Optimal Fiscal Policy in an Economy with the Possibility of Default*

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Abstract

In this paper we analyze how the tax-smoothing result obtained in models of optimal fiscal policy is altered by the introduction of the possibility of default. We build a standard stochastic optimal fiscal policy model 'a la Lucas and Stokey (1983) in a context of international risk sharing with limited enforcement. We consider the problem of a benevolent government that has to choose optimally distortive taxes on labor income and transfers from an international institution to finance a stochastic exogenous stream of public expenditure. The international institution is willing to provide funds to the country under the conditions that, at time 0, the expected discounted value of the sum of the implied transfers is zero and that, at each point in time, the country does not have an incentive to exit the contract. We use the Marcet and Marimon (1997) technique in order to write our problem in a recursive way. Secondly, we extend the model to allow both the government and the international institution to leave the contract. We solve the model numerically and obtain that the dynamics of fiscal variables are different to the case in which we do not allow for the possibility to leave the contract. In particular, we find that the volatility of the tax rate is higher than the volatility of the shock hitting the economy since the former responds strongly to the incentives to default of both the government and the international institution. Moreover, as it has been described in the literature of fiscal policy in developing countries, optimal taxes are procyclical.

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1 Introduction

A strong result in the optimal fiscal policy literature is that under complete markets, uncertainty and distortionary taxation, taxes should be smooth "across states" and the function of government debt is to smooth distortions across states and over time. Running a balanced budget period by period is not welfare-improving since market completeness implies that state-contingent debt plays the role of an insurance contract against shocks. Moreover, both taxes and debt inherit the same statistical properties (in terms of persistence) of the shocks that hits the economy. The result of tax smoothness is also obtained under incomplete markets, although for a different reason. Since in this latter case there is not a complete set of state contingent claims, in order to spread over time the burden of a higher taxation due to a negative shock, taxes are more persistent over time. In particular taxes contains a unit root component since shocks to the excess burden of taxation exert a permanent effect on them.

Nevertheless, this apparently strong theoretical result is at odds with what we observe in the data. Figures 1 and 2 show the evolution of government expenditure and tax revenues over GDP in USA and in Argentina, using quarterly data for the period $1993 - 2005$[1][2]. Table 1 shows some statistics for the series. We can observe that, although the variability of government expenditure is roughly the same in the two countries, tax rates in Argentina are much more volatile than in the USA: the standard deviation of the series for Argentina is almost 60% higher than the one for the US.

In this paper we show that the tax smoothing property does not hold in a framework in which the government does not have a commitment technology towards obligations with the rest of the world. In particular, our results show that the tax rate in a country depends inversely on the incentives to default on external transfers. We consider the case of a government that can use transfers received from an international institution in order to smooth the distortions caused by an exogenous government shock. Moreover, we assume that the country cannot commit to stay in the contract with its external creditors, but if it leaves it is excluded forever from the international financial community.

[1] Because the predictions of optimal fiscal policy models are in terms of smoothness of *tax rates*, we use $\frac{\text{tax revenues}_t}{\text{GDP}_t}$ as a proxy for the marginal tax rate. As we will see later, in our model this is exactly the marginal tax rate since $y_t = 1 - l_t$.

[2] The series for USA are from the Bureau of Economic Analysis of the US Department of Commerce. In the case of Argentina, the data we use is from the IMF, INDEC and Ministerio de Economia. We use quarterly series of current government expenditure net of interest payments plus gross government investment as our measure of government expenditure, and total tax revenues plus contributions to social security as our measure of tax revenues.

Table 1: Some facts: USA and Argentina

	USA		Argentina	
	Govt. expenditure	Tax rate	Govt. expenditure	Tax revenues
Mean	0.175456	0.184961	0.170375	0.182808
St. deviation	0.009205	0.013017	0.009765	0.021397
Coef. of variation	0.0525	0.07038	0.0573	0.117

How does the contract between the country and the rest of the world have to be formulated in order for the country not to default in equilibrium? The answer hinges on the optimal timing for the country to default. When a good shock hits the economy, the government is tempted not to pay back the transfers received in the past and to go into autarky, since the value of its outside option increases. In order to induce the country not to default, the contract has to specify that, contingent on a good shock, the tax rate has to decrease since in this way the higher consumption generates an increase in the utility of staying in the contract.

Although there are in the literature different explanations for the high volatility of the tax rate that we observe in the data, no study has focused on the possibility of default as one of them [3]. The underlying mechanism in our model is as follows. Imagine a risk-sharing setup in which there is a country that is risk-averse and an international institution that is risk-neutral. Both agents have full commitment. In the risk-averse country there is a government that has to set taxes in order to finance exogenous government expenditure. The policy prescription is that the tax rate in the risk-averse country should be completely flat. During periods of government expenditure higher than average, the country accumulates external debt and depletes it during periods of lower than average government expenditures. Now suppose that the country does not have a full commitment technology and the international institution is not willing to accumulates too much debt. If there is a credit constraint which prevents the government to use external transfers, the country has to increase revenues through distortionary taxes in order to finance its shocks and

[3]Scott (1999) extends the model of Lucas and Stokey (1983) and finds that the variance of the tax rate depends not only on the variance of the government shock, but on the variance of any shock which affects employment: the more sluggish the labor market, the less volatile the tax rate.

to pay back external transfers. Symmetrically, if the government shock is low and there is a higher incentive to default, in order to rule out this possibility the institution has to be willing to give the country more resources, in such a way that the tax rate decreases. In both cases, the increase (decrease) in the tax rate is more than proportional to the increase (decrease) of government expenditures, in such a way that the tax smoothing result does not apply.

The choice of the government to increase taxes instead of external transfers depends very much on the type of taxation available to the government, and consequently on the distortions associated with taxation. The higher these distortions, the higher the incentive for a government to rely more on external debt rather than on taxes in order to finance government consumption. The problem of the optimal choice between tax and transfers is not interesting in a world in which the government has access only to lump-sum taxes. In this case, since the Ricardian equivalence holds, there is no incentive to default simply because there is no incentive to use external transfers to smooth distortions. To rule out this possibility, we concentrate on the case of distortive labor income taxes.

Our research is mainly related to the issues of limited commitment and fiscal policy. These topics have already been the subject of numerous studies in the fields of international finance and macroeconomics. However, the focus has been on understanding these issues separately: on the one hand, the literature on limited commitment has focused on explaining how the limited commitment problem affects the degree of risk-sharing among the agents. On the other hand, macroeconomists have been concerned on studying fiscal policy in environments where sovereign default is ruled out from the setup. Up to our knowledge, the study of fiscal policy and its consequences over macroeconomic variables when there exists default risk has received little attention.

The rest of the paper is organized as follows. Section 2 contains a brief review of the literature. Section 3 describes the framework that we use in the analysis. Section 4 presents an example in which the tax-smoothing result is altered by the possibility of default from one of the agents in our economy. Section 5 extends the model to include relax the assumption of full commitment from the international institution. In section 6 we calibrate the model to the argentinean economy and show the results of our simulations. Section 7 concludes.

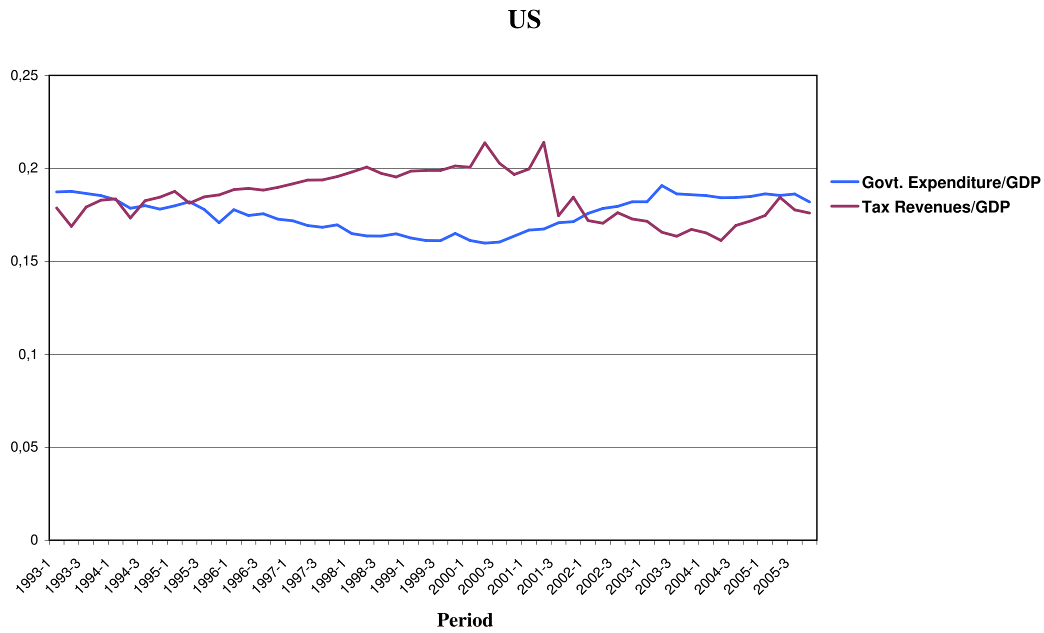

Figure 1: USA - Fiscal variables

2 Related literature

Our paper builds on existing models of efficient risk-sharing in limited commitment environments. One of the first contributions in this line of research is the paper by Marcet and Marimon (1992). The authors analyze the evolution of consumption, investment and output along the growth path under four different benchmarks: autarky (AU), full commitment and full information (PO), limited information and full commitment (PI) and full information and limited commitment (PC). In all these cases there are two agents: agent 1 (the manager) is risk averse and decides how much to invest, agent 2 (the investor) is risk neutral. The main finding is that in the (PC) case, although the steady state distribution of capital is the same as in the (PO) case, during the growth path capital is lower and borrowing and lending are used to smooth consumption against unforseen shocks. Consumption of the manager is smoother than under autarky.

Kocherlakota (1996) analyzes the properties of efficient allocations in a model with symmetric information and two-sided lack of commitment. He constructs the Pareto frontier of the economy as a solution of a Central Planner problem, who has to choose state-contingent consumption and future promised utility level given a certain level of utility promised in the past to agent one and a participation constraint for both agent one and two. He shows that if the discount factor is high enough to guarantee that a first best allocation can be achieved by a subgame-perfect strategy, then the correlation between individual consumption and contemporaneous and lagged individual income is zero. Therefore, the lack of commitment cannot be seen as the source of the inefficiency underlying the empirical evidence of a positive correlation between these two variables, and other sources of inefficiency have to be found (e.g., asymmetric information). Moreover, when the agent is hit by a high positive income shock his participation constraint binds and the consumption increases for more than one period.

Both Alvarez and Jermann (2000) and Abraham and Cárceles-Poveda (2006) papers deal with the problem of decentralizing an economy with participation constraints. The first paper analyzes an endowment economy in which participation constraints are seen as portfolio constraints. It is shown that it is possible to decentralize the constrained efficient allocation as a competitive equilibrium with endogenous solvency constraints that are not too tight, defined as the limits to debt accumulation that induce agents to pay back and at the same time allow as much risk sharing as possible. Abraham and Cárceles-Poveda (2006) study the same decentralization problem but for an economy with capital. They show that in this case it is necessary to introduce both endogenous borrowing constraints and capital accumulation constraints in order to decentralize the constrained efficient allocation.

Since our scope is to study optimal fiscal policy in a limited commitment setting, we briefly review the main findings in the vast literature of optimal fiscal policy in DSGE economies. See Barro (1988) for an authoritative overview.

One of the first contributions in the optimal fiscal policy literature in a dynamic setup is by Lucas and Stokey (1983). In their model they investigate optimal fiscal policy in a production economy with complete markets, no capital, exogenous Markov government expenditures and state contingent taxes and government debt. Contrary to what had been previously found by Barro (1979), they argue that optimal tax rates should follow closely the correlation of the government expenditure shock. Chari, Christiano and Kehoe (1994) extend the analysis to economies with capital and taxes on capital income as well as labor income. The conclusions in this case are in line with those of Lucas and Stokey (1983). They find that optimal labor taxes should not behave as a random walk, as Barro suggested, but should inherit the serial correlation properties of shocks. The expected tax rates on capital income should be close to zero and the role of the shock absorber in the model is played by the return on debt and the ex post tax on capital income.

Aiyagari, Marcet, Sargent and Seppälä (2002) study a similar problem to the one of Lucas and Stokey (1983) with the crucial difference that bond markets

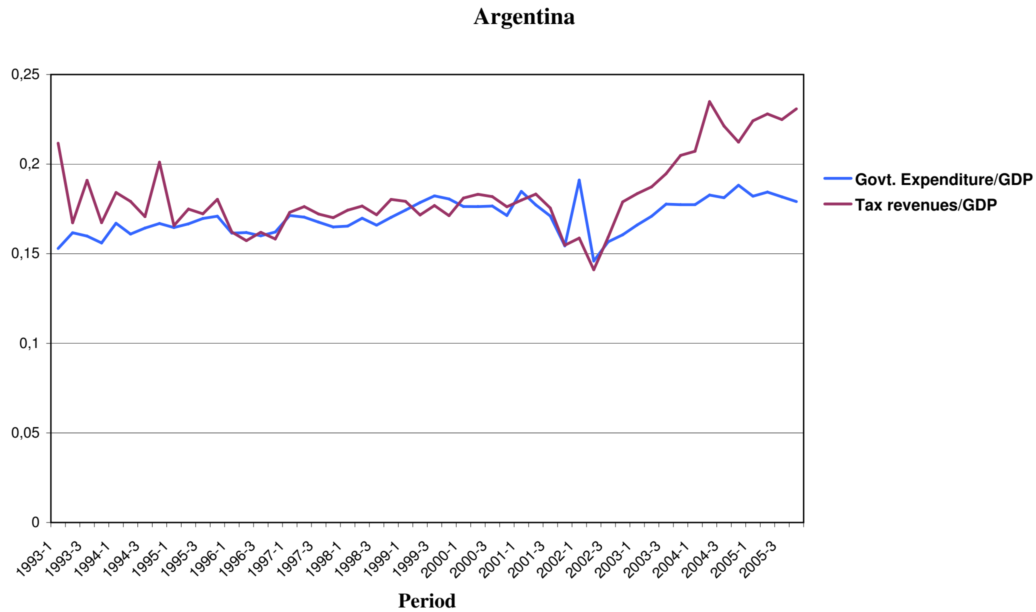

Figure 2: Argentina - Fiscal variables

are incomplete. In particular, they assume that both the government and the households buy and sell only a risk-free one-period bond. The results of the model change substantially when this modification is introduced. The authors find that the contemporaneous effects of a government expenditure shock on deficit and taxes are similar to those described by Lucas and Stokey, but the market incompleteness adds a near unit root component into government debt and taxes, which is in line with the results of Barro. Nevertheless, the impulse-response functions for consumption and leisure in the case of complete and incomplete markets are very similar, a result that assesses the ability of the Ramsey planner to self insure in the case of incomplete markets by using debt as a buffer stock.

Finally, in the recent years there have been some attempts to add default to dynamic macroeconomic models. A number of papers (Arellano (2005), Aguiar and Gopinath (2004), Hamann (2004)) have introduced sovereign default in otherwise standard business cycle models in order to quantitatively match some empirical regularities of small open developing economies. More specifically, they adapt the framework of Eaton and Gersovitz (1981) to a dynamic stochastic general equilibrium model. These models are usually able to explain with relative success the evolution of the interest rate, current account, output, consumption and the real exchange. Nevertheless, since they all consider endowment economies, they fail to capture the effects of default risk over the taxation scheme. Our intended contribution is to extend the analysis to be able to characterize the shape of fiscal policy and the links between the risk of default and taxes in a limited commitment framework.

Cuadra and Sapriza (2007) study the business cycle behavior of taxes in developed and developing countries and conclude that in the first group of countries taxes are counter-cyclical while in the second they are procyclical. In order to explain this observation, they build a model in which a benevolent government has to finance a public good through taxes on domestic consumption or one-period non-contingent debt traded with the rest of the world[4]. International lenders internalize the fact that the government can decide not to repay the external debt and hence charge for it the risk-free interest rate plus a risk premium reflecting the probability of default. With this framework the authors are able to explain three main stylized facts for developing economies, namely, the positive correlation between risk premia and the level of external debt, higher risk premia during recessions and the procyclicality of fiscal policy in developing economies.

Our work is different to Cuadra and Sapriza (2007) in several ways. First, we focus on the different behavior of tax rates in terms of volatility in developed and developing economies and provide an explanation for this observation based on the level of commitment of a country to external lenders. Second, the two frameworks differ substantially. In Cuadra and Sapriza (2007) when a bad productivity shock happens, or when the amount of inherited debt is high, the marginal benefit of paying back the debt is lower than the marginal cost of not paying back, and therefore default is optimal. If the government chooses to default it goes into autarky for some periods, during which labor productivity is a fixed value. In our model, when a bad shock happens the value of the outside option decreases and it is relatively less appealing for the country to go into autarky than when the shock is good. Finally, while in their model there is not an internal market for bonds, in our model agents can trade a complete set of contingent bonds with the government and among themselves. This has important implications for consumption smoothing as it allows the government to distribute the burden of taxation across time. It is important to notice that we also obtain procyclical fiscal policy in our setup[5], so our results are in line with the stylized facts described by Cuadra and Sapriza (2007).

Scholl (2007) develops a framework similar to ours to answer to a different question. She analyzes the problem of a donor that has to decide how much

[4]In this setup, agents derive utility from consumption of the public good.

[5]Fiscal policy is procyclical in the sense that tax rates increase when there is a bad shock (high government expenditure) and viceversa.

aid give to a government that gets utility both from household utility and from unproductive government consumption. The government has an incentive to use aid to increase only its own personal consumption without decreasing the distortive tax income it levies on private agents. In deciding the aid path, the donor has to ensure that the utility the government receives from staying in the contract and fulfilling the conditions the donor imposes on the use of the funds has to be higher than the utility it gets from receiving no aid from that time on. This outside option is endogenous because it depends on the capital stock the government has accumulated over time and that cannot be depleted by the donor because of sovereignty.

3 The model

We consider an economy in which households are risk-averse and receive utility from consumption and leisure. We assume that the country is a production economy where the only production factor is labor. The technology is such that one unit of labor produces one unit of the (only) final good. Every period, each household is endowed with one unit of time that she can devote to leisure or to work.

There is a benevolent government that has to finance an exogenous expenditure stream, which is the only shock of the economy, through distortionary labor income taxes, domestic bonds and transfers from an international institution. The government chooses these policy instruments to maximize the welfare of domestic households[6].

Finally, there is an international institution that provides transfers to the domestic government. The level of transfers is chosen by the government as long as certain conditions are met. In the basic setup we will assume that these conditions are two, namely, that the expected sum of transfers at time zero equals zero[7] and that the government finds it optimal to stay in the contract in all periods and states of nature.

At this point we need to be precise about the way in which we model the commitment of the government. We assume that the government has full commitment with respect to its own citizens but limited commitment with respect to the international institution providing the transfers. More specifically, we assume that at any point in time the government can leave the contract with

[6] We assume that transfers go directly to the government and not to the households. This seems to be a reasonable assumption since in reality governments are the usual recipients of international loans from the IMF and other institutions.

[7] For simplicity we assume that the intertemporal discount factor of the international institution is the same as the one of domestic households.

the international institution if the value of its outside option is higher than the value of staying in the contract. We will carefully define these two values in what follows.

3.1 Households and firms

Firms are competitive and produce an identical final good y_t with technology

$$y_t = 1 - l_t$$

Profit maximization implies that $w_t = 1 \ \forall t$. Given that the technology displays constant returns to scale, profits are zero.

The representative household solves

$$\max_{\{c_t, l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t) \tag{1}$$

subject to the period-by-period budget constraint

$$c_t + \sum_{g_{t+1}|g^t} p_t^b(g_{t+1}) b_t(g_{t+1}) = b_{t-1}(g_t) + (1 - \tau_t)(1 - l_t) \tag{2}$$

where $b_t(g_{t+1})$ is a bond which pays 1 contingent on the realization of the shock tomorrow, and τ_t is the labor tax rate.

The household optimality conditions are:

$$\frac{u_{l,t}}{u_{c^1,t}} = (1 - \tau_t) \tag{3}$$

$$p_t = \beta \frac{u_{c^1,t+1}}{u_{c^1,t}} \pi(s_{t+1}|s^t) \tag{4}$$

where $\pi(s_{t+1}|g^t)$ is the conditional probability of the government expenditure shock tomorrow given its realization today.

3.2 Government

The government in Country 1 has to finance an exogenous stochastic stream of public expenditure $\{g_t\}_{t=0}^{\infty}$. Contrary to what we have assumed for transfers, we consider that the institutional setup is such that the government cannot collect lump-sum taxes but instead has to set a distortionary tax on labor income. We also assume that the government can issue one-period state-contingent bonds that sells to the domestic households. The period by period budget constraint of the government is:

$$g_t + \sum_{g_{t+1}|g^t} p_t^b(g_{t+1}) b_t^G(g_{t+1}) = b_{t-1}^G(g_t) + \tau_t(1-l_t) + T_t \tag{5}$$

where T_t are transfers received from the international institution.

Iterating forward on the government's period by period budget constraint (5) and using equation (4) we obtain the intertemporal budget constraint at time $t = 0$:

$$E_0 \sum_{t=0}^{\infty} \beta^t (u_{c,t}(g_t - \tau_t(1-l_t) - T_t)) = u_{c^1,0}(b_{-1}^G) \tag{6}$$

3.3 The contract

We will first assume that the international institution can commit to stay in the contract in all states of the world; we will relax this assumption in the following sections. The government, on the other hand, can exit the contract at any point in time at the cost of going into autarky forever. We consider that, if the government chose to leave the contract, the domestic financial system would be destroyed, i.e., the government would have to run a balanced budget from that period onwards [8].

Under this setup, the government will remain in the contract only if, given the realization of the shock today, the continuation value of staying in the contract is higher than the continuation value of leaving it, i.e, if the expected sum of future discounted utilities from staying in the contract is higher than the expected sum of future discounted utilities from going into autarky from then on.

Given the government's lack of a commitment technology towards the international institution, transfers have to be chosen such that there are no incentives to default on the contract in any state of nature. This translates into imposing the following participation constraint when choosing the optimal fiscal plan:

$$E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j}) \geq V^a(g_t) \forall t$$

where $V^a(g_t)$ is the value of going into autarky contingent on the realization of the public consumption shock.

We can now state the problem of the government as:

[8] This assumption is made for simplicity, and it can be modified into a situation in which the government cannot borrow from abroad but still can issue domestic debt. We will later discuss this into further detail.

$$\max_{\{c_t,l_t,T_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

subject to

$$c_t + g_t = (1 - l_t) + T_t \tag{7}$$

$$E_0 \sum_{t=0}^{\infty} \beta^t (u_{c,t} c_t - u_{l,t}(1 - l_t)) = u_{c,0}(b_{-1}) \tag{8}$$

$$E_0 \sum_{t=0}^{\infty} \beta^t T_t = 0 \tag{9}$$

$$E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j}) \geq V^a(g_t) \forall t \tag{10}$$

Equation (7) is the resource constraint of the economy, equation (8) is the intertemporal budget constraint of households and can be obtained by substituting (3) into (6), equation (9) states that the expected discounted value of transfers received by the government has to be equal to zero at the beginning of the contract. The international institution is willing to provide insurance as far as this fairness condition is satisfied. By Walras' Law, the Ramsey planner can omit considering the its budget constraint.

Since the participation constraint (10) at time t includes endogenous variables dated in the future, the problem cannot be immediately written in a recursive way. We apply the approach of Marcet and Marimon (1997) and write the Lagrangian as:

$$\begin{aligned} \mathcal{L} = \max_{\{c_t,l_t,T_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t [(1+\gamma_t) u(c_t, l_t) - \psi_t (c_t + g_t - (1 - l_t) - T_t) \\ - \mu_t (V^a(g_t)) - \Delta(u_{c,t} c_t - u_{l,t}(1 - l_t)) - \lambda T_t] + \Delta(u_{c,0}(b_{-1})) \end{aligned} \tag{11}$$

s.t.

$$\gamma_t = \gamma_{t-1} + \mu_t$$

$$\gamma_{-1} = 0$$

where Δ is the Lagrange multiplier associated to the implementability constraint (8), ψ_t is the Lagrange multiplier associated to the resource constraint

(7), λ is the Lagrange multiplier associated to (9) and μ_t is the Lagrange multiplier associated to the participation constraint (10).

The optimality conditions for c_t, l_t, $t \geq 1$ are[9]

$$u_{c,t}(1+\gamma_t) - \lambda - \Delta(u_{cc,t}c_t + u_{c,t} - u_{cl,t}(1-l_t)) = 0 \qquad (12)$$

$$u_{l,t}(1+\gamma_t) - \lambda - \Delta(u_{cl,t}c_t + u_{l,t} - u_{ll,t}(1-l_t)) = 0 \qquad (13)$$

Given that the incentive to default depends on the realization of the shock, it is the case in some states of nature the participation constraint will be binding. The co-state variable γ_{t-1} keeps track of all the past events in which the constraint has been binding. In this sense it collects information about how much consumption and leisure had to adjust in the past in order for the government to find it optimal to stay in the contract. Marcet and Marimon (1997) show that the equilibrium allocations are time-invariant functions in the augmented state space that includes the exogenous state variable g_t and the co-state γ_{t-1}.

Lucas and Stokey (1983) show in a simple Ramsey problem of optimal taxation that the optimal allocation is a function only of the current realized quantity of government purchases g_t and does not depend upon the specific history leading to that outcome. In our case, because of the presence of the costate variable γ_{t-1} this is no longer true. The optimal allocation now depends on γ_{t-1} and, therefore, on the specific history that has determined the value of γ_{t-1}. Moreover, as we can observe from equations (12) and (13), given that the government can use transfers as a buffer stock for the expenditure shock, the allocations do not depend directly on g_t. If the government had a commitment technology towards the international institution (i.e, $\gamma_t = 0 \; \forall t$) both consumption and leisure would be constant in all states of nature and transfers would absorb completely the fiscal shock. We will study in depth the difference between the full commitment and limited commitment scenarios in the next section.

4 An example of labor tax-smoothing

In this section we study one example of labor tax smoothing discussed in Lucas and Stokey (1983). We show how the tax-smoothing result of standard models of optimal fiscal policy is altered by the introduction of limited commitment from the country towards its obligations with the rest of the world. Following Lucas and Stokey (1983) we assume that $b_{-1} = 0$.

[9]We derive the optimality conditions in detail in the appendix.

We assume that government spending is zero in every period except in period T in which there is a large anticipated increase in g. More formally, $g_t = 0$ for $t \neq T$, and $g_T > 0$. We assume that households have a logarithmic utility function of the form

$$u(c_t, l_t) = \alpha log(c_t) + \delta log(l_t) \tag{14}$$

with $\alpha > 0$ and $\delta > 0$. The analysis depends crucially on whether and when the participation constraint is binding. We can distinguish three possible cases that will be analyzed in detail:

- The participation constraint is never binding
- The participation constraint binds at $t = 0$ and $t = T + 1$
- The participation constraint binds at $t = T + 1$

4.1 The participation constraint is never binding

Consider first the case in which

$$\sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j}) > \sum_{j=0}^{\infty} \beta^j u(c_{t+j,A}, l_{t+j,A}) \forall t$$

Then $\mu_t = 0$ $\forall t$ and the analysis is equal to the one in the Lucas and Stokey model. From equation (40) in the appendix it is straightforward to see that $\Delta < 0$. Notice that, given that the allocations depend on the shock only through its effect on γ_t, in this case the optimal allocation $(c_t, l_t) = (\bar{c}, \bar{l})$ is constant for $\forall t$ and, consequently, from condition (3), the tax rate is also constant, $\tau_t = \bar{\tau}$ for $\forall t$. Moreover, it is straightforward to see that the tax rate needs to be positive. In fact, supposing $\tau_t < 0$ implies that by the intratemporal household optimality condition it has to be the case that $u_{l,t} > u_{c,t}$ Using equations (12) and (13) with $\gamma_t = 0 \forall t$ it follows that $-\Delta(u_{l,t} - u_{ll,t}(1 - l_t) < 0$. Given our utility function and the fact that $\Delta < 0$, this is obviously a contradiction.

4.2 The participation constraint binds at $t = 0$ and $t = T + 1$

Consider now the case in which, for $\mu_0 = 0$, the following holds:

$$
\sum_{t=0}^{T}\beta^t u(c_t,l_t)+\sum_{t'=T+1}^{\infty}\beta^{t'}u(c_{t'},l_{t'})
$$
$$
<\sum_{t=0}^{T-1}\beta^t u(c_{t,A},l_{t,A})+\beta^T u(c_{T,A},l_{T,A})+\sum_{t=T+1}^{\infty}\beta^t u(c_{t,A},l_{t,A}) \tag{15}
$$

Then the participation constraint will be binding at exactly periods $t = 0$ and $t = T + 1$, as is stated in the following proposition:

Proposition 1. *If* $\mu_0 > 0$*, then* $\mu_t = 0$ *for* $t = 1, 2, \dots, T$*,* $\mu_{T+1} > 0$ *and* $\mu_{t'} = 0$ *for* $t' = T + 2, T + 3, \dots$

Proof. See Appendix

□

This proposition is important because, as we will show next, the fact that the participation constraint binds at two different moments in time determines that the result of Lucas and Stokey for the tax schedule is altered.

Since $\gamma_t = \gamma_0$ for $t = 1, 2, \dots, T$ and $g_t = 0$ for $t = 1, 2, \dots, T - 1$, the allocations $\{c_t\}_{t=0}^{T}$, $\{l_t\}_{t=0}^{T}$ are constant. This implies that the tax rate $\tau_t = \bar{\tau}$ for $t = 0, 1, ..., T$. It is easy to see that, provided that $\Delta < 0$, the tax rate needs to be positive (see Appendix)[10].

The multiplier $\mu_{T+1} > 0$ and is zero in every subsequent period $T + 2, T + 3, \dots$. That is, μ_{T+1} will adjust to ensure that $u(c, l) = u(c_A, l_A)$ thereafter, so the allocations $\{c_j\}_{j=T+1}^{\infty} = \hat{c}, \{l_j\}_{j=T+1}^{\infty} = \hat{l}$ will be constant. Once more we can conclude that the tax rate has to be positive from $T + 1$ onwards. The following proposition compares the allocations and tax rates before and after the big shock:

Proposition 2. *Define* $t \leq T < t'$ *and assume that the participation constraint binds such that* $\gamma_t = \gamma < \gamma' = \gamma_{t'}$*. Given a logarithmic utility function as* (14) *then* $c_t < c_{t'}$*,* $l_t < l_{t'}$ *and* $\tau_t > \tau_{t'}$*.*

Proof. See Appendix

□

Proposition 2 states that, if the participation constraint binds in period $t = 0$, then the tax-smoothing result obtained in a standard model of the type

[10] The proof that $\Delta < 0$ in this example can be found in the Appendix. Notice again that, because in period T the participation constraint does not bind, the allocations and tax rate in period T are exactly equal to the ones in $t = 1, 2, \dots, T - 1$.

Table 2: Parameter values

Preferences	$\alpha = \beta = 1$	
Intertemporal discount factor	$\beta = 0.98$	
Government expenditure	T = 10	$g_T = 0.2$

of Lucas and Stokey (1983) is no longer valid. While in the latter case the tax rate remains the same before and after time T in which $g_T > 0$, in our setup the tax rate is higher at the beginning ($t \leq T$) and lower after the bad shock has taken place ($t > T$)[11]. It is important to bear in mind that the intertemporal budget constraint of the government has to be satisfied in both setups, which means that under both model specifications total discounted tax revenues have to be equal to g_T, discounted by the proper discount factor.

In order to obtain some intuition for the dynamics of the tax rate, notice that by the optimality condition of households (3) and knowing that $\tau_t > 0 \; \forall t$ then

$$\frac{u_{l,t}}{u_{c,t}} = (1 - \tau_t) \Rightarrow u_{l,t} < u_{c,t}$$

At $T+1$ the utility of staying in the contract has to increase up to the level of autarky. Taking into account the previous inequality, the Ramsey planner increases consumption relatively more than leisure, since the marginal utility of consumption is higher than the marginal utility of leisure. This implies that from the time the participation constraint starts to bind onwards, the ratio of marginal utility of leisure over the marginal utility of consumption increases, which implies a lower the tax rate. The possibility of default generates a change in the optimal allocation that changes permanently the tax rate. From $T+1$ onwards, the level of taxes never goes back to the same level they were before $T+1$.

4.3 The participation constraint binds at $t = T + 1$

Suppose now that

[11] Remember that in our setup, if we are in the first situation 4.1, the result obtained is identical to the one of Lucas and Stokey.

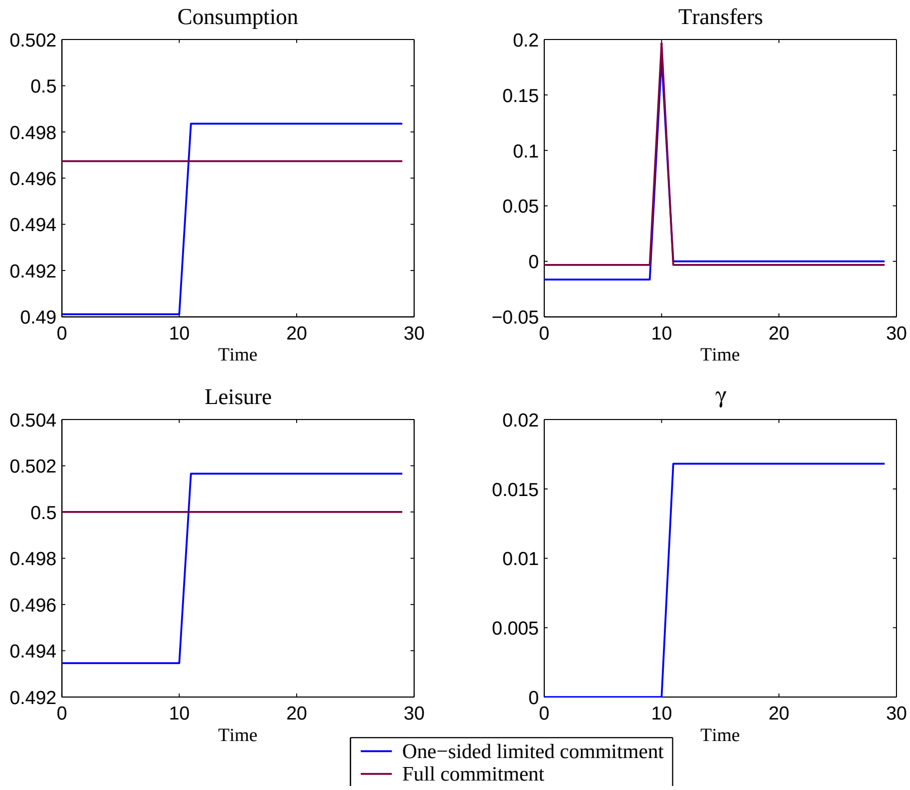

Figure 3: Example: $g_t = 0$ for $t \neq T$, and $g_T > 0$

$$\sum_{t=0}^{T} \beta^t u(c_t, l_t) + \sum_{t'=T+1}^{\infty} \beta^{t'} u(c_{t'}, l_{t'}) > \sum_{t=0}^{T-1} \beta^t u(c_A, l_A) + \beta^T u(c_{A'}, l_{A'}) + \sum_{t=T+1}^{\infty} \beta^t u(c_A, l_A)$$

but

$$\sum_{j=0}^{\infty} \beta^j u(c, l) = \frac{1}{1-\beta} u(c, l) < \frac{1}{1-\beta} u(c_A, l_A) = \sum_{j=0}^{\infty} u(c_A, l_A)$$

Then the participation constraint would be binding *only* in period $t = T+1$. Proposition 2 applies directly and the qualitative implications of this case are identical to the previous case 4.2.

4.4 The example in numbers

Having derived analytically the main results that come out in example 4.2, we now proceed to show numerically how the allocations and tax revenues evolve in the context of the example.

We assume, as before, a logarithmic utility function. In addition, we use the parameter specification shown in Table 2. It is a standard parametrization commonly used in the business cycles literature. We solve the problem in the following way: given arbitrary values for Δ and for λ we solve for the allocation. Then we search for the Δ which guarantees the intertemporal budget constraint of the government to be satisfied. Finally we look for the λ such that equation (9) is satisfied.

Figures 3 and 4 show the evolution of the allocations c_t, l_t, the fiscal policy variable τ_t and transfers T_t, domestic bonds b_t and the costate variable γ_t. The numerical exercise corresponds to the third case analyzed previously, that is, when the PC binds only in $T+1$. For comparison purposes, we plot the same variables in the case in which the government has full commitment towards the international institution[12].

Our analytical results are confirmed by the numerical example: after the negative government shock g_T, the participation constraint binds and this causes the consumption and leisure levels to increase thereafter. The tax rate is higher in the first periods up to the moment of the bad shock. From period $T+1$ onwards, taxes are constant and used to pay interests for the bonds sold to households. Government debt is used to attenuate the effect of the bad shock on tax rates and, consequently, on households. Transfers are negative in the first periods, which can be interpreted as if the government was accumulating assets in order to finance the large negative (and expected) shock in period T. In period T transfers are very high and positive because they play the role of insurance against the bad shock and after period $T+1$ are very close to zero.

Notice the difference between the limited commitment and full commitment cases. Under full commitment, the allocations are constant and transfers absorb completely the shock. The high positive transfer in period T is repaid forever by the government through small negative transfers in every $t \neq T$. Taxes remain constant *even in period* T and play the role of paying for the transfers the government has to give to the international institution, *but they do not react to the shock*. In this case there is perfect risk sharing between the government and the international institution. Under limited commitment taxes have to absorb part of the shock as now there is not perfect risk sharing and, consequently, the government cannot obtain full insurance for its shocks.

[12] In the appendix we describe the problem and optimality conditions when there is full commitment from the government.

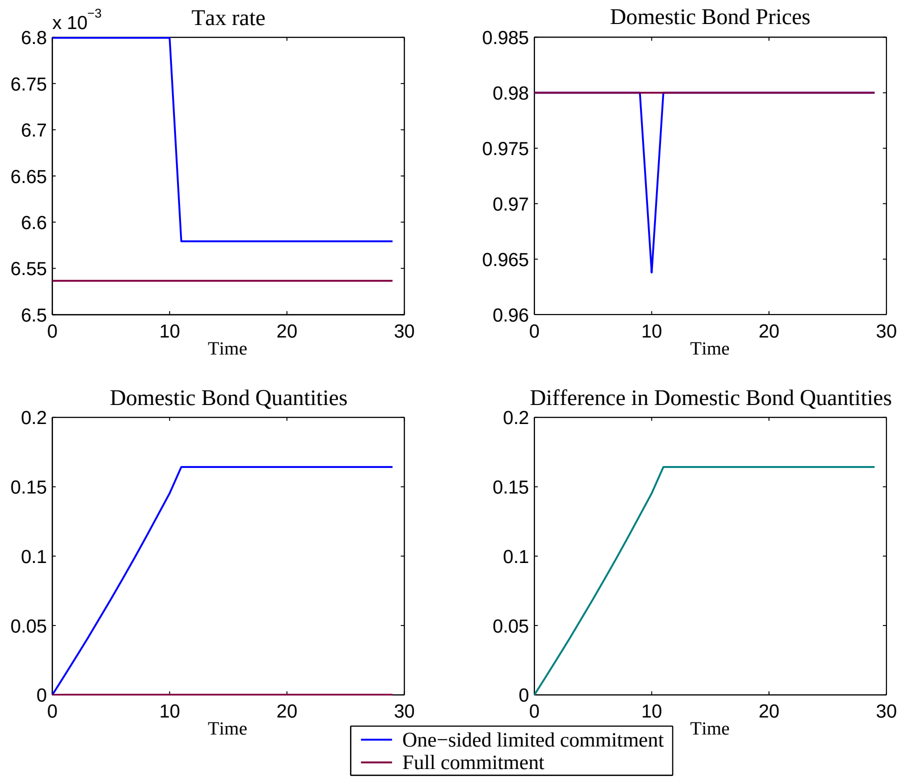

Figure 4: Example: $g_t = 0$ for $t \neq T$, and $g_T > 0$

5 The extended model: Two sided limited commitment

In the previous section we assumed that the only constraint related to the external transfers was that their expected discounted present value is equal to 0. One implication of this assumption is that it is feasible for the government to choose a sequence of very high positive transfers during the first periods and negative ones later on (or viceversa) to pay back the debt accumulated at the beginning. In the real world it his hard to imagine that this can occur, since there are many episodes of sudden stops that lead to think that there are limits to the amount of funds that a country can receive. To add some more realism, in this section we impose an additional constraint: not only the expected present discounted value of transfers to be zero at the beginning of the contract, but also at any point in time and for any state of the world, it has to be the case that there is a maximum amount of resources that the government can receive[13]. In this case the problem of the government is to choose consumption c_t, leisure l_t and transfers T_t to maximize

$$\max_{\{c_t,l_t,T_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

s.t.

$$c_t + g_t = (1 - l_t) + T_t \tag{16}$$

$$E_0 \sum_{t=0}^{\infty} \beta^t [u_{c,t} c_t - u_{l,t}(1 - l_t)] = u_{c,0} b_{-1}^G \tag{17}$$

$$E_0 \sum_{t=0}^{\infty} \beta^t T_t = 0 \tag{18}$$

$$E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j}) \geq V^a(g_t) \tag{19}$$

$$E_t \sum_{j=0}^{\infty} \beta^j T_{t+j} \leq \underline{B} \tag{20}$$

As before, the contract has to satisfy the condition that at the beginning of the contract, the expected present discounted value of the transfers is equal to 0, as stated in the third constraint. Constraint (20) expresses the fact that

[13] It can be showed that this framework is equivalent to consider a contract between two countries, one risk averse and the other risk neutral, both of which can default.

the expected discounted present value of future transfers has to be lower than $\underline{B}$. This is like saying that the government cannot accumulate assets such that future transfers are higher than $\underline{B}$.

The optimization problem is almost exactly equal to the one already discussed in section 3.3. The only difference is that now we have one more equation that includes future decision variables. Once more we apply the methodology of Marcet and Marimon (1997) and define a new co-state variable $\gamma_t^2 = \gamma_{t-1}^2 + \mu_t^2$ where μ_t^2 is the Lagrange multiplier associated with equation (20). The Lagrangean can be written as

$$\begin{aligned}\mathcal{L} = \max_{\{c_t,l_t,T_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t [(1+\gamma_t^1)u(c_t,l_t) - \psi_t(c_t + g_t - (1-l_t) - T_t) \\ - \mu_t^1(V^a(g_t)) - \Delta(u_{c,t}c_t - u_{l,t}(1-l_t)) - T_t(\lambda + \gamma_t^2)] + \Delta(u_{c,0}(b_{-1}))\end{aligned} \tag{21}$$

s.t.

$$\gamma_t^1 = \gamma_{t-1}^1 + \mu_t^1$$

$$\gamma_t^2 = \gamma_{t-1}^2 + \mu_t^2$$

$$\gamma_{-1}^1 = 0$$

$$\gamma_{-1}^2 = 0$$

The optimality conditions for c_t, l_t, $t \geq 1$ in this case are

$$u_{c,t}(1+\gamma_t^1) - (\lambda + \gamma_t^2) - \Delta(u_{cc,t}c_t + u_{c,t} - u_{cl,t}(1-l_t)) = 0 \tag{22}$$

$$u_{l,t}(1+\gamma_t^1) - (\lambda + \gamma_t^2) - \Delta(u_{cl,t}c_t + u_{l,t} - u_{ll,t}(1-l_t)) = 0 \tag{23}$$

where λ and Δ are the Lagrange multipliers associated to equations (18) and (17), and γ_t^1 and γ_t^2 are the cumulative sum of Lagrange multipliers until period t associated to (19) and (20) respectively.

From (22) and assuming a logarithmic specification for the utility function as in equation (14) we can state the following result:

Proposition 3. *Define $t < t'$ and assume that the participation constraint* (19) *binds such that $\gamma_t^1 = \gamma^1 < \gamma^{1'} = \gamma_{t'}^1$. Given a logarithmic utility function as* (14) *then $c_t < c_{t'}$, $l_t < l_{t'}$ and $\tau_t > \tau_{t'}$. If, on the other hand, the participation constraint* (20) *binds such that $\gamma_t^2 = \gamma^2 < \gamma^{2'} = \gamma_{t'}^2$ the allocations and tax rate change and $c_t > c_{t'}$, $l_t > l_{t'}$ and $\tau_t < \tau_{t'}$.*

Table 3: Parameter values

Preferences	$\alpha = \beta = 1$
Intertemporal discount factor	$\beta = 0.98$
Government expenditure process	$g_t = g * +\rho^g g_{t-1} + \epsilon_t$
$g*$	$0.1820 * 0.33$
ρ^g	0.9107
σ_g^2	$0,1320 * 0.0607$
$\underline{B}$	0.031
$b_{-1} = b_{-1}^G$	0

Proof. See Appendix. □

Proposition 3 states that the tax rate varies whenever one of the two participation constraints (19) and (20) binds[14]. Moreover, fiscal policy is procyclical: the tax rate increases in times of a bad government expenditure shock when equation (20) holds with equality. Conversely, it decreases when the realization of the government expenditure shock is low and constraint (19) binds. This is in line with what is described in the literature for developing countries (see Cuadra and Sapriza (2007)).

6 Numerical results

We proceed now to solve the model numerically for a government spending process calibrated to the Argentinean economy. We use quarterly series of current government expenditure net of interest payments plus gross government investment as our measure of government expenditure for the period 1993-I to 2005-IV.

Given that we need to calibrate the process for government expenditure we estimate an AR(1) process in levels for the argentinean data. We find that for the broader measure of real government expenditure, $\hat{\rho} = 0,9107$ for a specification as

$$g_t = \alpha + \rho g_{t-1} + \epsilon_t$$

[14]Note that it cannot be the case that the two participation constraints bind at the same time.

Table 4: One-sided limited commitment - Tax rates and costate variable

τ with PC	τ without PC	μ_t^1	g_t	γ_{t-1}
0.01736	0.0254	0.219	0.04	0
0.01978	0.0254	0.139	0.055	0
0.02114	0.0254	0.101	0.07	0
0.01655	0.0165	0	0.04	0.25
0.01655	0.0165	0	0.055	0.25
0.01655	0.0165	0	0.07	0.25
0.01161	0.0116	0	0.04	0.5
0.01161	0.0116	0	0.055	0.5
0.01161	0.0116	0	0.07	0.5

We also need to obtain a value for the variance of the shock associated to g_t. Given that the variance of g_t and, similarly, of ϵ_t are influenced by the units in which government expenditure is measured, we need to find a statistic that is not influenced by neither the currency in which expenditure is denominated nor the size of the government itself. We therefore use the coefficient of variation (CV), defined as

$$CV = \frac{\text{Std. Dev}}{\text{Mean}}$$

In the data, $CV = 0,1320$. We estimate the mean of g_t as the value of g_t in steady state, given the mean of $\frac{g_t}{GDP_t}$ in the data. This value for is $\overline{\frac{g}{GDP}} = 0,182$. Since our problem does not have a well defined steady state, we consider, as others in the literature, that $1 - l_t = \frac{1}{3}$ in steady state. Then $\overline{\frac{g}{GDP}} = \overline{\frac{g}{1-l}} = 0,182$. Therefore $\overline{g} = 0,33{*}0,182 = 0,0607$. Finally, the variance of $g_t = (0,1320 * 0.0607)^2 = 0,0000641$. We obtain the variance of ϵ_t in the following way:

$$\sigma_\epsilon^2 = \sigma_g^2(1 - \rho^2)$$

Table 4 illustrates how the possibility to default affects the tax rate in the case of one-sided limited commitment as in section 3. The last two columns are the extended vector of state variables, the government expenditure shock, which is the natural state variable, and the co-state variable that measures the binding pattern of the participation constraint, which is the state variable

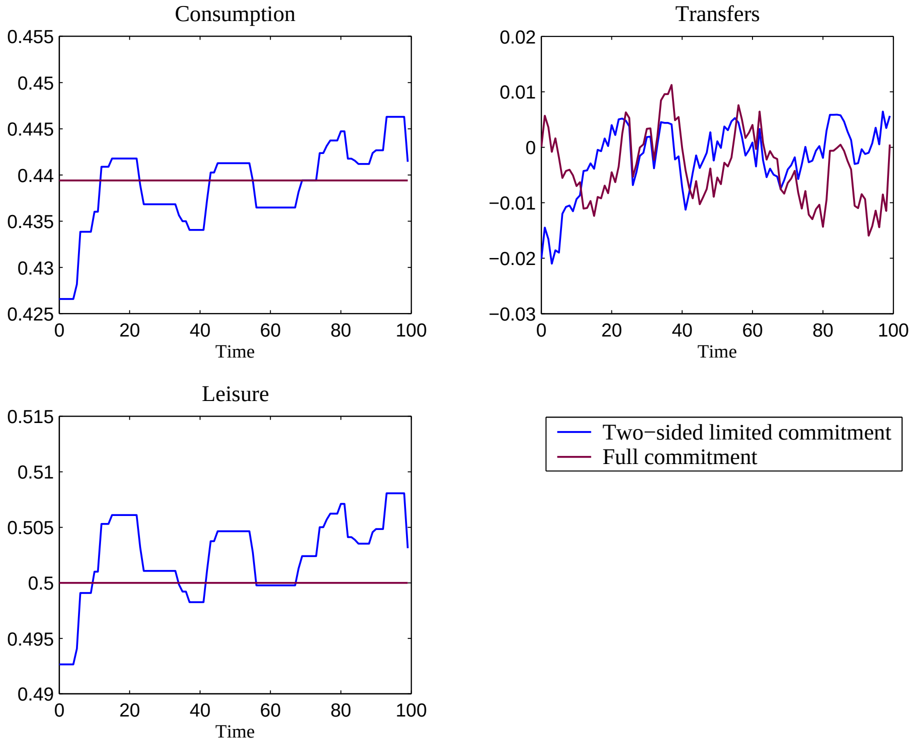

Figure 5: Two-sided limited commitment - Allocations

that makes the problem recursive. The second column shows the tax rate that would prevail in t if the PC for the government would not be binding, while the first column is the actual tax rate taking into account that the country has an incentive to default; when this happens, the PC is binding and in the next period the country will enter the contract with a higher γ_t.

Three observations are needed. First, the lower the government expenditure, the higher is the incentive for the government to default since the autarkic value increases. Similarly, the more binding has been in the past the PC, the lower the incentive to default today because the country has been promised to receive more resources today. Second, there is a negative correlation between the actual tax rate and the incentive to default today: the more the government finds it convenient to default today, the more the tax rate decreases with respect to a situation in which there would be no incentive to default at all. As a corollary of this, the tax rate only shifts downwards at times in which the participation constraint binds ($\mu_t > 0$) and remains constant in any other case. Our third observation is, consequently, that the tax rate presents a downward trend when we consider only limited commitment for the government.

We now proceed to show the results for the two-sided limited commitment framework of section 5 since it is more appropriate to explain the upward and

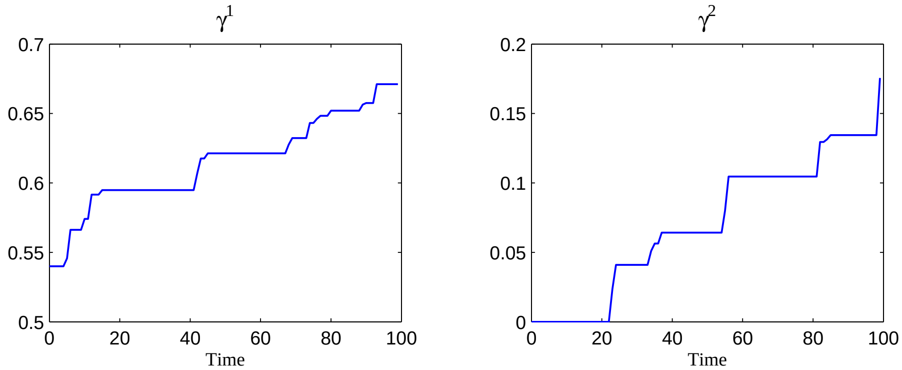

Figure 6: Two-sided limited commitment - Co-state variables

downward movements of the tax rate that are observed in the real world. Table 3 summarizes the parameter values described before. The algorithm used to solve the problem is described in the appendix. Figures 5, 6 and 7 show the allocations, co-state variables and fiscal variables respectively for an arbitrary government expenditure shock, for the case in which both the government and the international institution have limited commitment (blue line). For comparison purposes, we show in the same graphs the case of full commitment (red line).

Figure 5 shows that, as in the example of section 4.4, in the full commitment case consumption and leisure are constant because the government chooses transfers optimally to absorb completely the government expenditure shock. In the two-sided limited commitment case instead, optimal consumption and leisure depend on the outside option the government has and on the amount of external transfers. If the government is not constrained in the amount of transfers it can use, then in good times, when it is relatively more profitable to go into autarky (and therefore equation (19) holds with equality), the government uses transfers from abroad to increase both consumption and leisure. Conversely, when constraint (20) binds, it pays back the transfers received and consumption and leisure decrease.

Once more the possibility of default changes completely the way in which the government uses resources from abroad. While in the full commitment case they are used as perfect insurance from the aggregate shock, in the limited commitment case they serve only as a partial buffer mechanism. When the constraint on the transfers is binding, then the government is forced to increase the tax rate to pay back its debt. The increase in the tax rate discourages both consumption and leisure.

Figure 6 shows the evolution over time of the Lagrange multipliers associated

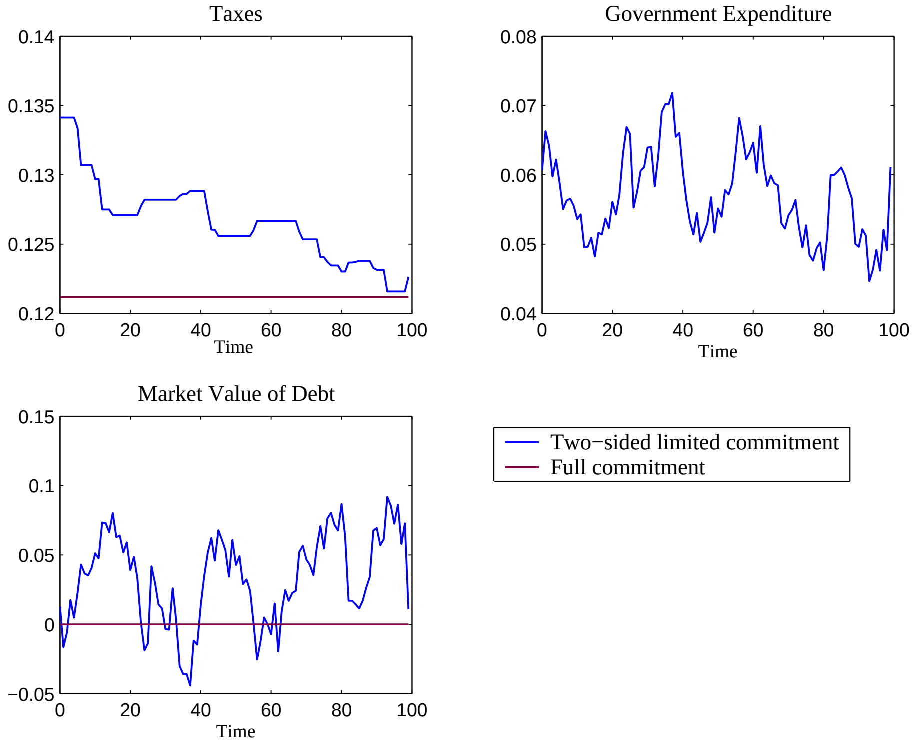

Figure 7: Two-sided limited commitment - Fiscal variables

with equations (19) and (20). Figure 7 shows the fiscal side of the economy. The fluctuations in consumption and leisure are such that the tax rate increases in bad times and decreases during good times, while under full commitment it is completely constant.

ADD COMMENT ON STATISTICS

7 Conclusions and further research

One stylized fact we observe is that the tax volatility varies substantially among countries: from empirical evidence it seems that countries that are subject to default risk have a higher tax rate volatility. The aim of this paper is to analyze the tax smoothing property in a context in which a country could default on its obligations with the rest of the world.

We build a model in which there is a country whose government can finance its exogenous stream of consumption through external transfers or distortionary taxes on labor income of its own citizens. There is an international institution that is risk-neutral and is willing to provide transfers if two conditions are met: at time 0, in expected value transfers have to be zero and the sequence of transfers has to be such that the country must have incentives to remain in the

Table 5: Statistics of the allocation under full Commitment

	Mean	Variance	Autocorr
consumption	0.4394	0	-
leisure	0.5	0	-
g	0.0608	0.0057	0.8038
labor tax rate	0.1212	0	-
transfers	-0.0153	0.0032	-
Debt	0	0	-

Table 6: Statistics of the allocation under two sided limited commitment

	Mean	Variance	Autocorr
consumption	0.4347	0.0039	0.9251
leisure	0.4996	0.0035	0.9250
g	0.0608	0.0057	0.8038
labor tax rate	0.1299	0.0019	0.9295
transfers	-0.0049	0.0065	0.8606
Debt	0.0130	0.0303	0.8068
γ^1	0.5764	0.0159	-
γ^2	0.0164	0.0155	-

contract in all states of nature.

The government behaves as a Ramsey planner and has to choose taxes, debt and transfers from abroad in order to minimize the loss distortions associated with distortionary taxation. In choosing the optimal fiscal plan he takes into account both restrictions imposed by the institution. We compare this framework with a standard model with full commitment on external transfers and we find that tax rate behavior is very different in the two cases. While in an extended Lucas and Stokey (1983) framework with two agents the tax rate is constant independently from the realization of the shock, in our setup the tax rate depends both on the incentive to default of the risk-adverse country and on the presence of credit constraints on external transfers. When the participation constraint of the country binds, the tax rate decreases; conversely, when the participation constraint of the international institution binds, the tax rate has to increase. Consequently, the implied variability of taxes is higher than what was prescribed in the previous literature on fiscal policy, due to the risk that the country may default on the contract signed with the external lender. The movement of the tax rate caused by the limited commitment feature of the model implies that the tax rate is procyclical, a feature described in the data.

The model can be extended in several directions. As a next step we plan to relax the assumption that, once in autarky, the country is forced to run a balanced budget to allow for the possibility of using debt to smooth the government expenditure shock. What we expect to find is that there would more incentives to default, and therefore the volatility of the tax rate would be even higher than what we find.

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A Appendix

A.1 The Ramsey problem with full commitment

In order to have a useful benchmark, we show the optimal allocation if the Ramsey planner could commit to pay back external transfers. In this case, the maximization problem would be

$$\max_{\{c_t, l_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

s.t.

$$b_{-1} u_{c,0} = E_0 \sum_{t=0}^{\infty} \beta^t (u_{c,t} c_t - u_{l,t}(1 - l_t)) \tag{24}$$

$$c_t + g_t = 1 - l_t + T_t \tag{25}$$

$$E_0 \sum_{t=0}^{\infty} \beta^t T_t = 0 \tag{26}$$

The optimality conditions for $t \geq 1$ are:

$$u_{c,t} + \Delta(u_{cc,t} c_t - u_{c \cdot t} + u_{cl,t}(1 - l_t)) = \lambda \tag{27}$$

$$u_{l,t} + \Delta(u_{cl,t} c_t + u_{l,t} - u_{ll,t}(1 - l_t)) = \lambda \tag{28}$$

where λ is the multiplier associated with constraint (26), and Δ is the multiplier associated with the implementability condition (24). From (27) and (28) we observe that both consumption and leisure, and therefore taxes, are constant across time and states. This is due to the fact that the government uses transfers from the international institution to absorb completely the exogenous shock. When g_t is higher than average, transfers are used to finance government expenditure; conversely, when g_t is below average, the government uses the proceeds from taxation to pay back transfers received in the past. In other words, the international institution, which is a risk neutral agent, provides full insurance to the domestic Ramsey planner.

Assume now, as we do in section 4, that $b_{-1} = b_{-1}^G = 0$. Then equation (6) in the text becomes

$$u_{\bar{c}} \left[E_0 \sum_{t=0}^{\infty} \beta^t (g_t - \bar{\tau}(1 - \bar{l}) - T_t) \right] = 0$$

which can be rewritten as

$$E_0 \sum_{t=0}^{\infty} \beta^t g_t - E_0 \sum_{t=0}^{\infty} \beta^t T_t = \frac{\bar{\tau}(1-\bar{l})}{1-\beta}$$

Notice that the second term of the left hand side is equal to zero because of constraint (26). Then

$$E_0 \sum_{t=0}^{\infty} \beta^t g_t = \frac{\bar{\tau}(1-\bar{l})}{1-\beta}$$

From this last expression it is clear that the sign of the tax rate depends on the sign of the expected discounted sum of public expenditures. If, as in section 4, we assume that this sum is positive, then obviously $\bar{\tau} > 0$. Although transfers can completely absorb government expenditure fluctuations, these ultimately have to be financed through taxation or initial government wealth.

A.2 The Ramsey problem with limited commitment

As stated in the text, the problem of the Ramsey planner when it cannot commit to stay in the contract with the international institution is

$$\max_{\{c_t, l_t, T_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

subject to

$$c_t + g_t = (1 - l_t) + T_t$$

$$E_0 \sum_{t=0}^{\infty} \beta^t (u_{c,t} c_t - u_{l,t}(1 - l_t)) = u_{c,0}(b_{-1})$$

$$E_0 \sum_{t=0}^{\infty} \beta^t T_t = 0$$

$$E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j}) \geq V^a(g_t) \forall t$$

Call μ_t the Lagrange multiplier associated to the last constraint. Defining a new co-state variable $\gamma_t = \gamma_{t-1} + \mu_t$, we can write the Lagrangean of the previous problem as

$$\mathcal{L} = \max_{\{c_t, l_t, T_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t [(1 + \gamma_t) u(c_t, l_t) - \psi_t (c_t + g_t - (1 - l_t) - T_t)$$
$$- \mu_t (V^a(g_t)) - \Delta(u_{c,t} c_t - u_{l,t}(1 - l_t)) - \lambda T_t] + \Delta(u_{c,0}(b_{-1}))$$

s.t.

$$\gamma_t = \gamma_{t-1} + \mu_t$$

$$\gamma_{-1} = 0$$

The optimality conditions for $t \geq 1$ are[15]:

- c_t:

$$u_{c,t}(1+\gamma_t) - \psi_t - \Delta(u_{cc,t}c_t + u_{c,t} - u_{cl,t}(1-l_t)) = 0 \tag{29}$$

- l_t:

$$u_{l,t}(1+\gamma_t) - \psi_t - \Delta(u_{cl,t}c_t + u_{l,t} - u_{ll,t}(1-l_t)) = 0 \tag{30}$$

- T_t:

$$\psi_t - \lambda = 0 \tag{31}$$

Other FOCs are:

$$c_t + g_t = (1-l_t) + T_t \tag{32}$$

$$E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j}) \geq V^a(g_t) \forall t \tag{33}$$

$$\mu_t (E_t \sum_{j=0}^{\infty} \beta^j u(c_{t+j}, l_{t+j}) - V^a(g_t)) = 0 \tag{34}$$

$$\gamma_t = \mu_t + \gamma_{t-1} \tag{35}$$

$$\mu_t \geq 0 \tag{36}$$

Multiplying equations (29) and (30) by c_t and $-(1-l_t)$ respectively, and summing:

[15] For $t = 0$ the FOCs of the problem are different, which is the source of time inconsistency of these type of Ramsey problems. Nevertheless, we will assume that the planner can commit to the policies promised at time 0.

$$
\begin{aligned}
&(1+\gamma_t-\Delta)(u_{c,t}c_t-u_{l,t}(1-l_t))-\psi_t(c_t-(1-l_t))\\
&-\Delta\underbrace{(u_{cc,t}c_t^2-2u_{cl,t}(1-l_t)c_t+u_{ll,t}(1-l_t)^2)}_{A_t}=0
\end{aligned}
\tag{37}
$$

Notice that given that the utility function is strictly concave, expression A is strictly negative. By a similar procedure we can write down an equivalent expression at $t=0$:

$$
\begin{aligned}
&(1+\gamma_0-\Delta)(u_{c,0}(c_0-b_{-1})-u_{l,0}(1-l_0))-\psi_0(c_0-(1-l_0)-b_{-1})\\
&-\Delta\underbrace{(u_{cc,0}(c_0-b_{-1})^2-2u_{cl,0}(1-l_0)(c_0-b_{-1})+u_{ll,0}(1-l_0)^2)}_{A_0}=0
\end{aligned}
\tag{38}
$$

Multiplying (37) by $\beta^t\pi(s^t)$[16], summing over t and s^t and adding expression (38):

$$
\begin{aligned}
&E_0\sum_{t=0}^{\infty}\beta^t(1+\gamma_t-\Delta)(u_{c,t}c_t-u_{l,t}(1-l_t))-(1+\gamma_0-\Delta)u_{c,0}b_{-1}\\
&-\Delta Q-E_0\sum_{t=0}^{\infty}\beta^t\psi_t(c_t-(1-l_t))+\psi_0 b_{-1}=0
\end{aligned}
$$

where Q is the expected value of the sum of negative quadratic terms A_t. Using the implementability constraint (8) and the resource constraint (32) we obtain equation (39), which will prove useful in later examples

$$
\begin{aligned}
&E_0\sum_{t=0}^{\infty}\beta^t(\gamma_t-\gamma_0)(u_{c,t}((1-l_t)+T_t-g_t)-u_{l,t}(1-l_t))\\
&-\Delta Q+E_0\sum_{t=0}^{\infty}\beta^t\psi_t(g_t-T_t)+\psi_0 b_{-1}=0
\end{aligned}
\tag{39}
$$

For later purposes, using the resource condition (3) we can reexpress this equation as[17].

$$
E_0\sum_{t=0}^{\infty}\beta^t(\gamma_t-\gamma_0)u_{c,t}(\tau_t(1-l_t)-g_t+T_t)-\Delta Q+E_0\sum_{t=0}^{\infty}\beta^t\lambda(g_t-T_t)+\lambda b_{-1}=0
\tag{40}
$$

[16] $\pi(s^t)$ is the probability of history s^t taking place given that the event s_0 has been observed.

[17] Notice that if the participation constraint was never binding, then $\gamma_t=\gamma_0=0$ and we would recover an identical condition to the one obtained in the Lucas and Stokey (1983) model.

From an expression equivalent to (39), Lucas and Stokey (1983) show in their setup that when the present value of all government expenditures exceeds the value of any initial wealth, the Lagrange multiplier $\Delta < 0$. However, in our case we have a term involving the costate variable γ_t which prevents us from applying the same reasoning. We will, nevertheless, assume that $\Delta < 0$; in the next subsections we will investigate the case in which $\Delta = 0$ and show that $\Delta < 0$ for a specific example. Finally, we will follow Ljungqvist and Sargent (2000) and assume that if the government's initial claims against the private sector were to exceed the present value of future government expenditures, the government would return its excess financial wealth as lump-sum transfers and Δ would remain to be zero.

A.3 Condition under which $\Delta = 0$

Setting $\Delta = 0$, from equations (29) and (30) we know that

$$u_{c,t}(1+\gamma_t) = u_{l,t}(1+\gamma_t) \tag{41}$$

$$u_{c,t} = u_{l,t} \tag{42}$$

This last expression and equation (3) in the text imply that $\tau_t = 0 \forall t$. Inserting these results into equation (40):

$$E_0 \sum_{t=0}^{\infty} \beta^t (\gamma_t - \gamma_0)(u_{c,t}(T_t - g_t) + E_0 \sum_{t=0}^{\infty} \beta^t \lambda (g_t - T_t) + \lambda b_{-1} = 0$$

Using (41)

$$\Rightarrow E_0 \sum_{t=0}^{\infty} \beta^t (-\gamma_t + \gamma_0 + 1 + \gamma_t) u_{c,t}(g_t - T_t) + u_{c,0}(1+\gamma_0) b_{-1} = 0$$

$$\Rightarrow E_0 \sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} (g_t - T_t) = -b_{-1} \tag{43}$$

Remembering equation (4), we can rewrite (43) as

$$\sum_{t=0}^{\infty} \sum_{s^t} p_t^0 g_t = -b_{-1} = b_{-1}^g \tag{44}$$

where p_t^0 is the price of a hypothetical bond issued in period 0 with maturity in period t contingent on the realization of s_t. Equation (44) states that when the government's initial claims b_{-1}^g against the private sector equal the present-value of all future government expenditures net of transfers, the Lagrange multiplier Δ is zero. Since the government does not need to resort to any distortionary taxation, the household's present-value budget does not exert any additional constraining effect on welfare maximization beyond what is already present through the economy's technology.

A.4 Proof of Proposition 1

Consider the case in which expression (15) in the text holds. We will show that, in this case, the participation constraint will be binding in periods $t = 0$ and $t = T + 1$. It is important to bear in mind that the allocations could change in time only due to a different γ_t. Since $\gamma_{t-1} \leq \gamma_t \ \forall t$, then $u(c_{t-1}) \leq u(c_t)$[18].

We begin by showing that the PC holds with strict inequality for $1 \leq t \leq T$. In period $t = 0$ the planner will adjust μ_0 such that (15) holds with equality. Assume now that $\mu_1 > 0$. This implies that, if μ_1 was equal to zero, the PC would be violated, that is,

$$u(c_0, l_0) + \sum_{t=2}^{T-1} \beta^{t-1} u(c_t, l_t) + \beta^{T-1} u(c_T, l_T) + \sum_{t'=T+1}^{\infty} \beta^{t'-1} u(c_{t'}, l_{t'}) < \sum_{t=0}^{T-2} \beta^t u(c_A, l_A) + \beta^{T-1} u(c_{A'}, l_{A'}) + \sum_{t=T}^{\infty} \beta^t u(c_A, l_A) \tag{45}$$

Subtracting (15) that holds with equality at $t = 0$ from (45):

$$\begin{aligned} &\beta[u(c_2, l_2) - u(c_1, l_1)] + \beta^2[u(c_3, l_3) - u(c_2, l_2)] + \ldots + \beta^{T-1}[u(c_T, l_T) - u(c_{T-1}, l_{T-1})] + \\ &\beta^T[u(c_{T+1}, l_{T+1}) - u(c_T, l_T)] + \beta^{T+1}[u(c_{T+2}, l_{T+2}) - u(c_{T+1}, l_{T+1})] + \ldots \\ &< \beta^{T-1}[u(c_{A'}, l_{A'}) - u(c_A, l_A)] + \beta^T[u(c_A, l_A) - u(c_{A'}, l_{A'})] \end{aligned} \tag{46}$$

Reordering terms we arrive at:

[18] this observation hinges on the fact that the utility of households has to be increasing in γ_t.

$$\beta \underbrace{[u(c_2,l_2)-u(c_1,l_1)]}_{\geq 0} + \beta^2 \underbrace{[u(c_3,l_3)-u(c_2,l_2)]}_{\geq 0} + \ldots + \beta^{T-1} \underbrace{[u(c_T,l_T)-u(c_{T-1},l_{T-1})]}_{\geq 0} +$$

$$+ \beta^T \underbrace{[u(c_{T+1},l_{T+1})-u(c_{T-1},l_{T-1})]}_{\geq 0} + \beta^{T+1} \underbrace{[u(c_{T+2},l_{T+2})-u(c_{T+1},l_{T+1})]}_{\geq 0} + \ldots$$

$$< \underbrace{[u(c_{A'},l_{A'})-u(c_A,l_A)]}_{<0} \underbrace{(\beta^{T-1}-\beta^T)}_{>0} \tag{47}$$

Note that expression (47) is a contradiction, since the LHS of the inequality is greater or equal to 0, but the RHS is strictly smaller than 0. We conclude then that it cannot be that $\mu_1 > 0$. Therefore, the PC is not binding in period $t = 1$. The same reasoning can be extended to periods $t = 2, 3, \ldots, T$. Therefore, $\gamma_t = \gamma_0$ for $t = 1, 2, ..., T$ and the allocations $\{c_t\}_{t=0}^T, \{l_t\}_{t=0}^T$ are constant.

Now we proceed to show that the PC is again binding in period $t = T + 1$. First notice that from period $T + 1$ onwards $g_t = 0$, so any necessary adjustment will take place in period $T + 1$ and the allocations will remain constant thereafter. Assume that $\mu_j = 0$ for $j = T + 1, T + 2, ..., \infty$. Then equation (15) becomes:

$$\sum_{t=0}^{\infty} \beta^t u(c,l) = \sum_{t=0}^{T-1} u(c_A,l_A) + \sum_{t=T+1}^{\infty} \beta^t u(c_A,l_A) + \beta^T u(c_{A'},l_{A'}) \tag{48}$$

which can be rewritten as

$$(\frac{1-\beta^T}{1-\beta} + \frac{\beta^{T+1}}{1-\beta})u(c,l) + \beta T u(c,l) = (\frac{1-\beta^T}{1-\beta} + \frac{\beta^{T+1}}{1-\beta})u(c_A,l_A) + \beta^T u(c_{T,A},l_{T,A})$$

We know that $u(c,l) > u(c_{T,A},l_{T,A})$, therefore

$$(\frac{1-\beta^T}{1-\beta} + \frac{\beta^{T+1}}{1-\beta})(u(c_A,l_A) - u(c,l)) > 0$$

Since $0 < \beta < 1$, this latter expression implies that $u(c_A,l_A) > u(c,l)$. Thus, at $T + 1$ and for a given γ_0, the participation constraint will be:

$$\sum_{j=0}^{\infty} \beta^j u(c,l) = \frac{1}{1-\beta}u(c,l) < \frac{1}{1-\beta}u(c_A,l_A) = \sum_{j=0}^{\infty} u(c_A,l_A) \tag{49}$$

which obviously contradicts the fact that $\mu_{T+1} = 0$.

We have shown that the planner will again adjust μ_{T+1} such that the PC (49) holds with equality[19]. Since equation (49) will be identical in every subsequent period, there will be no further need to adjust the multiplier associated to the participation constraint.

A.5 Proof that $\Delta < 0$ in section 4

If the participation constraint (10) never binds, then from equation (40) it is immediate to see that $\Delta < 0$ since $E_0 \sum_{t=0}^{\infty} \beta^t g_t = \beta^T g_T > 0$. Now we prove that $\Delta < 0$ in section 4 in the other two possible cases, namely when equation (10) holds with equality at $t = 0$ and $t = T + 1$ and when it holds with equality only at $t = T + 1$.

We need to work with a specific utility function. In particular, we use the logarithmic specification (14)

$$u(c_t, l_t) = \alpha log(c_t) + \delta log(l_t)$$

Optimality conditions (29) and (30) become:

$$\frac{\alpha}{c_t}(1 + \gamma_t) - \lambda - \Delta \left(-\frac{\alpha}{c_t^2} c_t + \frac{\alpha}{c_t} \right) = 0$$

$$c_t = \frac{\alpha}{\lambda}(1 + \gamma_t) \tag{50}$$

$$\frac{\delta}{l_t}(1 + \gamma_t) - \lambda - \Delta \left(\frac{\delta}{l_t} + \frac{\delta}{l_t^2}(1 - l_t) \right) = 0$$

$$l_t = \frac{\delta(1 + \gamma_t) \pm \sqrt{\delta^2(1 + \gamma_t)^2 - 4\Delta\delta\lambda}}{2\lambda} \tag{51}$$

Notice from equation (51) that if $\Delta < 0$ then we need to take the square root with positive sign in order to have $l_t > 0$. If, on the other hand, $\Delta > 0$, we also take the square root with the positive sign because a higher l_t yields higher utility for a given c_t. Then

$$l_t = \frac{\delta(1 + \gamma_t) + \sqrt{\delta^2(1 + \gamma_t)^2 - 4\Delta\delta\lambda}}{2\lambda} \tag{52}$$

We can compare l_t and $l_{t'}$ for $\gamma_t < \gamma_{t'}$:

[19] Notice that, given that our shock in this example is not a Markov process, neither γ_t nor the allocations c_t and l_t are time-invariant functions of the state variables g_t, γ_{t-1} but, on the contrary, the depend on t.

$$l_t - l_{t'} = \frac{\delta(\gamma_t - \gamma_{t'}) + \sqrt{[\delta(1+\gamma_t)]^2 - 4\lambda\Delta\delta} - \sqrt{[\delta(1+\gamma_{t'})]^2 - 4\lambda\Delta\delta)}}{2\lambda} < 0 \tag{53}$$

We conclude that labor increases with γ_t. Given our assumption about the government expenditure shock and the result from proposition 1[20], equation (40) can be written as

$$\sum_{t=T+1}^{\infty} \beta^t (\gamma_{T+1} - \gamma_0) u_{\bar{c}} (\bar{\tau}(1-\bar{l}) + \bar{T}) - \Delta Q + \beta^T \lambda g_T = 0 \tag{54}$$

where $\bar{c}, \bar{l}, \bar{\tau}$ and $\bar{T}$ are the constant allocations and fiscal variables from $t = T+1$ onwards. In order to determine the sign of the first term of the previous expression, we recall the period by period budget constraint of the government for $t \geq T+1$:

$$(\beta - 1)\bar{b}^G = \bar{\tau}(1-\bar{l}) + \bar{T}$$

The sign of the first term of equation (40) depends on whether government bonds are positive or negative after the big shock has taken place. From equation (8)

$$\sum_{j=0}^{T} \beta^t (u_{\tilde{c}} \tilde{c} - u_{\tilde{l}}(1-\tilde{l})) + \sum_{j=T+1}^{\infty} \beta^t (u_{\bar{c}} \bar{c} - u_{\bar{l}}(1-\bar{l})) = 0$$
$$\Rightarrow \frac{1-\beta^{T+1}}{1-\beta} \left(\alpha - \frac{\delta}{\tilde{l}}(1-\tilde{l}) \right) + \frac{\beta^{T+1}}{1-\beta} \left(\alpha - \frac{\delta}{\bar{l}}(1-\bar{l}) \right) = 0 \tag{55}$$

where $\tilde{c}$ and $\tilde{l}$ are the constant allocations from $t = 0$ to $t = T$. We know that the participation constraint binds in period $T+1$ and consequently $\bar{l} > \tilde{l}$. But this implies that

$$\alpha - \frac{\delta}{\tilde{l}}(1-\tilde{l}) < 0$$
$$\alpha - \frac{\delta}{\bar{l}}(1-\bar{l}) > 0 \tag{56}$$

because the two terms of (55) have to add up to zero. Now we recover b_t for $t \geq T+1$ from the intertemporal budget constraint (8) of households at time $T+1$:

[20] The following argument applies directly to the case in which equation (10) holds with equality in $T+1$ only.

$$
\begin{aligned}
u_{\bar{c}}\bar{b} =& \sum_{j=0}^{\infty} \beta^j (u_{\bar{c}}\bar{c} - u_{\bar{l}}(1-\bar{l})) = \frac{1}{1-\beta}(u_{\bar{c}}\bar{c} - u_{\bar{l}}(1-\bar{l})) \\
=& \frac{1}{1-\beta}\left(\alpha - \frac{\delta}{\bar{l}}(1-\bar{l})\right) > 0
\end{aligned}
$$

If $\bar{b} > 0$, $\bar{b}^G < 0$ so the first term in equation (54) is positive. But then from this equation it is immediate to see that $\Delta < 0$.

A.6 Proof of Proposition 2

From proposition 1 we know that $\gamma_t < \gamma_{t'}$ for $t \leq T < t'$. We have shown in equation (53) that leisure increases with γ_t, so $l_t < l_{t'}$. Now we proceed to show that consumption also increases. Recall equation (50) and compare c_t with $c_{t'}$:

$$
c_t - c_{t'} = \frac{\alpha}{\lambda}(\gamma_t - \gamma_{t'}) < 0
$$

Finally we need to show how the tax rate varies with γ_t. From expression (3) we can write the tax rate τ_t as

$$
\tau_t = \frac{u_{c,t} - u_{l,t}}{u_{c,t}}
$$

It follows that $\tau_t \gtreqless \tau_{t'}$ when $c_t l_{t'} \lesseqgtr c_{t'} l_t$. After some algebra the last condition translates into

$$
\left(\frac{1+\gamma_t}{1+\gamma_{t'}}\right)^2 \lesseqgtr \frac{[\delta(1+\gamma_t)]^2 - 4\lambda\Delta\delta}{[\delta(1+\gamma_{t'})]^2 - 4\lambda\Delta\delta}
$$

$$
(1+\gamma_t)^2 \lesseqgtr (1+\gamma_{t'})^2
$$

Since we know that $\gamma_t < \gamma t'$ we can conclude that $\tau_t > \tau_{t'}$ and therefore the tax rate is inversely related to γ. This completes the proof.

A.7 Proof of Proposition 3

The first part of the proposition is identical to proposition 2 so we omit the proof here. For the second part, we want to show that if $\gamma_{t'}^2 > \gamma_t^2$ then $\tau_{t'} > \tau_t$ for all $t^{'} > t$.

From equation (22) and (23) we get that

$$
c_t = \frac{\alpha(1+\gamma_t^1)}{\lambda + \gamma_t^2} \tag{57}
$$

and

$$l_t = \frac{\delta(1+\gamma_t^1) + \sqrt{[\delta(1+\gamma_t^1)]^2 - 4\delta\Delta(\lambda+\gamma_t^2)}}{2(\lambda+\gamma_t^2)} \tag{58}$$

.

Assume that $\gamma_{t'}^1 = \gamma_t^1$ and that $\gamma_{t'}^2 > \gamma_t^2$. From the intratemporal household optimality condition it follows that $\tau_{t'} > \tau_t$ is satisfied if and only if

$$u_{c,t'} u_{l,t} > u_{c,t} u_{l,t'} \tag{59}$$

Using (57) and (58) we get that (59) implies

$$\begin{aligned} &\frac{\delta(1+\gamma_{t'}^1) + \sqrt{[\delta(1+\gamma_{t'}^1)]^2 - 4\delta\Delta(\lambda+\gamma_{t'}^2)}}{2(\lambda+\gamma_{t'}^2)} \frac{\alpha(1+\gamma_t^1)}{\lambda+\gamma_t^2} > \\ &\frac{\alpha(1+\gamma_{t'}^1)}{\lambda+\gamma_{t'}^2} \frac{\delta(1+\gamma_t^1) + \sqrt{[\delta(1+\gamma_t^1)]^2 - 4\delta\Delta(\lambda+\gamma_t^2)}}{2(\lambda+\gamma_t^2)} \end{aligned} \tag{60}$$

Simplifying and remembering that $\Delta < 0$, the previous inequality is satisfied as far as $\gamma_{t'}^2 > \gamma_t^2$.

A.8 Computational algorithm