S&P 500 Index, an Option-Implied Risk Analysis

Giovanni Barone-Adesi *  Chiara Legnazzi †  Carlo Sala ‡§

December 1, 2016

∗Swiss Finance Institute at Università della Svizzera Italiana (USI), Institute of Finance, Via G. Buffi 13, CH-6900 Lugano, Switzerland.
E-mail: giovanni.baroneadesi@usi.ch

†Swiss Finance Institute at Università della Svizzera Italiana (USI), Institute of Finance, Via G. Buffi 13, CH-6900 Lugano, Switzerland.
E-mail: chiara.legnazzi@usi.ch Telephone: +41 58 666 4496

‡Department of Financial Management and Control, ESADE Business School, Ramon LLull University, Avenida de Torreblanca 59, 08172 Sant Cugat, Barcelona, Spain.
E-mail: carlo.sala@esade.edu Website: http://www.people.usi.ch/salaca

§We are grateful to support of the Swiss Finance Institute (SFI) and the Swiss National Foundation (SNF) grant 153135.
Abstract

The forward-looking nature of option prices provides a natural model-free way to extract several risk measures. The use of option data and the absence of any distributional assumption make the option-implied VaR and CVaR naturally anticipatory and solve the elicitability issue linked to the classical CVaR. Tested on the 2005-2015 S&P 500 index and on its corresponding options, the obtained results appear to be superior compared to the classical risk measures, especially during unstable periods, when a proper risk management is needed the most.

Keywords: Option Prices, VaR and CvaR, Long and Short-term Risk Measures, Elicitability, S&P 500 Index.

JEL classification: G13, G17
Introduction

The main goal of risk management is capital protection. As defined by the seminal book of Jorion (2007) \[31\] “Financial risk management refers to the design and implementation of procedures for identifying, measuring and managing financial risk”. Given the future time orientation and the high degree of randomness implicit in any financial market, the identification process is the most complex task to perform among the three steps.

Indeed, protection has to be provided with respect to possible negative future scenarios which can impact on firms or investors' capital. Despite the wide variety of risk measurement tools presented and used on a daily basis both in academia and in the business community, it is somehow surprising how the big majority of them are just statistically-based methodologies\[1\] which, inferring from past data, provide backward-looking final results with weak economic contents. Hence, by construction, statistically-based historical risk measures have strong time biases. Both from a theoretical and a practical viewpoint, these biases act and have an impact both back and forward in time. For example: how can past data be a reliable statistics for future forecasting? How many past data are needed for the estimation process? Does the choice of the length of the rolling window have an impact in quiet or unstable periods? And, above all, how much a one-day horizon risk estimate is helpful in preventing future losses?. Answering to these questions and proposing an alternative model when the classical ones fail in delivering an adequate answer is the main goal of the paper.

Due to the natural randomness inherent in any financial market, it is well-known that past events are rarely a good proxy of the future. From a risk management viewpoint, it follows naturally that an adequate capital protection cannot be guaranteed by just relying on historical records. This is especially true for crashes. These movements, being infrequent - thus not easily detectable through statistical induction - but very strong in magnitude, are by far the most dangerous ones from a risk management perspective. Due to the difficulty in forecasting these events with classical statistically-based historical techniques, it follows that an adequate protection against future crashes can only be achieved through a model that accounts for forward-looking information.

How strongly and unexpectedly the last financial crises\[2\] have impacted on the financial and economic sectors worldwide best demonstrate how the actual risk measures are still far from providing adequate capital protection. Or, said otherwise, how the classical backward-looking statistical risk measures have failed in their job.
Among several risk measures, the Value at Risk (henceforth: VaR) and Conditional Value at Risk (henceforth: CVaR) have emerged to be the leading ones. Both measures, aggregating the market risk into a single number, assess the maximal loss at a specified probability level over a fixed time horizon\(^3\).

Being coherent\(^4\) and providing the magnitude of losses over the chosen probability level, the CVaR has become the benchmark method for measuring unexpected future losses\(^5\). However, the classical CVaR turns out to be non elicitable (see Weber (2006)\(^38\) and Gneiting (2011)\(^30\)). Aside from technicalities\(^6\), elicibility is connected to a problem of model selection, since the use of historical data requires the choice of a probability model for the profits and losses distribution\(^7\). The lack of elicibility poses some challenges for the backtesting process, thus making the classical CVaR estimates less trustworthy and more difficult to evaluate in relation to other risk measures. As a consequence of the elicibility problem, the BCBS recommended to keep using the CVaR for risk estimation but only the VaR \([13]\) for backtesting, thus leaving the quest for a proper quantitative risk measure still wide open.

To date, there are essentially three approaches to estimate both measures: parametrically and numerically using historical data or running Monte Carlo simulations. Although parametric methods have the advantage of being less computationally intensive and more precise once the parametrization is correctly done, they do not perform well with financial market data\(^8\). The variance-covariance estimates may lead to risk underestimation as soon as the assumed parameters turn out to be unstable. While historical techniques produce unsatisfactory results as soon as data inputs are not i.i.d. (as it is the case for time varying volatilities), Monte Carlo techniques often incur into numerical errors\(^9\). Through the combination of the historical and the Monte Carlo simulations, the Filtered Historical Simulation technique of Barone Adesi et al. (1999)\(^11\) overcomes most of these issues. Despite the advantages, all the proposed numerical methods base their estimations on past data thus being unsatisfactory for forecasting purposes, especially at longer horizons.

Options market data can be of help to overcome the above problems. Barone Adesi (2016)\(^7\) shows how VaR and CVaR are naturally implied by option prices. The forward-looking nature of option prices makes the relative estimates superior in anticipating future crashes without relying on any statistical model in the estimation process. Given the absence of a pre-specified model, it follows that both the possible numerical errors and the elicitation requirements are automatically ruled out. Knowing that the pricing kernel approaches the unity as the time horizon goes to zero, the risk-neutral and the physical measures are almost identical for short and medium time periods, therefore the risk premium calibration is not needed.
Using data on the S&P 500 index and the relative European options written on it, this article presents and compares risk measures derived using the option-implied methodology and the classical statistically-based techniques. The main difference between the proposed technique and the classical ones is the type of information contained into the input data. While the option-implied risk measures forecast future scenarios using forward-looking financial data, the classical ones rely on historical returns, thus having a directional mismatch. By construction, classical risk measures are then strongly myopic. Producing a much larger number of backtesting exceedances, results show how the backward-looking risk measures systematically underestimate future risks at one month horizon. This phenomenon is remarkable during the crisis, where most of classical models failed to provide an adequate capital protection.

Differently than most of short-term historically based models, the proposed measures are not affected by the “risk that the risk will change” (Engle (2009)[25], (2011)[26] and Brownlees et al. (2011)[17]). It has in fact emerged that while risk forecasting at short-time horizons produces reliable estimates, it is intrinsically myopic and does not provide enough time for an adequate capital protection. Economically, estimates based on a too short time horizon are inadequate to properly manage future risks. Indeed, a one-day ahead alert at a given threshold produces a forecast of downside risk that is valid for one day only. While in quiet periods a short notice can still be a reasonable time buffer (for some asset classes and investors’ types) to implement an hedging strategy, risk measures based on short-time horizons are almost useless during high volatility periods.

To date, most of longer-time risk forecasts are based upon rather imprecise model assumptions. The most common technique to extend the one day forecasts to longer time horizons is to multiply them by the square root of the desired number of days. Lacking of any economical foundation, the resulting estimates are biased by large errors whose magnitude is enhanced when the underlying assets are illiquid and when the market is unstable.

Econometrically, a long-run risk measure has to deal with a proper estimation of the volatility. Among a family of different ARCH measures, Brownlees et al (2011)[17] show how, also during the 2007-2009 financial crisis, the best candidate for volatility forecasting is the asymmetric GJR GARCH model of Glosten-Jagannathan-Runkle[29]. While this is true for short time periods, their estimates fail at medium time horizons. The risk estimates derived under the asymmetric GARCH model coupled with empirical innovations are benchmark for our analysis. Backtesting results in section[7] show how historical based autoregressive models are slow in incorporating new market scenarios, thus under/overestimating risk while volatility is rising/falling.

The obtained results show how, inferring risk from option market data, has both theoretical and empirical
advantages. Indeed, not only option-implied risk measures models do not require density assumptions - thus avoiding simulations or unreal parametrizations - but also, given the nature of the data, they are able to anticipate unexpected future scenarios in high volatility periods. Just letting the option market data speak at the utmost and not relying on past information, option-implied risk model are thus of key importance when really needed: in periods of market turmoil.

It is well-known\textsuperscript{10} that the statistical properties of market prices depend on the general market outlook. In periods of market stability the heterogeneity among investors' behaviour increases, whereas during high volatility periods agents' actions become more similar (i.e. bank runs and flight to safety), thus having a strong impact on the tails and the asymmetry of the market prices distribution. As a consequence, the stochastic processes followed by the market prices are endogenous to the agents' actions and the statistical analysis conducted in stable periods can rarely be helpful during high volatility ones, and viceversa. Empirically, piling up past returns to estimate future profit and loss distributions, classical risk measures turn out to be almost unconditional with respect to the actual market conditions and strongly dependent on the time window used in the estimation process. For example: if a bank uses as an input for its model a 5 years time series of historical returns to estimate its future profit and losses distribution, the final estimates strongly depend on the length of the time window considered. More precisely, including or not an “extreme” event, such as the black Monday on October 17, 1987 impacts dramatically on the final results. Policy makers and media have agreed and realized that the historical nature of classical risk models has been one of the most important pitfalls of these estimation techniques during the recent crisis. The 1987-2006 Chairman of the US Federal Reserve (FED) Alan Greenspan blamed that at the heart of the risk underestimation there has been the use of a too short and optimistically biased time windows in estimation\textsuperscript{11}. Very similar concepts\textsuperscript{12} emerge from a collection of anonymous and official interviews collected in Ashby (2010)\textsuperscript{[6]} aiming to understand better what went wrong during the 2007-2009 financial crisis. It follows that, while these pitfalls seem to be well-known both at a theoretical and empirical level, the majority of existing risk models still base their estimates on historical records only.

Given the recent developments, option market data can be the natural solution to solve this issue. Options are non-redundant naturally forward-looking financial assets. While the former is true whenever the volatility is stochastic, the latter is such by construction. It is well-known, at least since Arrow (1964)\textsuperscript{[3]} and Arrow and Debreu (1954)\textsuperscript{[4]}, that the relevant and necessary information about investors present and future beliefs are naturally embedded into these assets. Abusing with words, option prices are daily usable and
easily accessible sufficient statistics. Since the beginning of the century, following the high trading success and availability on the market, the attention of academics and practitioners on the predictive content of different contingent claims has increased substantially. From a more technical viewpoint, the key difference and advantage of option surfaces over historical returns lies in the structure of the data. While the single stock return is a scalar, from which it is impossible to infer a density without making unreal assumptions or piling up many past returns (thus making the final result unconditional\textsuperscript{13}), any option surface is naturally a matrix. Indeed, for each day, the market prices a set of combinations of future states and time horizons. It is therefore not by chance that option markets data have been extensively used for several theoretical and empirical purposes both in academia and industry. Nevertheless, the use of option data in the risk management context is still unexplored. To the best of our knowledge, only two papers (Āit-Sahalia and Lo(2000)\textsuperscript{2} and Mitra (2015)\textsuperscript{34}) relate to the topic in a similar manner. This paper differs from the aforementioned ones both theoretically and empirically. Theoretically, the proposed model follows Barone Adesi (2016)\textsuperscript{7}, the only approach which is entirely density-free and does not need to rely on any change of measure. Empirically, this is the first paper proposing an accurate and comprehensive analysis on how option-implied risk measures perform compared to the classical ones.

Results could be of interest for regulators, single companies and central banks. Regulators, acting externally, do not know the precise composition of a company portfolio in terms of financial assets and relative exposure. Therefore, option market data can be helpful in assessing the degree of riskiness of the company perceived by the market. Individual companies and central banks could use option data to derive risk metrics to be compared with estimation outputs derived by their internal models. In both cases, the option-implied technique allows to derive risk measures without relying on any assumption on the data and model used by the company. Supported by the strong evidence\textsuperscript{14} that a large class of investors holds the market as a form of investment, the underlying of the analysis is the S&P 500 index\textsuperscript{15}. For large companies, which usually have a sufficient amount of traded options, this approach allows to encompass the segmentation of the risk estimation task, which is usually segmented among several divisions.

The article is organized as follows: section \textsuperscript{1} reviews the classical statistically-based historical VaR and CVaR risk measures. Section \textsuperscript{2} introduces the option-implied risk measures. For the empirical part, section \textsuperscript{3} describes the characteristics of the dataset. Section \textsuperscript{4} presents the empirical results on option-implied risk measures using the interpolation technique to derive $\alpha$ and section \textsuperscript{5} repeats the same analysis but relying on the Black and Scholes pricing equation. Section \textsuperscript{6} derives the physical VaR and CVaR estimates under the Gaussian and FHS distribution. Section \textsuperscript{7} presents and compares the backtesting results for all the
aforementioned risk estimates. Section concludes.

1 VaR and CVaR

Defined as the quantile of the projected profit and loss distribution, the VaR \((1-\alpha, T)\) determines the potential maximum loss that may affect a portfolio of losses \((L)\), associated with a generic portfolio value \((S)\), at a given time horizon \((T < \infty)\) and for a given probability \((1 - \alpha)\), where \(\alpha \in [0, 1]\).

Under this framework the VaR is defined as:

\[
L = S_0 - S = \text{VaR}
\]

Setting the reference value \(S_0\) at zero, the reflexivity property for the cumulative loss density function \(F(L)\) follows naturally:

\[
F(L) = F(S_0 - S)
= F(0) - F(-S)
= 1 - F(-S)
\]

where \(F(0) = 1\) as 0 is the upper bound of the loss function.

Replacing \(S\) with \(K\):

\[
\text{VaR} = S_0 - K
\]

or, more generally, in a continuous world:

\[
1 - c = \int_{-\infty}^{K} f(S) dS
= F(K)
= \alpha
\]

where \(F(K) = \Pr(S < K)\), \(\alpha\) represents the desired confidence level and:

\[
c = \int_{K}^{+\infty} f(S) dS
\]
Because of its forward-looking nature, the pricing probability distribution of the future portfolio value, \( f(S) \), is the most challenging element to estimate. Working with real data, the distribution is no longer continuous and:

$$\inf\{k \in \mathbb{R} | F(K) \geq \alpha \}$$

is the discrete version of the same VaR \((1 - \alpha, T)\).

Although empirically robust and easy to implement, the lack of sub-additivity makes the VaR an unsatisfactory and less tractable risk measure in controlling for tail risks. Not being sub-additive, the VaR discourages diversification and is not even a weakly coherent risk measure\(^{16}\). The VaR turns out to be sub-additive only if all marginals of the joint distribution are elliptical. This condition coincides with the assumption underneath the 1952 Markowitz’s variance-minimizing portfolio, as a consequence the VaR calculations are not even required once the variance is known.

Also known in literature as expected shortfall, expected tail loss or average value at risk, the CVaR, which is the conditional expected loss determined in \(^7\), has been proposed by the Basel Committee as an alternative risk measure:

$$\text{CVaR} = \frac{1}{\alpha} \int_{-\infty}^{K} L(S) f(S) dS$$

The main pitfall of the CVaR, which makes it technically inferior with respect to the VaR, is the lack of elicitability (Gneiting (2011)\(^{30}\) and Ziegel (2014)\(^{39}\)). Directly linked to the historical nature of the estimate, the general concept of elicitability allows to perform meaningful comparisons among models based on the backtesting results. While the VaR, being a quantile is naturally elicitable, the CVaR is not\(^{17}\). Not requiring any distributional assumption, options can naturally overcome this issue, thus making the option-implied CVaR an easily back-testable and coherent risk measure.

## 2 Linking VaR and CVaR to the option market

There is a natural link between option prices and VaR and CVaR under the pricing measure (Barone Adesi (2016)\(^7\)). Extracting the risk measures from option surfaces converts the classical statistical measures of downside risk into actual economic risk measures. The VaR is nothing but a quantile, a single numeric value determined at a particular threshold over the cumulative distribution of the profit and losses. Assuming a portfolio with bounded liability, thus changing the lower bound of the integral, the relation between the VaR
and the first derivative of the put price, \( p \), over the strike price is therefore immediate:

\[
x = \frac{dp}{dK} = \frac{d[e^{-rT} \int_0^K (K - S)f(S)dS]}{dK} = e^{-rT} \int_0^K f(S)dS
\]

\[
= e^{-rT} \int_0^K (K - S)f(S)dS
\]

\[
= e^{-rT}F(K)
\]

Adapting the lower bounds of (7):

\[
\int_0^K f(S)dS = F(K)
\]

\[
= \alpha
\]

and setting \( x = \alpha \), and \( r = 0 \), the equality follows.

Dealing with non negligible stochastic interest rates, the compounding for the time to maturity \( T \) is needed to achieve equality between (17) and (18). In a nutshell, the option-implied VaR is nothing but the difference between the initial portfolio value and the strike price of a European put option:

\[
\text{VaR}_\alpha = S - K_\alpha
\]

where the \( K_\alpha \) is the strike price identified by the value of alpha. Being alpha proportional to the probability that the portfolio value is below \( K_\alpha \), the obtained VaR is forward-looking and directly linked to the perceived tomorrow market’s beliefs. The use of put options is related to the analysis of the left tail of the distribution. With no loss of generality and by the same token, similar results hold also for the right tail of the distribution, the one linked to call options.

Following the same logic, the CVaR is related to the price of a put option:

\[
p = e^{-rT} \int_0^K (K - S)f(S)dS
\]

where \( (K - S) = L(S) \). Setting \( S_0 = K + S_0 - K \) it follows from (7), (9), (12) and (20) that the option-implied CVaR is:

\[
\text{CVaR} = \text{VaR} + e^{-rT} \frac{p}{\alpha}
\]
Equation 21 can be better understood recalling that the CVaR measure can also be defined as:

$$\text{CVaR} = \text{VaR} + E[L(S) - \text{VaR}|L(S) > \text{VaR}]$$  (22)

where the left tail of the distribution is represented by the conditional expectation of the portfolio future losses given that the losses are greater than the threshold, $K_\alpha$. The conditional expectation in 21 becomes forward-looking through the use of the market put price discounted by probability of falling into the deeper part of the left tail of the profit and loss distribution.

The structure of the data used in estimation allows to increase the degree of precision of the estimates, thus making our analysis naturally time-dependent:

$$\text{CVaR}_{t,T} = e^{r_{t,T} T} \frac{P_{t,T}}{\alpha_{t,T}} + \text{VaR}_{t,T}$$  (23)

where the subscript $t,T$ refers to the estimation of a today value, $t$, with respect to a given future value, $T$. With no need of any historical data, the option-implied CVaR is fully a forward-looking risk measure.

From an estimation viewpoint, the most challenging part is to properly determine $K_\alpha$. Following Barone-Adesi and Elliot (2007)\[8\], alpha is computed numerically as the average of first order conditions of three contiguous option prices:

$$x = \frac{1}{2} \left[ \frac{dp_{\text{UP}}}{dK_{\text{UP}}} + \frac{dp_{\text{DOWN}}}{dK_{\text{DOWN}}} \right]$$  (24)

$$= \frac{1}{2} \left[ \frac{p_3 - p_2}{K_3 - K_2} + \frac{p_2 - p_1}{K_2 - K_1} \right]$$  (25)

where $p_3 > p_2 > p_1$ and $K_3 > K_2 > K_1$. Going into the limit, as $K_3 \to K_2$ and $K_2 \to K_1$ the above differentials converge to $x$ defined at different strikes. Finally, $\alpha$ is corrected for the risk free rate and equals the compounded value of $x$. The proposed technique, eliminating the first-order error in the Taylor expansion of the derivative and the first-order error due to the implied volatility changing across strike prices, turns out to be numerically efficient and accurate. Moreover, it does not rely on any given pricing model.

To relate it to the probabilistic intuition underpinning the well-known Black and Scholes pricing equation, it is immediate to see that:

$$Pr(S < K) = N(-d_2)$$  (26)
Therefore, as a robustness check, alpha is also derived starting from the option pricing model. Defined as:

$$\alpha = N(-d_2) = 1 - N\left[\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T\right] \sigma\sqrt{T}$$

(27)

The cross-section of \(\alpha\) is computed by first extracting the implied volatility surface and then inverting the function to compute the required set of \(N(-d2)\). Discounting the obtained probabilities completes the estimation.

From a numerical viewpoint, the proposed options-implied VaR and CVaR are obtained from three option prices only, a characteristic which makes them “superior” with respect to other measures relying on the VIX and SKEW Indexes data. The theoretical pricing models based on the two indexes are indeed based upon the existence of an infinite amount of put and call options. In real world modelling a dense support of prices is rarely available, as a result the aforementioned measures imply large truncation errors which become larger in the tails estimation, where the liquidity dries up quicker, especially during crashes.

3 Dataset

The methodology presented in section 2 is tested on the European options (SPX) traded on the S&P 500 with 30-days time to maturity. To avoid issues linked to the use of overlapping data, single VaR and CVaR estimates are derived on a monthly basis. Given the monthly expiration convention, the presented risk measures refer to the third Thursday of each single month. Starting from January 1, 2005 to August 31, 2015 the final sample period consists of 128 monthly observations. To shorten the time to maturity, the same analysis can easily be replicated using weekly options. The choice of the sample period is motivated by the evolution of the option market which, since January 2005, has a sufficiently high volume of traded contracts. The real market situation of the sample period, which covers both the financial and credit crises in full, makes a risk management exercise challenging and interesting. Both the S&P 500 and its corresponding European options are among the most traded financial assets in the US market, hence the degree of liquidity and of data availability are always very high for the last 10 years. Also the wild-card and other delivery features are not an issue for these datasets. To discard possible mispricings, prices violating classical lower arbitrage bounds are excluded from the dataset:

$$p_{t,T} \geq \text{Max}[Ke^{-r(T-t)} - S_t + d_{t,T}, 0]$$

(28)

$$c_{t,T} \geq \text{Max}[S_t - Ke^{-r(T-t)} + d_{t,T}, 0]$$

(29)
where \( d_{r,T} \) is the continuously compounded dividend.

The SPX market contracts traded on the CBOE have strike price intervals of 5 and 25 points for deeply out of the money (OTM henceforth) options. From the time series of daily zero coupon, the curve of the daily risk-free interest rates is obtained by interpolation. For each time-to-maturity (\( \tau \)), the previous (\( \tau_- \)) and next (\( \tau_+ \)) time period zero-coupon whose values straddle the time \( \tau \) are linearly interpolated.

The option price literature treats dividends in several different ways with greater or minor impact on the final results. In our case, the time series of S&P 500 index dividend yield from January, 1, 2005 to August, 31, 2015 has been used to compute an ex-dividend spot index level.

To validate the proposed option-implied risk measures, the results are compared with the classical VaR and CVaR estimates. It is an empirical fact\(^{20}\) that filtered historical simulation method enjoys the advantages of both the historical simulation and the MonteCarlo techniques which in turn are usually superior to most of parametric models. To make our exercise more valuable, we compare the option-implied risk measures with the filtered historical simulation ones. As it appears clear from the comparison with the results using Gaussian innovations, the empirical feature of the filtered historical innovations embeds most of the extreme-value-theory models with no needs of placing any parametric assumption on the model. For the GARCH estimations we therefore extend the time series of other 10 years (figure 10.1) and we use the relative log returns (figure 10.1). The choice of an asymmetric GARCH model (details in 6) is validated by the results of the Lagrange Multiplier ARCH test by Engle (1984)\(^{27}\). All data are from OptionMetrics.

### 4 Option-Implied CVaR and VaR

Based on the methodology of section 4, we derive the option-implied CVaR and VaR at multiple \( \alpha \) levels, namely 5\%, 10\% and 15\%. In this section, the \( \alpha \) parameter is computed numerically as in Barone-Adesi and Elliot (2007)\(^{8}\). It follows that, for each value of \( \alpha \), this methodology requires the availability of three prices at consecutive strikes, thus ensuring a smaller estimation error but also failing to provide estimates during periods of low liquidity. For this reason, section 5 acts both as a robustness check and as an alternative approach for the (rare) periods of low liquidity.

#### 4.1 Put options

Having on the horizontal axis the time period and on the vertical one the dollar value, figures 1, 2 and 3 (Online Appendix) show the monthly VaR (in red) and CVaR (in thick blue) for the three risk levels.
Two vertical lines are placed at the boundaries of the financial crisis period (from June 2007 to December 2009, according to the NBER). During this time interval the CVaR and VaR exhibit several pronounced upward spikes across all $\alpha$ specifications, thus reflecting the negative outlook of the market in that period. In general, the CVaR is always larger than the VaR, where the distance between the two measures, defined as $\Delta_t = \text{CVaR}_t - \text{VaR}_t$, represents the future market beliefs with respect to that possible negative scenario. Mathematically, $\Delta$ equals the compounded put option price discounted for the corresponding $\alpha$ level. This quantity is highly sensitive to market expectations: in periods of turmoil, for example, investors are more risk averse and put options are likely to become highly valuable, thus implying a larger difference between the CVaR and VaR measures. Table 3 (Online Appendix) reports basic statistics information about the monthly $\Delta_{t,\alpha}$. The mean difference between the CVaR and the VaR does not change significantly across the $\alpha$ levels, with only exception at $\alpha = 5\%$ when the put option is almost worthless. As the methodology strongly relies on the liquidity of the option market, table 2 (Online Appendix) reports the number of missing values for each risk level.

### 4.2 Call options

The same analysis is repeated using call options. In this case the CVaR and VaR correspond to the maximal loss associated to a portfolio of short positions. The derivation of risk measures based on right tail provides a benchmark for the estimates obtained using the left one and allows to make inference on the degree of tail asymmetry. Both measures are decreasing in the level of $\alpha$ and display similar patterns across the sample period. However, keeping $\alpha$ fixed, we observe that the predicted values for the CVaR and the VaR based on the right tail are almost always below those obtained from the left one. This result can be motivated either by the larger volume of traded OTM put options, which are therefore more informative and sensitive to a change in the market situation, either by the relative overpricing of put with respect to call options, as the former ones act as an insurance in cases of market crashes. As for the put options, tables 4 and 5 (Online Appendix) report the number of missing data and the summary statistics on $\Delta_{t,\alpha}$, respectively. For $\alpha$ fixed at 5% the number of missing data is higher compared to the left tail, thus confirming the higher degree of liquidity of the put options market compared to the call options one. In addition, for each $\alpha$ level the average $\Delta_{t,\alpha}$s are much lower than the ones based on the left tail, thus meaning that, keeping $\alpha$ fixed, the OTM call options are considerably cheaper than the corresponding OTM put ones.
5 Black and Scholes option-implied CVaR and VaR

This section repeats the same procedure but relying on the Black and Scholes pricing equation (1973) [15]. The key difference between the two approaches is the determination of the cross section of $\alpha_{t,n}$ where $t$ represents the time and $n$ the different strike prices. As a first step the option-implied volatilities are backed out by numerical inversion from the daily S&P 500 option market prices:

$$\sigma^IV_t = f(S_t, K_{t,n}, r_t, q_t, \tau, p_t/c_t)$$  \hspace{1cm} (30)

where $S_t$ represents the underlying market value, $K_{t,n}$ the vector of strike prices, $r_t$ the daily risk-free rate, $q_t$ the daily market dividends, $\tau$ the options time-to-maturity and $p_t/c_t$ the option market prices for European put ($p_t$) and call ($c_t$) options, respectively.

Then, the Black and Scholes probability of the underlying being over a predetermined threshold $K_{t,n}$ is computed as:

$$Pr(S_{t+\tau} < K_{t,n}) = N(-d_2)$$  \hspace{1cm} (31)

$$= 1 - N \left[ \frac{\ln \left( \frac{S_t}{K_{t,n}} \right) + \left( r_t - q_t - \frac{\sigma^2_t}{2} \right) \tau}{\sigma_t \sqrt{T}} \right]$$  \hspace{1cm} (32)

$$= \alpha_{t,n}$$  \hspace{1cm} (33)

Once that the time series of $\alpha_{t,n}$ is available the daily VaR and CVaR are computed as before:

$$\text{CVaR}_{t,T} = e^{\gamma \cdot \tau} \cdot T \cdot \frac{p_t}{\alpha_{t,T}^{\text{BS}}} + \text{VaR}_{t,T}$$  \hspace{1cm} (34)

where:

$$\text{VaR}_{t,T} \begin{cases} S_t - K_i & \text{Put} \\ K_i - S_t & \text{Call} \end{cases}$$  \hspace{1cm} (35)

where subscript $t$ is the value today, at which, conditional on all the information available up to that date, a forecast relative to the future horizon $t + \tau = T$, is made.

The biggest advantage of this methodology, as for most of the Black and Scholes results, is the existence of a closed formula solution for the value of the threshold. It follows that the obtained results are much more stable. Technically, not requiring any contiguous option price for the interpolation, alpha can be estimated as
soon as there is at least one valid market price. This comes at the cost of the Black and Scholes assumptions. Tables 6 and 8 (Online Appendix) report the number of missing values for the entire sample. As expected, the amount of missing data across all $\alpha$ specifications is much smaller if compared with 3 and 4 (Online Appendix). Higher stability conflicts with the accuracy of the final results. Indeed, the Black and Scholes paradigm implies larger estimation errors if compared with the interpolation method of Barone and Elliot (2006)\textsuperscript{8}. A reduced smile effect is remarkable into the deepest area of the tails where the Black and Scholes model provides more biased estimates.

As a first comparison between the two methodologies, table 10 (Online Appendix) shows summary statistics of:

$$\Delta_{t,\alpha} = \alpha_{t,n}^{GBA} - \alpha_{t,n}^{BS}$$ (36)

Aside for rare extreme absolute values (max/min in the table) which are a consequence of almost-zero values of either $\alpha_{t}^{GBA}$ or $\alpha_{t}^{BS}$, the overall behaviour of the two estimates is comparable both for the mean and the median. Interestingly, also the highest and lowest statistical values show a similar behaviour across the sample for both put and call options.

5.1 CVaR and VaR results based on the left and right tail of the distribution

Figures 7, 8 and 9 (Online Appendix) show respectively the 5%, 10% and 15% time series of VaR (red line) and CVaR (thick blue line) computed with 34 and 35 using the daily cross section of put options \textsuperscript{25}. As expectable, while the overall values get lower, the distances between CVaR and VaR get bigger as $\alpha$ grows. Although at different magnitudes, results are comparable also with the CVaR and VaR extracted using the interpolated alphas. As for the estimates obtained under the Barone Adesi and Elliot methodology, the two vertical lines mark the start and the end of the last recession period. For all Black & Scholes option-implied risks measures, this recession period coincides with the highest VaR value. It is interesting to see how option prices are able to forecast the quiet period pre crisis and the subsequent big jumps. The biggest spike within the recession area represents the pre and post Lehman Brother failure (September 15, 2008). The subsequent jumps well represent the downward S&P 500 movements of June 2010 (S&P 500 down of $200 at $1022.58 from $1217.28 of two months before) and the turbulent period from July to November 2011 during which the S&P 500 index dropped from above $1300 losing more than $200. At the rightmost part of the figure, also the last summer crisis is well identified (Summer 2015). At different magnitudes, all turbulent periods are well represented also by the call risk measures. As already anticipated, the missing values represent days
of no trade in the OTM SPX options market.

6 Alternative Risk Estimations

To validate the efficiency of the proposed option-implied risk measures, and to better show the relative advantages and disadvantages of estimating the VaR and CVaR under the risk-neutral measure, this section compares the option-implied results with traditional statistical based historical risk models. The majority of the risk models present in the academic literature and used in industry are not defined in the risk-neutral setting but under the physical one. The main difference among the two measures is determined by the risk premium. Even though academics and practitioners are unceasingly producing an increasing number of theoretical and empirical papers relative to the risk premium, its estimation is still a very open debate (see Damodaran (2016)\[21\]). Investors are not risk neutral and the ex-ante quantification of their risk preferences is not a trivial task to perform. The subjective probabilities assigned by investors to future outcomes are fully non-linear and unobservable quantities to measure. In the physical setting, the absence of a reliable methodology to estimate the subjective risk premium is the main drawback faced by any risk measure. Indeed, by construction, statistical based methodologies are unable of properly capturing a reliable future risk premium.

At this point options can be helpful. Being naturally forward-looking measures, it is not necessary to correct them ex ante to anticipate a future scenario. Moreover, the risk neutral and the physical measure are almost equivalent as the time-to-maturity goes to zero. The absence of any a priori assumption and the convergence of the final estimates for short time horizons are the main advantages of option-implied risk measures. Moreover, while statistical-based historical measures are not helpful in anticipating the market since the estimates are highly dependent on the historical data used for the simulations, option-implied measures naturally embed investors’ expectations and hence are able to anticipate possible market movements.

In a nutshell: options implied risk measures, escaping from the classical real-world paradigm, are quick to estimate, density-free and forward-looking by construction. All these aspects become crucial in periods of high volatility, where statistical-based models may leave risk managers without a reliable answer.

To validate the above points this section presents, as statistical-based alternatives to the option-implied ones, the estimates of the monthly VaR and CVaR using a Gaussian and a filtered historical simulation (FHS) technique (see Barone Adesi et al. (1999)\[10\]). For both methods the market stochastic volatility is estimated by means of an asymmetric Glosten, Jagannathan and Runkle (1993)\[29\] GARCH (1,1) process. The main difference lies in the innovation term, while the Gaussian-based risk measures assume parametric
normal innovations, the FHS ones use empirically estimated innovations to perturb the simulated future returns. The different innovations distribution impacts the volatility estimates and therefore the simulated paths of the underlying.

For each time $t = 1, \ldots, 128$ an asymmetric GJR GARCH model is fitted to reconstruct the index dynamics under the real-world distribution. The model is fitted to historical daily log returns of the S&P 500. Starting from January 20, 2005 we go back up to 10 years of daily observations (January 2, 1996 for a total of 2279 daily observations, see figure 10.1:26:

$$ \log \frac{S_t}{S_{t-1}} = r_t = \mu + \epsilon_t $$  

$$ \epsilon_t = \sqrt{\sigma_t^2 z_t} $$  

$$ z_t | F_{t-1} \sim f(0, 1) $$  

$$ \sigma_t^2 = \omega + \beta \sigma_{t-1}^2 + \alpha \epsilon_{t-1}^2 + \gamma \epsilon_{t-1}^2 1_{t-1} $$  

$$ 1_{t-1} = \begin{cases} 1, & \text{if } \epsilon_{t-1} < 0, \\ 0 & \text{if } \epsilon_{t-1} \geq 0. \end{cases} $$  

where $\mu$ represents the time-invariant drift term$^{27}$ and $\{41\}$ accounts for the leverage effect (Black (1976)$^{14}$).

Table 1 (Online Appendix) shows summary statistics of the estimated physical parameters $\theta_t = f(\omega, \alpha, \beta, \gamma)$ obtained via Gaussian Pseudo Maximum Likelihood (PML) estimation.

The choice of the GJR GARCH (1,1) model is justified by its empirically demonstrated capacity in fitting the S&P 500 data well (see, among the others Ghysels et al.(1996)$^{28}$ and Christoffersen and Jacobs (2004)$^{20}$).

From the set of estimated physical parameters, $i^{28}$ log-returns are simulated:

$$ \hat{S}_{t,t} = S_t \exp \left( \text{Drift} - \frac{\hat{\sigma}_{t,\text{sim}}^2}{2} \right) dt + \hat{\sigma}_{t,\text{sim}}^2 dW_t $$  

where $S_t$ represents the S&P 500 price at day $t$, the drift input accounts for the risk premium, $\hat{\sigma}_{t,\text{sim}}^2$ is the simulated stochastic volatility, $dt$ is fixed at one day and $dW_t$ represents the canonical Brownian motion. As anticipated above, the physical measure strongly depends on the risk premium estimation, which impacts directly the drift. Following Merton (1980)$^{33}$ and accounting for multiple scenarios, the drift is set at a daily fixed percentages, namely 4,6 or 8%.

The parametric approach assumes $dW_t$ to be Gaussian while the non-parametric one draws randomly from the vector of previously computed filtered innovations. From the estimated log-returns the simulated probability
density functions (pdf) and cumulative density functions (cdf) are extracted by means of kernel regressions. Figures 15 and 16 (Online Appendix) plot the physical VaR and CVaR estimates using filtered historical innovations. The same exercise is repeated in the case of Gaussian innovations (see figures 13 and 14 in the Online Appendix). Compared to the option-implied measures, independently on the type of innovations used, the physical VaR lies below the option-implied one except for the crisis periods. Even though the difference among the two approaches reduces under the assumption of FHS innovations, which allow to capture the non-normal features of the data, the simulation based methodology tends to underestimate the potential losses associated to a specific negative scenario. This result, in contrast with the most part of the existing literature, is motivated by the fast mean reversion of the GJR GARCH, which makes measures at one-month horizon almost unconditional.

7 Backtesting

The next step beyond estimation is backtesting. Model validation is usually performed in the form of backtesting or, alternatively, through stress tests and/or independent reviews. Backtesting refers to validating an estimated risk measure based on realized data.

This section reports backtesting results for the VaR and CVaR measures based on the left and right tail of the distribution at several risk levels, namely $\alpha = 5\%, 10\% \text{ and } 15\%$. The backtesting measures for the VaR are the number of times that the options expire in the money and the average loss associated to the contracts expiring in the money at maturity $T$.

Each measure is derived based on the left (eq. 43) and right (eq. 44) tail:

$$\sum_{i=1}^{N} I[S_{t,T} < K_{t,T}]; \quad \sum_{i=1}^{N} K_{t,T} - S_{t,T}. \quad \text{(43)}$$

$$\sum_{i=1}^{N} I[S_{t,T} > K_{t,T}]; \quad \sum_{i=1}^{N} S_{t,T} - K_{t,T}. \quad \text{(44)}$$

where $S_{t,T}$ is the price of the underlying one month after date $t$, $K_{t,T}$ is the strike price of the option contract set at date $t$ expiring at time $T$ and $I[\cdot]$ is the indicator function. As concerns the CVaR, we use as
backtesting measure the average excess loss beyond the VaR, i.e.:

$$\frac{1}{N} \sum_{t=1}^{N} \text{CVaR}_{t,T} - \text{VaR}_{t,T} \mathbb{1}_{\{S_{t,T} < K_t\}}$$ (45)

$$\frac{1}{N} \sum_{t=1}^{N} \text{CVaR}_{t,T} - \text{VaR}_{t,T} \mathbb{1}_{\{K_t > S_{t,T}\}}$$ (46)

The CVaR backtesting measure equals the mean difference between the CVaR and the VaR conditional on the observation of a VaR exceedance.

Tables 1 and 2 show the backtesting results for the left and right tail, respectively. The first column reports the number of exceedances for each $\alpha$ level. Figures 3, 4 and 5 graphically indicate with black stars the dates on which an exceedance occurs in the left tail (Backtesting results based on the right tail are available in the Online Appendix - see figures 17, 18, 19). The second column reports the average loss linked to the option contract and the third one the average excess loss beyond the VaR. By definition the proportion of ITM contracts should be equal to the corresponding alpha level; however, even taking into account the number of missing data, in both tails the proportion of exceedances is always below the corresponding $\alpha$ level. Indeed for $\alpha = 5\%, 10\%, 15\%$, in the left tail the proportion of exceedances equals 1%, 5%, 7% and in the right one 2.6%, 8.6%, 12.5%, respectively. Therefore, the proposed risk measures are conservative forecasts of potential losses.

The same backtesting statistics are computed also on the VaR and CVaR estimates obtained under the B&S methodology. For each risk level, tables 3 and 4 report the backtesting results and figures 6, 7 and 8 show graphically when the exceedances occur for the left tail (Backtesting results based on the right tail are available in the Online Appendix - see figures 20, 21, 22). In the left tail for $\alpha = 5\%$ and 10%, compared to the first methodology the number of exceedances is always lower under the B&S assumption, thus meaning that the actual distribution is more concentrated towards 0 than assumed. In the right tail, there is not a remarkable difference between the two methodologies: only for $\alpha = 15\%$ the B&S approach has a lower forecasting power compared to the first one. Across all possible specifications the proportion of ITM contracts at maturity is always lower than the corresponding alpha levels, thus suggesting that the option based method achieves satisfying results also in the B&S framework.

Due to the different nature of the classical risk measures, another backtesting methodology is applied for the VaR and the CVaR derived under the Gaussian and FHS distributions. Tables 5 and 6 report the backtesting statistics at each risk level for Gaussian and FHS innovations, respectively. The first row reports
the backtesting measure for the VaR computed as the number of times the realized losses exceed the VaR measure. The second row shows the values of the Z-statistic proposed by Acerbi and Szekely (2014) as a backtesting measure for the CVaR.

For a specified $\alpha$ level, the Z-statistic equals:

$$Z(\mathbf{X}) = \sum_{t=1}^{T} \frac{X_{t,I_t}}{\text{CVaR}_{\alpha,t}} N_T + 1;$$

where $X_t$ is the realized loss at time $t$, $I_t$ is the indicator function of a VaR exceedance, $\text{CVaR}_{\alpha,t}$ is the time $t$ estimated CVaR under the assumed probability model and $N_T$ is the total number of exceedances. The proposed statistics does not test directly the CVaR, but it is subordinated to a preliminary (positive) VaR test. Under the null hypothesis that the forecasting model correctly predicts the risk, the statistic $Z$ should be equal to zero. When the alternative is true a positive value of $Z$ indicates a risk underestimation.

The proportion of exceedances is always much higher up to the end of the crisis than the corresponding $\alpha$ level, thus meaning that the density based estimation approach systematically underestimates risk over a monthly horizon. A possible explanation can be the low reactivity of the GJR GARCH volatility estimates at one-month horizon, due to the fast mean-reversion of the model. As a consequence, the model underestimates the volatility in the first part of the sample and overestimates it right after the end of the crisis. This mismatch produces a big (low) number of exceedances in the first (second) part of the sample (see figures for a graphical representation). For each risk level, figures plot the occurrence of exceedances using filtered historical innovations (Backtesting results using Gaussian innovations are available in the Online Appendix – see figures for). Independently on the type of innovations and for each alpha level, more than 97% of the exceedances occur in the first part of the sample period (from January 2005 to January 2009), when the S&P 500 index level experienced a dramatic fall. By construction, the CVaR backtesting statistic is computed only when the realized loss exceeds the VaR. For each type of distribution, the value of $Z$ is positive and increasing in $\alpha$, thus confirming that the physical CVaR always underestimates risk and the magnitude of the estimation error decreases with the confidence level.

8 Conclusion

Statistically based historical models underestimate future portfolio losses. Misestimation especially occurs during periods of turmoil, when the statistical characteristics of the market change in a quicker way and when a robust risk estimation is needed the most. The proposed option-implied risk measures, being naturally forward-looking and density-free, are able to anticipate the probabilities of future market movements,
providing a warning signal within a sufficient advance.

The article presents and compares the 2005-2015 backtesting results of VaR and CVaR estimated under the two different methodologies. The option-implied risk measures show quantitative and qualitative advantages with respect to those obtained with Gaussian and filtered historical simulations under the physical measure. Those results are confirmed under non parametric and parametric (Black and Scholes setting) assumptions and when the estimates are based both on the left and right tail of the distribution. Findings are of particular importance during the 2007-2009 financial crisis, when most of the classical measures failed to provide an adequate capital protection.
Notes

1. In contrast to the proposed option-implied risk measures, this text refers to the risk measures present to date in the market either as “classical risk measures” or, as it will be clarified in a few lines, as “statistically-based historical risk measures”.

2. Without any pretense to be exhaustive, the 1987 black Monday, the 2000 dot-com bubble burst, the 2007 financial crisis and the subsequent credit crises are just some examples of recent big standard deviations movements. At a lower scale, the 1998 Long Term Capital Management (LTCM) 4.68 billion loss and the subsequent effects on commercial and investment banks is a famous example of risk management weakness due to an unexpected random event.

3. The VaR risk measure has been introduced by the risk division of J.P. Morgan. See the J.P. Morgans RiskMetrics (1994) and CreditMetrics (1997) - Technical Documents for further details and Jorion (1997), Dowd (1999), Duffie and Pan (1997), Best (1998) and Penza and Bansal (2000) for an overview.

4. Since Artzner et al. (1999), a risk measure is defined as coherent if it satisfies four mathematical requirements: homogeneity, subadditivity, monotonicity and translation invariance. The subadditivity feature, which in economic terms relates to diversification, undermines the overall coherence of the VaR, thus making the CVaR (which meets all the aforementioned requirement) a preferable measure to the VaR.

5. Through the consultative paper, “Fundamental Review of Trading Book Capital Requirements Form” (May 2012), the Basel Committee on Banking Supervision (BCBS) has formally recommended to use the CVaR instead of the VaR.

6. See Ziegel (2015) for a complete review on coherence and elicitability and Fissler et al. (2016) for recent developments on the VaR-CVaR joint elicitability.

7. In statistics, only functionals for which meaningful point forecasts and forecast performance comparisons are possible are assumed to be elicitable. Quantiles are naturally elicitable functionals.

8. There are no generally accepted parametric forms of asset prices, volatility surfaces, or put/call price functions (Campbell, Lo and MacKinlay (1997), chapter 2).

9. Monte Carlo estimations often rely on factorization techniques that are highly sensitive to the ordering of the data.

10. See i.e.: Danielsson (2002) and references therein.

11. “...risk management models generally covered only the past two decades, a period of euphoria. Had instead the models been fitted more appropriately to historic periods of stress, capital requirements would have been much higher and the financial world would be in far better shape today.”

12. i.e.: an anonymous UK insurer claimed that “It wasn’t until you stepped back and looked at the whole thing you saw that it was a big movement. So you could have information but have controls that looked at too short a time window to be effective, looking at the right thing but not standing back far enough to see the overall trend for what it is.”

13. Even if the single market data would be fully informative of the today scenario, it would not be enough to infer a density. Summing up many past data would make the today information almost meaningless. Equally weighting all data leads to an unconditional final measure.

14. As reported by different surveys, a big fraction of U.S. investors relies on indexing policies for investment.
Analyzing 60 years of market data, Bogle (2005) shows how, since 2000, index funds accounted for roughly one-third of equity fund cash inflows and represented about one-seventh of the total amount of equity fund assets.

Beyond its high degree of liquidity, another advantage of the S&P 500 index is linked to its availability through index futures and exchange traded funds (ETF).

A risk measure is defined as weakly coherent if it lacks of the positive homogeneity (for details see Carr et al. (2001)).

Following Acerbi and Skezely (2015) the CVaR is only backtestable if the profit and loss distribution is known. Computations are similar in the case of a call option, the unique difference is that the sign must be reverted.

Launched on the CBOE in October 2005 weekly options, SPXW, are standard options but with shorter time to maturity. Since October 2015 also the relative VIX weekly is available.

See Acerbi and Skezely (2015).

Results are omitted but available upon request.

On that date indeed ∆5% equal its minimum, i.e. 2.36.

Plots of the two risk measures at each α level are available in the Online Appendix - see figures 4, 5 and 6.

By construction, the daily option surface is spanned for call and put options. For each observation date t, times-to-maturity τ and strike prices, values of K_t,n n = 1, . . . , N < ∞ define the matrix of option prices. For this paper the time-to-maturity is always fix at τ = 30 days, so that t + 30 = T, while the strike price is determined by α.

The corresponding VaR and CVaR estimates based on call options at each α level are also available in the Appendix - see figures 10, 11 and 12.

A common numerical issue here arises: how to determine how much to go back in time is in fact as much important as still unknown for these numerical problems.

Although all estimated factors are time-varying, to assume a constant value for µ in a small period of time has negligible effects on the final estimation. The same would not be true for the variance. In fact, for ∆t small enough:

\[ \sigma_t^2 = c_t^2 \Delta t = O(\Delta t) \]

while for the mean:

\[ \mu^2 = c^2 \Delta t^2 = O(\Delta t^2) \]

Although negligible, values from must be considered as approximations.

Where i represents the number of simulation. The choice of this parameter is subjective and may depend by different options: i.e. the computing power.

Extra details concerning the estimation in Barone Adesi, Engle and Mancini (2008) and Sala (2016).

See eq.(4) pp.3 of Acerbi and Szekely (2014).

To compute a backtesting measure for the CVaR at least one VaR exceedance must occur.

Negative values of Z are not a concern as they indicate a risk overestimation.
9 Bibliography


Backtesting results under the risk neutral measure.

9.1 Interpolated methodology - Backtesting results for $\alpha = 5\%, 10\%, 15\%$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th># exceedances</th>
<th>VaR mean loss</th>
<th>CVaR mean excess loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>15%</td>
<td>8</td>
<td>52.40</td>
<td>80.17</td>
</tr>
<tr>
<td>10%</td>
<td>6</td>
<td>47.61</td>
<td>98.63</td>
</tr>
<tr>
<td>5%</td>
<td>2</td>
<td>46.25</td>
<td>128.38</td>
</tr>
</tbody>
</table>

Table 1: Left tail backtesting results for $\alpha = 5\%, 10\%, 15\%$ - Interpolated methodology.
<table>
<thead>
<tr>
<th>$\alpha$</th>
<th># exceedances</th>
<th>VaR mean loss</th>
<th>CVaR mean excess loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>$15%$</td>
<td>14</td>
<td>12.40</td>
<td>73.21</td>
</tr>
<tr>
<td>$10%$</td>
<td>12</td>
<td>8.31</td>
<td>80.87</td>
</tr>
<tr>
<td>$5%$</td>
<td>3</td>
<td>7.15</td>
<td>74.76</td>
</tr>
</tbody>
</table>

Table 2: Right tail backtesting results for $\alpha = 5\%, 10\%, 15\%$ - Interpolated methodology.
9.2 B&S methodology - Backtesting results for \( \alpha = 5\%, 10\%, 15\% \).

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th># exceedances</th>
<th>VaR mean loss</th>
<th>CVaR mean excess loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>15%</td>
<td>9</td>
<td>43.32</td>
<td>86.77</td>
</tr>
<tr>
<td>10%</td>
<td>3</td>
<td>65.71</td>
<td>118.27</td>
</tr>
<tr>
<td>5%</td>
<td>1</td>
<td>56.18</td>
<td>156.46</td>
</tr>
</tbody>
</table>

Table 3: Left tail backtesting results for \( \alpha = 5\%, 10\%, 15\% \) - B&S methodology.
<table>
<thead>
<tr>
<th>$\alpha$</th>
<th># exceedances</th>
<th>VaR mean loss</th>
<th>CVaR mean excess loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>15%</td>
<td>19</td>
<td>14.68</td>
<td>68.17</td>
</tr>
<tr>
<td>10%</td>
<td>12</td>
<td>10.35</td>
<td>84.38</td>
</tr>
<tr>
<td>5%</td>
<td>2</td>
<td>5.93</td>
<td>70.02</td>
</tr>
</tbody>
</table>

Table 4: Right tail backtesting results for $\alpha = 5\%, 10\%, 15\%$ - B&S methodology.
Backtesting results under the physical measure.

9.3 Backtesting results physical VaR with Gaussian innovations for $\alpha = 5\%, 10\%, 15\%$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th># exceedances</th>
<th>Acerbi Z-stat.</th>
</tr>
</thead>
<tbody>
<tr>
<td>15%</td>
<td>40</td>
<td>3.93</td>
</tr>
<tr>
<td>10%</td>
<td>38</td>
<td>3.79</td>
</tr>
<tr>
<td>5%</td>
<td>35</td>
<td>3.59</td>
</tr>
</tbody>
</table>

Table 5: Backtesting results physical VaR with Gaussian innovations for $\alpha = 5\%, 10\%, 15\%$. 
9.4 Backtesting results physical VaR with FHS innovations for $\alpha = 5\%, 10\%, 15\%$.

<table>
<thead>
<tr>
<th></th>
<th># exceedances</th>
<th>Acerbi Z-stat.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 15%$</td>
<td>40</td>
<td>4.22</td>
</tr>
<tr>
<td>$\alpha = 10%$</td>
<td>38</td>
<td>4.21</td>
</tr>
<tr>
<td>$\alpha = 5%$</td>
<td>35</td>
<td>3.55</td>
</tr>
</tbody>
</table>

Table 6: Backtesting results physical VaR with FHS innovations for $\alpha = 5\%, 10\%, 15\%$. 
10 Figures

10.1 S&P 500 Index daily closing prices and log returns

![Figure 1: 1996-2015 S&P 500 Index daily closing prices](image1)

![Figure 2: 1996-2015 S&P 500 Index daily log returns](image2)
Backtesting results: Graphical representation of the exceedances under the risk neutral measure.

10.2 Backtesting results under the interpolated methodology.

![Graph showing CVaR and VaR exceedances for α=5%](image)

Figure 3: Left tail - CVaR and VaR exceedances for α=5%.
Figure 4: Left tail - CVaR and VaR exceedances for $\alpha=10\%$.

Figure 5: Left tail - CVaR and VaR exceedances for $\alpha=15\%$.
10.3 Backtesting results under the B&S methodology.

Figure 6: Left tail - CVaR and VaR exceedances for $\alpha=5\%$. 
Figure 7: Left tail - CVaR and VaR exceedances for $\alpha=10\%$.

Figure 8: Left tail - CVaR and VaR exceedances for $\alpha=15\%$. 
10.4 Backtesting results under FHS innovations.

Figure 9: VaR exceedances for $\alpha=5\%$.
Figure 10: VaR exceedances for $\alpha=10\%$.

Figure 11: VaR exceedances for $\alpha=15\%$. 