j_i$. The solution exists if and only if the following is true:

$$\sum_j \mu^j_i \left[ \pi(\alpha) - \pi(a^j_i, \alpha_{-i}) \right] = 0, \mu^j_i \geq 0 \text{ implies (4)}$$

$$\sum_j \mu^j_i \left[ v_i(\alpha_i) - v_i(a^j_i, \alpha_{-i}) \right] \leq 0 \text{ (5)}$$

There is no loss of generality in assuming that $\sum \mu^j_i = 1$. Let the mixed strategy with probability $\mu^j_i$ on action $a^j_i$ be denoted by $\alpha'_i$, then the assumption in (4) says that $\pi(\alpha) = \pi(\alpha'_i, \alpha_{-i})$, and the conclusion (5) is $v_i(\alpha'_i) \geq v_i(\alpha_i)$. This is exactly the statement that profitable deviations by player $i$ are detectable.

Let $s_{ik}$ be the incentive compatible contracts. To satisfy the weaker budget constraint, let $x_k = \sum_i s_{ik}$. We can choose $t \geq 0$ large enough so that

$$x_k - t \leq y_k \text{ for all } k$$
and let $s^*_{ik} = s_{ik} - t$, we have

$$\sum_i s^*_{ik} = -t + \sum_i s_{ik} = -t + x_k \leq y_k$$

and (2) is satisfied, while all the incentive properties are unchanged.  

Borrowing the terminology from the implementation literature, we can $\alpha$ implementable, or we can refer to it also as sustainable. The problem of finding optimal contracts can be broken down into two steps. The first step is to take an admissible $\alpha$ as given, and find the optimal incentive compatible $s_{ik}$ to implement $\alpha$. It is equivalent to minimizing the loss from undistributed revenue. Let $l(\alpha)$ be the the minimum cost of implementing $\alpha$. This step is a linear programming problem. The second step is to find $\alpha$ to maximize $m(\alpha) = W(\alpha) - l(\alpha)$. The function $W(\alpha)$ is well-understood. To find $l(\alpha)$, we need to convert the linear programming problem into its dual, and through further transformation, obtain a formula to compute $m(\alpha) = W(\alpha) - l(\alpha)$. This is the duality result to be developed in the next section.

4 Duality

Consider the collection of deviations $\alpha'_i, i = 1, 2 \ldots n$. We say that the collection of deviations $\alpha' = \{\alpha'_1, \alpha'_2, \ldots, \alpha'_n\}$ are unidentifiable deviations if $\pi(\alpha'_i, \alpha_{-i}), i = 1, 2, \ldots, n$ are all identical. The unidentifiable strategies play an important role in the duality result.

We shall extend the concept of deviation strategies $\alpha'_i$ to include any vectors $p_i = \alpha'_i(a_i)$ negative or positive with the only condition $\sum_j p_j = 1$. For such possibly infeasible deviation strategies $\pi(\alpha'_i, \alpha_{-i}), v_i(\alpha'_i)$ can be mathematically defined. This is done for easy mathematical expressions. It is possible to express our results without using such extended deviation strategies. However, the mathematical expressions will not be as elegant as the one given here. The simpler expressions facilitate computations when the result is applied to the analysis of economic games.

Let $\Delta$ denote the simplex of the probability distributions over the outcome space $Y$. We have the following result.

**Theorem 4** Let $\alpha$ be a completely mixed admissible strategy profile, and $\alpha'_i$ an extended deviation strategy of player $i$. Then we have

$$m(\alpha) = \min \eta y - \sum_i v_i(\alpha'_i)$$
subject to

\[ \pi(\alpha_i', \alpha_{-i}) = \eta \text{ for all } i \text{ and } \eta \in \Delta \]

**Proof.** The function \( m(\alpha) \) is the value of the following maximization problem

\[
\max - \sum_i v_i(\alpha_i) + \sum_i \sum_k \pi_k(\alpha) s_{ik}
\]

subject to

\[
\sum_i s_{ik} \leq y_k \text{ for all } k \quad (6)
\]

\[ -v_i(\alpha_i) + \sum_k \pi_k(\alpha) s_{ik} \geq -v_i(a_i) + \sum_k \pi_k(a_i, \alpha_{-i}) s_{ik} \text{ for all } i \text{ and } a_i \]

Denote the generic element of the set \( A_i \) by \( a_i^1 \). The maximization problem can be written as

\[
\max - \sum_i v_i(\alpha_i) + \sum_i \sum_k \pi_k(\alpha) s_{ik} \quad (7)
\]

subject to

\[
\sum_i s_{ik} \leq y_k \text{ for all } k \quad (8)
\]

\[
\sum_k [\pi_k(a_i^1, \alpha_{-i}) - \pi_k(\alpha)] s_{ik} \leq v_i(a_i^1) - v_i(\alpha_i) \text{ for all } a_i^1 \in A_i
\]

We shall refer to this as the primary problem. Because of the admissibility of \( \alpha \), the system of constraints has a solution. The objective function is bounded above by \( E(\alpha) - \sum_i v_i(\alpha_i) \), hence there exists a maximum for the objective function. Let \( e_{kk'} = 0 \) when \( k \neq k' \), and \( e_{kk} = 1 \) for all \( k \). To convert to the dual, use the following "switch box" for player \( i \)

<table>
<thead>
<tr>
<th>dual var.</th>
<th>the coefficient of ( s_{ik'} ) in the constraints</th>
<th>right hand side</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu_i )</td>
<td>( \pi_k'(a_i^1, \alpha_{-i}) - \pi_k(\alpha) )</td>
<td>( \leq v_i(a_i^1) - v_i(\alpha_i) )</td>
</tr>
<tr>
<td>( \eta_k )</td>
<td>( e_{kk'} )</td>
<td>( \leq y_k )</td>
</tr>
</tbody>
</table>

Hence the dual problem is the minimization problem

\[
\min - \sum_i v_i(\alpha_i) + \sum_k \eta_k y_k + \sum_i \sum_j \mu_i^j [v_i(a_i^j) - v_i(\alpha_i)] \quad (9)
\]

subject to \( \mu_i^j \geq 0, \eta_k \geq 0 \) for all \( i, j, k \) and

\[
\sum_j \mu_i^j [\pi_k(a_i^j, \alpha_{-i}) - \pi_k(\alpha)] + \eta_k = \pi_k(\alpha) \text{ for all } i, k \quad (10)
\]
Summing (10) over \( k \), we have
\[
\sum_k \eta_k = 1 \tag{11}
\]

We have the following minimization problem equivalent to the dual problem
\[
\min - \sum_i v_i(\alpha_i) + \sum_k \eta_k y_k + \sum_i \sum_j \mu^j_i[v_i(a^j_i) - v_i(\alpha_i)] \tag{12}
\]
subject to \( \mu^j_i \geq 0, \eta_k \geq 0 \) for all \( i, j, k \), \( \sum_k \eta_k = 1 \), and
\[
\sum_j \mu^j_i[\pi_k(a^j_i, \alpha - i) - \pi_k(\alpha)] + \eta_k = \pi_k(\alpha) \text{ for all } i, k \tag{13}
\]

We now make use of the dual variables \( \mu^j_i, \eta_k \) to construct the strategy variables of the players. Define the extended deviation strategy \( \alpha'_i \) with
\[
\alpha'_i(a^j_i) = \alpha_i(a^j_i) - \mu^j_i + (\sum_j \mu^j_i)\alpha_i(a^j_i)
\]
then we have
\[
\pi(\alpha'_i, \alpha - i) - \pi(\alpha) = \sum_j \mu^j_i[\pi(\alpha) - \pi(a^j_i, \alpha - i)]
\]
and
\[
v_i(\alpha'_i) - v_i(\alpha_i) = \sum_j \mu^j_i[v_i(\alpha_i) - v_i(a^j_i)]
\]
Hence the expression in (12) can be written as
\[
- \sum_i v_i(\alpha_i) + \sum_k \eta_k y_k + \sum_i [v_i(\alpha_i) - v_i(\alpha'_i)] = \sum_k \eta_k y_k - \sum_i v_i(\alpha'_i) \tag{14}
\]
and the constraint (13) can be written as
\[
\pi(\alpha) - \pi(\alpha'_i, \alpha - i) + \eta = \pi(\alpha) \text{ for all } i
\]
or
\[
\pi(\alpha'_i, \alpha - i) = \eta \text{ for all } i \tag{15}
\]
Hence the value of the objective function (12) at \( \mu^j_i, \eta_k \) is the value of (14) at a collection of deviation strategies satisfying (15), and hence is larger than the minimum value of the following optimization problem
\[
\min \eta y - \sum_i v_i(\alpha'_i) \tag{16}
\]
subject to
\[ \pi(\alpha'_i, \alpha_{-i}) = \eta \in \Delta \text{ for all } i \] (17)
where \( \eta \) is the vector with the components \( \eta_k \), and \( \Delta \) is the probability simplex of the outcome space.

Conversely, given any collection of extended deviations strategies \( \alpha'_i \) and \( \eta \in \Delta \) satisfying the constraint (17), define
\[ \alpha''_i = \alpha_i + \varepsilon(\alpha_i - \alpha'_i) = (1 + \varepsilon)\alpha_i - \varepsilon\alpha'_i \] (18)
Since \( \alpha_i(a_{ij}^i) > 0 \) for all \( i, j \), we must have \( p_j^i = \alpha''_i(a_{ij}^i) > 0 \) when \( \varepsilon \) is chosen to be small enough. We have
\[
\pi(\alpha''_i, \alpha_{-i}) - \pi(\alpha) = \varepsilon[\pi(\alpha) - \pi(\alpha'_i, \alpha_{-i})]
\]
or
\[
\frac{1}{\varepsilon}[\pi(\alpha''_i, \alpha_{-i}) - \pi(\alpha)] = \pi(\alpha) - \pi(\alpha'_i, \alpha_{-i}) \] (19)
Similarly, we have
\[
\frac{1}{\varepsilon}[v_i(\alpha''_i) - v_i(\alpha_i)] = v_i(\alpha_i) - v_i(\alpha'_i)
\]
We now define \( \mu_j^i = \frac{p_j^i}{\varepsilon} \) and \( \eta_k \) the \( k \)-th component of \( \eta \), then we have
\[
\sum_j \mu_j^i[v_i(a_{ij}^i) - v_i(\alpha_i)] = \frac{1}{\varepsilon}[v_i(\alpha''_i) - v_i(\alpha_i)] = v_i(\alpha_i) - v_i(\alpha'_i)
\]
or
\[
-v_i(\alpha'_i) = -v_i(\alpha_i) + \sum_j \mu_j^i[v_i(a_{ij}^i) - v_i(\alpha_i)]
\]
Hence the objective function in (16) can be written as
\[
\eta y - \sum_i v_i(\alpha'_i) = \eta y + \sum_i \left(-v_i(\alpha_i) + \sum_j \mu_j^i[v_i(a_{ij}^i) - v_i(\alpha_i)]\right)
\]
\[
= -\sum_i v_i(\alpha_i) + \sum_k \eta_k y_k + \sum_i \sum_j \mu_j^i[v_i(a_{ij}^i) - v_i(\alpha)]
\]
which is the same as in (12).

The constraint equation in (17) can be written as
\[
\pi(\alpha) - \pi(\alpha'_i, \alpha_{-i}) = \pi(\alpha) - \eta \text{ with } \eta \in \Delta
\]
and from (19), it becomes
\[
\frac{1}{\varepsilon}[\pi(\alpha''_i, \alpha_{-i}) - \pi(\alpha)] = \pi(\alpha) - \eta \text{ with } \eta \in \Delta
\]
By the definition of $\mu^j_i, \eta_k$, it is transformed further into

$$\sum_j \mu^j_i [\pi(a^j_i, \alpha_{-i}) - \pi(\alpha)] = \pi(\alpha) - \eta \text{ with } \eta \in \Delta$$

or

$$\sum_j \mu^j_i [\pi_k(a^j_i, \alpha_{-i}) - \pi_k(\alpha)] + \eta_k = \pi_k(\alpha) \text{ for all } i, k \text{ and } \eta \in \Delta$$

which is the same one in (13).

Therefore, we know that given any collection of extended deviation strategies $\alpha'_i$ and $\eta \in \Delta$ satisfying the constraint (17), the value of the objective function (16) is one of the value of the objective function in (12) at $\mu^j_i, \eta_k$ we just defined. This means that the minimum value of the objective function (16) is as large as the minimum value of the objective function (12). We have just shown that the two minimization problems are equivalent and have the same minimum value. The proof is now complete.

For boundary strategy profiles, we have the following result.

**Theorem 5** Let $\alpha$ be an admissible strategy profile and $\alpha'_i$ an extended deviation strategy for player $i$. We have

$$m(\alpha) = \min \eta y - \sum_i v_i(\alpha'_i)$$

subject to

$$\alpha'_i(a^j_i) \geq 1 \text{ whenever } \alpha_i(a^j_i) = 1 \quad (21)$$

$$\alpha'_i(a^j_i) \leq 0 \text{ whenever } \alpha_i(a^j_i) = 0 \quad (22)$$

$$\pi(\alpha'_i, \alpha_{-i}) = \eta \text{ for all } i \text{ and } \eta \in \Delta \quad (23)$$

If there exist no non-trivial collection of extended deviation strategies $\alpha'_i$ satisfying all the constraints in (21)$\sim$(23), then we must have $m(\alpha) = E(\alpha) - \sum_i v_i(\alpha_i)$.

**Proof.** The proof is a refinement of the arguments used in Theorem 4.

In the arguments for Theorem 4, given $\mu^j_i, \eta_k$, we define the extended deviation strategies as follows

$$\alpha'_i(a^j_i) = \alpha_i(a^j_i) - \mu^j_i + \left( \sum_j \mu^j_i \right) \alpha_i(a^j_i)$$
Follow the same arguments as in the proof of Theorem 4, we get a collection of deviation strategies satisfying the constraints (23). If \( \alpha_i(a_j^i) = 1 \), we have
\[
\alpha'_i(a_j^i) = \alpha_i(a_j^i) - \mu_j^i + \sum_j \mu_j^i \geq \alpha_i(a_j^i) = 1
\]
If \( \alpha_i(a_j^i) = 0 \), we have
\[
\alpha'_i(a_j^i) = \alpha_i(a_j^i) - \mu_j^i \leq \alpha_i(a_j^i) = 0
\]
Therefore constraints (21) and (22) are satisfied in this choice of deviation strategies. Follow the same step as in the proof of Theorem 4, and we obtain a collection of extended deviation strategies satisfying all the required constraints. In the construction in the other direction, we are given a collection of deviation strategies \( \alpha_0^i \) satisfying (21) (23). We defined the following deviation strategies in (18)
\[
\alpha''_i(a_j^i) = (1 + \varepsilon)\alpha_i - \varepsilon\alpha'_i
\]
and used the property of completely mixed strategy profile \( \alpha \) to insure that
\[
p_j^i = \alpha''(a_j^i) \geq 0
\]
and then use \( p_j^i \) to define the non-negative \( \mu_j^i \). From the constraints, we know that when \( \alpha_i(a_j^i) = 1 \), we have \( \alpha'_i(a_j^i) \geq \alpha_i(a_j^i) \). Hence we have
\[
\alpha''_i(a_j^i) = \alpha_i(a_j^i) - \varepsilon[\alpha_i(a_j^i) - \alpha_i(a_j^i)] \leq \alpha_i(a_j^i)
\]
and when \( \varepsilon > 0 \) is small enough, we also have \( \alpha''_i(a_j^i) > 0 \). When \( \alpha_i(a_j^i) = 0 \), we have the constraint property \( \alpha'_i(a_j^i) \leq 0 \). Hence we have
\[
\alpha''_i(a_j^i) = -\varepsilon\alpha'_i(a_j^i) \geq 0
\]
and again we can insure that \( p_j^i = \alpha''_i(a_j^i) \geq 0 \) from the constraint properties without using the completely mixed property. Hence our arguments used in the proof of the Theorem 4 also insure that \( \mu_j^i \geq 0, \eta_k \geq 0 \). This completes the proof that \( m(\alpha) \) can be written as stated in the theorem. If there exists no collection of non-trivial extended deviation strategies satisfying all the constraints in the theorem, then the minimum occurs at the trivial one \( \alpha'_i = \alpha_i \), and we must have \( m(\alpha) = \eta y - \sum_i v_i(\alpha_i) \).

We now give an example of the application of the theorems to determine the efficiency measure \( m(\alpha) \). The same method will be used in the next section to derive a general formula for \( m(\alpha) \) in the 2-by-2-by-2 games. Knowing the behavior of \( m(\alpha) \) is essential for finding the optimal contracts.
Example 6 We shall use the same partnership in example 1. Let \( \alpha = (\text{work, work}) \). Hence we need to apply Theorem 5. The extended deviation strategy \( \alpha'_1 \) should be of the form \((x, 1 - x)\) with \(x \geq 1\). To get the minimum possible value for the expression in that theorem, \(x\) should satisfy the following equation

\[
(x, 1 - x) \begin{pmatrix}
\frac{2}{3} & \frac{1}{6} \\
\frac{1}{6} & 0
\end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1
\]

or

\[
\frac{2}{3}x + \frac{1}{6}(1 - x) = 1
\]

and we get \(x = \frac{5}{3}\). The two extended deviations strategies \( \alpha'_1 = \left(\frac{5}{3}, -\frac{2}{3}\right) = \alpha'_2 \) are not identifiable, and satisfy all the constraints in Theorem 5. We have

\[
m(\alpha) = y_1 - v_1(\alpha'_1) - v_2(\alpha'_2) = 24 - \frac{5}{3} \times 3.5 \times 2 = 12\frac{1}{3}
\]

We have constructed the payment contracts earlier with the same total payoff for the players. Hence the share contract we constructed is an optimal one for the partners, conditional on the implementation of \( \alpha = (\text{work, work}) \). To know whether it is an optimal (second best) contract, we need to know how \(m(\alpha)\) varies with \(\alpha\). It will be shown near the end of the paper that \( (\text{work, work}) \) is indeed the optimal strategy to implement.

5 Two-by-Two-by-Two games

Let \(y_1, y_2\) be the two possible revenue signals. Let \(a_1, a_2; b_1, b_2\) be the two pure strategies of player one and two respectively. Let \(\pi_{ij}\) be the probability of signal \(y_1\) occurring when the chosen strategy profile is \((a_i, b_j)\). Let the costs of choosing \(a_i, b_j\) by players 1, 2 be \(c_i = v_i(a_i), d_j = v_j(b_j)\). We have made no assumptions about the relative sizes of these parameters. The general formulation will be useful for later applications in the general analysis.

Let \(\alpha_1 = (p_1, p_2)\) be the player one strategy with probabilities \(p_1, p_2\) of using the pure strategies \(a_1, a_2\) respectively. Similarly let \(\alpha_2 = (q_1, q_2)\) be the player two strategy. We will use the simplified notation \((p_1, q_1)\) for the strategy profile \(((p_1, 1 - p_1), (q_1, 1 - q_1))\) when the context is clear.

At the strategy profile \(\alpha = (\alpha_1, \alpha_2)\), the change of probability of \(y_1\) occurring due to player one’s "unitary" deviation in the direction of more effort is denoted by \(r_1\) and is given by

\[
r_1 = (1, -1) \begin{pmatrix}
\pi_{11} & \pi_{12} \\
\pi_{21} & \pi_{22}
\end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = (\pi_{11} - \pi_{21})q_1 + (\pi_{12} - \pi_{22})q_2
\]
Hence we have

\[ r_1 = \pi_{12} - \pi_{22} + (\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21})q_1 \quad (24) \]

Similarly, when player two deviates in the same "unitary" direction \((1, -1)\), the change of probability of \(y_1\) is given by

\[ r_2 = (p_1, p_2) \left( \frac{\pi_{11} \pi_{12}}{\pi_{21} \pi_{22}} \right) \left( \begin{array}{c} 1 \\ -1 \end{array} \right) = (\pi_{11} - \pi_{12})p_1 + (\pi_{21} - \pi_{22})p_2 \]

hence

\[ r_2 = \pi_{21} - \pi_{22} + (\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21})p_1 \quad (25) \]

When \( r_1 \neq 0, r_2 \neq 0 \), let

\[ T = y_1 - y_2 - \frac{c_1 - c_2}{r_1} - \frac{d_1 - d_2}{r_2} \]

In order to apply Theorem 4 to compute the efficiency index \( m(\alpha) \) of implementing the profile \( \alpha \), we need to (i) adopt individual directions of deviations (more or less effort in the strategy space) so that \( \pm r_1, \pm r_2 \) have the same signs: (ii) adjust individual intensity of deviations (probability of deviations in the strategy space) so that \( \pm k_1 r_1, \pm k_2 r_2 \) have the same magnitude; (iii) choose common direction of deviations in the probability of outcomes so that the total surplus from deviations becomes negative; (iv) choose common scale of deviations in the probability space of outcomes so that the extended deviation strategy leads to either sure outcome \( y_1 \) or \( y_2 \).

In the first two steps, we obtain unidentifiable deviations. In the last two steps, we choose the direction in which \( m(\alpha) \) is minimized in the dual problem, and the minimum is obtained when the outcome is certain after an extended deviation strategy.

It should be noted that in general \( r_i, T \) are functions of the strategy profile and will be denoted by \( r_i(\alpha), T(\alpha) \) when such dependence needs to be spelled out. We need another important notation to state our results. Let the synergy index be

\[ \Pi = \pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}. \]

When \( \Pi = 0 \), \( r_1 = \pi_{12} - \pi_{22}, r_2 = \pi_{21} - \pi_{22}, T \) are constants. Note that when \( \Pi = 0 \),

\[ \pi_{11} - \pi_{21} = \pi_{12} - \pi_{22} \]

and

\[ \pi_{11} - \pi_{12} = \pi_{21} - \pi_{22} \]
By the admissibility condition, we must have 

\[ \text{pro} \]

the two strategies 

how the duality result Theorem 4 is applied, and why it is so powerful.

is a linear function of the strategies. The proof is this case clearly shows 

e does not depend on the other player’s actions. There is no "synergy" 

Hence the change in the probabilities of success from a player’s actions 

does not depend on the other player’s actions when \( \Pi = 0 \). In this case, 

the analysis becomes easier. When \( \Pi > 0 \), there is positive synergy 

effect between the two players, and the analysis is more complicated.

First we shall assume that \( \Pi = 0 \), and take \( \alpha \) to be an admissible 

completely mixed strategy profile. The computations of the efficiency 

index \( m(\alpha) \) will depend on the sign of \( T \). In this no synergy case, \( m(\alpha) \) 

is a linear function of the strategies. The proof is this case clearly shows 

how the duality result Theorem 4 is applied, and why it is so powerful.

If \( \alpha \) is admissible, and \( r_1 = 0 \), then player one is indifferent between 

the two strategies \( a_1, a_2 \). The unidentifiable deviations must satisfy 

\[ \pi(\alpha'_1, \alpha_2) - \pi(\alpha) = \pi(\alpha_1', \alpha_2') - \pi(\alpha) = 0 \]

By the admissibility condition, we must have 

\[ v_i(\alpha'_i) - v_i(\alpha_i) \geq 0 \text{ for } i = 1, 2 \]

and hence \( m(\alpha) = W(\alpha) \) in this case. Similarly, if \( r_2 = 0 \), we also have 

\( m(\alpha) = W(\alpha) \). We will now assume that \( r_1 \neq 0, r_2 \neq 0 \).

**Theorem 7** Assume that \( \Pi = 0, r_1 = \pi_{12} - \pi_{22} \neq 0, r_2 = \pi_{21} - \pi_{22} \neq 0, \) 

and \( T = y_1 - y_2 - \frac{c_1 - c_2}{\pi_{12} - \pi_{22}} - \frac{d_1 - d_2}{\pi_{21} - \pi_{22}} \leq 0 \). For a completely mixed admissible 

strategy profile \( \alpha = (p_1, q_1) \), we have 

\[ m(\alpha) = y_1 - c_2 - d_2 - \frac{(c_1 - c_2)(1 - \pi_{22}) + (c_1 - c_2)(\pi_{21} - \pi_{22})}{\pi_{12} - \pi_{22}} q_1 \]

\[ - \frac{(d_1 - d_2)(1 - \pi_{22}) + (d_1 - d_2)(\pi_{12} - \pi_{22})}{\pi_{21} - \pi_{22}} p_1 \]

Proof: Let \( \pi_1(\alpha) \) be the probability of outcome \( y_1 \) at the strategy 

profile \( \alpha \). Let \( sgn(r_i) \) be the sign of \( r_i \).

Let 

\[ k_1 = \frac{1 - \pi_1(\alpha)}{|r_1|}, k_2 = \frac{1 - \pi_1(\alpha)}{|r_2|} \]  \hspace{1cm} (26)

Define the extended deviation strategies as follows 

\[ \alpha'_1 = \alpha_1 + k_1(sgn(r_1), -sgn(r_1)) = (p_1 + sgn(r_1)k_1, 1 - p_1 - sgn(r_1)k_1) \]

\[ \alpha'_2 = \alpha_2 + k_2(sgn(r_2), -sgn(r_2)) = (q_1 + sgn(r_2)k_2, 1 - q_1 - sgn(r_2)k_2) \]
We have
\[ \pi_1(\alpha', \alpha_2) = \pi_1(\alpha) + sgn(r_1)k_1r_1 = \pi_1(\alpha) + 1 - \pi_1(\alpha) = 1 \]
hence we have
\[ \pi(\alpha', \alpha_2) = (1, 0) \] (27)
and similarly
\[ \pi(\alpha_1, \alpha'_2) = (1, 0) \]
Thus \( \alpha'_1, \alpha'_2 \) are unidentifiable deviations. The total surplus from these deviations is
\[
(1 - \pi_1(\alpha))(y_1 - y_2) - sgn(r_1)(c_1 - c_2)k_1 - sgn(r_1)(d_1 - d_2)k_2 \\
= (1 - \pi_1(\alpha)) \left( y_1 - y_2 - \frac{c_1 - c_2}{sgn(r_1) |r_1|} - \frac{d_1 - d_2}{sgn(r_2) |r_2|} \right) \\
= (1 - \pi_1(\alpha)) \left( y_1 - y_2 - \frac{c_1 - c_2}{r_1} - \frac{d_1 - d_2}{r_2} \right) = (1 - \pi_1(\alpha)) T \leq 0
\]
and therefore we can apply Theorem 4 with \( \eta = (1, 0) \). We have
\[
m(\alpha) = y_1 - c_2 - d_2 - (c_1 - c_2)(p_1 + sgn(r_1)k_1) \\
-(d_1 - d_2)(q_1 + sgn(r_2)k_2) \] (28)
We want to compute \( m(\alpha) \). First we relate \( \pi_1(\alpha) \) to \( r_1 \),
\[
\pi_1(\alpha) = (p_1, 1 - p_1) \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}
= (0, 1) \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + p_1(1, -1) \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}
\]
Hence we have
\[ \pi_1(\alpha) = \pi_{21}q_1 + \pi_{22}q_2 + p_1r_1 \] (29)
From the definition for \( k_1 \), we have
\[
k_1 = \frac{1 - \pi_1(\alpha)}{|r_1|} = \frac{1 - \pi_{21}q_1 - \pi_{22}q_2}{|r_1|} - sgn(r_1)p_1
\]
and hence
\[
sgn(r_1)p_1 + k_1 = \frac{1 - \pi_{21}q_1 - \pi_{22}q_2}{|r_1|}
\]
Multiply by \( \text{sgn}(r_1) \), we have
\[
p_1 + \text{sgn}(r_1)k_1 = \frac{1 - \pi_{21}q_1 - \pi_{22}q_2}{r_1}
\]

We can substitute \( 1 - q_1 \) for \( q_2 \), and we have
\[
p_1 + \text{sgn}(r_1)k_1 = \frac{1 - \pi_{22} - (\pi_{21} - \pi_{22})q_1}{r_1}
\]

or
\[
p_1 + \text{sgn}(r_1)k_1 = \frac{1 - \pi_{22} - \pi_{21} - \pi_{22}q_1}{r_1}
\]

Similarly, for player two we have
\[
q_1 + \text{sgn}(r_2)k_2 = \frac{1 - \pi_{12}p_1 - \pi_{22}p_2}{r_2} = \frac{1 - \pi_{22} - (\pi_{12} - \pi_{22})p_1}{r_2}
\]

or
\[
q_1 + \text{sgn}(r_2)k_2 = \frac{1 - \pi_{22} - \pi_{12} - \pi_{22}p_2}{r_2}
\]

Substituting (31) and (33) into (28), and the theorem is proved.

Now we want to consider the case when \( T = y_1 - y_2 - \frac{c_1 - c_2}{\pi_{12} - \pi_{22}} - \frac{d_1 - d_2}{\pi_{21} - \pi_{22}} \geq 0 \). We have the following result by a similar argument.

**Theorem 8** Assume that \( \Pi = 0, r_1 = \pi_{12} - \pi_{22} \neq 0, r_2 = \pi_{21} - \pi_{22} \neq 0 \) and \( T = y_1 - y_2 - \frac{c_1 - c_2}{\pi_{12} - \pi_{22}} - \frac{d_1 - d_2}{\pi_{21} - \pi_{22}} \geq 0 \). For a completely mixed admissible strategy profile \( \alpha = (p_1, q_1) \), we have
\[
m(\alpha) = y_2 - c_2 - d_2 + \frac{(c_1 - c_2)\pi_{22}}{\pi_{12} - \pi_{22}} + \frac{(c_1 - c_2)(\pi_{21} - \pi_{22})}{\pi_{12} - \pi_{22}}q_1
\]
\[
+ \frac{(d_1 - d_2)\pi_{22}}{\pi_{21} - \pi_{22}} + \frac{(d_1 - d_2)(\pi_{12} - \pi_{22})}{\pi_{21} - \pi_{22}}p_1
\]

Now, we consider the more complex case when \( \Pi \neq 0 \).

**Theorem 9** Suppose \( \Pi \neq 0, r_1 \neq 0, r_2 \neq 0 \). For an admissible completely mixed strategy profile \( \alpha = (p_1, q_1), T \leq 0 \), we have
\[
m(\alpha) = y_2 - (y_1 - y_2)L - c_2 - d_2 - \frac{(c_1 - c_2)(\pi_{22} - \pi_{21})}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}} - \frac{(d_1 - d_2)(\pi_{22} - \pi_{12})}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}} + (L + 1)T
\]

where \( L = \frac{\pi_{12}\pi_{21} - \pi_{11}\pi_{22}}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}} \).
Proof: If $T \leq 0$, from (30), we have

$$p_1 + \text{sgn}(r_1)k_1 = \frac{1 - \pi_{22} - (\pi_{21} - \pi_{22})q_1}{r_1} = \frac{1 - \pi_{22} - (\pi_{21} - \pi_{22})q_1}{\pi_{12} - \pi_{22} + (\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21})q_1}$$

Reduce the numerator into a constant term, making use of the equation,

$$-\pi_{22} + \frac{(\pi_{22} - \pi_{21})(\pi_{22} - \pi_{12})}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}} = \frac{\pi_{12}\pi_{21} - \pi_{11}\pi_{22}}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}}$$

we get the following expression for $p_1 + \text{sgn}(r_1)k_1$

$$\frac{\pi_{22} - \pi_{21}}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}} + \frac{1 + \frac{\pi_{12}\pi_{21} - \pi_{11}\pi_{22}}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}}}{\pi_{21} - \pi_{22} + (\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21})q_1}$$

Similarly, $q_1 + \text{sgn}(r_2)k_2$ is given by

$$\frac{\pi_{22} - \pi_{12}}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}} + \frac{1 + \frac{\pi_{12}\pi_{21} - \pi_{11}\pi_{22}}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}}}{\pi_{21} - \pi_{22} + (\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21})q_1}$$

Substitute (36) and (37) into (28), we have

$$m(\alpha) = y_1 - c_2 - d_2 - \frac{(c_1 - c_2)(\pi_{22} - \pi_{21})}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}} - \frac{(d_1 - d_2)(\pi_{22} - \pi_{12})}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}}$$

$$- \frac{(c_1 - c_2)(1 + \frac{\pi_{12}\pi_{21} - \pi_{11}\pi_{22}}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}})}{\pi_{21} - \pi_{22} + (\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21})q_1}$$

$$- \frac{(d_1 - d_2)(1 + \frac{\pi_{12}\pi_{21} - \pi_{11}\pi_{22}}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}})}{\pi_{21} - \pi_{22} + (\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21})q_1}$$

hence

$$m(\alpha) = y_1 - c_2 - d_2 - \frac{(c_1 - c_2)(\pi_{22} - \pi_{21})}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}}$$

$$- \frac{(d_1 - d_2)(\pi_{22} - \pi_{12})}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}} - (1 + L)(-\frac{c_1 - c_2}{r_1} - \frac{d_1 - d_2}{r_2})$$

Using the definition of $T$, we have

$$T - y_1 + y_2 = -\frac{c_1 - c_2}{r_1} - \frac{d_1 - d_2}{r_2}$$
Substitute into (38), and the theorem is proved.

Note that it can be shown that when $\Pi$ converges to 0, the formula in Theorem 9 converges to the formula in Theorem 7. However, we still need different formulae applicable for the two cases. For the case $T \geq 0$, we have the following result with a similar proof.

**Theorem 10** Suppose $\Pi \neq 0, r_1 \neq 0, r_2 \neq 0$. For an admissible completely mixed strategy profile $\alpha = (p_1, q_1), T \geq 0$, we have

$$m(\alpha) = y_2 - (y_1 - y_2)L - c_2 - d_2 - \frac{(c_1 - c_2)(\pi_{22} - \pi_{21})}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}}$$

$$- \frac{(d_1 - d_2)(\pi_{22} - \pi_{12})}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}} + LT$$

where $L = \frac{\pi_{12}\pi_{21} - \pi_{11}\pi_{22}}{\pi_{11} + \pi_{22} - \pi_{12} - \pi_{21}}$.

Note that the formulas for the two cases $T \geq 0, T \leq 0$ only differ by $T$. This is true whether $\Pi = 0$ or $\Pi \neq 0$.

To find $\sup_{\alpha} m(\alpha)$, it is important that we consider boundary strategy profiles. We will see that it occurs often at pure strategy profiles. The above arguments can also be applied to "boundary" strategy profiles as well, as long as the unidentifiable deviations $\alpha'_1, \alpha'_2$ with a negative surplus also satisfy the boundary conditions in Theorem 5. This means that there is a feasible direction of unidentifiable deviations with a positive surplus at the strategy profile. In particular, if $r_1 > 0, T \leq 0$, deviations in the direction of lower effort is feasible and yields positive total surplus when $p_i > 0, q_i > 0$. Hence for the strategy profile with $p_1 = 1, 0 < q_1 < 1$, Theorem 7 applies only if $r_1 > 0, T \leq 0$; in other cases the total surplus from unidentifiable and feasible deviations is always $\leq 0$, hence we must have $m(\alpha) = W(\alpha)$. Similarly, for the strategy profile with $0 < p_1 < 1, q_1 = 1$, Theorem 7 applies only if $r_2 > 0, T \leq 0$; in other cases we must have $m(\alpha) = W(\alpha)$. For the strategy profile with $p_1 = 0, 0 < q_1 < 1$, Theorem 7 applies only if $r_1 < 0, T \leq 0$; in other cases we must have $m(\alpha) = W(\alpha)$. For the strategy profile with $0 < p_1 < 1, q_1 = 0$, Theorem 7 applies only if $r_2 < 0, T \leq 0$. For the profile $p_1 = 1 = q_1$, Theorem 7 applies only if $r_1 > 0, r_2 > 0, T \leq 0$; in other cases we must have $m(\alpha) = W(\alpha)$. For the strategy profile $p_1 = 1, q_1 = 0$, Theorem 7 applies only if $r_1 > 0, r_2 < 0, T \leq 0$; in other cases we must have $m(\alpha) = W(\alpha)$. For the strategy profile $p_1 = 0, q_1 = 1$, Theorem 7 applies only if $r_1 < 0, r_2 > 0, T \leq 0$; in other cases we must have
\( m(\alpha) = W(\alpha) \). For the strategy profile \( p_1 = 0 = q_1 \), Theorem 7 applies only if \( r_1 < 0, r_2 < 0, T \leq 0 \); in other cases we must have \( m(\alpha) = W(\alpha) \). Thus we have the following result for all strategy profiles.

**Theorem 11** Assume that \( \Pi = 0, r_1 = \pi_{12} - \pi_{22} \neq 0, r_2 = \pi_{21} - \pi_{22} \neq 0 \), and \( T = y_1 - y_2 - \frac{c_1 - c_2}{\pi_{12} - \pi_{22}} - \frac{d_1 - d_2}{\pi_{21} - \pi_{22}} \leq 0 \). For any mixed admissible strategy profile \( \alpha = (p_1, q_1) \), we have the same formula given in Theorem 7 if and only if all of the following conditions hold: (i) \( \pi_{12} - \pi_{22} > 0 \) whenever \( p_1 = 1 \); (ii) \( \pi_{12} - \pi_{22} < 0 \) whenever \( p_1 = 0 \); (iii) \( \pi_{21} - \pi_{22} > 0 \) whenever \( q_1 = 1 \); (iv) \( \pi_{21} - \pi_{22} < 0 \) whenever \( q_1 = 0 \). In other cases, we have \( m(\alpha) = W(\alpha) \).

Similarly, for the case \( \Pi = 0, T \geq 0 \), we have the following result.

**Theorem 12** Assume that \( \Pi = 0, r_1 = \pi_{12} - \pi_{22} \neq 0, r_2 = \pi_{21} - \pi_{22} \neq 0 \), and \( T = y_1 - y_2 - \frac{c_1 - c_2}{\pi_{12} - \pi_{22}} - \frac{d_1 - d_2}{\pi_{21} - \pi_{22}} \geq 0 \). For any mixed admissible strategy profile \( \alpha = (p_1, q_1) \), we have the same formula given in Theorem 8 if and only if all of the following conditions hold: (i) \( \pi_{12} - \pi_{22} < 0 \) whenever \( p_1 = 1 \); (ii) \( \pi_{12} - \pi_{22} > 0 \) whenever \( p_1 = 0 \); (iii) \( \pi_{21} - \pi_{22} < 0 \) whenever \( q_1 = 1 \); (iv) \( \pi_{21} - \pi_{22} > 0 \) whenever \( q_1 = 0 \). In other cases, we have \( m(\alpha) = W(\alpha) \).

For the case \( \Pi \neq 0, T \leq 0 \), we have

**Theorem 13** Suppose \( \Pi \neq 0, r_1 \neq 0, r_2 \neq 0 \). For an admissible mixed strategy profile \( \alpha = (p_1, q_1) \) with \( T = y_1 - y_2 - \frac{c_1 - c_2}{r_1} - \frac{d_1 - d_2}{r_2} \leq 0 \), we have the same formula in Theorem 9 if and only if all of the following conditions hold: (i) \( r_1 > 0 \) whenever \( p_1 = 1 \); (ii) \( r_1 < 0 \) whenever \( p_1 = 0 \); (iii) \( r_2 > 0 \) whenever \( q_1 = 1 \); (iv) \( r_2 < 0 \) whenever \( q_1 = 0 \). In other cases, we have \( m(\alpha) = W(\alpha) \).

For the case of \( \Pi \neq 0, T \geq 0 \), we have

**Theorem 14** Suppose \( \Pi \neq 0, r_1 \neq 0, r_2 \neq 0 \). For an admissible mixed strategy profile \( \alpha = (p_1, q_1) \) with \( T = y_1 - y_2 - \frac{c_1 - c_2}{r_1} - \frac{d_1 - d_2}{r_2} \geq 0 \), we have the same formula in Theorem 10 if and only if all of the following conditions hold: (i) \( r_1 < 0 \) whenever \( p_1 = 1 \); (ii) \( r_1 > 0 \) whenever \( p_1 = 0 \); (iii) \( r_2 < 0 \) whenever \( q_1 = 1 \); (iv) \( r_2 > 0 \) whenever \( q_1 = 0 \). In other cases, we have \( m(\alpha) = W(\alpha) \).
6 Second-Best Partnership Contracts

We are now ready to characterize the optimal contracts in two-by-two-by-two partnership games.

**Theorem 15** The supremum of $m(\alpha)$ occurs at a pure strategy profile or at $\alpha$ with $T = 0$ and $-1 < L < 0$. In the latter case any strategy profile $\alpha$ with $T = 0$ is optimal and $m(\alpha) = W(\alpha)$

**Proof.** It is convenient to say that the strategy profile $\alpha$ dominates another profile $\beta$ if $m(\alpha) \geq m(\beta)$.

If one of the $r_i$ is 0, the probability distribution of outputs is not affected by player $i$’s strategy. In this case, there is no loss of efficiency from the provision of incentives, and the efficiency index is given by $m(\alpha) = W(\alpha)$. When $r_i \neq 0$ for both $i$, the formulas in the last section for $m(\alpha)$ are applicable.

First consider the case $\Pi = 0$. Since all $r_i, T$ are constants, and do not change signs, a single formula for $m(\alpha)$ from either Theorem 7 or 8 applies to all completely mixed strategy profiles of the players. Furthermore, Theorem 11 or Theorem 12 implies that for the boundary strategy profiles, either the same formula applies or $m(\alpha) = W(\alpha)$. In either case, a strategy profile is dominated by one close to a pure strategy profile, and the supremum occurs at a pure strategy profile.

Consider now the case $\Pi \neq 0$. Note that if the signs of $r_1, r_2, T$ do not change, then there is a single formula applicable to all completely mixed strategy profiles. The theorem in this case is proved exactly like the case $\Pi = 0$. There are two ways in which the sign of $T$ changes. It can be due to the change in the signs of $r_1, r_2$. In this case, the border of change is $T = \infty$. It can also change signs when the signs of $r_i$ remain the same. In this case, the border of change is $T = 0$. Consider first the case when the border of change is $T = \infty$, and the sign of $T$ does not change whenever the signs of $r_1, r_2$ remain the same so that $T \neq 0$.

Assume that the border is $r_1 = 0$ or

$$q_1 = -\frac{\pi_{12} - \pi_{22}}{\Pi} \quad (40)$$

Near the border, $T$ is near $+\infty$ on the positive side, and $-\infty$ on the negative side. On the negative side, the formula in Theorem 7 applies. Since $m(\alpha)$ must be bounded above, it must be the case that $L + 1 \geq 0$. 
On the positive side, the conclusion is \( L \leq 0 \). Hence we have\(^2\)
\[
-1 \leq L \leq 0
\]
Hence in each region, any strategy is dominated by one near a pure strategy profile. The overall supremum is the larger of the two regional supremum and the supremum occurs at a pure strategy profile. The case when the border is defined by \( r_2 = 0 \) is similar.

We now consider the case when regions are divided by \( T = 0 \). If \( L \geq 0 \), we also have \( L + 1 > 0 \). This means that the two formulas for \( m(\alpha) \) in Theorems 7 and 8 are monotonic in the same direction. Note that the two formulas have the same value on the border \( T = 0 \). This means that the supremum occurs at a pure strategy profile. If \( L + 1 \leq 0 \), we have the same conclusion.

We now consider the case \(-1 < L < 0\). Among the completely mixed strategy profiles, the maximum \( m(\alpha) \) now occurs on the border \( T = 0 \). Since \( m(\alpha) \) is constant on the border \( T(\alpha) = 0 \), each strategy profile with \( T = 0 \) yields the same \( m(\alpha) \) and is optimal as well. ■

Remark 16 The condition \( L = 0 \) is the same as \( \frac{\pi_{12}}{\pi_{11}} = \frac{\pi_{22}}{\pi_{21}} \). The condition \( L = -1 \) is the same as \( \frac{1-\pi_{12}}{1-\pi_{11}} = \frac{1-\pi_{22}}{1-\pi_{21}} \).

Theorem 15 makes it a relatively simple matter to find an optimal contract. For the example 1, we have \( c_1 = d_1 = 3.5, c_2 = d_2 = 0, \pi_{11} = \frac{2}{3}, \pi_{12} = \frac{1}{6} = \pi_{21}, \pi_{22} = 0 \), and \( L = \frac{1}{12} \). Hence the optimal contract is near one of the pure strategy profiles. Since
\[
T = 12 - \frac{21}{1+2q_1} - \frac{21}{1+2p_1} < 0
\]
Theorem 9 applies, and we have
\[
m(\alpha) = 14.5 + \frac{13}{12}T
\]
(41)
We have the largest \( m(\alpha) \) when \( T \) is largest at \( p_1 = 1, q_1 = 1 \). Hence for all completely mixed strategy profiles
\[
m(\alpha) \leq 14.5 - \frac{13}{6} = 12 \frac{1}{3}
\]
(42)
\(^2\)This proof is more intuitive. A direct algebraic proof can be given based on the property that \( q_1 \) in (40) must be between 0 and 1. Multiply this by \( \pi_{21} - \pi_{22} \) when positive or \( \pi_{22} - \pi_{12} \) otherwise, and apply the equation (35), then we will get the same result.
and it can be easily verified, using Theorem 13, that (42) applies to all strategy profiles. The contract given in (3) is indeed optimal.

The possibility of an optimal contract implementing a strategy profile away from a pure strategy one is illustrated by the example

\[ y_1 = 12, y_2 = 0, c_1 = 1.5, d_1 = 1.4, c_2 = d_2 = 0 \]

\[ \pi_{11} = \frac{2}{3}, \pi_{12} = \frac{1}{2}, \pi_{21} = \pi_{22} = 0 \]

with \( L = -\frac{3}{4} \). An optimal strategy profile to implement is \( p_1 = 1, q_1 = 0.25 \) with \( T = 0 \). An optimal contract can be found by a linear programming problem with one solution given by \( s_{11} = 3.6, s_{12} = 0, s_{21} = 8.4, s_{22} = 0 \).

References


Laffont, J.J. and E. Maskin (1979), "A differentiable approach to expected utility maximizing mechanisms", in Aggregation and Revelation of Preferences, ed. by J.J. Laffont, North-Holland.

